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ON A PROBLEM OF L. ALCÓN CONCERNING PATH DOMINATION

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Abstract

A walk W between two non-adjacent vertices in a graph G is called tolled if the first vertex of W is among vertices from W adjacent only to the second vertex of W, and the last vertex of W is among vertices from W adjacent only to the second-last vertex of W. In this article, we solve a problem posed by Alcón that seeks to characterize the class of graphs such that for every pair of non-adjacent vertices u and v, every uv shortest path dominates every uv tolled walk.

Keywords: domination, walks, interval graphs.

2020 Mathematics Subject Classification: 05C38, 05C75, 05C69, 05C12.

Introduction

In [1] the concept of path domination was introduced to understand the structure of those graphs in which for every pair of non-adjacent vertices u and v, and every pair of uv walks W and W', each internal vertex of W' is adjacent to some internal vertex of W.

Based in this concept, new characterization of standard classes like chordal, interval, superfragile and HHD-free graphs [3] have been obtained in [1, 8, 10].

Also, in [1], p. 1032, Alcón suggests the following problem: determine if the graphs in which every uv shortest path dominates every uv tolled walk are exactly the ones in **Interval**$^+$.

A graph G is in **Interval**$^+$, if G is a chordal graph that contains none of the graphs F_2 or $F_4(n)_{n\geq 6}$ as induced subgraph, and satisfy the following condition. If G has an induced subgraph H isomorphic to F_1 ($F_3(n)_{n\geq 6}$), then the distance in G between the vertices of F_1 ($F_3(n)_{n\geq 6}$) labelled u and w in Figure 1 is 2, and any vertex of G adjacent to both u and w is universal to F_1 ($F_3(n)_{n\geq 6}$).

In this paper, we show that there exist graphs in which every uv shortest path dominates every uv tolled walk that are not in **Interval**$^+$. Thus, Alcón's Conjecture results false. We propose a reformulation of the problem. For this, we introduce the class **IntervalE**$^+$ that is the class of those chordal graphs G that contain none of the graphs F_2 or $F_4(n)_{n\geq 6}$ as induced subgraph, and satisfy the following condition. If G has an induced subgraph H isomorphic to $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$), then the distance in G between the vertices of $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$) labelled u and w in Figure 2 is 2, and any vertex of G adjacent to both u and w is universal to $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$). And then, we prove that the graphs in which every uv shortest path dominates every uv tolled walk are exactly the ones in **IntervalE**$^+$.

The paper is organized as follows. In Section 1, we give necessary definitions, in Section 2, the main result is presented. Conclusions are developed in Section 3.

1. Preliminaries

We introduce the necessary definitions in this section.

All the graphs in this paper are finite, undirected, simple, and connected. We use standard graph terminology [11].

Let G be a graph. The subgraph induced in G by a subset $S \subseteq V(G)$ is denoted by $G[S]$. For any vertex v of G, the *neighborhood* of v is denoted by $N[v] = \{u \in V(G) |\ uv \text{ is an edge of G}\} \cup \{v\}$. We denoted $|V(G)|$ by $|G|$ and by $G(n)$ a graph with n vertices.

Let us introduce the following definitions. A *uv walk* is a sequence $W : u = v_0, v_1, \ldots, v_{k-1}, v_k = v$ whose terms are vertices, not necessarily distinct, such that u is adjacent to v_1, v_i is adjacent to v_{i+1} for $i \in \{1, \ldots, k-2\}$, and v_{k-1} is adjacent to v. The vertices u and v are called *ends of the walk*, and the vertices $v_1, \ldots, v_{k-1}$ are its *internal vertices*. The integer k is the *length of the walk*.

A *uv path* is a uv walk with all its vertices distinct. The *distance* $d_G(u, v)$ between vertices u and v is the minimum number of edges on a path connecting these vertices. If no confusion can arise we will omit the index G.

Let $W : v_0, v_1, \ldots, v_{k-1}, v_k$ be a path, $W[a, b] : a = v_i, \ldots, v_j = b$ denote the section of the path W between a and b. Let $W(a, b] = W[a, b] \setminus \{a\}$, $W[a, b) = W[a, b] \setminus \{b\}$, and $W(a, b) = W[a, b] \setminus \{a, b\}$.

A *uv shortest path* (or *geodesic* [6]) is a uv path of length $d(u, v)$.

A *uv tolled walk* is a uv walk, $u = v_0, v_1, \ldots, v_{k-1}, v_k = v$, satisfying that u is adjacent only to the vertex v_1, v is adjacent only to the vertex v_{k-1}, $\{v_1\} \cap \{v_2, \ldots, v_{k-1}\} = \emptyset$ and $\{v_{k-1}\} \cap \{v_1, \ldots, v_{k-2}\} = \emptyset$ [2]. Note that v_1 may be v_{k-1}, but if $v_1 = v_{k-1}$ then $k = 2$.

It is clear that every shortest path is a tolled walk.

Remark 1. Every uv walk contains a uv induced path.

Definition 1. The uv walk $W : u, v_1, \ldots, v_{m-1}, v$ *dominates* the uv walk W' : $u, v'_1, \ldots, v'_{n-1}, v$ if every internal vertex of W' is adjacent to some internal vertex of W or belongs to W.

Now, we introduce the notation **SP** and **TW** to refer to the set of shortest paths and tolled walks respectively, which connects two non-adjacent vertices u and v of a graph G:

$\mathbf{SP}(u, v) = \{W : W \text{ is a } uv \text{ shortest path}\}$,

$\mathbf{TW}(u, v) = \{W : W \text{ is a } uv \text{ tolled walk}\}$.

A *cycle* of length k in a graph G is a path $C : v_1, v_2, \ldots, v_k$ plus and edge between v_1 and v_k. Each edge of G between two non-consecutive vertices of C is called a *chord*. The cycle of length k without chords is denoted by C_k.

Let P be an induced path of length at least two. We say that a graph obtained by adding an universal vertex to P is a *gem*.

A graph is *chordal* if every cycle of length at least 4 has a chord. Let **Chordal** denote the class of chordal graphs. Note that $\mathbf{Chordal} = \{\mathbf{C_k} : k > 3\}$**-free**.

A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line. Let **Interval** denote the class of interval graphs.

An *asteroidal triple* of a graph G is a set of 3 non-adjacent vertices of G such that each pair is connected by a path avoiding the neighborhood of the third vertex.

Lekkerkerker and Boland [9] proved that:

1. For any graph G: G is an interval graph if and only if G is chordal and contains no asteroidal triple.
2. $\mathbf{Interval} = \mathbf{Chordal} \cap \{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3(n)_{n\geq 6}}, \mathbf{F_4(n)_{n\geq 6}}\}$**-free** (see Figure 1).

Definition 2. $\mathbf{SP/TW}$ is the class formed by those graphs G such that for every pair of non-adjacent vertices u and v of G, every $W \in \mathbf{SP}(u, v)$ dominates every $W' \in \mathbf{TW}(u, v)$, i.e., $W \in \mathbf{SP}(u, v)$ and $W' \in \mathbf{TW}(u, v)$ implies W dominates W'.

In [1], the class of graphs called $\mathbf{Interval}^+$ was introduced, which is the class of those chordal graphs G that contain none of the graphs F_2 or $F_4(n)_{n\geq 6}$ as an induced subgraph, and satisfy the following condition. If G has an induced subgraph H isomorphic to F_1 $(F_3(n)_{n\geq 6})$, then the distance in G between the vertices of F_1 $(F_3(n)_{n\geq 6})$ labelled u and w in Figure 1 is 2, and any vertex of G adjacent to both u and w is universal to F_1 $(F_3(n)_{n\geq 6})$.

The following theorem shows the relationship between the classes $\mathbf{SP/TW}$ and $\mathbf{Interval}^+$.

Theorem 1 [1]**.** $\mathbf{SP/TW} \subseteq \mathbf{Interval}^+$.

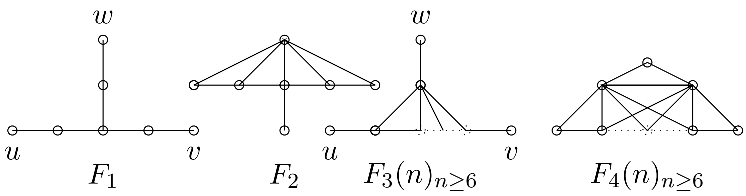

Figure 1. Chordal forbidden induced subgraphs for interval graphs.

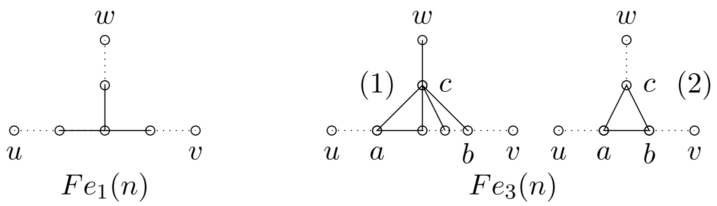

Figure 2. In Fe_1 the distance between each pair of pending vertices is at least four. In graph (1) the distance between u and v is at least four, and vertices in the induced path between a and b plus c induce a gem. In graph (2) the distance between each pair of pending vertices is at least three, and the vertices a, b, c induce only a triangle.

Alcón left open the problem of determining if the class **SP/TW** is exactly the class **Interval**$^+$ [1].

Conjecture 1 [1]**. SP/TW = Interval**$^+$.

In what follows, we show that the conjecture is false, and we introduce class **IntervalE**$^+$ that will turn out to be **SP/TW**.

Let $F'e_1$ be the graph in Figure 3. It is easy to check that $F'e_1 \in$ **Interval**$^+$. However, it is not in **SP/TW** since the shortest path u, x, v does not dominate the bold tolled walk $u, 1, 2, 4, w, y, y, w, 4, 2, 3, v$ between u and v, since $y \notin N[x]$.

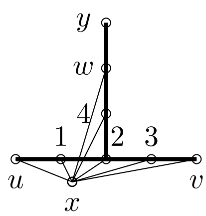

Figure 3. $F'e_1 \in$ **Interval**$^+ \setminus$ **SP/TW**.

In what follows we reformulate the conjecture, for this we will define class **IntervalE**$^+$. Let us abuse the notation, when defining the following family of graphs.

Let $Fe_1(n)$ be a tree with n vertices, such that its only pendent vertices are u, v and w, and the distance between each pendent vertex and the vertex of degree three is at least two. Note that $Fe_1(7) = F_1$ (see Figure 2).

Let $Fe_3(n)$ be a graph with n vertices, such that u, v and w are its pendent vertices, a, b, c are vertices of degree three such that the distance between a and u, b and v, c and w, respectively, is minimum (see Figure 2). If a, b, c induce a gem then the distance between c and w is exactly one (see (2) in Figure 2). In both cases, if no confusion can arise we will omit n.

Note that Fe_1 and Fe_3 is obtained from F_1 and F_3, respectively, through a possible increase in the paths between a pair of its pending vertices.

Let **IntervalE**$^+$ be the class of those chordal graphs G that contain none of the graphs F_2 or $F_4(n)_{n\geq 6}$ as induced subgraph, and satisfy the following condition. If G has an induced subgraph H isomorphic to $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$), then the distance in G between the vertices of $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$) labelled u and w in Figure 2 is 2, and any vertex of G adjacent to both u and w is universal to $Fe_1(n)_{n\geq 7}$ ($Fe_3(n)_{n\geq 6}$). It is important to notice that **IntervalE**$^+ \subseteq$ **Interval**$^+$.

Conjecture 2. $\mathbf{SP/TW} = \mathbf{IntervalE}^+$.

Lemma 2. $\mathbf{SP/TW} \subseteq \mathbf{IntervalE}^+$.

Proof. By Theorem 1, $\mathbf{SP/TW} \subseteq \mathbf{Interval}^+$. Clearly if $G \in \mathbf{SP/TW}$ then $G \in \mathbf{Chordal} \cap \{\mathbf{F_2}, \mathbf{F_4(n)_{n\geq 6}}\}$**-free**.

Let Fe_1 be a graph such that u, v, w are its pendent vertices, i be the vertex of degree three, $W_{u,w} : u, 1, \ldots, i, \ldots, j, w$ be the uw induced path of Fe_1, $W'_{i,v} : i, j{+}1, \ldots, k, v$ be the iv induced path of Fe_1 (see Figure 4). Note that if $|Fe_1| = 7$ then $Fe_1 = F_1$.

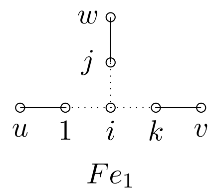

Figure 4.

Observe that if G contains Fe_1 as an induced subgraph (Figure 4), the distance between u and w cannot be $j + 1$, otherwise the uw shortest path $u, 1, \ldots, i, \ldots j, w$ does not dominate the tolled walk $u, 1 \ldots, i, \ldots, k, v, k, \ldots, i, \ldots, j, w$.

On the other hand, since $G \in \mathbf{Interval}^+$ and $Fe_1[\{i-2, i-1, i, i+1, i+2, j+1, j+2\}] = F_1$, there exists a vertex y adjacent to every vertex of $\{i-2, i-1, i, i+1, i+2, j+1, j+2\}$.

In what follows, we show that y must be adjacent to every vertex of $W_{u,w}[u, i-3] \cup W_{u,w}[i+3, w] \cup W'_{i,v}[j+3, v]$, whenever those vertices exist.

By the way of contradiction, suppose that there exists at least a vertex of $W_{u,w}[u, i-3]$ which is not adjacent to y. Among all, we choose the one closest to $i-3$, let us say a (a may be $i-3$). But then the shortest path $j+2, y, i+2$ does not dominate the tolled walk $j+2, \ldots, i, i-1, \ldots, a, a+1, \ldots, i, i+1, i+2$.

Thus y is adjacent to every vertex of Fe_1.

A reasoning analogous to the one applied in the case of Fe_1 shows that if G has an induced subgraph H isomorphic to $Fe_3(n)_{n \geq 6}$ then the distance in G between the vertices of $Fe_3(n)_{n \geq 6}$ labelled u and w in Figure 4 is 2. The reader should have no problem verifying the details. Therefore $\mathbf{SP/TW} \subseteq \mathbf{IntervalE^+}$. ∎

2. Main Results

In this section, we characterize the class **SP/TW**. We start with a preliminary observation that follows directly from the definition of tolled walks.

Observation 3. *Let G be a graph, and $T : u = y_0, y_1, \ldots, y_m = v$ be a uv tolled walk. It follows from the definition of tolled walk that if there exists a vertex in T such that $y_k \notin N[u] \cup N[v]$ then u and y_k are in the same connected component of $G[T] - N[v]$, and also v and y_k are in the same connected component of $G[T] - N[u]$.*

Note the following properties about chordal graphs.

Lemma 4. *Let u and v be non-adjacent vertices of a chordal graph G, and P and Q be induced paths between u and v. If P has a vertex $y \notin Q$, and $a \in P \cap Q$ is the vertex closest to y in $P[u, y]$ and $b \in P \cap Q$ is the vertex closest to y in $P[v, y]$, then $Q[a, b] \cap P = \{a, b\}$.*

Proof. Suppose that there exists at least a vertex $c \in Q(a, b) \cap P$. Clearly $c \notin P[a, b]$ because a and b are the vertices closest to y in $P[u, y]$ and $P[v, y]$. Hence there do not exist chords between c and vertices of $P[a, b]$. But then $G[P[a, b] \cup Q[a, b]]$ contains C_r ($r \geq 4$) as an induced subgraph, a contradiction. ∎

Lemma 5. *Let G be a chordal graph, P and Q be two induced path between u and v non-adjacent vertices of G. If Q is a shortest path, then every vertex in $P - Q$ is adjacent to at most three vertices of Q.*

Proof. Suppose that there exists at least a vertex $c \in P$ that is adjacent to a and b vertices in Q such that $|Q[a, b]| > 3$. Since $G[Q[a, b] \cup \{c\}]$ is a chordal graph, c is adjacent to every vertex of $Q[a, b]$. It is clear that $Q - Q(a, b) + ac + bc$

is a uv walk with at least two vertices fewer than Q. By Remark 1, it contains a uv induced path Q_1 such that $|Q_1| < |Q|$, a contradiction. ■

We are now able to prove the following.

Theorem 6. $\mathbf{SP/TW} = \mathbf{IntervalE}^+$.

Proof. By Lemma 2 $\mathbf{SP/TW} \subseteq \mathbf{IntervalE}^+$.

To prove that $\mathbf{IntervalE}^+ \subseteq \mathbf{SP/TW}$, let us suppose, on the contrary, that $G \in \mathbf{IntervalE}^+$, but $G \notin \mathbf{SP/TW}$.

As $G \notin \mathbf{SP/TW}$ there exist two non-adjacent vertices u and v, a uv shortest path W and a uv tolled walk $T : u = y_0, y_1, \ldots, y_m = v$ satisfying that W does not dominate T. Thus, there is some internal vertex of T that is neither a vertex of W nor adjacent to any internal vertex of W. Let y be a vertex of $T - W$ such that it is not adjacent to any vertex of W. We can assume that $y \neq y_1, y_{m-1}$, otherwise $G[W \cup T]$ contains as an induced subgraph a cycle of size at least four.

Let $P : u = z_0, z_1, \ldots, z_l = v$ be a shortest path in $G[T]$ from u to v. Note that $|P \cap W| \geq 2$. Since T is a uv tolled walk, $z_1 = y_1$ and $z_{l-1} = y_{m-1}$. Note that $z_1 \neq z_{l-1}$, and then $|V(P)| \geq 4$.

In the following claim, we show that y cannot be an internal vertex of P.

Claim 7. *$y \notin P$.*

Proof. Assume the contrary, $y \in P$.

Clearly, $|P \cap W| \geq 2$. Let a, b be two vertices of $P \cap W$ such that a, y, b appear in this order in P, $d_P(a, y)$ and $d_P(b, y)$ is minimum. Note that a may be u and b may be v.

Since y is not a vertex of W nor adjacent to any internal vertex of W, it follows that $d(a, y) > 1$ and $d(b, y) > 1$. By Lemma 4, $P \cap W[a, b] = \{a, b\}$. On the other hand, $G[P[a, b] \cup W[a, b]]$ is a chordal graph, there exist chords between vertices of $P[a, b]$ and vertices $W[a, b]$. Thus there exists at least a chord between a vertex of $W(a, b)$ and y, a contradiction.

Therefore $y \notin P$. □

In what follows we will analyze two cases, depending of y is or is not adjacent to vertices of P. Observe that if $a \in P$ is adjacent to y then $a \notin W$. We will present some figures that could help in the analysis of cases. In such figures, we will allow ourselves to omit some edges between two adjacent vertices.

Case 1. y is adjacent to some vertex of P. Note that if y is adjacent to two non-consecutive vertices of P, let a and b, then since $G[P[a, b] \cup \{y\}] \neq C_r$ (for some $r > 3$), it follows that y is adjacent to every vertex of $P[a, b]$.

By before exposed, we can analyze two situations: y is adjacent to two consecutive vertices or y is adjacent to one and only one vertex of P.

Case 1.1. y is adjacent to two consecutive vertices z_i, z_{i+1} of P. By the choice of y, $z_i, z_{i+1} \notin W$. Let $a \in P[u, z_i] \cap W$ and $b \in P[z_{i+1}, v] \cap W$ such that the distance in P between a (b) and z_i (z_{i+1}) is minimum. Note that z_i (z_{i+1}) may be adjacent to a (b), and a (b) might be u (v), respectively.

Since $G[P[a, b] \cup W[a, b]]$ is a chordal graph, there exist at least a vertex in $W[a, b]$ which is adjacent to z_i and z_{i+1}. In order to fix ideas, let us consider $W[a, b] : a, x_1, \ldots, x_{n-1}, b$.

Let p and q be the first and the last index such that z_i, z_{i+1} are simultaneously adjacent to x_p and x_q (see Figure 5). Since $W \in \mathbf{SP}$, by Lemma 5, $q \leq p + 2$.

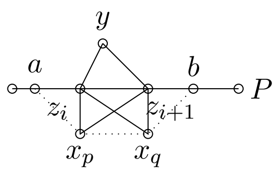

Figure 5. The dotted line between a and b represents $W[a, b]$.

Since $G[P[a, z_i] \cup W[a, x_p]]$ is a chordal graph, then z_i must be adjacent to x_{p-1} or x_p must be adjacent to z_{i-1}. By similarity, $G[P[b, z_{i+1}] \cup W[b, x_q]]$ is a chordal graph, then z_{i+2} must be adjacent to x_q or x_{q+1} must be adjacent to z_{i+1}. Thus $G[\{y, z_i, z_{i+1}\} \cup W[x_{p-1}, x_{q+1}]] = F_4$ or $G[\{y, z_{i-1}, z_i, z_{i+1}, z_{i+2}\} \cup W[x_p, x_q]] = F_4$ or $G[\{y, z_i, z_{i+1}, z_{i+2}\} \cup W[x_{p-1}, x_q]] = F_4$ or $G[\{y, z_{i-1}, z_i, z_{i+1}\} \cup W[x_p, x_{q+1}]] = F_4$, a contradiction.

Case 1.2. Suppose that y is adjacent to one and only one vertex of P. Let us consider two situations: y is adjacent to z_i for some $i \in \{2, \ldots, l-2\}$ or y is adjacent to z_1 or z_{l-1}.

Case 1.2.1. y is adjacent to z_i for $i \neq 1, l-1$. Let a and b be vertices in $W \cap P$ such that $a \in P[u, z_i]$, $b \in P[z_i, v]$, $d_P(a, z_i)$ and $d_P(z_i, b)$ is minimum.

Since $G[P[a, b] \cup W[a, b]]$ is a chordal graph, we have that there exist chords between vertices of $P[a, b]$ and $W[a, b]$. Let us consider $W[a, b] : a, x_1, \ldots, x_{n-1}, b$.

By Lemma 5, as $W \in \mathbf{SP}$, if z_i is adjacent to two vertices (x_p, x_q at a maximum distance) in $W[a, b]$, those vertices must be at a distance of at most 2 in W.

Let us consider two cases depending of z_i is or is not adjacent to one and only one vertex of $W[a, b]$.

Case 1.2.1.1. z_i is adjacent to one and only one vertex of $W[a, b]$. Thus $a \neq z_{i-1}$ and $b \neq z_{i+1}$. Let x_h be the vertex of $W[a, b]$ such that z_i is adjacent to x_h. Since G is a chordal graph, let $z_p \in P[a, z_i]$ and $z_q \in P[z_i, b]$ such that x_h is adjacent to z_p and z_q, and the distance in P between z_i and z_p, z_q is maximum (see Figure 6).

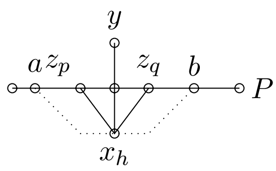

Figure 6. The dotted line between a and b represents $W[a, b]$.

If $p \leq i - 2$ and $q \geq i + 2$ then $G[P[z_p, z_q] \cup \{y, x_h\}]$ contains F_2 as induced subgraph, a contradiction.

Suppose that $p = i - 1$ and $q = i + 1$. By the choice of p and q, x_h is not adjacent to any vertex of $P[a, z_p) \cup P(z_q, b]$. Hence $G[P[a, z_p] \cup W[a, x_h]]$ contains C_r $(r > 3)$ as induced subgraph, which is a contradiction, or there exists x_{h-1} in $W(a, x_h]$ such that it adjacent to z_p. Analogously there exists x_{h+1} in $W[x_h, b)$ such that z_q must be adjacent to x_{h+1}. Thus $G[\{z_p, z_q, z_i, y, x_h, x_{h+1}, x_{h-1}\}] = F_2$, a contradiction.

Suppose that $p = i - 1$ and $q = i$ (by symmetry $p = i$ and $q = i + 1$).

Observe that z_{p-1} may be a but $z_{q+1} \neq b$. Since x_h is not adjacent to z_{q+1} and z_i is not adjacent to any vertex of $W[x_h, b]$, $G[P[z_i, b] \cup W[x_h, b]]$ contains C_r $(r > 3)$ as induced subgraph, a contradiction.

Case 1.2.1.2. z_i is adjacent at least two vertices of $W[a, b]$. Note that those vertices may be a and b.

Let p and q be the first and the last index such that z_i is adjacent to x_p and x_q. By Lemma 5, since $W \in \mathbf{SP}$, $q \leq p + 2$.

Note that $G[W[x_p, x_q] \cup \{z_i\}]$ induces a triangle or a gem. We can affirm that $G[W[a, b] \cup \{y\} \cup P]$ contains F_3 as induced subgraph whose pending vertices are y, u and v (see Figure 7). Since $G \in \mathbf{IntervalE}^+$, there exists $w \in G$ universal vertex of F_3. Clearly every shortest path between u and v has 3 vertices. Thus, we arrive to a contradiction if $|W| > 3$, or if $|W| = 3$, results $x \in W \setminus \{u, v\}$ must be adjacent to y, which is also a contradiction.

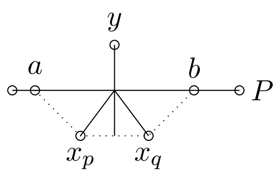

Figure 7. The dotted line between a and b represents $W[a, b]$.

Case 1.2.2. Suppose that y is adjacent only to $z_1 = y_1$ (by symmetry $z_{l-1} = y_{m-1}$). As $T \in \mathbf{TW}$, there exists an induced path between y and v in $G[T]$ that does not have vertices neighbors of u. Assume that P_1 is a shortest among the paths join y to a vertex in P such that $P_1 \cap N[u] = \emptyset$. Let $P_1 : y = t_1, t_2, \ldots, t_i$ with $t_i \in P$. Note that $t_2 \neq z_1$ and t_i may be adjacent to z_1.

No vertex in $P_1[y, t_{i-1})$ is adjacent to a vertex in $P(z_1, t_i)$ because of the minimality of P_1, thus every vertex in $P_1[y, t_{i-1})$ is adjacent to z_1, t_{i-1} is adjacent to z_1 as well, and every vertex in $P_1[y, t_{i-1}]$, and every vertex in $P[z_1, t_i]$ is adjacent to t_{i-1}.

In what follows, we can assume that $t_i = z_2$.

Let a be a vertex in $P \cap W$ such that $d_P(u, a)$ is minimum. By the choice of y, $a \neq z_1$. Note that a may be z_2. Let $W[u, a] : u, x_1, \ldots, a$.

Note that if $|W| = 3$ then x_1 must be adjacent to every vertex of P (see Figure 8). Let p the first index such that x_1 is adjacent to t_p. Clearly $p \neq 1$. Thus $G[P_1[t_{p-1}, t_i = z_2] \cup \{u, z_1, x_1, z_3\}] = F_4$, a contradiction.

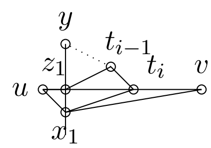

Figure 8. The dotted line between y and t_{i-1} represents the induced path P_1, and the filled line between t_i and v represents $P[t_i, v]$.

Suppose that $|W| \neq 3$. Note that x_1 must be adjacent to z_1.

First, we assume that at least a vertex of $W[u, a]$ is adjacent to a vertex of P_1. Let x_j be the first index such that x_j is adjacent to a vertex of P_1, and let p and q the first and the last index such that x_j is adjacent to t_p and t_q. Note that since $G[\{t_p, z_1, x_j\} \cup W[x_1, x_j]]$ is a chordal graph, x_j must be adjacent to z_1.

Suppose that $p \neq q$. Note that $G[P_1[t_p, t_q] \cup \{x_j\}]$ is a triangle or a gem whose universal vertex is x_j. Assume that x_j is not adjacent to any vertex of $P[t_i, v]$. Note that x_{j-1} cannot be adjacent to a vertex of $P[t_i, v]$, since $G[P_1[t_q, t_i] \cup P \cup \{x_{j-1}, x_j\}]$ is a chordal graph. We can affirm that there exists an induced path in $G[P_1 \cup P[t_i, v]]$, it follows that $G[W[x_1, x_j] \cup \{u\} \cup P_1[t_{p-1}, t_i] \cup P[t_i, v]]$ contains as induced subgraph to Fe_3 or Fe_1 with pending vertices u, v, t_{p-1} (see Figure 9).

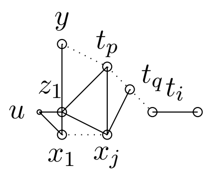

Figure 9. The dotted line between x_1 and x_j represents the induced path $W[x_1, x_j]$, and the filled line between t_i and v represents $P[t_i, v]$.

But then as $G \in \mathbf{IntervalE}^+$, there exists w such u, w, v is a shortest path, a contradiction because $|W| > 3$. Thus x_j must be adjacent to a vertex of $P[t_i, v]$. Moreover, it must be adjacent to every vertex of $P[t_i, v]$ otherwise as before exposed $G[W[x_1, x_j] \cup \{u\} \cup P_1[t_{p-1}, t_i] \cup P[t_i, v]]$ contains as induced subgraph Fe_3 or Fe_1 with pending vertices u, v, t_{p-1} and we arrive to a contradiction.

Since x_j is adjacent to every vertex of $P[t_i, v]$, results $q = i$, and then $G[\{x_{j-1}, x_j, z_1\} \cup P_1[t_{p-1}, t_i]] = F_4$, a contradiction.

Now, assume that $p = q$. Note that x_j cannot be adjacent to any vertex of $P[t_i, v]$, otherwise $p = i$ and then z_1 is adjacent to t_i both non-consecutive vertices of W.

On the other hand, there exists x_{j+1} and it must be adjacent to a vertex of $P_1[t_p, t_i] \cup P[t_i, v]$ since $G[W[x_j, a] \cup P_1[t_p, t_i] \cup P[a, t_i]]$ is a chordal graph. Moreover it must adjacent to t_p (see Figure 10).

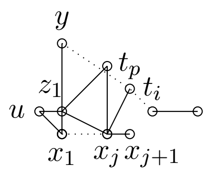

Figure 10.

But then $G[\{z_1, t_p, t_{p-1}, t_p, x_{j-1}, x_j, x_{j+1}\}] = F_4$, a contradiction.

Now, none vertex of $W[u, a]$ is adjacent to a vertex of P_1. Thus, $G[\{z_1, t_{i-1}\} \cup P[t_i, a] \cup W[x_1, a]]$ or $G[\{z_1, t_{i-1}\} \cup P[z_1, a] \cup W[x_1, a]]$ contains $C_r (r > 3)$ depending of $a \in P[t_i, v]$ or $a \in P[z_1, t_i]$, a contradiction.

Case 1.2.2.2. $t_i = z_2$. This case can be treated similarly to *Case* 1.2.2.1.

Case 2. y is adjacent to no vertex of P. Note that $P \cap N[y] = \emptyset$. Since T is a uv tolled walk, by Observation 3, there exists an induced path between u and y in $G[T]$ avoiding the neighborhood of v, and also there exists an induced path between v and y in $G[T]$ avoiding the neighborhood of u. Those paths, together with P allow us to state that u, v, y is an asteroidal triple.

Hence, we assume that there exist three induced paths of $G[T]$: P between u and v; P_1 between u and y; P_2 between v and y; and three vertices $a \in V(P) \cap V(P_1)$; $b \in V(P) \cap V(P_2)$; and $c \in V(P_1) \cap V(P_2)$; such that $V(P) \cap N[y] = \emptyset$, $V(P_1) \cap N[v] = \emptyset$, $V(P_2) \cap N[u] = \emptyset$, and the distance between a and u, between b and v, and between c and y in the respective paths is maximum.

Note that $u, y_1 \in P_1$ and $v, y_{m-1} \in P_2$ since T is a uv tolled walk.

Without loss of generality, we can assume that $(P \cap P_1)[u, a]$, $(P \cap P_2)[b, v]$ and $(P_1 \cap P_2)[c, y]$ are induced paths (see Figure 11). By the choice of a, b and c, and since G contains no F_2 and F_4 as induced subgraphs, we can affirm that $y \neq c$.

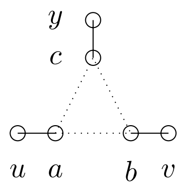

Figure 11.

Case 2.1. Suppose that $P \cup P_1 \cup P_2$ is a tree. Clearly $a = b = c$. Note that since y is not adjacent to any vertex of P, the distance in $P \cup P_1 \cup P_2$ between y and a is at least two. Also the distance between u, v and a is at least two because u, y, v is an asteroidal triple. Thus $P \cup P_1 \cup P_2 = Fe_1$.

On the other hand, as $G \in \mathbf{IntervalE}^+$, there exists $w \in G$ such that w is an universal vertex of $G[P \cup P_1 \cup P_2 \cup \{w\}]$. Thus the distance in G between u and v is two, and then since W is a shortest path it follows that $|W|$ must be three. More clearly, $W = u, x, v$. Again as a result of $G \in \mathbf{IntervalE}^+$, we can conclude that x must be an universal vertex of $G[P \cup P_1 \cup P_2 \cup \{x\}]$. But then y must be adjacent to x, which is a contradiction by the choice of y.

Case 2.2. Suppose that $P \cup P_1 \cup P_2$ is not a tree. We can conclude that $G[P \cup P_1 \cup P_2]$ must contain $Fe_3(n)$ as induced subgraph being y, u and v its pending vertices, and a, b c are the labelled vertices in Figure 2.

Let us consider two cases, depending of a is or is not adjacent to b.

Case 2.2.1. Suppose that a is not adjacent to b. Then a is adjacent to c and b is adjacent to c. Note that y must be adjacent to c, otherwise $G[P - P[a, b] \cup P_1 \cup P_2]$ contains Fe_1 as induced subgraph.

Since $G \in \mathbf{IntervalE}^+$, there exists a vertex x in G such that x is an universal vertex of $Fe_3(n)$, which has y, u, v as its pendent vertices. Thus, the distance between u and v in G must be two. Hence $|W| = 3$. Let $W = u, x, v$. Clearly x must be adjacent to every vertex of $Fe_3(n)$, in particular x is adjacent to y, a contradiction.

Case 2.2.2. Suppose that a is adjacent to b. We can assume that a is adjacent to c and b is adjacent to c. This case can be treated similarly to *Case* 2.2.1.

Therefore $\mathbf{SP/TW} = \mathbf{IntervalE}^+$. ■

3. Conclusions

In this article, we have obtained a characterization of the graphs in which, for every pair of non-adjacent vertices u and v, every uv shortest path dominates every uv tolled walk. Thus, we solve a problem of Alcón concerning the class $\mathbf{SP/TW}$.

In [1, 8, 10] it was displayed that the notion of domination between different types of walks plays a central role in characterizations of graph classes. It is interesting to ask what other classes of graphs can be characterized by path domination. We summarize the results obtained so far in Table 1, and then we will show that other problems related to path domination could be studied. For this, let us introduce the notation $\mathbf{IP}$, $\mathbf{P}$, $\mathbf{W}$, $\mathbf{WTW}$ and $\mathbf{l_k}$ for $k = 2, 3$ to refer to the set of different types of walks connecting two non-adjacent vertices u and v of a graph G.

$\mathbf{IP}(u, v) = \{W : W$ is an uv induced path (or monophonic [6]), i.e., a uv path such that two of its vertice are adjacent if and only if are consecutive$\}$,

$\mathbf{P}(u, v) = \{W : W$ is a uv path$\}$,

$\mathbf{W}(u, v) = \{W : W$ is a uv walk$\}$,

$\mathbf{WTW}(u, v) = \{W : W$ is a uv weakly toll walk, i.e., a uv walk such that $u(v)$ has a single neighbor in W that can appear more than once in $W\}$ [5],

$\mathbf{l_k}(u, v) = \{W : W$ is a uv l_k path, i.e., a uv induced path with length at most $k\}$ [7],

$\mathbf{m_3}(u, v) = \{W : W$ is a uv m_3 path, i.e., a uv induced path with length at least three$\}$ [4].

Definition 3. Let $\mathbf{A}, \mathbf{B} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}, \mathbf{l_2}, \mathbf{l_3}, \mathbf{m_3}\}$. $\mathbf{A/B}$ is the class formed by those graphs G such that for every pair of non-adjacent vertices u and v of G, every $W \in \mathbf{A}(u, v)$ dominates every $W' \in \mathbf{B}(u, v)$, i.e., $W \in \mathbf{A}(u, v)$ and $W' \in \mathbf{B}(u, v)$ implies W dominates W'.

We denote by $\mathbf{Ch}$ the class of chordal graphs, by $\mathbf{Int}$ the class of interval graphs, by $\mathbf{Sup}$ the class of superfragile graphs, by $\mathbf{Pt^-}$ the class $\mathbf{Ptolematic^-}$, by $\mathbf{F_{2,4,5,6,7}} = \{\mathbf{F_2}, \mathbf{F_4(n)_{n\geq 6}}, \mathbf{F_5(n)_{n\geq 8}}, \mathbf{F_6(n)_{n\geq 7}}, \mathbf{F_7(n)_{n\geq 7}}\}$, and by $\mathbf{F_{2,3,4,5}} = \{\mathbf{F_2}, \mathbf{F_3(n)_{n\geq 6}}, \mathbf{F_4(n)_{n\geq 6}}, \mathbf{F_5(n)_{n\geq 8}}\}$, by $\mathbf{M_3W} = \{\mathbf{P_4}, \mathbf{A}, \overline{\mathbf{gem} \cup \mathbf{K_2}}, \mathbf{C_5}, \overline{\mathbf{X_{58}}}, \mathbf{X_{96}}, \mathbf{F_3(6)}\}$**-free**, by $\mathbf{M_3P} = \{\mathbf{H}, \mathbf{D}, \mathbf{Antenna}, \mathbf{X_5}\}$**-free**, by $\mathbf{M_3M_3} = \{\mathbf{C_{n>5}}, \mathbf{D}, \mathbf{Antenna}, \mathbf{X_5}, \mathbf{5-pan}, \mathbf{X_{37}}\}$**-free** [8, 10]. The classes $\mathbf{Ptolemaic-}$ and $\mathbf{g - Chordal}$ are defined in [1].

The classes $\mathbf{SP/SP}$, $\mathbf{SP/TW}$, $\mathbf{l_3/SP}$ are not hereditary classes. In [1] the class $\mathbf{SP/SP}$ was characterized as the class $\mathbf{g - Chordal}$ and in this paper we characterize the class $\mathbf{SP/TW}$ as the class $\mathbf{IntervalE^+}$.

Natural question arise.

1. We know that $\mathbf{l_3/SP} \subseteq \{\mathbf{C_4}, \mathbf{C_5}\}$**-free** and $\mathbf{l_3/SP} \neq \mathbf{g - Chordal}$ [8]. Is it possible to characterize the non-hereditary class of graphs $\mathbf{l_3/SP}$?
2. Do $\mathbf{A/m_3}$ and $\mathbf{m_3/A}$, for $\mathbf{A} \in \{\mathbf{l_k}, \mathbf{SP}, \mathbf{TW}, \mathbf{WTW}\}$ give rise to characterizations of some classes of graphs?

-	$\mathbf{l_2}$	$\mathbf{l_3}$	SP	IP	P	TW	WTW	W	$\mathbf{m_3}$
$\mathbf{l_2}$	$\{\mathbf{C_4}\}$-free	Ch	$\{\mathbf{C_4}\}$-free	Ch	$\mathbf{Pt^-}$	$\mathbf{Ch \cap F_{2,4,5,6,7}}$-free	$\mathbf{Ch \cap \{chair, dart, F_4(6)\}}$-free	Sup	
$\mathbf{l_3}$	$\{\mathbf{C_4, C_5, C_6}\}$-free	Ch		Ch	$\mathbf{Pt^-}$	$\mathbf{Ch \cap F_{2,3,4,5}}$-free	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	
SP	$\{\mathbf{C_4}\}$-free	$\{\mathbf{C_4, C_5, C_6}\}$-free	$\mathbf{g - Ch}$	Ch	$\mathbf{Pt^-}$	$\mathbf{IntE^+}$	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	
IP	Ch	Ch	Ch	Ch	$\mathbf{Pt^-}$	Int	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	HHD-free
P	Ch	Ch	Ch	Ch	$\mathbf{Pt^-}$	Int	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	HHD-free
TW	Ch	Ch	Ch	Ch	$\mathbf{Pt^-}$	Int	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	
WTW	Ch	Ch	Ch	Ch	$\mathbf{Pt^-}$	Int	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	
W	Ch	Ch	Ch	Ch	$\mathbf{Pt^-}$	Int	$\mathbf{Int \cap \{chair, dart\}}$-free	Sup	HHD-free
$\mathbf{m_3}$				$\mathbf{M_3P}$	$\mathbf{M_3P}$			$\mathbf{M_3W}$	$\mathbf{M_3M_3}$

Table 1. With $\mathbf{A} \in \{\mathbf{l_2}, \mathbf{l_3}, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}, \mathbf{m_3}\}$ in the first column and $\mathbf{B} \in \{\mathbf{l_2}, \mathbf{l_3}, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}, \mathbf{m_3}\}$ in the first row, the table describes each one of the graph classes $\mathbf{A/B}$.

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