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HAMILTONIAN PROPERTIES IN GENERALIZED LEXICOGRAPHIC PRODUCTS

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Abstract

The lexicographic product $G[H]$ of two graphs G and H is obtained from G by replacing each vertex with a copy of H and adding all edges between any pair of copies corresponding to adjacent vertices of G. We consider also the generalized lexicographic product such that we replace each vertex of G with arbitrary graph on the same number of vertices. We present sufficient and necessary conditions for traceability, hamiltonicity and hamiltonian connectivity of $G[H]$ if G is a path and hence we improved and extended results in [M. Kriesell, *A note on Hamiltonian cycles in lexicographical products*, J. Autom. Lang. Comb. 2 (1997) 135–138].

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1. Introduction

A product of graphs is well known graph operation (e.g. Cartesian, direct, lexicographic) and study hamiltonian properties in some product of graphs is standard problem in graph theory. In this paper we denote to a lexicographic product of graphs.

The lexicographic product $G[H]$ of two graphs G and H is defined by a vertex set $V(G[H]) = V(G) \times V(H)$ and an edge set $E(G[H]) = \{(g,h)(g',h') : gg' \in E(G) \text{ or } g = g' \wedge hh' \in E(H)\}$. In other words, the lexicographic product $G[H]$ of two graphs G and H is obtained from G by replacing each vertex with a copy of H and adding all edges between any pair of copies corresponding to adjacent vertices of G. A typical sufficient condition for the existence of a hamiltonian cycle or a hamiltonian path in a lexicographic product $G[H]$ forces G to contain a hamiltonian cycle or a hamiltonian path and H to have some additional properties. Hamiltonian cycles and paths in lexicographic products have been studied in [1, 5, 6, 7] and [9].

Clearly, $G[H]$ contains a cycle of length 3 if both of G and H contain at least one edge. Kaiser and Kriesell proved in [5] that concepts of pancyclicity and hamiltonicity coincide in the case of lexicographic products of graphs with at least one edge. Recall that the graph G is weakly pancyclic or pancyclic, if it contains cycles of every length between the length of a shortest cycle and that of a longest one, hamiltonian, respectively.

Theorem 1 [5]**.** *If G, H are graphs with at least one edge each, then $G[H]$ is weakly pancyclic.*

In this paper we consider also the concept of the generalized lexicographic product mentioned in [3, 4] (defined as an expansion), and [8]. Basically, the generalized lexicographic product is the graph $G[H_1, H_2, \ldots, H_m]$, which will be a graph like a lexicographic product with the difference that every vertex of G can be replaced by a different graph H_i. Precisely, let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_m\}$ and H_i be an arbitrary graph $i = 1, 2, \ldots, m$. Then generalized lexicographic product $G[H_1, H_2, \ldots, H_m]$ of a graph G and $H_1, H_2, \ldots, H_m$ is obtained from G by replacing each vertex u_i with the graph H_i and adding all edges between graphs H_i and H_j if the corresponding vertices u_i, u_j are adjacent in G. We say that $G[H_1, H_2, \ldots, H_m]$ is lex-regular if the number of vertices of H_i is the same for $i = 1, 2, \ldots, m$.

For a given graph G, we define $\pi(G)$ to be the maximum number of edges of a spanning linear forest of G (a forest is linear if its components are paths).

Main results of this paper are the following theorems which generalize and improve some results from [6].

Theorem 2. *Let P_{2k+1} be a path with odd number of vertices, $k \geq 1$. Let $H_1, H_2, \ldots, H_{2k+1}$ be graphs with n vertices. The graph $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ is*

- (i) *hamiltonian if and only if $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$, and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n$;*
- (ii) *traceable if and only if $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n - 1$;*
- (iii) *hamiltonian connected if and only if $\pi(H_1) \geq 2$; $\pi(H_{2k+1}) \geq 2$, and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n + 1$.*

Theorem 3. *Let P_{2k} be a path with even number of vertices, $k \geq 1$. Let $H_1, H_2, \ldots, H_{2k}$ be graphs with n vertices. Then the graph $P_{2k}[H_1, H_2, \ldots, H_{2k}]$ is*

(i) *hamiltonian if and only if $k = 1$ or $\pi(H_1) \geq 1$ and $\pi(H_{2k}) \geq 1$;*

(ii) *traceable;*

(iii) *hamiltonian connected if and only if*

- $\pi(H_1) \geq 1$ *and* $\pi(H_{2k}) \geq 1$ *for* $k = 1$*;*
- $\pi(H_1) \geq 2$ *and* $\pi(H_2) \geq 2$ *for* $k > 1$.

Observe that Theorems 2 and 3 give a complete characterization of hamiltonicity of $P_n[H]$, traceability of $P_n[H]$, and hamiltonian connectivity of $P_n[H]$.

2. Preliminaries

As for standard terminology, we refer to the book by Bondy and Murty [2]. However, before proving Theorem 2 and Theorem 3 we mention several concepts and results which we need to make use of.

For a multigraph G and $x, y \in V(G)$ let $[x, y]_G$ be the set of edges between x and y and let $m_G(xy) = |[x, y]_G|$ be the multiplicity of the edge xy in G. In particular, $\ell_G(x) = |[x, x]_G|$ denotes the number of loops at x and $\ell(G) = \max\{\ell_G(x) : x \in V(G)\}$. Note that the degree of a vertex x denoted by $d_G(x) = \sum_{y \in V(G)} m_G(xy) + 2\ell_G(x)$. A multigraph G is k regular if $d_G(x) = k$ for every vertex x in $V(G)$. Moreover, let $|G| = |V(G)|$ and $||G|| = |E(G)|$. For any $X \subseteq V(G)$, let $G(X)$ be the submultigraph induced by X.

A multigraph G' is said to be a multiple of a graph G if $V(G') = V(G)$ and for all $x \neq y \in V(G)$, $m_{G'}(xy) > 0$ holds only if $xy \in E(G)$. This means that from a given graph G, we can obtain a multiple G' by adding loops or by replacing a single edge in G by an arbitrary number of edges.

In [6], Kriesell proved that $G[H]$ is hamiltonian if G has a connected, k-regular multiple with additional properties.

Theorem 4 [6]**.** *Let G and H be graphs. If G has a connected, $2|H|$-regular multiple G' satisfying $\pi(H) \geq \ell_{G'}(x)$ for all $x \in V(G')$, then $G[H]$ contains a hamiltonian cycle that contains exactly $m_{G'}(xy)$ edges between $V(G(\{x\})[H])$ and $V(G(\{y\})[H])$ for all $x \neq y$ in $V(G)$.*

By a uv-path we mean a path from u to v in G. If a uv-path is hamiltonian, we call it a uv-hamiltonian path. The graph G is traceable, if G contains a hamiltonian path. The graph G is hamiltonian connected, if every two vertices of G are connected by a hamiltonian path.

Teichert in [9] and also Kriesell as corollary of Theorem 4 in [6] proved the following.

Theorem 5 ([6] and [9]). *Let G and H be graphs, $|G| \geq 2$. Suppose that G contains a hamiltonian path.*

(i) *If $|G|$ is even, then $G[H]$ is traceable.*

(ii) *If $|G|$ is even and $||H|| \geq 1$, then $G[H]$ is hamiltonian.*

(iii) *If $|G|$ is odd and $\frac{|H|-1}{2} \leq \pi(H)$, then $G[H]$ is traceable.*

(iv) *If $|G|$ is odd and $\frac{|H|}{2} \leq \pi(H)$, then $G[H]$ is hamiltonian.*

Note that the first two statements are in some sense necessary. Let G be only a path with n vertices, i.e., $G = P_n$. Clearly, $P_2[H]$ is hamiltonian. If $n \geq 4$ is even, then $P_n[H]$ is hamiltonian if and only if $||H|| \geq 1$ because if H has no edge, then $P_n[H]$ cannot have a hamiltonian cycle.

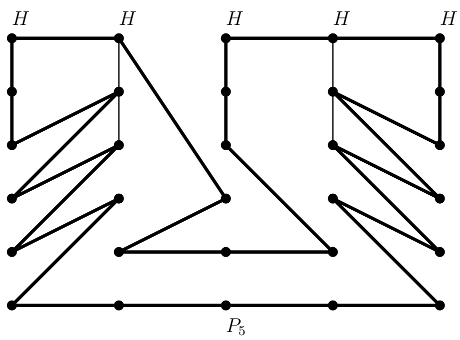

Figure 1. Hamiltonian cycle in $P_5[P_3 + 3K_1]$ (bold edges).

But the last two statements are not necessary. For example, take the graph H on 6 vertices with $\pi(H) = 2$ (e.g. the graph $H = P_3 + 3K_1$). Thus this graph does not satisfy the conditions (iii) and (iv) in the previous theorem. If $G = P_3$, then $G[H]$ is neither hamiltonian nor traceable (see the proof of Theorem 2). But if we instead of P_3 take P_5 as a graph G, then $G[H]$ is hamiltonian (see Figure 1, edges of $P_5[P_3+3K_1]$ between consecutive copies of H are missing for the clarity). For longer odd paths $G = P_{2k+1}$, $k \geq 3$, the lexicographic product $G[H]$ is also hamiltonian.

3. Proofs

Let P_{2k+1} be a path with odd number of vertices consecutively denoted by $u_1, u_2, \ldots, u_{2k+1}$, $k \geq 1$, and edges denoted by e_m where $e_m = u_m u_{m+1}$, $m = 1, 2, \ldots, 2k$.

Proof of Theorem 2.

(i) First suppose that $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n$. Clearly, we have $n \geq 2$ because of $\pi(H_1) \geq 1$. Now we find a connected $2n$-regular multiple of P_{2k+1}. Then we prove the hamiltonicity of $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ similarly as in [6].

We define the number of loops at each vertex u_i of multiple G' of P_{2k+1}. For even vertices $u_2, u_4, \ldots, u_{2k}$ we define $\ell_{G'}(u_{2i}) = 0$, $i = 1, 2, \ldots, k$, and for odd vertices $\ell_{G'}(u_{2i+1}) = \pi(H_{2i+1})$, $i = 0, 1, \ldots, k$. If $\sum_{i=0}^{k} \pi(H_{2i+1}) > n$ (the multiple G' has more than n loops), then we remove arbitrary loops from G' in such a way that $\ell_{G'}(u_1) \geq 1$, $\ell_{G'}(u_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n$. Note that $\ell_{G'}(u_j) \leq \pi(H_j)$ for $j = 1, 2, \ldots, 2k+1$.

Now we define the multiplicity of every edge e_m of P_{2k+1}, $m = 1, 2, \ldots, 2k$,

$$\begin{aligned}
m_{G'}(e_1) &= 2n - 2\ell_{G'}(u_1), \\
m_{G'}(e_2) &= 2\ell_{G'}(u_1), \\
m_{G'}(e_3) &= 2n - 2\ell_{G'}(u_1) - 2\ell_{G'}(u_3), \\
m_{G'}(e_4) &= 2\ell_{G'}(u_1) + 2\ell_{G'}(u_3), \\
m_{G'}(e_5) &= 2n - 2\ell_{G'}(u_1) - 2\ell_{G'}(u_3) - 2\ell_{G'}(u_5), \\
m_{G'}(e_6) &= 2\ell_{G'}(u_1) + 2\ell_{G'}(u_3) + 2\ell_{G'}(u_5),
\end{aligned}$$

in general we have, $i = 1, 2, \ldots, k$,

$$m_{G'}(e_{2i}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) \quad \text{and} \quad m_{G'}(e_{2i-1}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}).$$

Clearly, by the construction the multiplicity of every edge is at least 2, and the degree of every vertex $u_1, u_2, \ldots, u_{2k}$ is exactly $2n$ and for the last vertex of our path we have

$$\begin{aligned}
d_{G'}(u_{2k+1}) &= 2\ell_{G'}(u_{2k+1}) + m_{G'}(e_{2k}) = 2\ell_{G'}(u_{2k+1}) + \sum_{j=1}^{k} 2\ell_{G'}(u_{2j-1}) \\
&= 2\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = 2n.
\end{aligned}$$

Now we prove that the graph $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ contains a hamiltonian cycle using exactly $\ell_{G'}(u_i)$ edges of H_i and exactly $m_{G'}(e_j)$ edges between $V(P_{2k+1}(\{u_j\})[H_j])$ and $V(P_{2k+1}(\{u_{j+1}\})[H_{j+1}])$ for $i = 1, 2, \ldots, 2k+1$ and $j = 1, 2, \ldots, 2k$.

For every vertex u_i and every graph H_i, there exists a spanning linear subforest of $P_{2k+1}(\{u_i\})[H_i] \cong H_i$ with components $P_1(u_i), P_2(u_i), \ldots, P_{j_i}(u_i)$ satisfying

$$j_i = |H_i| - \ell_{G'}(u_i) \quad \text{and} \quad \sum_{t=1}^{j_i} ||P_t(u_i)|| = \ell_{G'}(u_i),$$

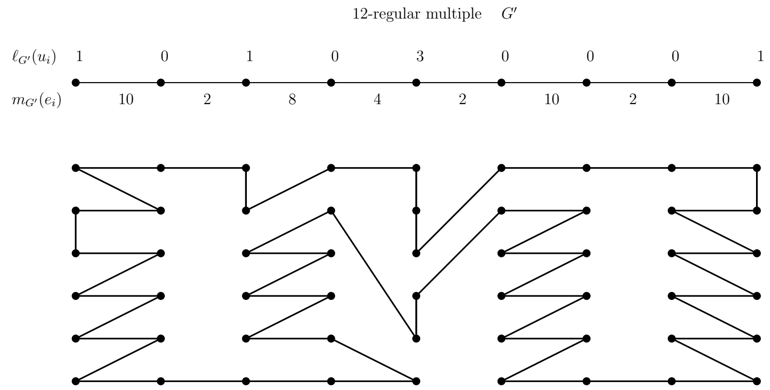

Figure 2. Hamiltonian cycle in $P_9[H_1, H_2, \ldots, H_9]$ from a multiple G'.

because $\ell_{G'}(u_i) \leq \pi(H_i)$.

Futhermore, after removing all the loops from multiple G', there exists a closed eulerian trail C in the graph G'. We obtain the hamiltonian cycle of $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ as required by replacing simultaneously the vertices u_i at their t-th occurence in C by the component $P_t(u_i)$ for $t = 1, 2, \ldots, j_i$ and $i = 1, 2, \ldots, 2k+1$ (for illustration see Figure 2).

Now suppose that $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ is hamiltonian. If $||H_1|| = 0$ or $||H_{2k+1}|| = 0$, then $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ cannot contain a hamiltonian cycle. Hence assume that $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$ and $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ has a hamiltonian cycle C. Note that if H_i has at most $\pi(H_i)$ edges in linear forest, then the number of components of a linear forest of H_i is at least $n - \pi(H_i)$. Now we count the number of edges between graphs $H_1, H_2, \ldots, H_{2k+1}$ in C.

The graph H_1 has at least $n - \pi(H_1)$ components. Therefore there are at least $2(n - \pi(H_1))$ edges between H_1 and H_2 in C.

Since H_2 has only n vertices, there are at most $2n$ edges in C from H_2. Thus, there are at most $2n - 2(n - \pi(H_1)) = 2\pi(H_1)$ edges between H_2 and H_3 in C.

Again, H_3 has at least $n - \pi(H_3)$ components. Therefore, there are at least $2(n - \pi(H_3)) - 2\pi(H_1) = 2n - 2\pi(H_3) - 2\pi(H_1)$ edges between H_3 and H_4 in C.

Since H_4 has only n vertices, there are at most $2n$ edges in C from H_4. Thus, there are at most $2n - (2n - 2\pi(H_3) - 2\pi(H_1)) = 2\pi(H_3) + 2\pi(H_1)$ edges between H_4 and H_5 in C.

If we continue step by step, we get that between H_{2k} and H_{2k+1} there are at most $2\pi(H_{2k-1}) + 2\pi(H_{2k-3}) + \cdots + 2\pi(H_3) + 2\pi(H_1)$ edges in C and from the other side H_{2k+1} has at least $n - \pi(H_{2k+1})$ components. Therefore there should be at least $2(n - \pi(H_{2k+1}))$ edges between H_{2k+1} and H_{2k} in C. Now we get that

$$2(n - \pi(H_{2k+1})) \leq 2\pi(H_{2k-1}) + 2\pi(H_{2k-3}) + \cdots + 2\pi(H_3) + 2\pi(H_1)$$

$$n \le \pi(H_{2k+1}) + \pi(H_{2k-1}) + \pi(H_{2k-3}) + \cdots + \pi(H_3) + \pi(H_1)$$
$$n \le \sum_{i=0}^{k} \pi(H_{2i+1}).$$

Thus we finish the proof of Theorem 2(i). $\square$

Before the proofs of Theorem 2, statements (ii) and (iii), we define functions $A(t)$, $B(t)$ and state the following lemmas. Let $a, b, t \in \{1, 2, \ldots, 2k+1\}$.

$$A(t) = 0 \text{ for } t < a; \quad B(t) = 0 \text{ for } t < b;$$
$$A(t) = 1 \text{ for } t \ge a; \quad B(t) = 1 \text{ for } t \ge b.$$

Lemma 6. *Let P_{2k+1} be a path with odd number of vertices, $k \ge 1$, $a, b \in \{1, 2, \ldots, 2k+1\}$. Let $H_1, H_2, \ldots, H_{2k+1}$ be graphs with n vertices such that one of the following conditions holds.*

(I) *a, b are even and $\pi(H_1) \ge 2$, $\pi(H_{2k+1}) \ge 2$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) = n+1$;*

(II) *a is odd, b is even and $\pi(H_1) \ge 1$, $\pi(H_{2k+1}) \ge 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) = n$;*

(III) *a, b are odd and $\pi(H_1) \ge 1$, $\pi(H_{2k+1}) \ge 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) = n-1$; moreover for $a = 1$, $b = 2k+1$ we have only $\sum_{i=0}^{k} \pi(H_{2i+1}) = n-1$.*

Then there exists a connected multiple G' of P_{2k+1} such that $d_{G'}(u_l) = 2n$ for $l \in \{1, 2, \ldots, 2k+1\} \setminus \{a, b\}$ and either $d_{G'}(u_a) = d_{G'}(u_b) = 2n-1$ if $a \ne b$ or $d_{G'}(u_a) = 2n-2$ if $a = b$.

Proof. (I) a, b are even. We have $n \ge 3$ because of $\pi(H_1) \ge 2$.

Similarly, as in the previous proof we define the number of loops at each vertex u_i of multiple G' of P_{2k+1}. For even vertices $u_2, u_4, \ldots, u_{2k}$ we define $\ell_{G'}(u_{2i}) = 0$, $i = 1, 2, \ldots, k$, and for odd vertices $\ell_{G'}(u_{2i+1}) = \pi(H_{2i+1})$, $i = 0, 1, \ldots, k$. Note that we have $\ell_{G'}(u_1) \ge 2$, $\ell_{G'}(u_{2k+1}) \ge 2$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n+1$.

Now we define the multiplicity of every edge e_m of P_{2k+1}, $m = 1, 2, \ldots, 2k$,

$$m_{G'}(e_1) = 2n - 2\ell_{G'}(u_1) + A(1) + B(1),$$
$$m_{G'}(e_2) = 2\ell_{G'}(u_1) - A(2) - B(2),$$
$$m_{G'}(e_3) = 2n - 2\ell_{G'}(u_1) - 2\ell_{G'}(u_3) + A(3) + B(3),$$
$$m_{G'}(e_4) = 2\ell_{G'}(u_1) + 2\ell_{G'}(u_3) - A(4) - B(4),$$

in general we have

$$m_{G'}(e_{2i}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i) - B(2i) \quad \text{and}$$

$$m_{G'}(e_{2i-1}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i-1) + B(2i-1), \text{ for } i = 1, 2, \ldots, k.$$

Clearly, by the construction, the multiplicity of every edge is at least 2 and the degree of every vertex of G' is the following

$$d_{G'}(u_1) = 2\ell_{G'}(u_1) + m_{G'}(e_1) = 2\ell_{G'}(u_1) + 2n - 2\ell_{G'}(u_1) + A(1) + B(1) = 2n.$$

Clearly, $A(1) = B(1) = 0$.

$$\begin{aligned} d_{G'}(u_{2i}) &= m_{G'}(e_{2i-1}) + m_{G'}(e_{2i}) \\ &= 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i-1) + B(2i-1) + \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i) - B(2i) \\ &= 2n + A(2i-1) + B(2i-1) - A(2i) - B(2i), \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

Then $d_{G'}(u_{2i}) = 2n$ for $2i \notin \{a, b\}$, $d_{G'}(u_{2i}) = 2n - 1$ for $2i \in \{a, b\}$, $a \neq b$, and $d_{G'}(u_{2i}) = 2n - 2$ for $2i = a = b$.

$$\begin{aligned} d_{G'}(u_{2i+1}) &= m_{G'}(e_{2i}) + m_{G'}(e_{2i+1}) + 2\ell_{G'}(u_{2i+1}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) \\ &- A(2i) - B(2i) + 2n - \sum_{j=1}^{i+1} 2\ell_{G'}(u_{2j-1}) + A(2i+1) + B(2i+1) + 2\ell_{G'}(u_{2i+1}) \\ &= 2n - A(2i) - B(2i) + A(2i+1) + B(2i+1) = 2n, \text{ for } i = 1, 2, \ldots, k-1. \end{aligned}$$

Note that $A(2i) = A(2i+1)$, $B(2i) = B(2i+1)$.

$$\begin{aligned} d_{G'}(u_{2k+1}) &= m_{G'}(e_{2k}) + 2\ell_{G'}(u_{2k+1}) = \sum_{j=1}^{k+1} 2\ell_{G'}(u_{2j-1}) - A(2k) - B(2k) \\ &= 2(n+1) - A(2k) - B(2k) = 2n. \end{aligned}$$

Clearly, $A(2k) = B(2k) = 1$.

Resulting G' is the connected multiple of P_{2k+1} as required.

(II) a is odd and b is even. We have $n \geq 2$ because of $\pi(H_1) \geq 1$.

We define the number of loops at each vertex u_i of multiple G' of P_{2k+1} as in (I) such that we have $\ell_{G'}(u_1) \geq 1$, $\ell_{G'}(u_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n$.

Now we define the multiplicity of every edge e_m of P_{2k+1}, $m = 1, 2, \ldots, 2k$,

$$m_{G'}(e_{2i}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) - B(2i) \quad \text{and}$$

$$m_{G'}(e_{2i-1}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i-1) + B(2i-1), \text{ for } i = 1, 2, \ldots, k.$$

Clearly, by the construction the multiplicity of every edge is at least 1 and the degree of every vertex of G' is the following

$$d_{G'}(u_1) = 2\ell_{G'}(u_1)+m_{G'}(e_1) = 2\ell_{G'}(u_1)+2n-2\ell_{G'}(u_1)-A(1)+B(1) = 2n-A(1).$$

Clearly, $B(1) = 0$, $d_{G'}(u_1) = 2n$ for $a \neq 1$ and $d_{G'}(u_1) = 2n - 1$ for $a = 1$.

$$\begin{aligned} d_{G'}(u_{2i}) = m_{G'}(e_{2i-1}) + m_{G'}(e_{2i}) &= 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i-1) + B(2i-1) \\ &+ \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) - B(2i) \\ &= 2n - A(2i-1) + B(2i-1) + A(2i) - B(2i) \\ &= 2n + B(2i-1) - B(2i), \text{ for } i = 1, 2, \dots, k. \end{aligned}$$

Then $A(2i-1) = A(2i)$, $d_{G'}(u_{2i}) = 2n$ for $2i \neq b$ and $d_{G'}(u_{2i}) = 2n-1$ for $2i = b$.

$$\begin{aligned} d_{G'}(u_{2i+1}) &= m_{G'}(e_{2i}) + m_{G'}(e_{2i+1}) + 2\ell_{G'}(u_{2i+1}) \\ &= \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) - B(2i) \\ &+ 2n - \sum_{j=1}^{i+1} 2\ell_{G'}(u_{2j-1}) - A(2i+1) + B(2i+1) + 2\ell_{G'}(u_{2i+1}) \\ &= 2n + A(2i) - B(2i) - A(2i+1) + B(2i+1) \\ &= 2n + A(2i) - A(2i+1), \text{ for } i = 1, 2, \dots, k-1. \end{aligned}$$

Then $B(2i) = B(2i+1)$, $d_{G'}(u_{2i+1}) = 2n$ for $2i+1 \neq a$ and $d_{G'}(u_{2i+1}) = 2n - 1$ for $2i + 1 = a$.

$$\begin{aligned} d_{G'}(u_{2k+1}) &= m_{G'}(e_{2k}) + 2\ell_{G'}(u_{2k+1}) \\ &= \sum_{j=1}^{k+1} 2\ell_{G'}(u_{2j-1}) + A(2k) - B(2k) = 2n + A(2k) - B(2k). \end{aligned}$$

Clearly, $B(2k) = 1$, $d_{G'}(u_{2k+1}) = 2n$ for $a \neq 2k+1$ and $d_{G'}(u_{2k+1}) = 2n - 1$ for $a = 2k + 1$.

Resulting G' is the connected multiple of P_{2k+1} as required.

(III) a, b are odd. From $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) = n - 1$ we have $n > 2$.

We define the number of loops at each vertex u_i of multiple G' of P_{2k+1} as in (I) such that we have $\ell_{G'}(u_1) \geq 1$, $\ell_{G'}(u_{2k+1}) \geq 1$, and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n-1$.

Now we define the multiplicity of every edge e_m of P_{2k+1}, $m = 1, 2, \ldots, 2k$,

$$m_{G'}(e_{2i}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) + B(2i) \quad \text{and}$$
$$m_{G'}(e_{2i-1}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i-1) - B(2i-1), \text{ for } i = 1, 2, \ldots, k.$$

Clearly, by the construction the multiplicity of every edge is at least 2 and the degree of every vertex of G' is the following

$$\begin{aligned} d_{G'}(u_1) &= 2\ell_{G'}(u_1) + m_{G'}(e_1) = 2\ell_{G'}(u_1) + 2n - 2\ell_{G'}(u_1) - A(1) - B(1) \\ &= 2n - A(1) - B(1). \end{aligned}$$

Then $d_{G'}(u_1) = 2n$ for $1 \notin \{a, b\}$, $d_{G'}(u_1) = 2n - 1$, for $1 \in \{a, b\}$, $a \neq b$, and $d_{G'}(u_1) = 2n - 2$ for $1 = a = b$.

$$\begin{aligned} d_{G'}(u_{2i}) &= m_{G'}(e_{2i-1}) + m_{G'}(e_{2i}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - A(2i-1) - B(2i-1) \\ &\quad + \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) + B(2i) = 2n - A(2i-1) - B(2i-1) \\ &\quad + A(2i) + B(2i) = 2n, \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

Note that $A(2i-1) = A(2i)$, $B(2i-1) = B(2i)$.

$$\begin{aligned} d_{G'}(u_{2i+1}) &= m_{G'}(e_{2i}) + m_{G'}(e_{2i+1}) + 2\ell_{G'}(u_{2i+1}) \\ &= \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + A(2i) + B(2i) \\ &\quad + 2n - \sum_{j=1}^{i+1} 2\ell_{G'}(u_{2j-1}) - A(2i+1) - B(2i+1) + 2\ell_{G'}(u_{2i+1}) \\ &= 2n + A(2i) + B(2i) - A(2i+1) - B(2i+1), \text{ for } i = 1, 2, \ldots, k-1. \end{aligned}$$

Then $d_{G'}(u_{2i+1}) = 2n$ for $2i+1 \notin \{a, b\}$, $d_{G'}(u_{2i+1}) = 2n - 1$ for $2i+1 \in \{a, b\}$, $a \neq b$, and $d_{G'}(u_{2i+1}) = 2n - 2$ for $2i + 1 = a = b$.

$$d_{G'}(u_{2k+1}) = m_{G'}(e_{2k}) + 2\ell_{G'}(u_{2k+1}) = \sum_{j=1}^{k+1} 2\ell_{G'}(u_{2j-1}) + A(2k) + B(2k)$$

$$= 2(n-1) + A(2k) + B(2k) = 2n - 2 + A(2k) + B(2k).$$

Then $d_{G'}(u_{2k+1}) = 2n$ for $2k+1 \notin \{a,b\}$, $d_{G'}(u_{2k+1}) = 2n-1$ for $2k+1 \in \{a,b\}$, $a \neq b$, and $d_{G'}(u_{2k+1}) = 2n-2$ for $2k+1 = a = b$.

Now let $a = 1$ and $b = 2k+1$. From $\sum_{i=0}^{k} \pi(H_{2i+1}) = n-1$ we have $n \geq 1$ and again we define the number of loops at each vertex u_i of multiple G' of P_{2k+1} as in (I) such that $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n-1$. Clearly, $A(j) = 1$ for $j = 1, 2, \ldots, 2k+1$, $B(j) = 0$ for $j = 1, 2, \ldots, 2k$ and $B(2k+1) = 1$.

As in general case, we define specifically the multiplicity of every edge e_m of P_{2k+1}, $m = 1, 2, \ldots, 2k$,

$$m_{G'}(e_{2i}) = \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + 1 \text{ for } i = 1, 2, \ldots, k,$$

$$m_{G'}(e_{2i-1}) = 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - 1 \text{ for } i = 1, 2, \ldots, k.$$

Clearly, by the construction the multiplicity of every edge is at least 1 and the degree of every vertex of G' is the following:

$$d_{G'}(u_1) = 2\ell_{G'}(u_1) + m_{G'}(e_1) = 2\ell_{G'}(u_1) + 2n - 2\ell_{G'}(u_1) - 1 = 2n - 1.$$

$$\begin{aligned} &d_{G'}(u_{2i}) = m_{G'}(e_{2i-1}) + m_{G'}(e_{2i}) \\ &= 2n - \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) - 1 + \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + 1 = 2n, \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

$$\begin{aligned} &d_{G'}(u_{2i+1}) = m_{G'}(e_{2i}) + m_{G'}(e_{2i+1}) + 2\ell_{G'}(u_{2i+1}) \\ &= \sum_{j=1}^{i} 2\ell_{G'}(u_{2j-1}) + 1 + 2n - \sum_{j=1}^{i+1} 2\ell_{G'}(u_{2j-1}) - 1 + 2\ell_{G'}(u_{2i+1}) = 2n, \\ &\text{for } i = 1, 2, \ldots, k-1. \end{aligned}$$

$$d_{G'}(u_{2k+1}) = m_{G'}(e_{2k}) + 2\ell_{G'}(u_{2k+1}) = \sum_{j=1}^{k+1} 2\ell_{G'}(u_{2j-1}) + 1 = 2(n-1) + 1 = 2n-1.$$

In both cases the resulting G' is the connected multiple of P_{2k+1} as required. $\square$

Lemma 7. *Let P_{2k+1} be a path with odd number of vertices, $k \geq 1$, $a, b \in \{1, 2, \ldots, 2k+1\}$. Let $H_1, H_2, \ldots, H_{2k+1}$ be graphs with n vertices. Assume that $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ contains a hamiltonian path P starting in vertex x from $V(P_{2k+1}(\{u_a\})[H_a])$ and ending in a vertex y from $V(P_{2k+1}(\{u_b\})[H_b])$, where $u_a, u_b \in V(P_{2k+1})$.*

(I) *If* a, b *are even, then* $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n+1$*;*

(II) *If* a *is odd and* b *is even, then* $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n$*;*

(III) *If* a, b *are odd, then* $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n-1$*.*

Proof. (I) a, b are even. Clearly, $A(1) = B(1) = 0$ and $A(2k) = B(2k) = 1$.

The graph H_1 has at least $n - \pi(H_1)$ components. Therefore there are at least $2(n-\pi(H_1)) + A(1) + B(1) = 2(n-\pi(H_1))$ edges between H_1 and H_2 in P.

Since H_2 has only n vertices, there are at most $2n$ edges in P from H_2. Thus, there are at most $2n - 2(n-\pi(H_1)) - A(2) - B(2) = 2\pi(H_1) - A(2) - B(2)$ edges between H_2 and H_3 in P. Note that $A(2) = 0$ and $B(2) = 0$, if $x, y \notin V(P_{2k+1}(\{u_2\})[H_2])$, respectively.

In general we have at most

$$\sum_{j=1}^{i} 2\pi(H_{2j-1}) - A(2i) - B(2i), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i} and H_{2i+1} in P and we have at least

$$2n - \sum_{j=1}^{i} 2\pi(H_{2j-1}) + A(2i-1) + B(2i-1), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i-1} and H_{2i} in P.

The last subgraph H_{2k+1} has $n - \pi(H_{2k+1})$ components in its path covering. Therefore there are at least $2(n - \pi(H_{2k+1}))$ edges between H_{2k+1} and H_{2k} in P. Thus we get

$$2(n - \pi(H_{2k+1})) \leq \sum_{j=1}^{k} 2\pi(H_{2j-1}) - A(2k) - B(2k) \leq \sum_{j=1}^{k} 2\pi(H_{2j-1}) - 2,$$

$$2n + A(2k) + B(2k) = 2n + 2 \leq \sum_{j=0}^{k} 2\pi(H_{2j+1}),$$

$$n + 1 \leq \sum_{j=0}^{k} \pi(H_{2j+1}).$$

(II) a is odd and b is even. Clearly, $B(1) = 0$ and $B(2k) = 1$.

Similarly as in (I), we have at most

$$\sum_{j=1}^{i} 2\pi(H_{2j-1}) + A(2i) - B(2i), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i} and H_{2i+1} in P and we have at least

$$2n - \sum_{j=1}^{i} 2\pi(H_{2j-1}) - A(2i-1) + B(2i-1), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i-1} and H_{2i} in P.

The last subgraph H_{2k+1} has $n - \pi(H_{2k+1})$ components in its path covering. Therefore there are at least $2(n - \pi(H_{2k+1})) - A(2k+1) + A(2k)$ edges between H_{2k+1} and H_{2k} in P. Clearly, $A(2k+1) = 1$. Thus we get

$$2(n - \pi(H_{2k+1})) - A(2k+1) + A(2k) \leq \sum_{j=1}^{k} 2\pi(H_{2j-1}) + A(2k) - B(2k),$$

$$2n - A(2k+1) + B(2k) = 2n \leq \sum_{j=0}^{k} 2\pi(H_{2j+1}),$$

$$n \leq \sum_{j=0}^{k} \pi(H_{2j+1}).$$

(III) a, b are odd.

Again similarly as in (I), we have at most

$$\sum_{j=1}^{i} 2\pi(H_{2j-1}) + A(2i) + B(2i), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i} and H_{2i+1} in P and we have at least

$$2n - \sum_{j=1}^{i} 2\pi(H_{2j-1}) - A(2i-1) - B(2i-1), \text{ for } i = 1, 2, \ldots, k,$$

edges between H_{2i-1} and H_{2i} in P.

The last subgraph H_{2k+1} has $n - \pi(H_{2k+1})$ components in its path covering. Therefore there should be at least $2(n - \pi(H_{2k+1})) - A(2k+1) + A(2k) - B(2k+1) + B(2k)$ edges between H_{2k+1} and H_{2k} in P. Clearly, $A(2k+1) = B(2k+1) = 1$. Thus we get

$$\begin{aligned} 2(n - \pi(H_{2k+1})) - A(2k+1) + A(2k) - B(2k+1) + B(2k) \\ \leq \sum_{j=1}^{k} 2\pi(H_{2j-1}) + A(2k) + B(2k), \end{aligned}$$

$$2n - A(2k+1) - B(2k+1) = 2n - 2 \leq \sum_{j=0}^{k} 2\pi(H_{2j+1}),$$

$$n - 1 \leq \sum_{j=0}^{k} \pi(H_{2j+1}).$$

□

Now we are ready to prove Theorem 2, statements (ii) and (iii).

(ii) First suppose that $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n-1$. Let x, y be vertices in $P_{2k+1}(\{u_1\})[H_1]$, $P_{2k+1}(\{u_{2k+1}\})[H_{2k+1}]$ such that vertices of H_1, H_{2k+1} corresponding to x, y are not vertices of degree 2 in some component (path) of a spanning linear forest of H_1, H_{2k+1} with $\pi(H_1), \pi(H_{2k+1})$ edges, respectively. We show that $P_{2k+1}[H_1, H_2, \dots, H_{2k+1}]$ contains an xy-hamiltonian path.

We set $a = 1$ and $b = 2k+1$. By Lemma 6(III), we find a connected multiple G' of P_{2k+1} such that $d_{G'}(u_l) = 2n$ for $l \in \{2, 3, \dots, 2k\}$ and $d_{G'}(u_1) = d_{G'}(u_{2k+1}) = 2n-1$. Note that if $\sum_{i=0}^{k} \pi(H_{2i+1}) > n-1$ (the multiple G' has more than $n-1$ loops), then we remove arbitrary loops from G' in such a way that $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n-1$. Clearly, $\ell_{G'}(u_j) \leq \pi(H_j)$ for $j = 1, 2, \dots, 2k+1$.

As before, we prove that $P_{2k+1}[H_1, H_2, \dots, H_{2k+1}]$ contains an xy-hamiltonian path using exactly $\ell_{G'}(u_i)$ edges of H_i and exactly $m_{G'}(e_j)$ edges between $V(P_{2k+1}(\{u_j\})[H_j])$ and $V(P_{2k+1}(\{u_{j+1}\})[H_{j+1}])$ for $i = 1, 2, \dots, 2k+1$ and $j = 1, 2, \dots, 2k$.

For every vertex u_i and every graph H_i, there exists a spanning linear subforest of $P_{2k+1}(\{u_i\})[H_i] \cong H_i$ with components $P_1(u_i), P_2(u_i), \dots, P_{j_i}(u_i)$ satisfying

$$j_i = |H_i| - \ell_{G'}(u_i) \quad \text{and} \quad \sum_{t=1}^{j_i} ||P_t(u_i)|| = \ell_{G'}(u_i),$$

because $\ell_{G'}(u_i) \leq \pi(H_i)$.

Futhermore, after removing all the loops from multiple G', there exists an open eulerian trail C in G' from u_1 to u_{2k+1} of P_{2k+1}. We obtain the xy-hamiltonian path P in $P_{2k+1}[H_1, H_2, \dots, H_{2k+1}]$ as required by replacing simultaneously the vertices u_i at their t-th occurence in C by the component $P_t(u_i)$, for $t = 1, 2, \dots, j_i$ and $i = 1, 2, \dots, 2k+1$, such that x is the first vertex and y is the last vertex of P. Note that x, y are endvertices of different paths $P_t(u_i)$ or isolated vertices.

Now we suppose that $P_{2k+1}[H_1, H_2, \dots, H_{2k+1}]$ contains some hamiltonian path P. We may assume that the hamiltonian path starts in H_1 and ends in H_{2k+1}. By Lemma 7(III), we get that $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n-1$. □

(iii) First suppose that $\pi(H_1) \geq 2$, $\pi(H_{2k+1}) \geq 2$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n+1$. Let x, y be vertices in $P_{2k+1}(\{u_a\})[H_a]$, $P_{2k+1}(\{u_b\})[H_b]$, $a, b \in \{1, 2, \dots, 2k+1\}$, respectively. We show that $P_{2k+1}[H_1, H_2, \dots, H_{2k+1}]$ contains an xy-hamiltonian path for every x, y.

Suppose that a, b are even. By Lemma 6(I), we find a connected multiple G' of P_{2k+1} such that $d_{G'}(u_l) = 2n$ for $l \in \{1, 2, \dots, 2k+1\} \setminus \{a, b\}$ and either $d_{G'}(u_a) = d_{G'}(u_b) = 2n-1$ if $a \neq b$ or $d_{G'}(u_a) = 2n-2$ if $a = b$. Note that if $\sum_{i=0}^{k} \pi(H_{2i+1}) > n+1$ (the multiple G' has more than $n+1$ loops), then we remove arbitrary loops from G' in such a way that $\ell_{G'}(u_1) \geq 2$, $\ell_{G'}(u_{2k+1}) \geq 2$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n+1$. Clearly, $\ell_{G'}(u_j) \leq \pi(H_j)$ for $j = 1, 2, \ldots, 2k+1$.

Suppose that a is odd and b is even (the case a is even and b is odd is symmetrical). If x is a vertex of degree 2 in some component (path) P of a spanning linear forest of H_a with $\pi(H_a)$ edges, then we remove one edge of P incident with x from this spanning linear forest. Hence we have $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n$. By Lemma 6(II), we find a connected multiple G' of P_{2k+1} such that $d_{G'}(u_l) = 2n$ for $l \in \{1, 2, \ldots, 2k+1\} \setminus \{a, b\}$ and $d_{G'}(u_a) = d_{G'}(u_b) = 2n-1$. Note that if $\sum_{i=0}^{k} \pi(H_{2i+1}) > n$ (the multiple G' has more than n loops), then we remove arbitrary loops from G' in such a way that $\ell_{G'}(u_1) \geq 1$, $\ell_{G'}(u_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n$. Clearly, $\ell_{G'}(u_j) \leq \pi(H_j)$ for $j = 1, 2, \ldots, 2k+1$.

Suppose that a, b are odd. We remove at most 2 edges from spanning linear forests of H_a and H_b such that now x and y are not vertices of degree 2 in some component (path) of a spanning linear forest of H_a or H_b and x, y are not in the same component (path) of a spaning linear forest of $H_a = H_b$. Hence we have $\pi(H_1) \geq 1$, $\pi(H_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n-1$ even if a, b are in the same component. By Lemma 6(III), we find a connected multiple G' of P_{2k+1} such that $d_{G'}(u_l) = 2n$ for $l \in \{1, 2, \ldots, 2k+1\} \setminus \{a, b\}$ and either $d_{G'}(u_a) = d_{G'}(u_b) = 2n-1$ if $a \neq b$ or $d_{G'}(u_a) = 2n-2$ if $a = b$. Note that if $\sum_{i=0}^{k} \pi(H_{2i+1}) > n-1$ (the multiple G' has more than $n-1$ loops), then we remove arbitrary loops from G' in such a way that $\ell_{G'}(u_1) \geq 1$, $\ell_{G'}(u_{2k+1}) \geq 1$ and $\sum_{i=0}^{k} \ell_{G'}(u_{2i+1}) = n-1$. Clearly, $\ell_{G'}(u_j) \leq \pi(H_j)$ for $j = 1, \ldots, 2k+1$.

Similarly as in the previous proof, we prove that $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ contains an xy-hamiltonian path using exactly $\ell_{G'}(u_i)$ edges of H_i and exactly $m_{G'}(e_j)$ edges between $V(P_{2k+1}(\{u_j\})[H_j])$ and $V(P_{2k+1}(\{u_{j+1}\})[H_{j+1}])$ for $i = 1, 2, \ldots, 2k+1$, $j = 1, 2, \ldots, 2k$.

For every vertex u_i and every graph H_i, there exists a spanning linear subforest of $P_{2k+1}(\{u_i\})[H_i] \cong H_i$ with components $P_1(u_i), P_2(u_i), \ldots, P_{j_i}(u_i)$ satisfying

$$j_i = |H_i| - \ell_{G'}(u_i) \quad \text{and} \quad \sum_{t=1}^{j_i} ||P_t(u_i)|| = \ell_{G'}(u_i),$$

because $\ell_{G'}(u_i) \leq \pi(H_i)$.

Futhermore, after removing all the loops from multiple G', there exists an open eulerian trail C in G' from u_a to u_b if $a \neq b$ and a closed eulerian trail C in G' if $a = b$. We obtain the xy-hamiltonian path in $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ as required by replacing simultaneously the vertices u_i at their t-th occurence in C by the component $P_t(u_i)$ for $t = 1, 2, \ldots, j_i$, $i = 1, 2, \ldots, 2k+1$ such that x is the first vertex and y is the last vertex of P. Note that x, y are endvertices of different paths $P_t(u_i)$ or isolated vertices.

Now suppose that $P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ is hamiltonian connected. Clearly, if $\pi(H_1) \leq 1$, $\pi(H_{2k+1}) \leq 1$, then there is no hamiltonian path starting and ending in H_2, H_{2k}, respectively. Thus $\pi(H_1) \geq 2$, $\pi(H_{2k+1}) \geq 2$ and $n > 2$ ($P_{2k+1}[H_1, H_2, \ldots, H_{2k+1}]$ has to be 3-connected). Since this graph has a hamilonian path between two arbitrary vertices, by Lemma 7, we immediately get that $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq n + 1$. ∎

Proof of Theorem 3.

(i) First assume that the graph $P_{2k}[H_1, H_2, \ldots, H_{2k}]$ has a hamiltonian cycle C and $\pi(H_1) = 0$. Since every vertex in C has degree 2 and there is no edge in $E(H_1)$, there are exactly $2n$ edges between H_1 and H_2. Thus there is no edge of C between H_2 and H_3 and we get that $k = 1$. Similarly for $\pi(H_{2k}) = 0$.

Now assume that $k = 1$ or $\pi(H_1) \geq 1$ and $\pi(H_{2k}) \geq 1$. Then the hamiltonicity of $P_{2k}[H_1, H_2, \ldots, H_{2k}]$ follows immediately from the proof of Theorem 6 from [6]. The author proved in [6] the hamiltonicity of lexicographic product of $G[H]$ where G is traceable by finding a $2n$-regular multiple of P_{2k} which uses only one loop at the first and last vertex of P_{2k}.

(ii) Again, the proof is an easy consequence of the proof of Theorem 6 from [6].

(iii) Let $k = 1$. If $\pi(H_1) = 0$ or $\pi(H_2) = 0$, then clearly there is no hamiltonian path between some vertices of H_2 or H_1, respectively. If $\pi(H_1) \geq 1$ and $\pi(H_{2k}) \geq 1$, then clearly there exists a hamiltonian path between every two vertices of $P_2[H_1, H_2]$.

Let $k > 1$. First suppose that $P_{2k}[H_1, H_2, \ldots, H_{2k}]$ is hamiltonian connected. If $\pi(H_1) \leq 1$ or $\pi(H_{2k}) \leq 1$, then clearly there is no hamiltonian path between some two vertices of H_2 or H_{2k-1}, respectively.

Now assume that $\pi(H_1) \geq 2$ and $\pi(H_{2k}) \geq 2$. Let x be a vertex of $V(P_{2k}(\{u_a\})[H_a])$ and y be a vertex of $V(P_{2k}(\{u_b\})[H_b])$. We may assume that $a \leq b$. Let F be 1-factor of P_{2k}, S_1 a multigraph with only one vertex u_1 and one loop, S_2 a multigraph with only vertex u_{2k} and one loop, and let $P' \subseteq P_{2k}$ be a path from u_a to u_b in P_{2k} and $P'' \subseteq P_{2k}$ a path from u_b to u_{2k} in P_{2k} if any.

(I) Let $b - a$ be odd. Then P' is an even path and let F_1 be 1-factor of P' and set $F_2 = P' - E(F_1)$. Then we define multiple G'

$$G' = 2P_{2k} + (2n - 4)F + S_1 + S_2 - F_1 + F_2.$$

Clearly, the degree of every vertex of G' except from u_a and u_b is $2n$ and $d_{G'}(u_a) = d_{G'}(u_b) = 2n - 1$ and the multiplicity of every edge of G' is at least 1.

(II) Let $b - a$ be even. We may assume that a is odd (otherwise we relabel all the vertices: $u_1 \to u_{2k}, u_2 \to u_{2k-1}, \ldots, u_{2k} \to u_1$). Let F_1 be 1-factor of $P' - u_b$ and F_2 be 1-factor of $P' - u_a$. Since a is odd, b is odd as well. Thus P'' is an even path. Let F_3 be 1-factor of P'' and set $F_4 = P'' - E(F_3)$. Then we define multiple G'

$$G' = 2P_{2k} + (2n-4)F + S_1 + 2S_2 - F_1 + F_2 - 2F_3 + 2F_4.$$

Clearly, the degree of every vertex of G' except from u_a and u_b is $2n$, $d_{G'}(u_a) = d_{G'}(u_b) = 2n-1$ if $a \neq b$, $d_{G'}(u_a) = d_{G'}(u_b) = 2n-2$ if $a = b$, and the multiplicity of every edge of G' is at least 1. Note that $n \geq 3$ because of $\pi(H_1) \geq 2$.

From such a multiple G' in both cases we get a hamiltonian path between two arbitrary vertices of the graph $P_{2k}[H_1, H_2, \ldots, H_{2k}]$ similarly as in the proof of Theorem 2. ∎

4. Next Results

Now we easily get results concerning the lexicographic product of P_{2k+1} and given graph H.

Theorem 8. *Let P_{2k+1} be a path with $2k+1$ vertices, $k \geq 1$, and H be a graph. Then the lexicographic product $P_{2k+1}[H]$ is pancyclic if and only if $\pi(H) \geq 1$ and $\left\lceil \frac{|H|}{\pi(H)} \right\rceil \leq k+1$.*

Proof. Set $H_1 = \cdots = H_{2k+1} = H$. By Theorem 2(i), $P_{2k+1}[H]$ is hamiltonian $\Leftrightarrow \sum_{i=0}^{k} \pi(H_{2i+1}) \geq |H| \Leftrightarrow (k+1)\pi(H) \geq |H| \Leftrightarrow \left\lceil \frac{|H|}{\pi(H)} \right\rceil \leq k+1$ because $k+1$ is an integer. The pancyclicity follows from Theorem 1. ∎

Theorem 9. *Let P_{2k+1} be a path with $2k+1$ vertices, $k \geq 1$, and H be a graph. Then the lexicographic product $P_{2k+1}[H]$ is traceable if and only if $\left\lceil \frac{|H|-1}{\pi(H)} \right\rceil \leq k+1$.*

Proof. Set $H_1 = \cdots = H_{2k+1} = H$. By Theorem 2(ii), $P_{2k+1}[H]$ is traceable if and only if $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq |H|-1 \Leftrightarrow (k+1)\pi(H) \geq |H|-1 \Leftrightarrow \left\lceil \frac{|H|-1}{\pi(H)} \right\rceil \leq k+1$ because $k+1$ is an integer. ∎

Corollary 10. *Let G and H be graphs, $|G| \geq 2$. Suppose that G contains a hamiltonian path.*

- *If $|G| = 2k+1$ and $\frac{|H|}{k+1} \leq \pi(H)$, then $G[H]$ is hamiltonian.*
- *If $|G| = 2k+1$ and $\frac{|H|-1}{k+1} \leq \pi(H)$, then $G[H]$ is traceable.*

Thus we improved Theorem 5 and the bounds are the best possible. Moreover we get a similar result also for hamiltonian connectivity.

Theorem 11. *Let P_{2k+1} be a path with $2k+1$ vertices, $k \geq 1$, and H be a graph. Then the lexicographic product $P_{2k+1}[H]$ is hamiltonian connected if and only if $\pi(H) \geq 2$ and $\left\lceil \frac{|H|+1}{\pi(H)} \right\rceil \leq k+1$.*

Proof. Set $H_1 = \cdots = H_{2k+1} = H$. By Theorem 2(iii), $P_{2k+1}[H]$ is hamiltonian connected if and only if $\sum_{i=0}^{k} \pi(H_{2i+1}) \geq |H| + 1 \Leftrightarrow (k+1)\pi(H) \geq |H| + 1 \Leftrightarrow \left\lceil \frac{|H|+1}{\pi(H)} \right\rceil \leq k+1$ because $k+1$ is an integer. ■

Corollary 12. *Let G and H be graphs, $|G| \geq 2$. Suppose that G contains a hamiltonian path. If $|G| = 2k+1$ and $\frac{|H|+1}{k+1} \leq \pi(H)$, then $G[H]$ is hamiltonian connected.*

Theorem 13. *Let P_{2k} be a path with $2k$ vertices, $k \geq 1$, and H be a graph. Then the lexicographic product $P_{2k}[H]$ is hamiltonian connected if and only*

- $\pi(H) \geq 1$ *for* $k = 1$*;*
- $\pi(H) \geq 2$ *for* $k > 1$*.*

Proof. Easy corollary of Theorem 3(iii). ■

Corollary 14. *Let G and H be graphs, $|G| \geq 2$. Suppose that G contains a hamiltonian path.*

If $|G| = 2$ and $||H|| \geq 1$, then $G[H]$ is hamiltonian connected.

If $|G| = 2k$, for $k > 1$, and $||H|| \geq 2$, then $G[H]$ is hamiltonian connected.

5. Conclusion

In this paper we finished a complete characterization of hamiltonicity (Theorem 5 and Theorem 8), traceability (Theorem 5 and Theorem 9) and hamiltonian connectedness (Theorem 11 and Theorem 13) of $G[H]$, where G is a path. Hence we improved and extended results in [6]. Moreover we proved these results also for lex-regular generalized lexicographic products. If G has no hamiltonian path, then for general graphs G it seems to be complicated to characterize when $G[H]$ is traceable, hamiltonian or hamiltonian connected. Let us mention that Kaiser and Kriesell proved in [5] that if G is 4-tough and $||H|| \geq 1$, then $G[H]$ is hamiltonian. Since G is 4-tough implies that G has a 2-walk, G is not so far from being hamiltonian. Clearly,

$$\text{hamiltonicity} \Longrightarrow \text{traceability} \Longrightarrow \text{2-walk} \Longrightarrow \text{3-tree} \Longrightarrow \text{3-walk} \Longrightarrow \cdots.$$

Hence it could be interesting to study hamiltonian paths and cycles in $G[H]$ if G is a 3-tree.

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