

## ON A PROBLEM OF L. ALCÓN CONCERNING PATH DOMINATION

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### Abstract

A walk  $W$  between two non-adjacent vertices in a graph  $G$  is called tolled if the first vertex of  $W$  is among vertices from  $W$  adjacent only to the second vertex of  $W$ , and the last vertex of  $W$  is among vertices from  $W$  adjacent only to the second-last vertex of  $W$ . In this article, we solve a problem posed by Alcón that seeks to characterize the class of graphs such that for every pair of non-adjacent vertices  $u$  and  $v$ , every  $uv$  shortest path dominates every  $uv$  tolled walk.

**Keywords:** domination, walks, interval graphs.

**2020 Mathematics Subject Classification:** 05C38, 05C75, 05C69, 05C12.

### INTRODUCTION

In [1] the concept of path domination was introduced to understand the structure of those graphs in which for every pair of non-adjacent vertices  $u$  and  $v$ , and every pair of  $uv$  walks  $W$  and  $W'$ , each internal vertex of  $W'$  is adjacent to some internal vertex of  $W$ .

Based in this concept, new characterization of standard classes like chordal, interval, superfragile and HHD-free graphs [3] have been obtained in [1, 8, 10].

Also, in [1], p. 1032, Alcón suggests the following problem: determine if the graphs in which every  $uv$  shortest path dominates every  $uv$  tolled walk are exactly the ones in **Interval**<sup>+</sup>.

A graph  $G$  is in **Interval**<sup>+</sup>, if  $G$  is a chordal graph that contains none of the graphs  $F_2$  or  $F_4(n)_{n \geq 6}$  as induced subgraph, and satisfy the following condition. If  $G$  has an induced subgraph  $H$  isomorphic to  $F_1$  ( $F_3(n)_{n \geq 6}$ ), then the distance in  $G$  between the vertices of  $F_1$  ( $F_3(n)_{n \geq 6}$ ) labelled  $u$  and  $w$  in Figure 1 is 2, and any vertex of  $G$  adjacent to both  $u$  and  $w$  is universal to  $F_1$  ( $F_3(n)_{n \geq 6}$ ).

In this paper, we show that there exist graphs in which every  $uv$  shortest path dominates every  $uv$  tolled walk that are not in **Interval**<sup>+</sup>. Thus, Alc3n's Conjecture results false. We propose a reformulation of the problem. For this, we introduce the class **Intervale**<sup>+</sup> that is the class of those chordal graphs  $G$  that contain none of the graphs  $F_2$  or  $F_4(n)_{n \geq 6}$  as induced subgraph, and satisfy the following condition. If  $G$  has an induced subgraph  $H$  isomorphic to  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ), then the distance in  $G$  between the vertices of  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ) labelled  $u$  and  $w$  in Figure 2 is 2, and any vertex of  $G$  adjacent to both  $u$  and  $w$  is universal to  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ). And then, we prove that the graphs in which every  $uv$  shortest path dominates every  $uv$  tolled walk are exactly the ones in **Intervale**<sup>+</sup>.

The paper is organized as follows. In Section 1, we give necessary definitions, in Section 2, the main result is presented. Conclusions are developed in Section 3.

## 1. PRELIMINARIES

We introduce the necessary definitions in this section.

All the graphs in this paper are finite, undirected, simple, and connected. We use standard graph terminology [11].

Let  $G$  be a graph. The subgraph induced in  $G$  by a subset  $S \subseteq V(G)$  is denoted by  $G[S]$ . For any vertex  $v$  of  $G$ , the *neighborhood* of  $v$  is denoted by  $N[v] = \{u \in V(G) \mid uv \text{ is an edge of } G\} \cup \{v\}$ . We denoted  $|V(G)|$  by  $|G|$  and by  $G(n)$  a graph with  $n$  vertices.

Let us introduce the following definitions. A  $uv$  walk is a sequence  $W : u = v_0, v_1, \dots, v_{k-1}, v_k = v$  whose terms are vertices, not necessarily distinct, such that  $u$  is adjacent to  $v_1$ ,  $v_i$  is adjacent to  $v_{i+1}$  for  $i \in \{1, \dots, k-2\}$ , and  $v_{k-1}$  is adjacent to  $v$ . The vertices  $u$  and  $v$  are called *ends of the walk*, and the vertices  $v_1, \dots, v_{k-1}$  are its *internal vertices*. The integer  $k$  is the *length of the walk*.

A  $uv$  path is a  $uv$  walk with all its vertices distinct. The *distance*  $d_G(u, v)$  between vertices  $u$  and  $v$  is the minimum number of edges on a path connecting these vertices. If no confusion can arise we will omit the index  $G$ .

Let  $W : v_0, v_1, \dots, v_{k-1}, v_k$  be a path,  $W[a, b] : a = v_i, \dots, v_j = b$  denote the section of the path  $W$  between  $a$  and  $b$ . Let  $W(a, b) = W[a, b] \setminus \{a\}$ ,  $W[a, b) = W[a, b] \setminus \{b\}$ , and  $W(a, b) = W[a, b] \setminus \{a, b\}$ .

A  $uv$  shortest path (or *geodesic* [6]) is a  $uv$  path of length  $d(u, v)$ .

A  $uv$  tolled walk is a  $uv$  walk,  $u = v_0, v_1, \dots, v_{k-1}, v_k = v$ , satisfying that  $u$  is adjacent only to the vertex  $v_1$ ,  $v$  is adjacent only to the vertex  $v_{k-1}$ ,  $\{v_1\} \cap \{v_2, \dots, v_{k-1}\} = \emptyset$  and  $\{v_{k-1}\} \cap \{v_1, \dots, v_{k-2}\} = \emptyset$  [2]. Note that  $v_1$  may be  $v_{k-1}$ , but if  $v_1 = v_{k-1}$  then  $k = 2$ .

It is clear that every shortest path is a tolled walk.

**Remark 1.** Every  $uv$  walk contains a  $uv$  induced path.

**Definition 1.** The  $uv$  walk  $W : u, v_1, \dots, v_{m-1}, v$  *dominates* the  $uv$  walk  $W' : u, v'_1, \dots, v'_{n-1}, v$  if every internal vertex of  $W'$  is adjacent to some internal vertex of  $W$  or belongs to  $W$ .

Now, we introduce the notation **SP** and **TW** to refer to the set of shortest paths and tolled walks respectively, which connects two non-adjacent vertices  $u$  and  $v$  of a graph  $G$ :

$$\mathbf{SP}(u, v) = \{W : W \text{ is a } uv \text{ shortest path}\},$$

$$\mathbf{TW}(u, v) = \{W : W \text{ is a } uv \text{ tolled walk}\}.$$

A *cycle* of length  $k$  in a graph  $G$  is a path  $C : v_1, v_2, \dots, v_k$  plus an edge between  $v_1$  and  $v_k$ . Each edge of  $G$  between two non-consecutive vertices of  $C$  is called a *chord*. The cycle of length  $k$  without chords is denoted by  $C_k$ .

Let  $P$  be an induced path of length at least two. We say that a graph obtained by adding an universal vertex to  $P$  is a *gem*.

A graph is *chordal* if every cycle of length at least 4 has a chord. Let **Chordal** denote the class of chordal graphs. Note that **Chordal** =  $\{C_k : k > 3\}$ -free.

A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line. Let **Interval** denote the class of interval graphs.

An *asteroidal triple* of a graph  $G$  is a set of 3 non-adjacent vertices of  $G$  such that each pair is connected by a path avoiding the neighborhood of the third vertex.

Lekkerkerker and Boland [9] proved that:

1. For any graph  $G$ :  $G$  is an interval graph if and only if  $G$  is chordal and contains no asteroidal triple.
2. **Interval** = **Chordal**  $\cap$   $\{F_1, F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}\}$ -free (see Figure 1).

**Definition 2.** **SP/TW** is the class formed by those graphs  $G$  such that for every pair of non-adjacent vertices  $u$  and  $v$  of  $G$ , every  $W \in \mathbf{SP}(u, v)$  dominates every  $W' \in \mathbf{TW}(u, v)$ , i.e.,  $W \in \mathbf{SP}(u, v)$  and  $W' \in \mathbf{TW}(u, v)$  implies  $W$  dominates  $W'$ .

In [1], the class of graphs called **Interval**<sup>+</sup> was introduced, which is the class of those chordal graphs  $G$  that contain none of the graphs  $F_2$  or  $F_4(n)_{n \geq 6}$  as an induced subgraph, and satisfy the following condition. If  $G$  has an induced subgraph  $H$  isomorphic to  $F_1$  ( $F_3(n)_{n \geq 6}$ ), then the distance in  $G$  between the vertices of  $F_1$  ( $F_3(n)_{n \geq 6}$ ) labelled  $u$  and  $w$  in Figure 1 is 2, and any vertex of  $G$  adjacent to both  $u$  and  $w$  is universal to  $F_1$  ( $F_3(n)_{n \geq 6}$ ).

The following theorem shows the relationship between the classes **SP/TW** and **Interval**<sup>+</sup>.

**Theorem 1** [1]. **SP/TW**  $\subseteq$  **Interval**<sup>+</sup>.

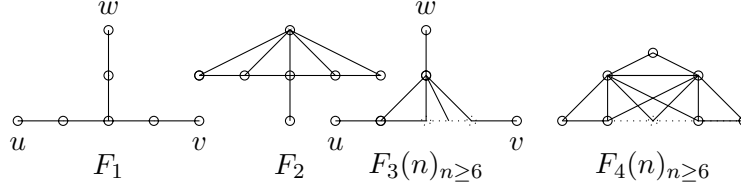


Figure 1. Chordal forbidden induced subgraphs for interval graphs.

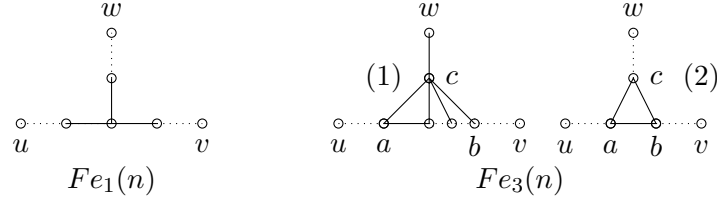


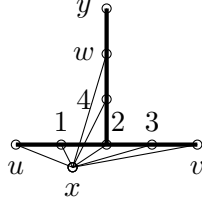
Figure 2. In  $Fe_1$  the distance between each pair of pending vertices is at least four. In graph (1) the distance between  $u$  and  $v$  is at least four, and vertices in the induced path between  $a$  and  $b$  plus  $c$  induce a gem. In graph (2) the distance between each pair of pending vertices is at least three, and the vertices  $a, b, c$  induce only a triangle.

Alcón left open the problem of determining if the class **SP/TW** is exactly the class **Interval<sup>+</sup>** [1].

**Conjecture 1** [1]. **SP/TW** = **Interval<sup>+</sup>**.

In what follows, we show that the conjecture is false, and we introduce class **IntervalE<sup>+</sup>** that will turn out to be **SP/TW**.

Let  $F'e_1$  be the graph in Figure 3. It is easy to check that  $F'e_1 \in \mathbf{Interval}^+$ . However, it is not in **SP/TW** since the shortest path  $u, x, v$  does not dominate the bold tolled walk  $u, 1, 2, 4, w, y, w, 4, 2, 3, v$  between  $u$  and  $v$ , since  $y \notin N[x]$ .

Figure 3.  $F'e_1 \in \mathbf{Interval}^+ \setminus \mathbf{SP/TW}$ .

In what follows we reformulate the conjecture, for this we will define class **IntervalE<sup>+</sup>**. Let us abuse the notation, when defining the following family of graphs.

Let  $Fe_1(n)$  be a tree with  $n$  vertices, such that its only pendent vertices are  $u, v$  and  $w$ , and the distance between each pendent vertex and the vertex of degree three is at least two. Note that  $Fe_1(7) = F_1$  (see Figure 2).

Let  $Fe_3(n)$  be a graph with  $n$  vertices, such that  $u, v$  and  $w$  are its pendent vertices,  $a, b, c$  are vertices of degree three such that the distance between  $a$  and  $u$ ,  $b$  and  $v$ ,  $c$  and  $w$ , respectively, is minimum (see Figure 2). If  $a, b, c$  induce a gem then the distance between  $c$  and  $w$  is exactly one (see (2) in Figure 2). In both cases, if no confusion can arise we will omit  $n$ .

Note that  $Fe_1$  and  $Fe_3$  is obtained from  $F_1$  and  $F_3$ , respectively, through a possible increase in the paths between a pair of its pending vertices.

Let  $\mathbf{IntervalE}^+$  be the class of those chordal graphs  $G$  that contain none of the graphs  $F_2$  or  $F_4(n)_{n \geq 6}$  as induced subgraph, and satisfy the following condition. If  $G$  has an induced subgraph  $H$  isomorphic to  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ), then the distance in  $G$  between the vertices of  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ) labelled  $u$  and  $w$  in Figure 2 is 2, and any vertex of  $G$  adjacent to both  $u$  and  $w$  is universal to  $Fe_1(n)_{n \geq 7}$  ( $Fe_3(n)_{n \geq 6}$ ). It is important to notice that  $\mathbf{IntervalE}^+ \subseteq \mathbf{Interval}^+$ .

**Conjecture 2.**  $\mathbf{SP/TW} = \mathbf{IntervalE}^+$ .

**Lemma 2.**  $\mathbf{SP/TW} \subseteq \mathbf{IntervalE}^+$ .

**Proof.** By Theorem 1,  $\mathbf{SP/TW} \subseteq \mathbf{Interval}^+$ . Clearly if  $G \in \mathbf{SP/TW}$  then  $G \in \mathbf{Chordal} \cap \{\mathbf{F}_2, \mathbf{F}_4(n)_{n \geq 6}\}$ -free.

Let  $Fe_1$  be a graph such that  $u, v, w$  are its pendent vertices,  $i$  be the vertex of degree three,  $W_{u,w} : u, 1, \dots, i, \dots, j, w$  be the  $uw$  induced path of  $Fe_1$ ,  $W'_{i,v} : i, j+1, \dots, k, v$  be the  $iv$  induced path of  $Fe_1$  (see Figure 4). Note that if  $|Fe_1| = 7$  then  $Fe_1 = F_1$ .

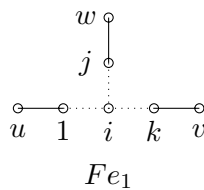


Figure 4.

Observe that if  $G$  contains  $Fe_1$  as an induced subgraph (Figure 4), the distance between  $u$  and  $w$  cannot be  $j+1$ , otherwise the  $uw$  shortest path  $u, 1, \dots, i, \dots, j, w$  does not dominate the tolled walk  $u, 1, \dots, i, \dots, k, v, k, \dots, i, \dots, j, w$ .

On the other hand, since  $G \in \mathbf{Interval}^+$  and  $Fe_1[\{i-2, i-1, i, i+1, i+2, j+1, j+2\}] = F_1$ , there exists a vertex  $y$  adjacent to every vertex of  $\{i-2, i-1, i, i+1, i+2, j+1, j+2\}$ .

In what follows, we show that  $y$  must be adjacent to every vertex of  $W_{u,w}[u, i-3] \cup W_{u,w}[i+3, w] \cup W'_{i,v}[j+3, v]$ , whenever those vertices exist.

By the way of contradiction, suppose that there exists at least a vertex of  $W_{u,w}[u, i-3]$  which is not adjacent to  $y$ . Among all, we choose the one closest to  $i-3$ , let us say  $a$  ( $a$  may be  $i-3$ ). But then the shortest path  $j+2, y, i+2$  does not dominate the tolled walk  $j+2, \dots, i, i-1, \dots, a, a+1, \dots, i, i+1, i+2$ .

Thus  $y$  is adjacent to every vertex of  $Fe_1$ .

A reasoning analogous to the one applied in the case of  $Fe_1$  shows that if  $G$  has an induced subgraph  $H$  isomorphic to  $Fe_3(n)_{n \geq 6}$  then the distance in  $G$  between the vertices of  $Fe_3(n)_{n \geq 6}$  labelled  $u$  and  $w$  in Figure 4 is 2. The reader should have no problem verifying the details. Therefore  $\mathbf{SP}/\mathbf{TW} \subseteq \mathbf{IntervalE}^+$ . ■

## 2. MAIN RESULTS

In this section, we characterize the class  $\mathbf{SP}/\mathbf{TW}$ . We start with a preliminary observation that follows directly from the definition of tolled walks.

**Observation 3.** *Let  $G$  be a graph, and  $T : u = y_0, y_1, \dots, y_m = v$  be a  $uv$  tolled walk. It follows from the definition of tolled walk that if there exists a vertex in  $T$  such that  $y_k \notin N[u] \cup N[v]$  then  $u$  and  $y_k$  are in the same connected component of  $G[T] - N[v]$ , and also  $v$  and  $y_k$  are in the same connected component of  $G[T] - N[u]$ .*

Note the following properties about chordal graphs.

**Lemma 4.** *Let  $u$  and  $v$  be non-adjacent vertices of a chordal graph  $G$ , and  $P$  and  $Q$  be induced paths between  $u$  and  $v$ . If  $P$  has a vertex  $y \notin Q$ , and  $a \in P \cap Q$  is the vertex closest to  $y$  in  $P[u, y]$  and  $b \in P \cap Q$  is the vertex closest to  $y$  in  $P[v, y]$ , then  $Q[a, b] \cap P = \{a, b\}$ .*

**Proof.** Suppose that there exists at least a vertex  $c \in Q(a, b) \cap P$ . Clearly  $c \notin P[a, b]$  because  $a$  and  $b$  are the vertices closest to  $y$  in  $P[u, y]$  and  $P[v, y]$ . Hence there do not exist chords between  $c$  and vertices of  $P[a, b]$ . But then  $G[P[a, b] \cup Q[a, b]]$  contains  $C_r$  ( $r \geq 4$ ) as an induced subgraph, a contradiction. ■

**Lemma 5.** *Let  $G$  be a chordal graph,  $P$  and  $Q$  be two induced path between  $u$  and  $v$  non-adjacent vertices of  $G$ . If  $Q$  is a shortest path, then every vertex in  $P - Q$  is adjacent to at most three vertices of  $Q$ .*

**Proof.** Suppose that there exists at least a vertex  $c \in P$  that is adjacent to  $a$  and  $b$  vertices in  $Q$  such that  $|Q[a, b]| > 3$ . Since  $G[Q[a, b] \cup \{c\}]$  is a chordal graph,  $c$  is adjacent to every vertex of  $Q[a, b]$ . It is clear that  $Q - Q(a, b) + ac + bc$

is a  $uv$  walk with at least two vertices fewer than  $Q$ . By Remark 1, it contains a  $uv$  induced path  $Q_1$  such that  $|Q_1| < |Q|$ , a contradiction. ■

We are now able to prove the following.

**Theorem 6.**  $\mathbf{SP/TW} = \mathbf{IntervalE}^+$ .

**Proof.** By Lemma 2  $\mathbf{SP/TW} \subseteq \mathbf{IntervalE}^+$ .

To prove that  $\mathbf{IntervalE}^+ \subseteq \mathbf{SP/TW}$ , let us suppose, on the contrary, that  $G \in \mathbf{IntervalE}^+$ , but  $G \notin \mathbf{SP/TW}$ .

As  $G \notin \mathbf{SP/TW}$  there exist two non-adjacent vertices  $u$  and  $v$ , a  $uv$  shortest path  $W$  and a  $uv$  tolled walk  $T : u = y_0, y_1, \dots, y_m = v$  satisfying that  $W$  does not dominate  $T$ . Thus, there is some internal vertex of  $T$  that is neither a vertex of  $W$  nor adjacent to any internal vertex of  $W$ . Let  $y$  be a vertex of  $T - W$  such that it is not adjacent to any vertex of  $W$ . We can assume that  $y \neq y_1, y_{m-1}$ , otherwise  $G[W \cup T]$  contains as an induced subgraph a cycle of size at least four.

Let  $P : u = z_0, z_1, \dots, z_l = v$  be a shortest path in  $G[T]$  from  $u$  to  $v$ . Note that  $|P \cap W| \geq 2$ . Since  $T$  is a  $uv$  tolled walk,  $z_1 = y_1$  and  $z_{l-1} = y_{m-1}$ . Note that  $z_1 \neq z_{l-1}$ , and then  $|V(P)| \geq 4$ .

In the following claim, we show that  $y$  cannot be an internal vertex of  $P$ .

**Claim 7.**  $y \notin P$ .

**Proof.** Assume the contrary,  $y \in P$ .

Clearly,  $|P \cap W| \geq 2$ . Let  $a, b$  be two vertices of  $P \cap W$  such that  $a, y, b$  appear in this order in  $P$ ,  $d_P(a, y)$  and  $d_P(b, y)$  is minimum. Note that  $a$  may be  $u$  and  $b$  may be  $v$ .

Since  $y$  is not a vertex of  $W$  nor adjacent to any internal vertex of  $W$ , it follows that  $d(a, y) > 1$  and  $d(b, y) > 1$ . By Lemma 4,  $P \cap W[a, b] = \{a, b\}$ . On the other hand,  $G[P[a, b] \cup W[a, b]]$  is a chordal graph, there exist chords between vertices of  $P[a, b]$  and vertices  $W[a, b]$ . Thus there exists at least a chord between a vertex of  $W(a, b)$  and  $y$ , a contradiction.

Therefore  $y \notin P$ . □

In what follows we will analyze two cases, depending of  $y$  is or is not adjacent to vertices of  $P$ . Observe that if  $a \in P$  is adjacent to  $y$  then  $a \notin W$ . We will present some figures that could help in the analysis of cases. In such figures, we will allow ourselves to omit some edges between two adjacent vertices.

*Case 1.*  $y$  is adjacent to some vertex of  $P$ . Note that if  $y$  is adjacent to two non-consecutive vertices of  $P$ , let  $a$  and  $b$ , then since  $G[P[a, b] \cup \{y\}] \neq C_r$  (for some  $r > 3$ ), it follows that  $y$  is adjacent to every vertex of  $P[a, b]$ .

By before exposed, we can analyze two situations:  $y$  is adjacent to two consecutive vertices or  $y$  is adjacent to one and only one vertex of  $P$ .

*Case 1.1.*  $y$  is adjacent to two consecutive vertices  $z_i, z_{i+1}$  of  $P$ . By the choice of  $y$ ,  $z_i, z_{i+1} \notin W$ . Let  $a \in P[u, z_i] \cap W$  and  $b \in P[z_{i+1}, v] \cap W$  such that the distance in  $P$  between  $a$  ( $b$ ) and  $z_i$  ( $z_{i+1}$ ) is minimum. Note that  $z_i$  ( $z_{i+1}$ ) may be adjacent to  $a$  ( $b$ ), and  $a$  ( $b$ ) might be  $u$  ( $v$ ), respectively.

Since  $G[P[a, b] \cup W[a, b]]$  is a chordal graph, there exist at least a vertex in  $W[a, b]$  which is adjacent to  $z_i$  and  $z_{i+1}$ . In order to fix ideas, let us consider  $W[a, b] : a, x_1, \dots, x_{n-1}, b$ .

Let  $p$  and  $q$  be the first and the last index such that  $z_i, z_{i+1}$  are simultaneously adjacent to  $x_p$  and  $x_q$  (see Figure 5). Since  $W \in \mathbf{SP}$ , by Lemma 5,  $q \leq p + 2$ .

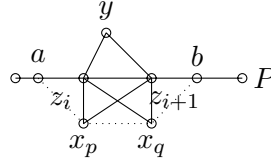


Figure 5. The dotted line between  $a$  and  $b$  represents  $W[a, b]$ .

Since  $G[P[a, z_i] \cup W[a, x_p]]$  is a chordal graph, then  $z_i$  must be adjacent to  $x_{p-1}$  or  $x_p$  must be adjacent to  $z_{i-1}$ . By similarity,  $G[P[b, z_{i+1}] \cup W[b, x_q]]$  is a chordal graph, then  $z_{i+2}$  must be adjacent to  $x_q$  or  $x_{q+1}$  must be adjacent to  $z_{i+1}$ . Thus  $G[\{y, z_i, z_{i+1}\} \cup W[x_{p-1}, x_{q+1}]] = F_4$  or  $G[\{y, z_{i-1}, z_i, z_{i+1}, z_{i+2}\} \cup W[x_p, x_q]] = F_4$  or  $G[\{y, z_i, z_{i+1}, z_{i+2}\} \cup W[x_{p-1}, x_q]] = F_4$  or  $G[\{y, z_{i-1}, z_i, z_{i+1}\} \cup W[x_p, x_{q+1}]] = F_4$ , a contradiction.

*Case 1.2.* Suppose that  $y$  is adjacent to one and only one vertex of  $P$ . Let us consider two situations:  $y$  is adjacent to  $z_i$  for some  $i \in \{2, \dots, l-2\}$  or  $y$  is adjacent to  $z_1$  or  $z_{l-1}$ .

*Case 1.2.1.*  $y$  is adjacent to  $z_i$  for  $i \neq 1, l-1$ . Let  $a$  and  $b$  be vertices in  $W \cap P$  such that  $a \in P[u, z_i]$ ,  $b \in P[z_i, v]$ ,  $d_P(a, z_i)$  and  $d_P(z_i, b)$  is minimum.

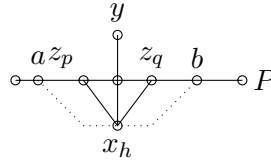
Since  $G[P[a, b] \cup W[a, b]]$  is a chordal graph, we have that there exist chords between vertices of  $P[a, b]$  and  $W[a, b]$ . Let us consider  $W[a, b] : a, x_1, \dots, x_{n-1}, b$ .

By Lemma 5, as  $W \in \mathbf{SP}$ , if  $z_i$  is adjacent to two vertices ( $x_p, x_q$  at a maximum distance) in  $W[a, b]$ , those vertices must be at a distance of at most 2 in  $W$ .

Let us consider two cases depending of  $z_i$  is or is not adjacent to one and only one vertex of  $W[a, b]$ .

*Case 1.2.1.1.*  $z_i$  is adjacent to one and only one vertex of  $W[a, b]$ . Thus  $a \neq z_{i-1}$  and  $b \neq z_{i+1}$ . Let  $x_h$  be the vertex of  $W[a, b]$  such that  $z_i$  is adjacent to  $x_h$ . Since  $G$  is a chordal graph, let  $z_p \in P[a, z_i]$  and  $z_q \in P[z_i, b]$  such that  $x_h$  is adjacent to  $z_p$  and  $z_q$ , and the distance in  $P$  between  $z_i$  and  $z_p, z_q$  is maximum (see Figure 6).



Figure 6. The dotted line between  $a$  and  $b$  represents  $W[a, b]$ .

If  $p \leq i - 2$  and  $q \geq i + 2$  then  $G[P[z_p, z_q] \cup \{y, x_h\}]$  contains  $F_2$  as induced subgraph, a contradiction.

Suppose that  $p = i - 1$  and  $q = i + 1$ . By the choice of  $p$  and  $q$ ,  $x_h$  is not adjacent to any vertex of  $P[a, z_p] \cup P[z_q, b]$ . Hence  $G[P[a, z_p] \cup W[a, x_h]]$  contains  $C_r$  ( $r > 3$ ) as induced subgraph, which is a contradiction, or there exists  $x_{h-1}$  in  $W(a, x_h]$  such that it adjacent to  $z_p$ . Analogously there exists  $x_{h+1}$  in  $W[x_h, b)$  such that  $z_q$  must be adjacent to  $x_{h+1}$ . Thus  $G[\{z_p, z_q, z_i, y, x_h, x_{h+1}, x_{h-1}\}] = F_2$ , a contradiction.

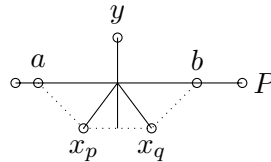
Suppose that  $p = i - 1$  and  $q = i$  (by symmetry  $p = i$  and  $q = i + 1$ ).

Observe that  $z_{p-1}$  may be  $a$  but  $z_{q+1} \neq b$ . Since  $x_h$  is not adjacent to  $z_{q+1}$  and  $z_i$  is not adjacent to any vertex of  $W[x_h, b]$ ,  $G[P[z_i, b] \cup W[x_h, b]]$  contains  $C_r$  ( $r > 3$ ) as induced subgraph, a contradiction.

*Case 1.2.1.2.*  $z_i$  is adjacent at least two vertices of  $W[a, b]$ . Note that those vertices may be  $a$  and  $b$ .

Let  $p$  and  $q$  be the first and the last index such that  $z_i$  is adjacent to  $x_p$  and  $x_q$ . By Lemma 5, since  $W \in \mathbf{SP}$ ,  $q \leq p + 2$ .

Note that  $G[W[x_p, x_q] \cup \{z_i\}]$  induces a triangle or a gem. We can affirm that  $G[W[a, b] \cup \{y\} \cup P]$  contains  $F_3$  as induced subgraph whose pending vertices are  $y, u$  and  $v$  (see Figure 7). Since  $G \in \mathbf{IntervalE}^+$ , there exists  $w \in G$  universal vertex of  $F_3$ . Clearly every shortest path between  $u$  and  $v$  has 3 vertices. Thus, we arrive to a contradiction if  $|W| > 3$ , or if  $|W| = 3$ , results  $x \in W \setminus \{u, v\}$  must be adjacent to  $y$ , which is also a contradiction.

Figure 7. The dotted line between  $a$  and  $b$  represents  $W[a, b]$ .

*Case 1.2.2.* Suppose that  $y$  is adjacent only to  $z_1 = y_1$  (by symmetry  $z_{l-1} = y_{m-1}$ ). As  $T \in \mathbf{TW}$ , there exists an induced path between  $y$  and  $v$  in  $G[T]$  that does not have vertices neighbors of  $u$ . Assume that  $P_1$  is a shortest among the

paths join  $y$  to a vertex in  $P$  such that  $P_1 \cap N[u] = \emptyset$ . Let  $P_1 : y = t_1, t_2, \dots, t_i$  with  $t_i \in P$ . Note that  $t_2 \neq z_1$  and  $t_i$  may be adjacent to  $z_1$ .

No vertex in  $P_1[y, t_{i-1})$  is adjacent to a vertex in  $P(z_1, t_i)$  because of the minimality of  $P_1$ , thus every vertex in  $P_1[y, t_{i-1})$  is adjacent to  $z_1$ ,  $t_{i-1}$  is adjacent to  $z_1$  as well, and every vertex in  $P_1[y, t_{i-1}]$ , and every vertex in  $P[z_1, t_i]$  is adjacent to  $t_{i-1}$ .

In what follows, we can assume that  $t_i = z_2$ .

Let  $a$  be a vertex in  $P \cap W$  such that  $d_P(u, a)$  is minimum. By the choice of  $y$ ,  $a \neq z_1$ . Note that  $a$  may be  $z_2$ . Let  $W[u, a] : u, x_1, \dots, a$ .

Note that if  $|W| = 3$  then  $x_1$  must be adjacent to every vertex of  $P$  (see Figure 8). Let  $p$  the first index such that  $x_1$  is adjacent to  $t_p$ . Clearly  $p \neq 1$ . Thus  $G[P_1[t_{p-1}, t_i = z_2] \cup \{u, z_1, x_1, z_3\}] = F_4$ , a contradiction.

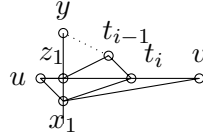


Figure 8. The dotted line between  $y$  and  $t_{i-1}$  represents the induced path  $P_1$ , and the filled line between  $t_i$  and  $v$  represents  $P[t_i, v]$ .

Suppose that  $|W| \neq 3$ . Note that  $x_1$  must be adjacent to  $z_1$ .

First, we assume that at least a vertex of  $W[u, a]$  is adjacent to a vertex of  $P_1$ . Let  $x_j$  be the first index such that  $x_j$  is adjacent to a vertex of  $P_1$ , and let  $p$  and  $q$  the first and the last index such that  $x_j$  is adjacent to  $t_p$  and  $t_q$ . Note that since  $G[\{t_p, z_1, x_j\} \cup W[x_1, x_j]]$  is a chordal graph,  $x_j$  must be adjacent to  $z_1$ .

Suppose that  $p \neq q$ . Note that  $G[P_1[t_p, t_q] \cup \{x_j\}]$  is a triangle or a gem whose universal vertex is  $x_j$ . Assume that  $x_j$  is not adjacent to any vertex of  $P[t_i, v]$ . Note that  $x_{j-1}$  cannot be adjacent to a vertex of  $P[t_i, v]$ , since  $G[P_1[t_q, t_i] \cup P \cup \{x_{j-1}, x_j\}]$  is a chordal graph. We can affirm that there exists an induced path in  $G[P_1 \cup P[t_i, v]]$ , it follows that  $G[W[x_1, x_j] \cup \{u\} \cup P_1[t_{p-1}, t_i] \cup P[t_i, v]]$  contains as induced subgraph to  $Fe_3$  or  $Fe_1$  with pending vertices  $u, v, t_{p-1}$  (see Figure 9).

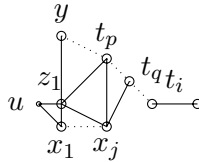


Figure 9. The dotted line between  $x_1$  and  $x_j$  represents the induced path  $W[x_1, x_j]$ , and the filled line between  $t_i$  and  $v$  represents  $P[t_i, v]$ .

But then as  $G \in \mathbf{IntervalE}^+$ , there exists  $w$  such  $u, w, v$  is a shortest path, a contradiction because  $|W| > 3$ . Thus  $x_j$  must be adjacent to a vertex of  $P[t_i, v]$ . Moreover, it must be adjacent to every vertex of  $P[t_i, v]$  otherwise as before exposed  $G[W[x_1, x_j] \cup \{u\} \cup P_1[t_{p-1}, t_i] \cup P[t_i, v]]$  contains as induced subgraph  $Fe_3$  or  $Fe_1$  with pending vertices  $u, v, t_{p-1}$  and we arrive to a contradiction.

Since  $x_j$  is adjacent to every vertex of  $P[t_i, v]$ , results  $q = i$ , and then  $G[\{x_{j-1}, x_j, z_1\} \cup P_1[t_{p-1}, t_i]] = F_4$ , a contradiction.

Now, assume that  $p = q$ . Note that  $x_j$  cannot be adjacent to any vertex of  $P[t_i, v]$ , otherwise  $p = i$  and then  $z_1$  is adjacent to  $t_i$  both non-consecutive vertices of  $W$ .

On the other hand, there exists  $x_{j+1}$  and it must be adjacent to a vertex of  $P_1[t_p, t_i] \cup P[t_i, v]$  since  $G[W[x_j, a] \cup P_1[t_p, t_i] \cup P[a, t_i]]$  is a chordal graph. Moreover it must adjacent to  $t_p$  (see Figure 10).

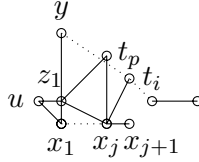


Figure 10.

But then  $G[\{z_1, t_p, t_{p-1}, t_p, x_{j-1}, x_j, x_{j+1}\}] = F_4$ , a contradiction.

Now, none vertex of  $W[u, a]$  is adjacent to a vertex of  $P_1$ . Thus,  $G[\{z_1, t_{i-1}\} \cup P[t_i, a] \cup W[x_1, a]]$  or  $G[\{z_1, t_{i-1}\} \cup P[z_1, a] \cup W[x_1, a]]$  contains  $C_r$  ( $r > 3$ ) depending of  $a \in P[t_i, v]$  or  $a \in P[z_1, t_i]$ , a contradiction.

*Case 1.2.2.2.*  $t_i = z_2$ . This case can be treated similarly to *Case 1.2.2.1*.

*Case 2.*  $y$  is adjacent to no vertex of  $P$ . Note that  $P \cap N[y] = \emptyset$ . Since  $T$  is a  $uv$  tolled walk, by Observation 3, there exists an induced path between  $u$  and  $y$  in  $G[T]$  avoiding the neighborhood of  $v$ , and also there exists an induced path between  $v$  and  $y$  in  $G[T]$  avoiding the neighborhood of  $u$ . Those paths, together with  $P$  allow us to state that  $u, v, y$  is an asteroidal triple.

Hence, we assume that there exist three induced paths of  $G[T]$ :  $P$  between  $u$  and  $v$ ;  $P_1$  between  $u$  and  $y$ ;  $P_2$  between  $v$  and  $y$ ; and three vertices  $a \in V(P) \cap V(P_1)$ ;  $b \in V(P) \cap V(P_2)$ ; and  $c \in V(P_1) \cap V(P_2)$ ; such that  $V(P) \cap N[y] = \emptyset$ ,  $V(P_1) \cap N[v] = \emptyset$ ,  $V(P_2) \cap N[u] = \emptyset$ , and the distance between  $a$  and  $u$ , between  $b$  and  $v$ , and between  $c$  and  $y$  in the respective paths is maximum.

Note that  $u, y_1 \in P_1$  and  $v, y_{m-1} \in P_2$  since  $T$  is a  $uv$  tolled walk.

Without loss of generality, we can assume that  $(P \cap P_1)[u, a]$ ,  $(P \cap P_2)[b, v]$  and  $(P_1 \cap P_2)[c, y]$  are induced paths (see Figure 11). By the choice of  $a, b$  and  $c$ , and since  $G$  contains no  $F_2$  and  $F_4$  as induced subgraphs, we can affirm that  $y \neq c$ .

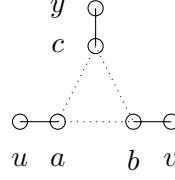


Figure 11.

*Case 2.1.* Suppose that  $P \cup P_1 \cup P_2$  is a tree. Clearly  $a = b = c$ . Note that since  $y$  is not adjacent to any vertex of  $P$ , the distance in  $P \cup P_1 \cup P_2$  between  $y$  and  $a$  is at least two. Also the distance between  $u, v$  and  $a$  is at least two because  $u, y, v$  is an asteroidal triple. Thus  $P \cup P_1 \cup P_2 = Fe_1$ .

On the other hand, as  $G \in \mathbf{IntervalE}^+$ , there exists  $w \in G$  such that  $w$  is an universal vertex of  $G[P \cup P_1 \cup P_2 \cup \{w\}]$ . Thus the distance in  $G$  between  $u$  and  $v$  is two, and then since  $W$  is a shortest path it follows that  $|W|$  must be three. More clearly,  $W = u, x, v$ . Again as a result of  $G \in \mathbf{IntervalE}^+$ , we can conclude that  $x$  must be an universal vertex of  $G[P \cup P_1 \cup P_2 \cup \{x\}]$ . But then  $y$  must be adjacent to  $x$ , which is a contradiction by the choice of  $y$ .

*Case 2.2.* Suppose that  $P \cup P_1 \cup P_2$  is not a tree. We can conclude that  $G[P \cup P_1 \cup P_2]$  must contain  $Fe_3(n)$  as induced subgraph being  $y, u$  and  $v$  its pending vertices, and  $a, b, c$  are the labelled vertices in Figure 2.

Let us consider two cases, depending of  $a$  is or is not adjacent to  $b$ .

*Case 2.2.1.* Suppose that  $a$  is not adjacent to  $b$ . Then  $a$  is adjacent to  $c$  and  $b$  is adjacent to  $c$ . Note that  $y$  must be adjacent to  $c$ , otherwise  $G[P - P[a, b] \cup P_1 \cup P_2]$  contains  $Fe_1$  as induced subgraph.

Since  $G \in \mathbf{IntervalE}^+$ , there exists a vertex  $x$  in  $G$  such that  $x$  is an universal vertex of  $Fe_3(n)$ , which has  $y, u, v$  as its pendent vertices. Thus, the distance between  $u$  and  $v$  in  $G$  must be two. Hence  $|W| = 3$ . Let  $W = u, x, v$ . Clearly  $x$  must be adjacent to every vertex of  $Fe_3(n)$ , in particular  $x$  is adjacent to  $y$ , a contradiction.

*Case 2.2.2.* Suppose that  $a$  is adjacent to  $b$ . We can assume that  $a$  is adjacent to  $c$  and  $b$  is adjacent to  $c$ . This case can be treated similarly to *Case 2.2.1*.

Therefore  $\mathbf{SP}/\mathbf{TW} = \mathbf{IntervalE}^+$ . ■

### 3. CONCLUSIONS

In this article, we have obtained a characterization of the graphs in which, for every pair of non-adjacent vertices  $u$  and  $v$ , every  $uv$  shortest path dominates

every  $uv$  tolled walk. Thus, we solve a problem of Alcón concerning the class **SP/TW**.

In [1, 8, 10] it was displayed that the notion of domination between different types of walks plays a central role in characterizations of graph classes. It is interesting to ask what other classes of graphs can be characterized by path domination. We summarize the results obtained so far in Table 1, and then we will show that other problems related to path domination could be studied. For this, let us introduce the notation **IP**, **P**, **W**, **WTW** and  $\mathbf{l}_k$  for  $k = 2, 3$  to refer to the set of different types of walks connecting two non-adjacent vertices  $u$  and  $v$  of a graph  $G$ .

$\mathbf{IP}(u, v) = \{W : W \text{ is an } uv \text{ induced path (or monophonic [6]), i.e., a } uv \text{ path such that two of its vertices are adjacent if and only if they are consecutive}\}$ ,

$\mathbf{P}(u, v) = \{W : W \text{ is a } uv \text{ path}\}$ ,

$\mathbf{W}(u, v) = \{W : W \text{ is a } uv \text{ walk}\}$ ,

$\mathbf{WTW}(u, v) = \{W : W \text{ is a } uv \text{ weakly toll walk, i.e., a } uv \text{ walk such that } u(v) \text{ has a single neighbor in } W \text{ that can appear more than once in } W\}$  [5],

$\mathbf{l}_k(u, v) = \{W : W \text{ is a } uv \text{ } l_k \text{ path, i.e., a } uv \text{ induced path with length at most } k\}$  [7],

$\mathbf{m}_3(u, v) = \{W : W \text{ is a } uv \text{ } m_3 \text{ path, i.e., a } uv \text{ induced path with length at least three}\}$  [4].

**Definition 3.** Let  $\mathbf{A}, \mathbf{B} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}, \mathbf{l}_2, \mathbf{l}_3, \mathbf{m}_3\}$ .  $\mathbf{A/B}$  is the class formed by those graphs  $G$  such that for every pair of non-adjacent vertices  $u$  and  $v$  of  $G$ , every  $W \in \mathbf{A}(u, v)$  dominates every  $W' \in \mathbf{B}(u, v)$ , i.e.,  $W \in \mathbf{A}(u, v)$  and  $W' \in \mathbf{B}(u, v)$  implies  $W$  dominates  $W'$ .

We denote by **Ch** the class of chordal graphs, by **Int** the class of interval graphs, by **Sup** the class of superfragile graphs, by **Pt**<sup>−</sup> the class **Ptolemaic**<sup>−</sup>, by  $\mathbf{F}_{2,4,5,6,7} = \{\mathbf{F}_2, \mathbf{F}_4(\mathbf{n})_{\mathbf{n} \geq 6}, \mathbf{F}_5(\mathbf{n})_{\mathbf{n} \geq 8}, \mathbf{F}_6(\mathbf{n})_{\mathbf{n} \geq 7}, \mathbf{F}_7(\mathbf{n})_{\mathbf{n} \geq 7}\}$ , and by  $\mathbf{F}_{2,3,4,5} = \{\mathbf{F}_2, \mathbf{F}_3(\mathbf{n})_{\mathbf{n} \geq 6}, \mathbf{F}_4(\mathbf{n})_{\mathbf{n} \geq 6}, \mathbf{F}_5(\mathbf{n})_{\mathbf{n} \geq 8}\}$ , by  $\mathbf{M}_3\mathbf{W} = \{\mathbf{P}_4, \mathbf{A}, \mathbf{gem} \cup \mathbf{K}_2, \mathbf{C}_5, \mathbf{X}_{58}, \mathbf{X}_{96}, \mathbf{F}_3(6)\}$ -free, by  $\mathbf{M}_3\mathbf{P} = \{\mathbf{H}, \mathbf{D}, \mathbf{Antenna}, \mathbf{X}_5\}$ -free, by  $\mathbf{M}_3\mathbf{M}_3 = \{\mathbf{C}_{\mathbf{n} > 5}, \mathbf{D}, \mathbf{Antenna}, \mathbf{X}_5, \mathbf{5-pan}, \mathbf{X}_{37}\}$ -free [8, 10]. The classes **Ptolemaic**<sup>−</sup> and **g** – **Chordal** are defined in [1].

The classes **SP/SP**, **SP/TW**,  $\mathbf{l}_3/\mathbf{SP}$  are not hereditary classes. In [1] the class **SP/SP** was characterized as the class **g** – **Chordal** and in this paper we characterize the class **SP/TW** as the class **IntervalE**<sup>+</sup>.

Natural questions arise.

1. We know that  $\mathbf{l}_3/\mathbf{SP} \subseteq \{\mathbf{C}_4, \mathbf{C}_5\}$ -free and  $\mathbf{l}_3/\mathbf{SP} \neq \mathbf{g} - \mathbf{Chordal}$  [8]. Is it possible to characterize the non-hereditary class of graphs  $\mathbf{l}_3/\mathbf{SP}$ ?
2. Do  $\mathbf{A/m}_3$  and  $\mathbf{m}_3/\mathbf{A}$ , for  $\mathbf{A} \in \{\mathbf{l}_k, \mathbf{SP}, \mathbf{TW}, \mathbf{WTW}\}$  give rise to characterizations of some classes of graphs?

-	$l_2$	$l_3$	SP	IP	P	TW	WTW	W	$m_3$
$l_2$	$\{C_4\}$ -free	Ch	$\{C_4\}$ -free	Ch	$Pt^-$	$Ch \cap F_{2,4,5,6,7}$ -free	$Ch \cap \{\text{chair, dart, } F_4(6)\}$ -free	Sup	
$l_3$	$\{C_4, C_5, C_6\}$ -free	Ch		Ch	$Pt^-$	$Ch \cap F_{2,3,4,5}$ -free	$Int \cap \{\text{chair, dart}\}$ -free	Sup	
SP	$\{C_4\}$ -free	$\{C_4, C_5, C_6\}$ -free	$g - Ch$	Ch	$Pt^-$	$IntE^+$	$Int \cap \{\text{chair, dart}\}$ -free	Sup	
IP	Ch	Ch	Ch	Ch	$Pt^-$	Int	$Int \cap \{\text{chair, dart}\}$ -free	Sup	HHD-free
P	Ch	Ch	Ch	Ch	$Pt^-$	Int	$Int \cap \{\text{chair, dart}\}$ -free	Sup	HHD-free
TW	Ch	Ch	Ch	Ch	$Pt^-$	Int	$Int \cap \{\text{chair, dart}\}$ -free	Sup	
WTW	Ch	Ch	Ch	Ch	$Pt^-$	Int	$Int \cap \{\text{chair, dart}\}$ -free	Sup	
W	Ch	Ch	Ch	Ch	$Pt^-$	Int	$Int \cap \{\text{chair, dart}\}$ -free	Sup	HHD-free
$m_3$				$M_3P$	$M_3P$			$M_3W$	$M_3M_3$

Table 1. With  $A \in \{l_2, l_3, SP, IP, P, TW, WTW, W, m_3\}$  in the first column and  $B \in \{l_2, l_3, SP, IP, P, TW, WTW, W, m_3\}$  in the first row, the table describes each one of the graph classes  $A/B$ .

### Acknowledgment

We express our gratitude to the anonymous referees for their thorough reviewing, which has resulted in an improved paper.

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Received 3 April 2023

Revised 30 October 2023

Accepted 4 November 2023

Available online 16 November 2023