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SUFFICIENT CONDITIONS FOR SPANNING TREES WITH CONSTRAINED LEAF DISTANCE IN A GRAPH

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Abstract

The leaf distance of a tree is the minimum of distances between any two leaves of a tree. It is well known that seeking sufficient conditions for a graph to have some special kinds of spanning trees is an interesting and popular problem. In this paper, we first provide a lower bound on the size of a graph G to guarantee that G has a spanning tree with leaf distance at least 4. Moreover, for any graph G with minimum degree δ, we also deduce a lower bound on the spectral radius (or the signless Laplacian spectral radius) of G to ensure the existence of a spanning tree with leaf distance of at least 4 in G.

Keywords: spanning tree, leaf distance, (signless Laplacian) spectral radius.

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1. Introduction

All graphs considered in this paper are undirected, connected and simple. Let $G = (V, E)$ be a graph with vertex set $V(G)$ of *order* $|V(G)| = n$ and edge set $E(G)$ of *size* $|E(G)| = m$. For $v \in V(G)$, let $d_G(v)$ (or $d(v)$ for short) be the degree of v. The minimum degrees of G is denoted by $\delta(G)$ (or δ for short). For a subset $S \subseteq V(G)$, we use $G[S]$ and $G - S = G[V(G)\backslash S]$ to denote the subgraphs of G induced by S and $V(G)\backslash S$, respectively. For $S \subseteq V(G)$, let $i(G-S)$ denote the numbers of isolated vertices in $G-S$. Let G_1 and G_2 be two vertex–disjoint graphs. We denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2. The join $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges between $V(G_1)$ and $V(G_2)$. Other undefined notations can be found in [4].

A *spanning tree* T of a connected graph G is a subgraph of G that is a tree includes all vertices of G. For an integer $k \geq 2$, a *k-tree* is a tree with the maximum degree at most k, a tree with at most k leaves is called a *k-ended tree.* In particular, a spanning 2-ended tree is also called a Hamilton path. Let G be a connected graph and T be its spanning tree. For a vertex $v \in V(G)$, the leaf degree of v in T is defined to be the number of leaves adjacent to v, and the leaf degree of T is the maximum leaf degree among all vertices of T. The *leaf distance* of T is defined to be the minimum of distances between any two leaves of T.

There are several sufficient conditions (involving toughness, independent number and degree sum *etc.*) for a graph G to have a spanning tree with bounded degree or leaf. Win [11] made a connection between the existence of spanning k-trees in a graph and its toughness. Win [10] provided a Chvátal-Erdős type condition to ensure that a t-connected graph contains a spanning k-ended tree. Kaneko [6] gave the following necessary and sufficient condition to guarantee that a graph G has a spanning tree with bounded leaf degree.

Theorem 1 [6]**.** *Let G be a connected simple graph and $k \geq 1$ be an integer. Then G has a spanning tree with leaf degree at most k if and only if $i(G-S) < (k+1)|S|$ for every $\emptyset \neq S \subseteq V(G)$.*

In the same paper, he proposed the following conjecture on graphs containing a spanning tree with constrained leaf distance.

Conjecture 2 [6]**.** Let $d \geq 3$ be an integer and G be a connected graph of order $n \geq d+1$. If $i(G-S) < \frac{2|S|}{d-2}$ for every $\emptyset \neq S \subseteq V(G)$, then G has a spanning tree with leaf distance at least d.

Notice that a tree with leaf degree 1 has leaf distance at least 3. Therefore Conjecture 2 for $d = 3$ was confirmed by Theorem 1 when $k = 1$. Recently, Kaneko, Kano and Suzuki [7] confirmed Conjecture 2 for $d = 4$ by showing the following result. But for $d \geq 5$ Conjecture 2 is still open.

Theorem 3 [7]**.** *Let G be a connected graph of order $n \geq 5$. If $i(G-S) < |S|$ for every $\emptyset \neq S \subseteq V(G)$, then G has a spanning tree with leaf distance at least 4.*

In view of Theorem 3, we first provide the following sufficient condition to ensure that a graph G has a spanning tree with leaf distance at least 4 in terms of the size of G.

Theorem 4. *Let G be a graph of order $n \geq 5$. We have*

(i) *for $n = 5$ or $n \geq 7$, if $|E(G)| > \binom{n-1}{2} + 1$, then G has a spanning tree with leaf distance at least 4;*

(ii) *for $n = 6$, if $|E(G)| > 12$, then G has a spanning tree with leaf distance at least 4.*

Remark 5. Theorem 4 provides a sufficient condition to guarantee the existence of a spanning tree with leaf distance at least 4 in a graph G. In other words, we do not know whether a graph G has a spanning tree with leaf distance at least 4 if G does not satisfy the edge condition in Theorem 4. In fact, there exist some graphs G with fewer edges (with 4 edges) has a spanning tree with leaf distance at least 4.

Recently, the problem of seeking the conditions for the existence of some special kinds of spanning trees in a graph has also been studied from spectral viewpoints. Recall that the *signless Laplacian* matrix of a graph G is defined as $Q(G) = D(G) + A(G)$, where $A(G)$ and $D(G)$ are, respectively, the adjacency matrix and the diagonal matrix of vertex degrees of G. The largest eigenvalues of $A(G)$ and $Q(G)$ are, respectively, called the *spectral radius* and the *signless Laplacian spectral radius* of G, denoted by $\rho(G)$ and $q(G)$. Fan, Goryainov, Huang and Lin [5] established sufficient conditions for the existence of a spanning k-tree in a connected graph with fixed order in terms of $\rho(G)$ and $q(G)$, respectively. Zheng, Huang and Wang [13] provided a tight spectral radius condition for the existence of a spanning k-ended tree in a t-connected graph, which generalizes a result of Ao, Liu and Yuan [1], etc. We will not list them all here, but will not focus primarily on those related to a spanning tree with constrained leaf distance in the following. We refer the readers to see the nice survey [9] and the book [2] for more details on spanning trees, respectively.

Recently, in view of Theorem 1, Ao, Liu and Yuan [1] established the following tight spectral conditions for the existence of a spanning tree with leaf degree at most k in a connected graph.

Theorem 6 [1]**.** *Let G be a connected graph of order $n \geq 2k + 12$, where $k \geq 1$ is an integer. If $\rho(G) \geq \rho(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))$ (or $q(G) \geq q(K_1 \vee (K_{n-k-2} \cup (k+1)K_1))$), then G has a spanning tree with leaf degree at most k unless $G \cong K_1 \vee (K_{n-k-2} \cup (k+1)K_1)$.*

Remark 7. Note that $K_1 \vee (K_{n-3} \cup 2K_1)$ has a spanning tree with leaf distance at least 3. Then, for $k = 1$ in Theorem 6, it provides a spectral condition for the existence of a spanning tree with leaf distance at least 3 in G. That is for any connected graph G of order $n \geq 14$, if $\rho(G) \geq \rho(K_1 \vee (K_{n-3} \cup 2K_1))$ (or $q(G) \geq q(K_1 \vee (K_{n-3} \cup 2K_1))$), then G has a spanning tree with leaf distance at least 3.

Motivated by [1, 7], it is natural and interesting to ask "*whether or not there are some spectral conditions to ensure the existence of a spanning tree with leaf distance at least* 4 *in a graph* G*?*" Inspired by the ideas from O [8] and using the typical spectral techniques and Theorem 3, we also establish the following spectral conditions (involving $\rho(G)$ or $q(G)$) for the existence of a spanning tree with leaf distance at least 4 in a graph G with minimum degree δ.

Theorem 8. *Let G be a graph of order $n \geq 6\delta + 2$, where δ is the minimum degree of G. If*

$$\rho(G) \geq \rho(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)),$$

then G has a spanning tree with leaf distance at least 4.

Theorem 9. *Let G be a graph of order $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$, where δ is the minimum degree of G. If*

$$q(G) \geq q(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)),$$

then G has a spanning tree with leaf distance at least 4.

The remainder of the paper is organized as follows. In Section 2, we present some preliminary results, which will be used in the subsequent section. In Section 3, we will give the proofs of Theorems 4, 8 and 9, respectively.

2. Tools

In this section, we introduce some preliminary results and lemmas which are useful. For an $n \times n$ real symmetric matrix M, let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ and $\phi_M(x) := det(xI_n - M)$ be the eigenvalues and the characteristic polynomial of M, where I_n is the identity matrix of order n.

Consider an $n \times n$ real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix}$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The *quotient matrix* $\mathcal{B}$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 10 [12]**.** *Let M be a square matrix with an equitable partition π and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M. Furthermore, if M is nonnegative and M_π is irreducible, then the largest eigenvalues of M and M_π are equal.*

Lemma 11 [3]**.** *If H is a spanning subgraph of a graph G, then $\rho(H) \leq \rho(G)$ (or $q(H) \leq q(G)$), with equality if and only if $G \cong H$. Moreover, if H is a proper subgraph of G, then $\rho(H) < \rho(G)$ (or $q(H) < q(G)$).*

3. Proofs of Theorems 4, 8 and 9

We now give the proofs of Theorems 4, 8 and 9, respectively.

***Proof of Theorem* 4.** Suppose to the contrary that G has no spanning tree with leaf distance at least 4. Then Theorem 3 implies that there exists a non-empty subset $S \subseteq V(G)$ satisfying $i(G-S) \geqslant |S|$. We choose such a connected graph G of order n so that its size is as large as possible. According to the choice of G, we see that the induced subgraph $G[S]$ and each connected component of $G-S$ are complete graphs, respectively, and $G \cong G[S] \vee (G-S)$.

First, we claim that there is at most one non-trivial connected component in $G-S$. Otherwise, we can add edges among all nontrivial connected components to get a bigger non-trivial connected component, which is a contradiction to the choice of G. For convenience, let $i(G-S)=i$ and $|S|=s$. We proceed by considering the following two possible cases.

Case 1. $G-S$ has only one non-trivial connected component, say G_1. In this case, let $|V(G_1)| = n_1 \geq 2$. We are to show $i=s$. If $i \geq s+1$, let H_1 be a new graph obtained from G by joining each vertex of G_1 with one vertex in $I(G-S)$ by an edge, where $I(G-S)$ is a set of isolated vertices in $G-S$. Then we have $|E(H_1)| = |E(G)| + n_1 > |E(G)|$ and $i(H_1 - S) \geq s$, a contradiction to the choice of G. Hence $i \leq s$. Recall that $i \geq s$. Therefore, we have $i=s$ and $G = K_s \vee (K_{n-2s} \cup sK_1)$.

Bear in mind that $n = s+s+n_1 \geq 2s+2 \geq 4$ and $|E(G)| = s^2 + \binom{n-s}{2}$. By a directed calculation, we have

$$\begin{aligned}\binom{n-1}{2} + 1 - |E(G)| &= \frac{1}{2}(s-1)(2n-3s-4)\\ &\geq \frac{1}{2}(s-1)(4s+4-3s-4) = \frac{1}{2}s(s-1) \geq 0.\end{aligned}$$

Thus, $|E(G)| \leq \binom{n-1}{2}+1$ for $n \geq 4$, and $\binom{n-1}{2}+1 < 12$ for $n=6$, a contradiction.

Case 2. $G-S$ has no non-trivial connected component. In this case, we are to prove $i \leq s+1$. If $i \geq s+2$, let H_2 be a new graph obtained from G by adding an edge in $I(G-S)$. Clearly, $i(H_2-S) \geq s$, H_2-S has exactly one non-trivial connected component and $|E(G)| < |E(H_2)|$, contradicting to the choice of G. Bear in mind that $i \geq s$, it suffices to consider $i = s$ (i.e., $n = 2s$) and $i = s+1$ (i.e., $n = 2s+1$).

For $i = s$, we have $G \cong K_s \vee sK_1$. That is, $n = 2s$ and $|E(G)| = s^2 + \binom{s}{2}$. By a directed calculation, we have

$$\binom{n-1}{2} + 1 - |E(G)| = \binom{2s-1}{2} + 1 - s^2 - \binom{s}{2} = \frac{1}{2}(s-1)(s-4).$$

Thus, $|E(G)| \leq \binom{n-1}{2} + 1$ for $s = 1$ or $s \geq 4$, which is a contradiction to $n \geq 8$. For $s \in \{2,3\}$ (or $n \in \{4,6\}$), we have

$$|E(G)| = s^2 + \binom{s}{2} = \frac{3s^2 - s}{2} = \begin{cases} 5 & n = 4, \\ 12 & n = 6, \end{cases}$$

a contradiction.

For $i = s+1$, we have $G \cong K_s \vee (s+1)K_1$. Therefore, $n = 2s+1$ and $|E(G)| = s(s+1) + \binom{s}{2}$. By a directed calculation, we have

$$\binom{n-1}{2} + 1 - |E(G)| = \binom{2s}{2} + 1 - s(s+1) - \binom{s}{2} = \frac{1}{2}(s-1)(s-2).$$

Thus, $|E(G)| \leq \binom{n-1}{2} + 1$ for $s = 1$ or $s \geq 2$, which is a contradiction to $n \geq 5$.

In view of Cases 1 and 2, the proof of Theorem 4 is complete. ■

***Proof of Theorem* 8.** Let G be a graph satisfying the conditions in Theorem 8, and let $H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. It is easy to check that $K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$ has a spanning tree with leaf distance at least 4. From now on, we assume that $G \neq H$. We shall prove the contrapositive of the theorem. That means, we assume that G has no spanning tree with leaf distance at least 4, and show that $\rho(G) < \rho(H)$. First, Theorem 3 implies that there exists a nonempty subset $S \subseteq V(G)$ such that $i(G-S) \geq |S|$. Let $|S| = s$. Then $s \geq \delta$ and G is a spanning subgraph of $G_1 = K_s \vee (K_{n-2s} \cup sK_1)$. Hence Lemma 11 implies that

$$\rho(G) \leq \rho(G_1), \tag{1}$$

with equality if and only if $G \cong G_1$. We now consider the following two cases.

Case 1. $s = \delta$. Then $G_1 \cong H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Combining this with (1), we conclude that

$$\rho(G) \leq \rho(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)),$$

where the equality holds if and only if $G \cong K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Since $G \neq H$, $\rho(G) < \rho(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)) = \rho(H)$, as desired.

Case 2. $s \geq \delta + 1$. The vertex set of G_1 can be partitioned as $V(G_1) = V(K_s) \cup V(K_{n-2s}) \cup V(sK_1)$, where $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V(K_{n-2s}) = \{v_1, v_2, \ldots, v_{n-2s}\}$ and $V(sK_s) = \{w_1, w_2, \ldots, w_s\}$. Let

$$G_2 = G_1 + \sum_{i=1}^{n-2s} \sum_{j=\delta+1}^{s} v_i w_j + \sum_{i=\delta+1}^{s-1} \sum_{j=i+1}^{s} w_i w_j - \sum_{i=\delta+1}^{s} \sum_{j=1}^{\delta} u_i w_j.$$

Clearly, $G_2 \cong H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Let $\rho = \rho(G_1)$ and $\rho^* = \rho(G_2)$. Let $\boldsymbol{x}$ be the Perron vector of $A(G_1)$ with respect to ρ. By symmetry, $\boldsymbol{x}$ takes the same value (say x_1, x_2 and x_3) on the vertices of $V(K_s)$, $V(K_{n-2s})$ and $V(sK_1)$, respectively. Then, by $A(G_1)\boldsymbol{x} = \rho\boldsymbol{x}$, we have $\rho x_3 = s x_1$. Since $\rho > 0$, we have

$$x_3 = \frac{s x_1}{\rho}. \tag{2}$$

Similarly, let $\boldsymbol{y}$ be the Perron vector of $A(G_2)$ with respect to ρ^*. By symmetry, $\boldsymbol{y}$ takes the same value (say y_1, y_2 and y_3) on the vertices of $V(K_\delta)$, $V(K_{n-2\delta})$ and $V(\delta K_1)$, respectively. Then, by $A(G_2)\boldsymbol{y} = \rho^*\boldsymbol{y}$, we have

$$\rho^* y_2 = \delta y_1 + (n - 2\delta - 1) y_2, \tag{3}$$

$$\rho^* y_3 = \delta y_1. \tag{4}$$

Note that Lemma 11 implies that $\rho^* > \rho(K_{n-2\delta}) = n - 2\delta - 1$ since $K_{n-2\delta}$ is a proper subgraph of G_2. Putting (4) into (3) with taking into account $\rho^* > n - 2\delta - 1$, we then have

$$y_2 = \frac{\rho^* y_3}{\rho^* - (n - 2\delta - 1)}. \tag{5}$$

Recall that $n \geq 2s$. Then $\delta + 1 \leq s \leq \frac{n}{2}$. Since G_1 is not a regular graph, it follows that $\rho < n - 1$.

Now, suppose to the contrary that $\rho \geq \rho^*$. Then, by (2) and (5), we have

$$\begin{aligned}
&\boldsymbol{y}^T(\rho^* - \rho)\boldsymbol{x} \\
&= \boldsymbol{y}^T(A(G_2) - A(G_1))\boldsymbol{x} = \sum_{i=1}^{n-2s} \sum_{j=\delta+1}^{s} (x_{v_i} y_{w_j} + x_{w_j} y_{v_i}) \\
&+ \sum_{i=\delta+1}^{s-1} \sum_{j=i+1}^{s} (x_{w_i} y_{w_j} + x_{w_j} y_{w_i}) - \sum_{i=\delta+1}^{s} \sum_{j=1}^{\delta} (x_{u_i} y_{w_j} + x_{w_j} y_{u_i})
\end{aligned}$$

$$
\begin{aligned}
&= (s-\delta)\left[(n-2s)(x_2y_2+x_3y_2)+(s-\delta-1)x_3y_2-\delta(x_1y_3+x_3y_2)\right]\\
&= (s-\delta)\left[(n-s-2\delta-1)x_3y_2+(n-2s)x_2y_2-\delta x_1y_3\right]\\
&\geq (s-\delta)\left[(n-s-2\delta-1)x_3y_2-\delta x_1y_3\right] \ \text{ (as } n\geq 2s)\\
&\geq (s-\delta)x_1y_3\left((n-s-2\delta-1)\cdot\frac{s}{\rho}\cdot\frac{\rho^*}{\rho^*-(n-2\delta-1)}-\delta\right)\\
&= \frac{(s-\delta)x_1y_3}{\rho(\rho^*-(n-2\delta-1))}\left(s(n-s-2\delta-1)\rho^*-\delta\rho(\rho^*-(n-2\delta-1))\right)\\
&= \frac{(s-\delta)\rho^*x_1y_3}{\rho(\rho^*-(n-2\delta-1))}\left(s(n-s-2\delta-1)+\delta(n-2\delta-1-\rho)\right) \ \text{ (as } \rho^*\leq\rho)\\
&> \frac{(s-\delta)\rho^*x_1y_3}{\rho(\rho^*-(n-2\delta-1))}\left(s(n-s-2\delta-1)-2\delta^2\right) \ \text{ (as } \rho<n-1).
\end{aligned}
$$

Let $\varphi(n,s)=s(n-s-2\delta-1)-2\delta^2=(n-2\delta-1)s-s^2-2\delta^2$. We assert that $\varphi(n,s)>0$ for $\delta+1\leq s\leq\frac{n}{2}$. Note that $\varphi(n,s)$ is a convex function on s. So it suffices to prove that $\varphi(n,\delta+1)>0$ and $\varphi\left(n,\frac{n}{2}\right)>0$, respectively. Indeed

$$
\begin{aligned}
\varphi(n,\delta+1)&=(n-2\delta-1)(\delta+1)-(\delta+1)^2-2\delta^2\\
&\geq(6\delta+2-2\delta-1)(\delta+1)-(\delta+1)^2-2\delta^2=(\delta+3)\delta>0
\end{aligned}
$$

and

$$
\varphi\left(n,\frac{n}{2}\right)=(n-2\delta-1)\cdot\frac{n}{2}-\left(\frac{n}{2}\right)^2-2\delta^2=\frac{1}{4}n^2-\left(\delta+\frac{1}{2}\right)n-2\delta^2.
$$

Note that $\varphi\left(n,\frac{n}{2}\right)$ is a monotonically increasing function with respect to $n\geq 6\delta+2$ since $2\delta+1<6\delta+2$. Then

$$
\begin{aligned}
\varphi\left(n,\frac{n}{2}\right)&=\frac{1}{4}n^2-\left(\delta+\frac{1}{2}\right)n-2\delta^2\\
&=\frac{1}{4}(6\delta+2)^2-\left(\delta+\frac{1}{2}\right)(6\delta+2)-2\delta^2=(\delta+1)\delta>0.
\end{aligned}
$$

Then $\varphi(n,s)>0$ for $\delta+1\leq s\leq\frac{n}{2}$. Thus

$$
\boldsymbol{y}^T(\rho^*-\rho)\boldsymbol{x}=\boldsymbol{y}^T(A(G_2)-A(G_1))\boldsymbol{x}=\frac{(s-\delta)\rho^*x_1y_3}{\rho(\rho^*-(n-2\delta-1))}\varphi(n,s)>0.
$$

It follows that $\rho^*>\rho$ as $\boldsymbol{y}^T\boldsymbol{x}>0$ and $\rho^*>n-2\delta-1$, which contradicts the assumption that $\rho^*\leq\rho$. Therefore $\rho^*>\rho$. This together with (1) implies that $\rho(G)\leq\rho(G_1)<\rho^*=\rho(H)$, as desired.

The proof of Theorem 8 is completed. ∎

***Proof of Theorem* 9.** Let G be a graph satisfying the conditions in Theorem 9, and let $H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. It is easy to check that $K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$ has a spanning tree with leaf distance at least 4. From now on, we assume that $G \neq H$. We shall prove the contrapositive of the theorem. That means, we assume that G has no spanning tree with leaf distance at least 4, and show that $q(G) < q(H)$. First, Theorem 3 implies that there exists a nonempty subset $S \subseteq V(G)$ such that $i(G - S) \geq |S|$. Let $|S| = s$. Then $s \geq \delta$ and G is a spanning subgraph of $G_1 = K_s \vee (K_{n-2s} \cup sK_1)$. Hence Lemma 11 implies that

$$q(G) \leq q(G_1), \tag{6}$$

with equality if and only if $G \cong G_1$. We now consider the following two cases.

Case 1. $s = \delta$. Then $G_1 \cong H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Combining this with (6), we conclude that

$$q(G) \leq q(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)),$$

where the equality holds if and only if $G \cong K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Since $G \neq H$, $q(G) < q(K_\delta \vee (K_{n-2\delta} \cup \delta K_1)) = q(H)$, as desired.

Case 2. $s \geq \delta + 1$. The vertex set of G_1 can be partitioned as $V(G_1) = V(K_s) \cup V(K_{n-2s}) \cup V(sK_1)$, where $V(K_s) = \{u_1, u_2, \ldots, u_s\}$, $V(K_{n-2s}) = \{v_1, v_2, \ldots, v_{n-2s}\}$ and $V(sK_s) = \{w_1, w_2, \ldots, w_s\}$. Let

$$G_2 = G_1 + \sum_{i=1}^{n-2s} \sum_{j=\delta+1}^{s} v_i w_j + \sum_{i=\delta+1}^{s-1} \sum_{j=i+1}^{s} w_i w_j - \sum_{i=\delta+1}^{s} \sum_{j=1}^{\delta} u_i w_j.$$

Clearly, $G_2 \cong H = K_\delta \vee (K_{n-2\delta} \cup \delta K_1)$. Let $q = q(G_1)$ and $q^* = q(G_2)$. Let $\boldsymbol{x}$ be the Perron vector of $A(G_1)$ with respect to q. By symmetry, $\boldsymbol{x}$ takes the same value (say x_1, x_2 and x_3) on the vertices of $V(K_s)$, $V(K_{n-2s})$ and $V(sK_1)$, respectively. Then, by $Q(G_1)\boldsymbol{x} = q\boldsymbol{x}$, we have

$$qx_3 = sx_3 + sx_1.$$

Since $q - s > 0$, we have

$$x_3 = \frac{s}{q - s} x_1. \tag{7}$$

Similarly, let $\boldsymbol{y}$ be the Perron vector of $Q(G_2)$ with respect to q^*. By symmetry, $\boldsymbol{y}$ takes the same value (say y_1, y_2 and y_3) on the vertices of $V(K_\delta)$, $V(K_{n-2\delta})$ and $V(\delta K_1)$, respectively. Then, by $Q(G_2)\boldsymbol{y} = q^*\boldsymbol{y}$, we have

$$q^* y_2 = (n - 2\delta - 1 + \delta) y_2 + \delta y_1 + (n - 2\delta - 1) y_2 = (2n - 3\delta - 2) y_2 + \delta y_1 \tag{8}$$

$$q^* y_3 = \delta y_3 + \delta y_1. \tag{9}$$

Putting (9) into (8), and considering that $q^* > 2(n - 2\delta - 1)$, we have

$$y_2 = \frac{q^* - \delta}{q^* - (2n - 3\delta - 2)} y_3. \tag{10}$$

Recall that $n \geq 2s$. Then $\delta + 1 \leq s \leq \frac{n}{2}$. Since both G_1 and G_2 are not regular graphs, it follows that $q < 2(n-1)$ and $q^* < 2(n-1)$. Note that G_2 contains $K_{n-2\delta}$ as a proper subgraph. Then Lemma 11 implies that $q^* > q(K_{n-2\delta}) = 2(n-2\delta-1)$. We now consider

$$\begin{aligned}
&\boldsymbol{y}^T(q^* - q)\boldsymbol{x}\\
&= \boldsymbol{y}^T(Q(G_2) - Q(G_1))\boldsymbol{x} = \sum_{i=1}^{n-2s}\sum_{j=\delta+1}^{s}(x_{v_i} + x_{w_j})(y_{v_i} + y_{w_j})\\
&+ \sum_{i=\delta+1}^{s-1}\sum_{j=i+1}^{s}(x_{w_i} + x_{w_j})(y_{w_i} + y_{w_j}) - \sum_{i=\delta+1}^{s}\sum_{j=1}^{\delta}(x_{u_i} + x_{w_j})(y_{u_i} + y_{w_j})\\
&= (s - \delta)\left[2(n - 2s)(x_2 + x_3)y_2 + 2(s - \delta - 1)x_3y_2 - \delta(x_1 + x_3)(y_2 + y_3)\right]\\
&= (s - \delta)\left[(2(n - s - \delta - 1) - \delta)x_3y_2 + 2(n - 2s)x_2y_2 - \delta(y_2 + y_3)x_1 - \delta x_3y_3\right]\\
&\geq (s - \delta)\left[(2(n - s - \delta - 1) - \delta)x_3y_2 - \delta(y_2 + y_3)x_1 - \delta x_3y_3\right] \ (\text{ as } n \geq 2s)\\
&\geq (s - \delta)x_1y_3\left(\frac{s(2(n - s - \delta - 1) - \delta)(q^* - \delta)}{(q - s)(q^* - (2n - 3\delta - 2))} - \frac{2\delta(q^* + \delta - n + 1)(q - s)}{(q - s)(q^* - (2n - 3\delta - 2))}\right.\\
&\left.- \frac{s\delta(q^* - (2n - 3\delta - 2))}{(q - s)(q^* - (2n - 3\delta - 2))}\right)\\
&= \frac{(s - \delta)x_1y_3}{(q - s)(q^* - (2n - 3\delta - 2))}\big(s(2n - 2s - 3\delta - 2)(q^* - \delta)\\
&- 2\delta(q^* + \delta - n + 1)(q - s) - s\delta(q^* - 2n + 3\delta + 2)\big)\\
&> \frac{(s-\delta)x_1y_3}{(q-s)(q^*-(2n-3\delta-2))}\big(s(2n-2s-3\delta-2)(q^*-\delta) - 2\delta(n-1+\delta)(q-s) - 3s\delta^2\big)\\
&> \frac{(s - \delta)x_1y_3}{(q - s)(q^* - (2n - 3\delta - 2))}\big(s(2n - 2s - 3\delta - 2)(2n - 5\delta - 2)\\
&- 2\delta(n - 1 + \delta)(2n - 2 - s) - 3s\delta^2\big)\\
&= \frac{2(s - \delta)x_1y_3}{(q - s)(q^* - (2n - 3\delta - 2))}\big(-(2n - 5\delta - 2)s^2 + s(2(n - 1)^2 - 7\delta(n - 1) + 7\delta^2)\\
&- 2\delta(n - 1)(n + \delta - 1)\big).
\end{aligned}$$

Let $\varphi(n, s) = -(2n-5\delta-2)s^2 + s(2(n-1)^2 - 7\delta(n-1) + 7\delta^2) - 2\delta(n-1)(n+\delta-1)$. We assert that $\varphi(n, s) > 0$ for $\delta+1 \leq s \leq \frac{n}{2}$. Note that $\varphi(n, s)$ is a convex function on s as $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$. So it suffices to prove that $\varphi(n, \delta+1) > 0$ and $\varphi\left(n, \frac{n}{2}\right) > 0$, respectively. Indeed

$$\begin{aligned}\varphi(n, \delta+1) &= -(2n - 5d - 2)(\delta+1)^2 + (\delta+1)\big(2(n-1)^2 - 7\delta(n-1) + 7\delta^2\big)\\ &\quad - 2\delta(n-1)(n+\delta-1)\\ &= 2n^2 - (11\delta^2 + 11\delta + 6)n + 4\delta(\delta+1)(3\delta+4) + 4.\end{aligned}$$

Note that $\varphi(n, \delta+1)$ is a monotonically increasing function with respect to $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$ since $\frac{1}{4}(11\delta^2 + 11\delta + 6) < \frac{1}{2}(11\delta^2 + 9\delta + 4)$. Then

$$\begin{aligned}\varphi(n, \delta+1) &= 2n^2 - (11\delta^2 + 11\delta + 6)n + 4\delta(\delta+1)(3\delta+4) + 4\\ &\geq \frac{1}{2}(11\delta^2 + 9\delta + 4)^2 - \frac{1}{2}(11\delta^2 + 11\delta + 6)(11\delta^2 + 9\delta + 4)\\ &\quad + 4\delta(\delta+1)(3\delta+4) + 4 = (\delta^2 + 8\delta + 3)\delta > 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\varphi\left(n, \frac{n}{2}\right) &= -\frac{1}{4}(2n - 5d - 2)n^2 + \frac{1}{2}n(2(n-1)^2 - 7\delta(n-1) + 7\delta^2)\\ &\quad - 2\delta(n-1)(n+\delta-1)\\ &= \frac{1}{4}(2n^3 - (17\delta+6)n^2 + (6\delta^2 + 30\delta + 4)n + 8\delta(\delta-1)).\end{aligned}$$

Let $\varphi(n) = 2n^3 - (17\delta+6)n^2 + (6\delta^2 + 30\delta + 4)n + 8\delta(\delta-1)$. Then

$$\varphi'(n) = 6n^2 - (34\delta + 12)n + 6\delta^2 + 30\delta + 4.$$

Note that $\varphi'(n)$ is a monotonically increasing function with respect to $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$ since $\frac{1}{6}(17\delta + 6) < \frac{1}{2}(11\delta^2 + 9\delta + 4)$. Then

$$\begin{aligned}\varphi'(n) &= 6n^2 - (34\delta + 12)n + 6\delta^2 + 30\delta + 4\\ &\geq \frac{3}{2}(11\delta^2 + 9\delta + 4)^2 - \frac{1}{2}(34\delta + 12)(11\delta^2 + 9\delta + 4) + 6\delta^2 + 30\delta + 4\\ &= \frac{1}{2}(363\delta^4 + 220\delta^3 + 81\delta^2 + 32\delta + 8) > 0.\end{aligned}$$

Thus $\varphi\left(n, \frac{n}{2}\right)$ is a monotonically increasing function with respect to $n \geq \frac{1}{2}(11\delta^2 + 9\delta + 4)$. Then we have

$$\begin{aligned}\varphi\left(n, \frac{n}{2}\right) &= \frac{1}{4}(2n^3 - (17\delta+6)n^2 + (6\delta^2 + 30\delta + 4)n + 8\delta(\delta-1))\\ &\geq \frac{1}{4}\varphi\left(\frac{1}{2}(11\delta^2 + 9\delta + 4)\right)\\ &= \frac{1}{16}\delta(1331\delta^5 + 1210\delta^4 + 165\delta^3 - 188\delta^2 - 30\delta + 8) > 0.\end{aligned}$$

Then $\varphi(n,s) > 0$ for $\delta + 1 \le s \le \frac{n}{2}$. Thus

$$\boldsymbol{y}^T(q^* - q)\boldsymbol{x} = \boldsymbol{y}^T(Q(G_2) - Q(G_1))\boldsymbol{x} = \frac{2(s-\delta)x_1y_3}{(q-s)(q^* - (2n-3\delta-2))}\varphi(n,s) > 0.$$

Therefore $q^* > q$ as $\boldsymbol{y}^T\boldsymbol{x} > 0$. This together with (6) implies that $q(G) \le q(G_1) < q^* = q(H)$, as desired.

The proof of Theorem 9 is completed. ∎

4. Further Discussions

In view of Theorems 8 and 9, in this section, we deduce the lower bound on $\rho(G)$ (or $q(G)$) to guarantee a connected graph G to have a spanning tree with leaf distance at least 4. For this purpose, we need the following spectral radius (or the signless Laplacian spectral radius) of $K_1 \vee (K_{n-2} \cup K_1)$.

Lemma 12. *Let n be a positive integer and $\theta(n)$ be the largest root of $x^3 - (n-3)x^2 - (n-1)x + n - 3 = 0$. Then we have $\rho(K_1 \vee (K_{n-2} \cup K_1)) = \theta(n)$ for $n \ge 8$.*

Proof. For $n \ge 8$, we consider the partition $V(K_1 \vee (K_{n-2} \cup K_1)) = V(K_1) \cup V(K_{n-2}) \cup V(K_1)$. Then the corresponding quotient matrix of $A(K_1 \vee (K_{n-2} \cup K_1))$ is

$$B_1 = \begin{pmatrix} 0 & n-2 & 1 \\ 1 & n-3 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence we have

$$\phi_{B_1}(x) = x^3 - (n-3)x^2 - (n-1)\,x + n - 3.$$

Note that the partition is equitable. Then Lemma 10 implies that the largest root of $x^3 - (n-3)x^2 - (n-1)\,x + n - 3 = 0$ is $\rho(K_1 \vee (K_{n-2} \cup K_1))$. That is $\rho(K_1 \vee (K_{n-2} \cup K_1)) = \theta(n)$.

This completes the proof. ∎

Lemma 13. *Let n be a positive integer and $\eta(n)$ be the largest root of $x^3 - (3n-5)x^2 + (2n^2 - 5n)x - 2n^2 + 10n - 12 = 0$. Then we have $q(K_1 \vee (K_{n-2} \cup K_1)) = \eta(n)$ for $n \ge 12$.*

Proof. For $n \ge 12$, we consider the partition $V(K_1 \vee (K_{n-2} \cup K_1)) = V(K_1) \cup V(K_{n-2}) \cup V(K_1)$. Then the corresponding quotient matrix of $Q(K_1 \vee (K_{n-2} \cup K_1))$ is

$$B_1 = \begin{pmatrix} n-1 & n-2 & 1 \\ 1 & 2n-5 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Hence we have

$$\phi_{B_1}(x) = x^3 - (3n-5)x^2 + (2n^2 - 5n)x - 2n^2 + 10n - 12.$$

Note that the partition is equitable. Then Lemma 10 implies that the largest root of

$$x^3 - (3n-5)x^2 + (2n^2 - 5n)x - 2n^2 + 10n - 12 = 0$$

is $q(K_1 \vee (K_{n-2} \cup K_1))$. That is $q(K_1 \vee (K_{n-2} \cup K_1)) = \eta(n)$.

This completes the proof. ■

Remark 14. For $\delta = 1$ in Theorem 8, we have that for any connected graph G of order $n \geq 8$, if $\rho(G) \geq \rho(K_1 \vee (K_{n-2} \cup K_1))$, then G has a spanning tree with leaf distance at least 4.

Remark 15. For $\delta = 1$ in Theorem 9, we have that for any connected graph G of order $n \geq 12$, if $q(G) \geq q(K_1 \vee (K_{n-2} \cup K_1))$, then G has a spanning tree with leaf distance at least 4.

From Remarks 14 and 15, we then have the following corollaries immediately.

Corollary 16. *Let G be a graph of order $n \geq 8$. If $\rho(G) \geq \theta(n)$, then G has a spanning tree with leaf distance at least 4, where $\theta(n)$ is the largest root of $x^3 - (n-3)x^2 - (n-1)x + n - 3 = 0$.*

Corollary 17. *Let G be a graph of order $n \geq 12$. If $q(G) \geq \eta(n)$, then G has a spanning tree with leaf distance at least 4, where $\eta(n)$ is the largest root of $x^3 - (3n-5)x^2 + (2n^2 - 5n)x - 2n^2 + 10n - 12 = 0$.*

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References

[1] G. Ao, R. Liu and J. Yuan, *Spectral radius and spanning trees of graphs*, Discrete Math. **346(8)** (2023) 113400.
https://doi.org/10.1016/j.disc.2023.113400

[2] J. Akiyama and M. Kano, Factors and Factorizations of Graphs: Proof Techniques in Factor Theory, Lecture Notes in Math. **2031** (Springer, Berlin Heidelberg, 2011).
https://doi.org/10.1007/978-3-642-21919-1

[3] R. Bapat, Graphs and Matrices (Springer, London, 2010).
https://doi.org/10.1007/978-1-84882-981-7

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (The Macmillan, Press Ltd., London, 1976).

[5] D. Fan, S. Goryainov, X. Huang and H. Lin, *The spanning k-trees, perfect matchings and spectral radius of graphs*, Linear Multilinear Algebra **70** (2022) 7264–7275.
https://doi.org/10.1080/03081087.2021.1985055

[6] A. Kaneko, *Spanning trees with constraints on the leaf degree*, Discrete Appl. Math. **115** (2001) 73–76.
https://doi.org/10.1016/S0166-218X(01)00216-5

[7] A. Kaneko, M. Kano and K. Suzuki, *Spanning trees with leaf distance at least four*, J. Graph Theory **55** (2007) 83–90.
https://doi.org/10.1002/jgt.20221

[8] S. O, *Spectral radius and matchings in graphs*, Linear Algebra Appl. **614** (2021) 316–324.
https://doi.org/10.1016/j.laa.2020.06.004

[9] K. Ozeki and T. Yamashita, *Spanning trees: A survey*, Graphs Combin. **27** (2011) 1–26.
https://doi.org/10.1007/s00373-010-0973-2

[10] S. Win, *On a conjecture of Las Vergnas concerning certain spanning trees in graphs*, Results Math. **2** (1979) 215–224.
https://doi.org/10.1007/BF03322958

[11] S. Win, *On a connection between the existence of k-trees and the toughness of a graph*, Graphs Combin. **5** (1989) 201–205.
https://doi.org/10.1007/BF01788671

[12] L. You, M. Yang, W. So and W. Xi, *On the spectrum of an equitable quotient matrix and its application*, Linear Algebra Appl. **577** (2019) 21–40.
https://doi.org/10.1016/j.laa.2019.04.013

[13] J. Zheng, X. Huang and J. Wang, *Spectral condition for spanning k-ended trees in t-connected graphs* (2023).
arXiv:2305.04211.pdf

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