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TIGHT DESCRIPTION OF FACES IN TOROIDAL GRAPHS WITH MINIMUM DEGREE AT LEAST 4

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Abstract

The degree $d(x)$ of a vertex or face x in a graph G is the number of incident edges. A face $f = v_1 \cdots v_{d(f)}$ in a graph G on the plane or other orientable surface is of type $(k_1, k_2, \ldots)$ if $d(v_i) \leq k_i$ for each i. By δ we denote the minimum vertex-degree of G.

It follows from the classical theorem by Lebesgue (1940) that every plane triangulation with $\delta \geq 4$ has a 3-face of types $(4, 4, \infty)$, $(4, 5, 19)$, $(4, 6, 11)$, $(4, 7, 9)$, $(5, 5, 9)$, or $(5, 6, 7)$. In 1999, Jendrol' gave a similar description: "$(4, 4, \infty)$, $(4, 5, 13)$, $(4, 6, 17)$, $(4, 7, 8)$, $(5, 5, 7)$, $(5, 6, 6)$" and conjectured that "$(4, 4, \infty)$, $(4, 5, 10)$, $(4, 6, 15)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)$" holds. In 2002, Lebesgue's description was strengthened by Borodin to "$(4, 4, \infty)$, $(4, 5, 17)$, $(4, 6, 11)$, $(4, 7, 8)$, $(5, 5, 8)$, $(5, 6, 6)$". In 2014, we obtained the following tight description, which, in particular, disproves the above mentioned conjecture by Jendrol': "$(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)$", and recently proved another tight description of faces in plane triangulations with $\delta \geq 4$: "$(4, 4, \infty)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 5, 8)$, $(5, 6, 7)$".

It follows from Lebesgue's theorem of 1940 that every plane 3-connected quadrangulation has a face of one of the types $(3, 3, 3, \infty)$, $(3, 3, 4, 11)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$. Recently, we improved this description to "$(3, 3, 3, \infty)$,

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$(3,3,4,9)$, $(3,3,5,6)$, $(3,4,4,5)$", where all parameters except possibly 9 are best possible and 9 cannot go down below 8.

In 1995, Avgustinovich and Borodin proved the following tight description of the faces of torus quadrangulations with $\delta \geq 3$: "$(3,3,3,\infty)$, $(3,3,4,10)$, $(3,3,5,7)$, $(3,3,6,6)$, $(3,4,4,6)$, $(4,4,4,4)$".

Recently, we proved that every triangulation with $\delta \geq 4$ of the torus has a face of one of the types $(4,4,\infty)$, $(4,6,12)$, $(4,8,8)$, $(5,5,8)$, $(5,6,7)$, or $(6,6,6)$, which description is tight.

The purpose of this paper is to prove that every graph with $\delta \geq 4$ that admits a closed 2-cell embedding on the torus has a face of one of the types $(4,4,4,4)$, $(4,4,\infty)$, $(4,5,16)$, $(4,6,12)$, $(4,8,8)$, $(5,5,8)$, $(5,6,7)$, or $(6,6,6)$, where all parameters are best possible.

Keywords: plane graph, toroidal graph, degree, face, structure.

2020 Mathematics Subject Classification: 05C75.

1. Introduction

The *degree* $d(x)$ of a vertex or face x in a a plane or torus graph G is the number of incident edges. A k-*vertex* and k-*face* is one of degree k, a k^+-*vertex* has degree at least k, and so on. A face f in a graph G on the plane or torus is of *type* $(k_1, k_2, \ldots)$, or a $(k_1, k_2, \ldots)$-*face* if $d(v_i) \leq k_i$ for each i. By δ and w denote the minimum vertex degree and smallest degree-sum of faces in G, respectively.

We now recall some results on the structure of faces in plane graph with $\delta \geq 3$, beginning with the fundamental theorem of Lebesgue [19] from 1940.

Theorem 1 (Lebesgue [19]). *Every plane graph with $\delta \geq 3$ has a face of one of the following types:*

$$(3,6,\infty),\ (3,7,41),\ (3,8,23),\ (3,9,17),\ (3,10,14),\ (3,11,13),$$
$$(4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),$$
$$(3,3,3,\infty),\ (3,3,4,11),\ (3,3,5,7),\ (3,4,4,5),\ (3,3,3,3,5).$$

The classical Theorem 1, along with other ideas in Lebesgue [19], has a lot of applications to plane graph coloring problems (the first example of such applications and recent surveys can be found in [4,13,17,21]).

Some parameters of Lebesgue's theorem were improved for several narrow classes of plane graphs. Back in 1963, Kotzig [18] proved that every plane triangulation with $\delta = 5$ satisfies $w \leq 18$ and conjectured that $w \leq 17$ holds. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form by proving that every such a graph has a $(5,5,7)$-face or a $(5,6,6)$-face, where all parameters are tight. This result also confirmed a conjecture of Grünbaum [14] from 1975 on the cyclic 11-connectivity of 5-connected planar graphs, and it has been extended to several classes of plane graphs over the last decades; see, for example, recent surveys [7, 13, 17] and also [3–5, 15, 16, 20].

It follows from the classical theorem by Lebesgue [19] that every plane triangulation with $\delta \geq 4$ has a 3-face of types $(4,4,\infty)$, $(4,5,19)$, $(4,6,11)$, $(4,7,9)$, $(5,5,9)$, or $(5,6,7)$. In 1999, Jendrol' [16] gave a similar description: "$(4,4,\infty)$, $(4,5,13)$, $(4,6,17)$, $(4,7,8)$, $(5,5,7)$, $(5,6,6)$" and conjectured that "$(4,4,\infty)$, $(4,5,10)$, $(4,6,15)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$" holds. In 2002, Lebesgue's description was strengthened by Borodin [3] to "$(4,4,\infty)$, $(4,5,17)$, $(4,6,11)$, $(4,7,8)$, $(5,5,8)$, $(5,6,6)$". In 2014, we obtained [6] the following tight description, which, in particular, disproves the above mentioned conjecture by Jendrol': "$(4,4,\infty)$, $(4,5,11)$, $(4,6,10)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$", and recently proved [11] another tight description of faces in plane triangulations with $\delta \geq 4$: "$(4,4,\infty)$, $(4,6,10)$, $(4,7,7)$, $(5,5,8)$, $(5,6,7)$".

In particular, precise descriptions of the structure of faces were obtained for plane graphs with $\delta \geq 4$ (Borodin, Ivanova [5]) and for plane triangulations (Borodin, Ivanova, Kostochka [12]). It follows from Theorem 1 that every plane quadrangulation with $\delta \geq 3$ has a face of one of the types $(3,3,3,\infty)$, $(3,3,4,11)$, $(3,3,5,7)$, $(3,4,4,5)$. Recently, we improved [8] this result to the following description: "$(3,3,3,\infty)$, $(3,3,4,9)$, $(3,3,5,6)$, $(3,4,4,5)$", where all parameters except possibly 9 are best possible, while 9 cannot go down below 8.

In 1995, Avgustinovich and Borodin gave the following tight description of faces in quadrangulations of the torus.

Theorem 2 (Avgustinovich, Borodin [1]). *Every torus quadrangulation with $\delta \geq 3$ has a face of one of the following types:* $(3,3,3,\infty)$, $(3,3,4,10)$, $(3,3,5,7)$, $(3,3,6,6)$, $(3,4,4,6)$, $(4,4,4,4)$, *where all parameters are best possible.*

Recently, we proved [9] that every toroidal triangulation with $\delta \geq 5$ has a face of one of the types $(5,5,8)$, $(5,6,7)$, or $(6,6,6)$, and later on we extended [10] this description to $\delta \geq 4$ as follows: "$(4,4,\infty)$, $(4,6,12)$, $(4,8,8)$, $(5,5,8)$, $(5,6,7)$, $(6,6,6)$", which is also tight.

The purpose of our paper is to prove the following further extension of the result in [10].

Theorem 3. *Every graph with $\delta \geq 4$ that admits a closed 2-cell embedding on the torus has a face of one of the types:*

(Ta) $(4,4,4,4)$,

(Tb) $(4,4,\infty)$,

(Tc) $(4,5,16)$,

(Td) $(4,6,12)$,

(Te) $(4,8,8)$,

(Tf) $(5, 5, 8)$,

(Tg) $(5, 6, 7)$, *or*

(Th) $(6, 6, 6)$,

where all parameters are best possible.

2. The Tightness of Theorem 3

It is easy to construct a 4-regular quadrangulation of the torus; for example, by deleting all 4-vertices from a graph on Figure 4, so the item (Ta) in our description is necessary. Now to justify (Tc), it suffices to replace each face of such a quadrangulation by a construction in Figure 1.

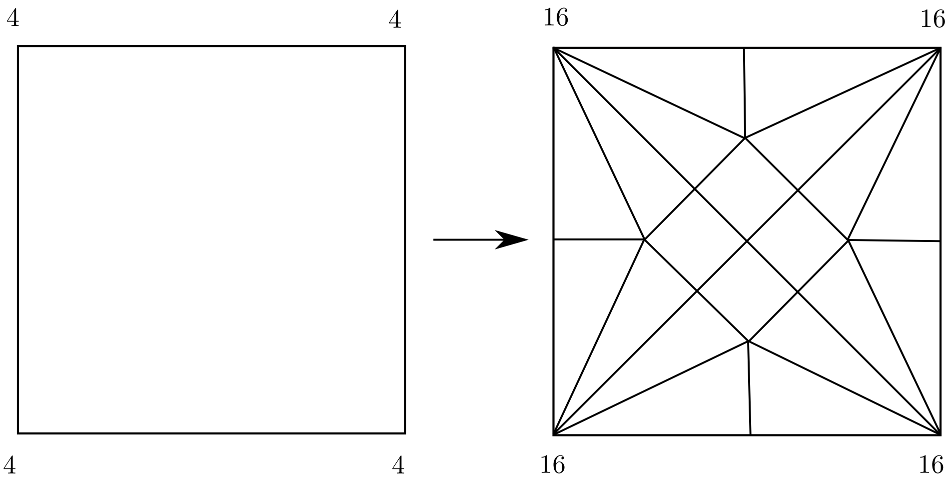

Figure 1. All 3-faces are of type $(4, 5, 16)$, and there are no $(4, 4, 4, 4)$-faces.

The tightness of (Tb) is confirmed by the double n-pyramid augmented by an edge joining its n-vertices. Figure 2 represents a bipartite torus graph with four 6-faces. Putting a vertex inside each its face and joining it with the six boundary vertices produces a 6-regular triangulation $T(6, 6, 6)$, which confirms the necessity and sharpness of (Th).

Next we put a vertex on every edge of $T(6, 6, 6)$, followed by putting a vertex $v(f)$ inside each 6-face f obtained and joining $v(f)$ with the six vertices of f; this results in a triangulation $T(4, 6, 12)$ confirming the tightness of (Td).

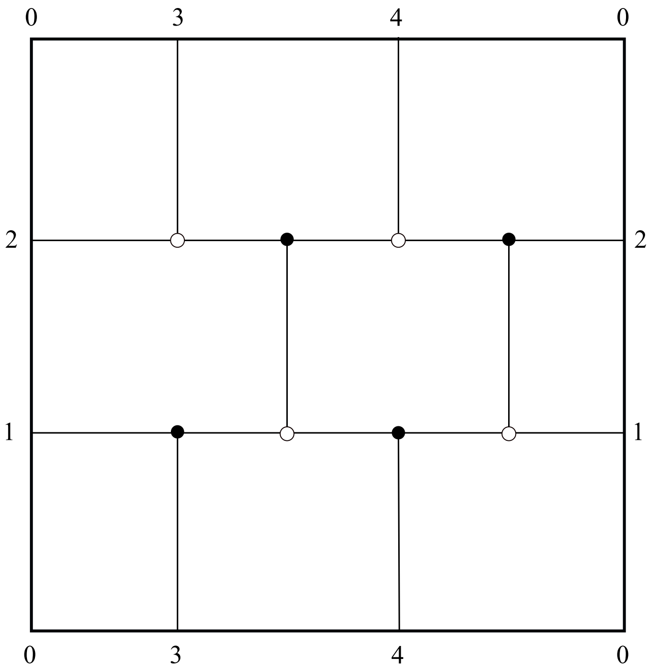

Figure 2. A bipartite torus graph with four 6-faces ([10]).

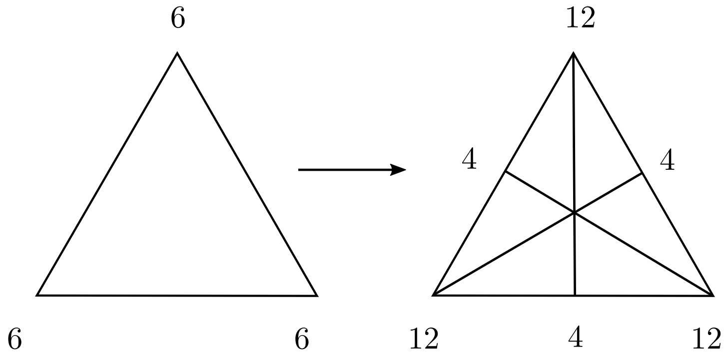

Figure 3. Producing a torus graph with all faces of type $(4, 6, 12)$ ([10]).

The next two constructions confirm the tightness of (Te) and (Tf), respectively.

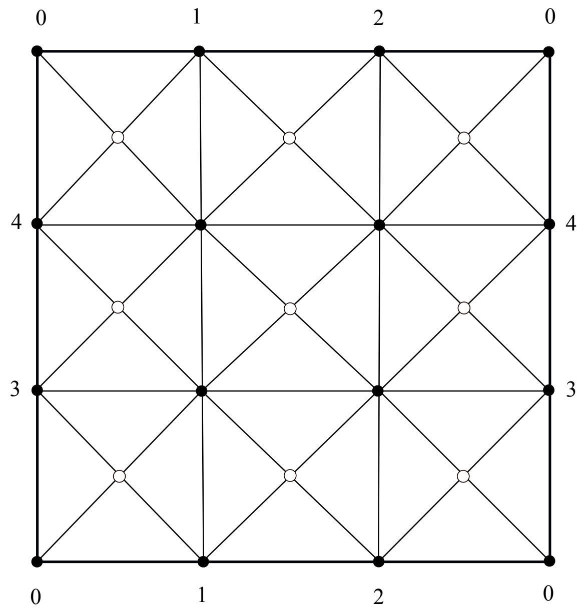

Figure 4. All faces are of type $(4, 8, 8)$ ([10]).

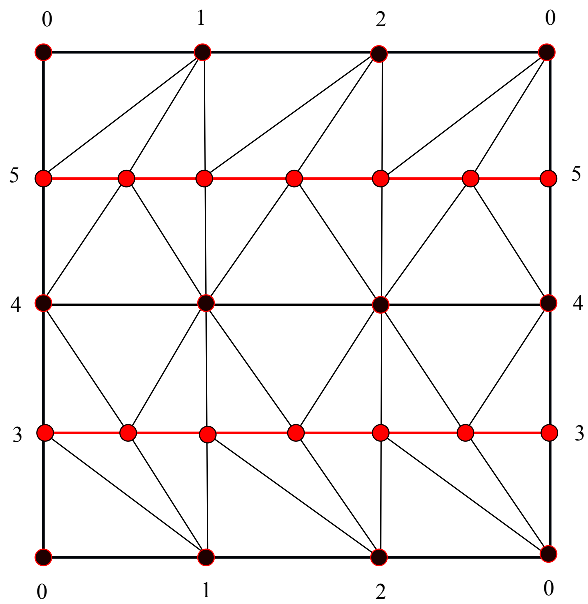

Figure 5. A torus graph with all $(5, 5, 8)$-faces ([9]).

Finally, replacing each 6-face in Figure 2 by the construction shown in Figure 6 produces a torus triangulation which confirms that the term (Th) in Theorem 3 is also best possible.

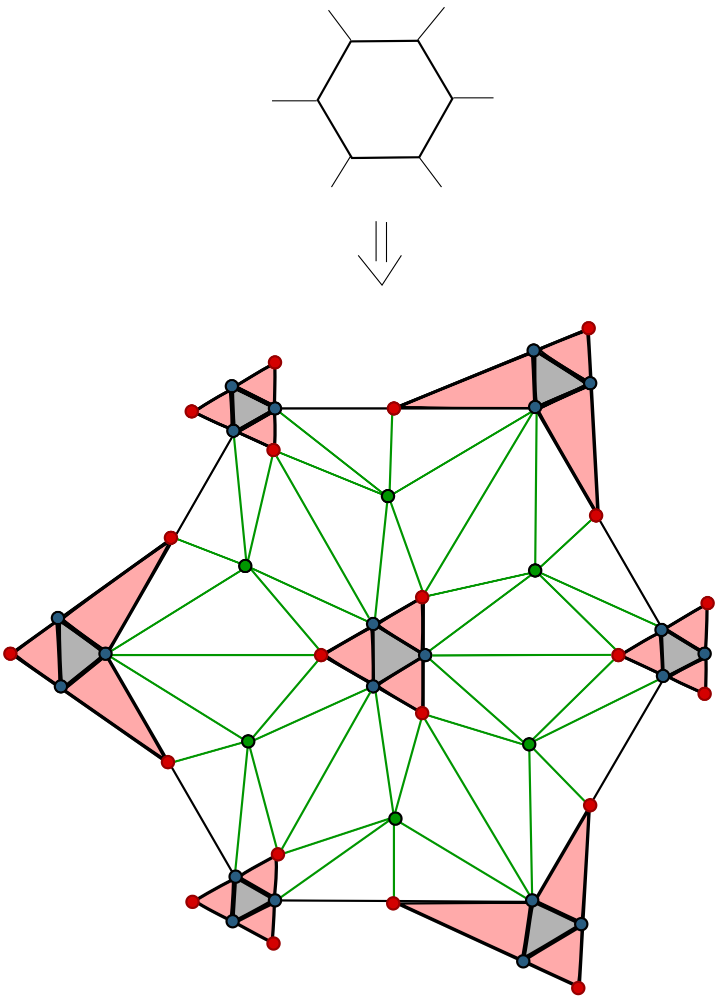

Figure 6. A replacement for each 6-face in Figure 1 which results in all faces of type $(5^+, 6^+, 7^+)$ ([9]).

3. Proving the Existence of Face-Types in Theorem 3

Suppose G is a counterexample to Theorem 3. Euler's formula $|V|-|E|+|F|=0$ for G, where V and F are the sets of its vertices and faces, respectively, can be rewritten as follows.

$$\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2d(f)-6)=0. \tag{1}$$

We assign a *charge* $\mu(v)=d(v)-6$ to every vertex v and $\mu(f)=2d(f)-6$ to every face f of G, so that only 5^{-}-vertices have a negative charge. Using the properties of G as a counterexample, we define a local redistribution of charges, preserving their sum, such that the *new charge* $\mu'(x)$ satisfies $\mu'(x)\geq 0$ whenever $x\in V\cup F$ and there is at least one x in $V\cup F$ with $\mu'(x)>0$. This will contradict the fact that the sum of the new charges is by (1) equal to 0.

In what follows, by "non-(k,l,m)!" we mean a short-hand for "since T has no (k,l,m)-faces". The neighbors of a vertex or face x in a cyclic order are denoted by $v_1,v_2,\dots,v_{d(x)}$.

We use the following rules R1–R3 of discharging (see Figure 7).

R1. Every 4^{+}-face gives $\frac{1}{2}$ to each incident vertex of degree 4 or 5.

R2. Each 4-vertex v in a face $f_1=v_1vv_2$ with $d(v_1)\leq d(v_2)$ (where $d(v_1)\geq 5$ due to non-$(4,4,\infty)$!) receives the following charges from v_1,v_2 through f_1.

(R2a) If $d(v_1)=5$ or $d(v_1)=6$, then $d(v_2)\geq 17$ due to non-$(4,5,16)$! or $d(v_2)\geq 13$ due to non-$(4,6,12)$!, respectively, and v_2 gives $\frac{1}{2}$ to v.

(R2b) Suppose $7\leq d(v_1)\leq 8$; then $d(v_2)\geq 9$ by non-$(4,8,8)$!, and now v_2 gives $\frac{1}{3}$ to v, while v_1 gives $\frac{1}{6}$.

(R2c) If $d(v_1)\geq 9$, then each of v_1 and v_2 gives $\frac{1}{3}$ to v.

R3. Each 5-vertex v in a face $f_1=v_1vv_2$ receives the following charges from v_1,v_2 through f.

(R3a) If $d(v_1)=7$, then v_1 gives $\frac{1}{7}$.

(R3b) If $d(v_1)=8$ (and hence $d(v_2)\geq 6$ due to non-$(5,5,8)$!), then v_1 gives $\frac{5}{24}$.

(R3c) If $9\leq d(v_1)\leq 12$ and $d(v_2)\geq 6$, then v_1 gives $\frac{1}{3}$ to v.

(R3d) Suppose $d(v_2)\geq 6$; then v_1 gives to v:

(R3d1) $\frac{1}{3}$ if $13\leq d(v_1)\leq 16$, or

(R3d2) $\frac{1}{2}$ if $d(v_1)\geq 17$.

(R3e) If $d(v_2)=5$ (and hence $d(v_1)\geq 9$ by non-$(5,5,8)$!), then v_1 gives $\frac{1}{7}$ to v and, by symmetry, also to v_2.

(R3f) If $d(v_2)=4$ (and hence $d(v_1)\geq 17$ by non-$(4,5,16)$!), then v_1 gives $\frac{1}{8}$ to v.

We now check that $\mu'(x)\geq 0$ whenever $x\in V\cup F$ and at least one vertex or face has a strictly positive new charge μ'.

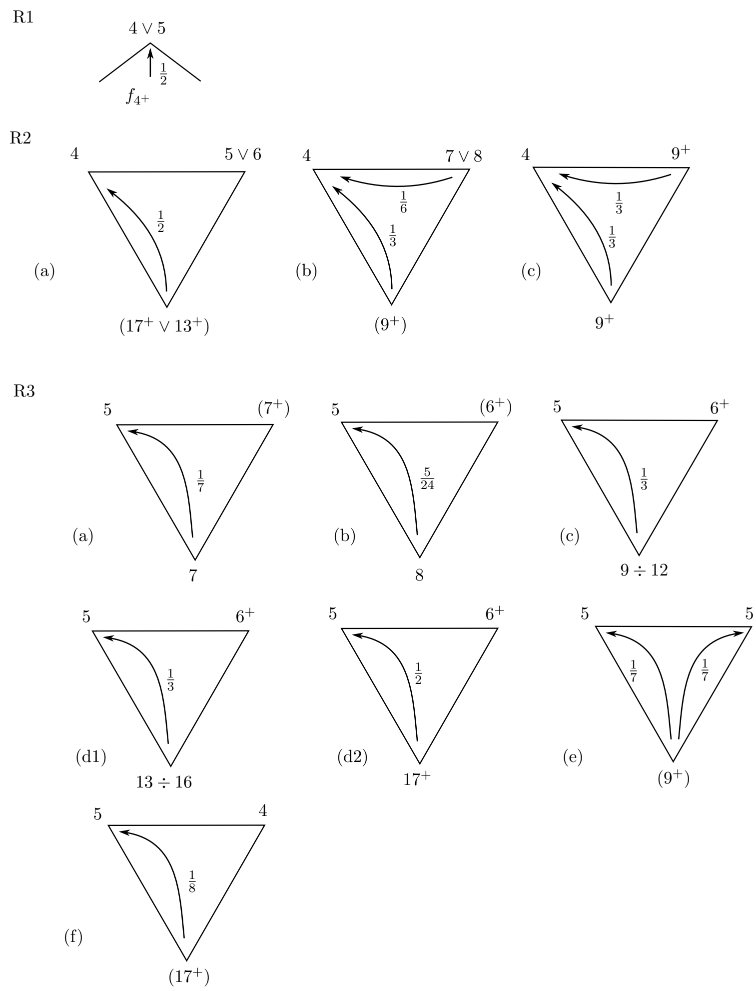

Figure 7. Rules of discharging.

First consider $f \in F$. If $d(f) = 3$, then $\mu'(f) = \mu(f) = 0$ as f does not participate in discharging. If $d(f) \geq 4$, then $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{1}{2} = \frac{3(d(v)-4)}{2} \geq 0$ in view of R1. In particular, $\mu'(f) > 0$ when $d(f) \geq 5$.

From now on suppose $v \in V$. Here, our proof splits.

Case 1. $d(v) = 4$. Note that v receives the total of at least $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ through each incident face by R1, R2 due to non-$(4,4,\infty)$!, non-$(4,5,16)$!, non-$(4,6,12)$!, and non-$(4,8,8)$!. Thus we already have $\mu'(v) \geq 4 - 6 + 4 \times \frac{1}{2} = 0$. Furthermore, it is not difficult to see that in fact $\mu'(v) > 0$ if v participates at least once in R2c.

Case 2. $d(v) = 5$. We know that v receives $\frac{1}{2}$ from each 4^+-face by R1. Through each 3-face, v receives $\frac{1}{8}$ by R3f and at least $\frac{1}{7}$ by R3a–R3e. It is easy to see that $\mu'(v) \geq 5 - 6 + 2 \times \frac{1}{2} + 3 \times \frac{1}{8} > 0$ if v is incident with at least two 4^+-faces. If v belongs to the boundary of precisely one 4^+-face, then $\mu'(v) \geq -1 + \frac{1}{2} + 4 \times \frac{1}{8} \geq 0$, with the equality only when each of its incident 3-faces participates in R3f. In particular, $\mu'(v) > 0$ unless v has at least two 17^+-neighbors due to non-$(4, 5, 16)!$.

From now on suppose v is completely surrounded by 3-faces. Note that v has at most two 6^--neighbors by non-$(5, 6, 6)!$, so we can assume that $d(v_1) \geq 7$, $d(v_3) \geq 7$ and $d(v_4) \geq 7$.

To ensure $\mu'(v) \geq 0$, it suffices, due to symmetry, to check that v receives the total of at least $\frac{1}{2}$ from v_1, v_2 and v_3 through the faces $f_1 = v_1vv_2$, $f_2 = v_2vv_3$ and $f_3 = v_3vv_4$.

If $d(v_2) = 4$, then $d(v_1) \geq 17$ and $d(v_3) \geq 17$ by non-$(4, 5, 16)!$, so we are done already due to the contribution of $\frac{1}{2}$ from v_3 by R3d2.

For $d(v_2) = 5$, we have $d(v_1) \geq 9$ and $d(v_3) \geq 9$ by non-$(5, 5, 8)!$, so v receives $\frac{1}{3} + \frac{1}{7}$ from v_3 by R3c or R3d1 combined with R3e. Furthermore, v receives $\frac{1}{7}$ from v_1 by R3e. Thus v receives $\frac{1}{3} + 2 \times \frac{1}{7} > \frac{1}{2}$ from v_1 and v_3 through the faces f_1, f_2 and f_3.

Next suppose $d(v_2) = 6$; now $d(v_1) \geq 8$ and $d(v_3) \geq 8$ by non-$(5, 6, 7)!$. This means that v receives from v_3 either $2 \times \frac{5}{24}$ by R3b if $d(v_3) = 8$, or $2 \times \frac{1}{3}$ by R3c combined with R3d1 when $d(v_3) \geq 9$. From v_1, our v receives through f_1 either $\frac{5}{24}$ by R3b if $d(v_1) = 8$, or $\frac{1}{3}$ by R3c when $9 \leq d(v_1) \leq 12$. Now if $d(v_1) \geq 13$, then v_1 sends to v through f_1 either $\frac{1}{3}$ by R3d1 or $\frac{1}{2}$ by R3d2. In total, v receives at least $3 \times \frac{5}{24} > \frac{1}{2}$ through the faces f_1, f_2 and f_3.

Finally, suppose $d(v_2) \geq 7$. Here, v receives from v_2 through the faces f_1 and f_2 either at least $2 \times \frac{1}{7}$ by R3a–R3c when $d(v_2) \leq 12$, or $2 \times \frac{1}{3}$ by R3d1 otherwise. By symmetry, the same donation occurs from v_3 through f_2 and f_3, and so v receives at least $\frac{4}{7} > \frac{1}{2}$ from v_2 and v_3 through the faces f_1, f_2 and f_3, as desired.

In particular, we have proved that a 5-vertex v not only satisfies $\mu'(v) \geq 0$, but in fact $\mu'(v) > 0$ holds unless v is incident with precisely one 4^+-face and has two or three 17^+-neighbors along with three or two 4-neighbors, respectively.

Case 3. $d(v) = 6$. Since v does not participate in discharging, we have $\mu'(v) = 6 - 6 = 0$.

Case 4. $d(v) = 7$. Such a vertex v sends at most $\frac{1}{6}$ through each incident 3-face which is incident with a 5^--vertex; namely, by R2b and R3a. If v is incident with at least one 4^+-face, then $\mu'(v) \geq 7 - 6 - 6 \times \frac{1}{6} = 0$. Otherwise, due to non-$(5, 5, 8)!$, our v has at most three 5^--neighbors, and hence again we have $\mu'(v) \geq 1 - 2 \times 3 \times \frac{1}{6} = 0$, as desired.

Case 5. $d(v) = 8$. Now v can only send at most $\frac{5}{24}$ through each incident 3-face by R2b or R3b, in view of non-$(5,5,8)$!, which means that $\mu'(v) \geq 8 - 6 - 8 \times \frac{5}{24} > 0$.

Case 6. $9 \leq d(v) \leq 12$. Note that v sends $\frac{1}{3}$ through each incident 3-face to a unique 5^--vertex in the boundary of that face by R2b, R2c and R3c or $2 \times \frac{1}{7}$ through a face incident with two 5-vertices by R3e due to non-$(4,5,16)$!. This implies $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{3} = \frac{2(d(v)-9)}{3} \geq 0$. In particular, $\mu'(v) > 0$ whenever $10 \leq d(v) \leq 12$.

Now suppose $d(v) = 9$. If v is incident with a 4^+-face or has two consecutive 6^+-neighbors in an incident 3-face, then $\mu'(v) \geq 9 - 6 - 8 \times \frac{1}{3} > 0$. Otherwise, v is completely surrounded by 3-faces and, by parity reasons, has two consecutive 5-neighbors, which implies $\mu'(v) \geq 3 - 8 \times \frac{1}{3} - \frac{2}{7} > 0$.

Case 7. $13 \leq d(v) \leq 16$. Such a v sends through each incident 3-face at most $\frac{1}{2}$ to a unique 5^--vertex by R2, R3d1 or $2 \times \frac{1}{7}$ by R3e, since R3f is not applicable in view of non-$(4,5,16)$!. This implies that $\mu'(v) \geq d(v) - 6 - \frac{d(v)}{2} > \frac{d(v)-12}{2} > 0$.

Case 8. $d(v) \geq 17$. Here, v sends through each incident 3-face vv_1v_2 either $\frac{1}{2} + \frac{1}{8}$ by R2a combined with R3f if $d(v_1) = 4$ and $d(v_2) = 5$, or at most $\frac{1}{2}$ by R2, R3 otherwise, which yields $\mu'(v) \geq d(v) - 6 - \frac{5d(v)}{8} = \frac{3(d(v)-16)}{8} > 0$.

Remark 4. Thus we have proved $\mu'(v) \geq 0$ for every $v \in V$. Furthermore, as shown in Cases 5–8, a vertex v satisfies $\mu'(v) > 0$ if $d(v) \geq 8$.

To arrive at a final contradiction with (1), it suffices to show, assuming $\delta(G) \leq 7$, that in fact there is an $x \in V \cup F$ with $\mu'(x) > 0$, since this will imply

$$0 = \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = \sum_{x \in V \cup F} \mu'(x) > 0.$$

To this aim, first consider a vertex v. If $d(v) = 5$, then, as shown in Case 2, $\mu'(v) > 0$ unless v has a 17^+-neighbor, which possibility is already excluded. Thus it remains to assume that G has no 5-vertices.

Next suppose $d(v) = 7$. It now follows from non-$(4,8,8)$! that v cannot send a positive charge through a 3-face by R2b due to the absence of 9^+-vertices in G and since R3a is not applicable because of the absence of 5-vertices, which means that $\mu'(v) = \mu(v) = 1$. Thus it remains to assume, moreover, that G has no 7-vertices either.

Furthermore, now each 3-face of G must be incident with three 6^--vertices, contrary to non-$(6,6,6)$!. This implies that G has no 3-faces.

Finally, as follows from an inequality two lines above Case 1, G cannot have 5^+-faces since they have $\mu' > 0$. Hence all faces of G are 4-faces. However, each

4-face f must be incident with at least one 6-vertex by non-$(4,4,4,4)$!, which 6-vertex does not receive charge from f by R1, and so $\mu'(f) \geq 2\times 4 - 6 - 3\times\frac{1}{2} > 0$.

This contradiction completes the proof of Theorem 3.

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