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BIPARTITE RAMSEY NUMBER PAIRS INVOLVING CYCLES

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Abstract

For bipartite graphs $G_1, G_2, \ldots, G_k$, the bipartite Ramsey number $b(G_1, G_2, \ldots, G_k)$ is the least positive integer b, so that any coloring of the edges of $K_{b,b}$ with k colors, will result in a copy of G_i in the ith color, for some i. We determine all pairs of positive integers r and t, such that for a sufficiently large positive integer s, any 2-coloring of $K_{r,t}$ has a monochromatic copy of C_{2s}. Let a and b be positive integers with $a \geq b$. For bipartite graphs G_1 and G_2, the bipartite Ramsey number pair (a, b), denoted by $b_p(G_1, G_2) = (a, b)$, is an ordered pair of integers such that for any blue-red coloring of the edges of $K_{r,t}$, with $r \geq t$, either a blue copy of G_1 exists or a red copy of G_2 exists if and only if $r \geq a$ and $t \geq b$. In [*Path-path Ramsey-type numbers for the complete bipartite graph*, J. Combin. Theory Ser. B 19 (1975) 161–173], Faudree and Schelp showed that $b_p(P_{2s}, P_{2s}) = (2s-1, 2s-1)$, for $s \geq 1$. In this paper we will show that for a sufficiently large positive integer s, any 2-coloring of $K_{2s,2s-1}$ has a monochromatic C_{2s}. This will imply that $b_p(C_{2s}, C_{2s}) = (2s, 2s-1)$, if s is sufficiently large.

Keywords: bipartite graph, Ramsey, cycle.

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1. Introduction

In this paper, we will follow the basic graph theory terminology and notation as prescribed by [2]. Specifically, let $G = (V, E)$ be a graph with vertex set V (or $V(G)$) and edge set E (or $E(G)$). We will use the notation $n(G)$ ($m(G)$, respectively) to signify the order (size, respectively) of a graph G or just n (m, respectively) if the context is clear. For a set $S \subseteq V$, the subgraph *induced* by S in G is denoted by $\langle S\rangle_G$ or just $\langle S\rangle$ if the context is clear. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V \mid uv \in E(G)\}$ (or $N(v)$ if the context is clear), and the *closed neighborhood of* v is defined as $N_G[v] = \{v\} \cup N(v)$ (or $N[v]$ if the context is clear). The degree of v, denoted $\deg_G(v)$ (or $\deg(v)$, if the context is clear), is defined as $|N_G(v)|$. If $S \subset V(G)$ and $v \in V(G) - S$, then $N_S(v) = N_G(v) \cap S$. Furthermore, $\deg_S(v) = |N_G(v) \cap S|$. If H is a bipartite graph then $\mathcal{L}(H)$ ($\mathcal{R}(H)$, respectively) will denote the left (right, respectively) partite set of H. If v is a vertex that has degree one in G, then v is called an *end* vertex. If $H = K_{1,2}$, then the one degree two vertex will be referred to as the *central* vertex.

For bipartite graphs $G_1, G_2, \ldots, G_k$, the *bipartite Ramsey number* $b(G_1, G_2, \ldots, G_k)$ is the least positive integer b so that any coloring of the edges of $K_{b,b}$ with k colors will result in a copy of G_i in the ith color, for some i. The existence of all numbers $b(G_1, G_2, \ldots, G_k)$ follows from a result of Erdős and Rado [3]. The case where G_i is an even cycle for all $1 \le i \le k$ has been considered in [1, 6, 7, 8] and [9]. Recent papers have mainly produced asymptotic bounds. In [1], Bicić, Letzter and Sudakov show that the asymptotic value of $b(C_{2s}, C_{2s}, C_{2s})$ is $(3 + o(1))s$. The authors Lin, Liu and Shen prove, in [8], that $b(C_{2s}, C_{2s}) = (2+o(1))s$. Exact values for bipartite Ramsey numbers and their variations are rare as the problem is notoriously difficult. This is because complete bipartite graphs have far fewer edges than complete graphs, and so in the bipartite case there are fewer edges to work with. In this paper we present an exact value for a bipartite Ramsey-type number that involves cycles.

Let a and b be positive integers with $a \ge b$. For bipartite graphs G_1 and G_2, the bipartite Ramsey number pair (a, b), denoted by $b_p(G_1, G_2) = (a, b)$, is an ordered pair of integers such that for any blue-red coloring of the edges of $K_{r,t}$, with $r \ge t$, either a blue copy of G_1 exists or a red copy of G_2 exists if and only if $r \ge a$ and $t \ge b$. In [4], Faudree and Schelp considered the bipartite Ramsey number pair problem for paths. In this paper, we will focus on bipartite Ramsey number pairs that involve cycles. We will show that $b_p(C_{2s}, C_{2s}) = (2s, 2s - 1)$, if s is a sufficiently large integer.

2. Known Results

We rely heavily on the 1984 result due to Gyàrfàs, Rousseau and Schelp [5], who established the maximum number of edges in a spanning subgraph of $K_{s,t}$ that contains no specified path.

Theorem 1 [5]. *Let c be a positive non-zero integer and $G(S,T)$ be a bipartite graph with $|T| = a$ and $|S| = b$ $(a \leq b)$. If G contains no path $P_{2\ell}$ for $\ell > c$, then*

$$m(G(S,T)) \leq \begin{cases} ab & \text{if } a \leq c, \\ bc & \text{if } c < a < 2c, \\ (a+b-2c)c & \text{if } a \geq 2c. \end{cases}$$

In order to make the paper easy to read, we provide as much detail in the arguments as possible. In Section 3, the main result is presented and proved using an essential lemma. This lemma is proved in Section 5. The proof of this vital lemma requires foundational lemmas and results, and they are presented and proved in Section 4.

3. Main Result

In this paper we will show that if s is a sufficiently large positive integer then $b_p(C_{2s}, C_{2s}) = (2s, 2s-1)$. Before proving our result, we state the following essential lemma.

Lemma 2. *Let $s \geq 18$ be an integer. If a blue-red coloring of the edges of $K_{2s,2s-1}$ results in a red copy of C_{2s-2}, then there either exists a red copy of C_{2s} or a blue copy of C_{2s}.*

The proof of Lemma 2 will be given in Section 5. We are now ready to prove our main result.

Theorem 3. *If s is an integer such that $s \geq \max\{18, (b(C_{34}, C_{34}) + 1)/2\}$, then every blue-red coloring of the edges of $G = K_{2s,2s-1}$ will result in a monochromatic copy of C_{2s}.*

Proof. Assume that s is an integer such that $s \geq \max\{18, (b(C_{34}, C_{34}) + 1)/2\}$. Consider a blue-red coloring C of the edges of $G = K_{2s,2s-1}$. Let G_1 (G_2, respectively) denote the graph with vertex set $V(G)$ and blue (red, respectively) edges. Assume that G_1 (G_2, respectively) has no C_{2s} (C_{2s}, respectively). The fact that $2s - 1 \geq 2(b(C_{34}, C_{34}) + 1)/2 - 1 = b(C_{34}, C_{34})$, implies that C has a monochromatic C_{34}. Assume, without loss of generality, that G_2 has a C_{34}.

It follows that, in G_2, we can pick a monochromatic cycle $C' = C_{2q}$, with $17 \le q \le s-1$, such that q is as large as possible. Set $q' = q+1$ and observe that $17 < q' \le s$. Pick a subgraph $G' = K_{2q',2q'-1}$ of G, such that if we restrict the blue-red coloring C to G', G' contains the monochromatic cycle C'. As $q' \ge 18$ we have, by Lemma 2, that since G' has the monochromatic $C' = C_{2q} = C_{2q'-2}$, G' (and G) has a monochromatic cycle $C'' = C_{2q'}$. If $q \le s-2$, then $18 \le q' \le s-1$, and so our choice of q is contradicted, whence $q = s-1$, and so $q' = s$, implying that G has a monochromatic C_{2s}. ■

Theorem 4. *If $s \ge \max\{18, (b(C_{34}, C_{34})+1)/2\}$, then $b_p(C_{2s}, C_{2s}) = (2s, 2s-1)$.*

Proof. Let $b_p(C_{2s}, C_{2s}) = (a, b)$, and recall that $a \ge b$. By Theorem 3, $a \le 2s$ and $b \le 2s-1$. Consider $H = K_{a,2s-2}$. Partition $\mathcal{R}(H)$ into sets Y and W, such that $|Y| = |W| = s-1$. Join each vertex in Y (W, respectively) to each vertex in $\mathcal{L}(H)$ with a blue (red, respectively) edge. This produces a blue-red coloring of H that contains neither a blue C_{2s}, nor a red C_{2s}, whence $b \ge 2s-1$ and so $b = 2s-1$.

Now consider $H = K_{2s-1,2s-1}$. Let $x \in \mathcal{L}(H)$ ($y \in \mathcal{R}(H)$, respectively). Partition $\mathcal{L}(H) - \{x\}$ ($\mathcal{R}(H) - \{y\}$, respectively) into disjoint sets X and U (Y and W, respectively) such that $|X| = |Y| = s-1$ and $|W| = |U| = s-1$. With red edges, join each vertex in X (U, respectively) to each vertex in Y (W, respectively), and x to every vertex in $\mathcal{R}(H)$. With blue edges, join each vertex in X (Y, respectively) to each vertex in W (U, respectively), and y to each vertex in $\mathcal{L}(H) - \{x\}$. This produces a coloring with no blue C_{2s} and no red C_{2s}, whence $a \ge 2s$ and so $a = 2s$. ■

4. Proofs of Foundation Lemmas

Lemma 5. *Let S and T be the partite sets of a bipartite graph, with $u, v \in S$. If every two vertices in S have at least $k > 0$ common neighbors in T, then, if $|S| > k$, there exists a $u-v$ path P that alternates between S and T, on $2k+1$ vertices.*

Proof. Let $s_2, s_3, \ldots, s_k \in S - \{u, v\}$. Pick the following sequence of vertices in T: $t_1 \in N(u) \cap N(s_2), t_2 \in N(s_2) \cap N(s_3) - \{t_1\}, \ldots, t_k \in N(s_k) \cap N(v) - \{t_1, t_2, \ldots, t_{k-1}\}$. We can form $P : u, t_1, s_2, t_2, s_3, \ldots, s_k, t_k, v$, which produces the desired result. ■

Let S and T be two disjoint vertex sets (with $|S| \ge 2$ and $|T| \ge 2$) where every vertex in S is adjacent to at least $|T|-2$ vertices in T. Let Z_1 be a set of k (≥ 0) disjoint copies of $K_{1,2}$. Observe that $|Z_1| = k$. For the purpose of the proof of Lemma 6, we label the vertices of the copies of $K_{1,2}$ within Z_1 as follows. If

$k \geq 1$, let the vertices z_i and z_i' (z_i'', respectively) denote the end vertices (central vertex, respectively) of the i'th copy. Let $V(Z_1)$ denote the set of vertices of all the copies of $K_{1,2}$ within Z_1, where $V(Z_1) = \emptyset$ if $k = 0$. Define Z_2 as a set that contains a single copy of P_5, or set $Z_2 = \emptyset$. If $Z_2 \neq \emptyset$, label vertices of the P_5 as $z_1, z_1'', z_1', z_2'', z_2'$, with end vertices z_1 and z_2'. If $Z_2 \neq \emptyset$ ($Z_2 = \emptyset$, respectively), define $V(Z_2) = \{z_1, z_1'', z_1', z_2'', z_2'\}$ ($V(Z_2) = \emptyset$, respectively).

For the set Z_1, join each vertex in $\bigcup_{i=1}^{k}\{z_i, z_i'\}$ to at least $|T| - 1$ vertices in T. For the set Z_2, join each vertex in $\{z_1, z_1', z_2'\}$ to at least $|T| - 1$ vertices in T. For an odd integer $\ell \geq 3$, let $P' : w_1, w_2, w_3, \ldots, w_{\ell-2}, w_{\ell-1}, w_\ell$ denote the vertices of a path. For every odd integer j with $1 \leq j \leq \ell$, join the vertex w_j to at least $|T| - 1$ vertices in T. Let $G = \langle S \cup T \cup V(Z_1) \cup V(Z_2) \cup V(P') \rangle$ (the sets $S, T, V(Z_1), V(Z_2)$ and $V(P')$ are all disjoint). We have the following important lemma.

Lemma 6. *Let $u, v \in S$, $u', v' \in S - \{u, v\}$ and $|T| \geq 4$. For the graph G, the following holds.*

1. *If $|T| - 1 \geq k + 1$, $|S| + k \geq |T|$, and every vertex in S has at least $|T| - 1$ neighbors in T, then for the set Z_1, there exists a $u - v$ path on $2|T| - 1 + 2k$ vertices.*

2. *If $|S| \geq |T| - 1$, and every vertex in S has at least $|T| - 1$ neighbors in T, then there exists a cycle on $2|T| - 2$ vertices.*

3. *If $|Z_1| = 2$ ($Z_2 \neq \emptyset$, respectively), $|S| + 2 \geq |T| - 1$ ($|S| + 1 \geq |T| - 1$, respectively), $|T| \geq 6$, and every vertex in S has at least $|T| - 1$ neighbors in T, then there exists a cycle on $2|T| + 2$ vertices.*

4. *If $|T| \geq k + 3$, $|S| + k - 1 \geq |T|$, every vertex in S has at least $|T| - 1$ neighbors in T, and u' and v' both have $|T|$ neighbors in T, then for the set Z_1 there exists a $u - v$ path on $2|T| + 1 + 2k$ vertices.*

5. *If $|T| - 1 \geq k + 2$, $|S| + k + 1 \geq |T|$ and every vertex in S has at least $|T| - 1$ neighbors in T, then for the set Z_1 and the path P', there exists a $u - v$ path on $2|T| - 1 + 2k + \ell - 1$ vertices.*

6. *If $|T| \geq k + 4$, $|S| + k + 1 \geq |T| + 1$, every vertex in S has at least $|T| - 1$ neighbors in T, and u' and v' both have $|T|$ neighbors in T, then for the set Z_1 and the path P' there exists a $u - v$ path on $2|T| + 2k + \ell$ vertices.*

7. *If $|T| \geq 8$, $|S| + k \geq |T| - 2$, $k \in \{0, 1\}$, and, if $k = 1$, the vertices z_1 and z_1' are joined to at least $|T| - 2$ vertices in T, then there exists a $u - v$ path on $2|T| - 5 + 2k$ vertices.*

Proof of Part 1. Suppose that $|T| - 1 \geq k + 1$. Recall that each vertex in $S' = S \cup \left(\bigcup_{i=1}^{k}\{z_i, z_i'\}\right)$ has $|T| - 1$ neighbors in T. Observe, by the pigeonhole principle, that each pair of vertices in S' have at least $|T| - 2$ common neighbors in T. If $|T|-2>k$, then pick vertices $s_{k+1},s_{k+2},\ldots,s_{|T|-2}\in S-\{u,v\}$. If $|T|-2=k$, then label z'_k as $s_{|T|-2}$.

If $k\geq 1$, pick a sequence of vertices $t_1,t_2,\ldots,t_{|T|-3},t_{|T|-2}\in T$ such that $t_1\in N(u)\cap N_T(z_1)$, $t_2\in N_T(z'_1)\cap N_T(z_2)-\{t_1\},\ldots,t_k\in N_T(z'_{k-1})\cap N_T(z_k)-\{t_1,t_2,\ldots,t_{k-1}\}$, $t_{k+1}\in N_T(z'_k)\cap N(s_{k+1})-\{t_1,t_2,\ldots,t_k\},\ldots,t_{|T|-2}\in N(s_{|T|-3})\cap N(s_{|T|-2})-\{t_1,t_2,\ldots,t_{|T|-3}\}$. If $k=0$, then pick $t_{k+1}\in N(u)\cap N(s_{k+1}),t_{k+2}\in N(s_{k+1})\cap N(s_{k+2})-\{t_{k+1}\}$,..., $t_{|T|-2}\in N(s_{|T|-3})\cap N(s_{|T|-2})-\{t_{k+1},t_{k+2},\ldots,t_{|T|-3}\}$.

Let $x\in N(s_{|T|-2})-\{t_1,t_2,\ldots,t_{|T|-2}\}$ and $y\in N(v)-\{t_1,t_2,\ldots,t_{|T|-2}\}$. If $k\geq 1$ ($k=0$, respectively), then if $x=y$, the path $P:u,t_1,z_1,z''_1,z'_1,t_2,\ldots,t_k,z_k,z''_k,z'_k,t_{k+1},s_{k+1},t_{k+2},s_{k+2},\ldots,s_{|T|-3},t_{|T|-2},s_{|T|-2},x,v$ ($P{:}\,u,t_{k+1},s_{k+1},t_{k+2},s_{k+2},\ldots,s_{|T|-3},\ t_{|T|-2},s_{|T|-2},x,v$, respectively) is a $u-v$ path on $2|T|+2k-1$ vertices. We may assume that $x\neq y$, whence $N(s_{|T|-2})\supseteq\{t_1,t_2,\ldots,t_{|T|-2},x\}$ and $N(v)\supseteq\{t_1,t_2,\ldots,t_{|T|-2},y\}$. If u is adjacent to x, then the path $P:u,x,s_{|T|-2},t_{|T|-2},s_{|T|-3},\ldots,s_{k+2},t_{k+2},s_{k+1},t_{k+1},z'_k,z''_k,z_k,t_k,\ldots,t_3,z'_2,z''_2,z_2,t_2,z'_1,z''_1,z_1,t_1,v$ ($P:u,x,s_{|T|-2},t_{|T|-2},s_{|T|-3},\ldots,s_{k+2},t_{k+2},s_{k+1},t_{k+1},v$, respectively) is a $u{-}v$ path on $2|T|{+}2k{-}1$ vertices. It follows that $N(u)\supseteq\{t_1,t_2,\ldots,t_{|T|-2},y\}$.

If $k\geq 1$ ($k=0$, respectively), suppose that z'_1 (s_{k+1}, respectively) is adjacent to x. The path $P:u,t_1,z_1,z''_1,z'_1,x,s_{|T|-2},\ t_{|T|-2},\ldots,s_{k+1},t_{k+1},z'_k,z''_k,z_k,\ldots,z'_2,z''_2,z_2,t_2,v$ ($P:u,t_{k+1},s_{k+1},x,s_{|T|-2}\ ,t_{|T|-2},\ldots,t_{k+3},s_{k+2},t_{k+2},v$, respectively) is a $u{-}v$ path on $2|T|{+}2k{-}1$ vertices. We may conclude that if $k\geq 1$ ($k=0$, respectively), then z'_1 (s_{k+1}, respectively) (and u) is adjacent to y, and as $N(v)\supseteq\{t_1,t_2,\ldots,t_{|T|-2},y\}$, the path $P:u,y,z'_1,z''_1,z_1,t_1,s_{|T|-2},t_{|T|-2},\ldots,t_{k+1},z'_k,z''_k,z_k,\ t_k,\ldots,z'_2,z''_2,z_2,t_2,v$ ($P:u,y,s_{k+1},t_1,s_{|T|-2},t_{|T|-2},\ldots,t_{k+3},s_{k+2},t_{k+2},v$, respectively) is a $u-v$ path on $2|T|+2k-1$ vertices. The proofs of parts 2 to 7 are similar and can be found in the appendix contained in Section 6. ■

Lemma 7. *Let S' and T' be the partite sets of a bipartite graph, where, for some integer $k'\geq 0$, $|S'|-k'>|T'|$. If each vertex in S' has more that k' neighbors in T', then there exist $k'+1$ disjoint copies of $K_{1,2}$, where each copy has its central vertex (two end vertices, respectively) in T' (S', respectively).*

Proof. Choose the largest amount of disjoint copies of $K_{1,2}$ such that each copy has its central vertex (two end vertices, respectively) in T' (S', respectively). Let T_1 (S_1, respectively) denote the set of central vertices (end vertices, respectively) of all these copies of $K_{1,2}$. Define $T_2=T'-T_1$ and $S_2=S'-S_1$. Observe that $|T_1|\leq k'$, since otherwise we are done. Hence $|S_1|\leq 2k'$.

As $|S'|-k'>|T'|$, we have that $|S_1|+|S_2|-k'>|T_1|+|T_2|=|S_1|/2+|T_2|$. This, together with the fact that $|S_1|/2\leq k'$, implies that $|S_2|>k'-|S_1|+|S_1|/2+|T_2|\geq|T_2|$. The pigeonhole principle and the fact that each vertex in S' has more than k' neighbors in T', implies that each vertex in S_2 has a neighbor in T_2. By our choice of T_1, observe that each vertex in T_2 can have at most one neighbor in S_2 (since otherwise there will be $|T_1|+1$ copies of $K_{1,2}$). It follows that each vertex in S_2 is matched to a vertex in T_2. This is impossible as $|S_2| > |T_2|$. ■

Lemma 8. *Let S and T be the partite sets of a bipartite graph, with $|S| \geq |T| \geq 8$. If u and v are vertices in S, and each vertex in S has at least $\left\lceil \frac{1}{2}|T| \right\rceil + 1$ neighbors in T, then there exists a $u-v$ path on 5 vertices, that alternates between S and T.*

Proof. Let $u, v \in S$. Pick sets $W_1 \subseteq N(u)$ and $W_2 \subseteq N(v)$, such that $|W_1| = |W_2| = \left\lceil \frac{1}{2}|T| \right\rceil + 1$. Define $W = W_1 \cap W_2$, $T_1 = W_1 \cup W_2$, $T_2 = T - T_1$ and observe that, by the pigeonhole principle, $W \neq \emptyset$. Let $w \in S - \{u, v\}$. If w has a neighbor $w_1 \in W_1$ and a neighbor $w_2 \in W_2$ ($w_1 \neq w_2$), then $P : u, w_1, w, w_2, v$ is a path with the desired property. Observe that $|T_1| = |W_1| + |W_2| - |W| = \left\lceil \frac{1}{2}|T| \right\rceil + \left\lceil \frac{1}{2}|T| \right\rceil + 2 - |W| \geq |T| + 2 - |W| = |T_1| + |T_2| + 2 - |W|$, whence $|W| - 2 \geq |T_2|$.

Clearly, $|T_2| \leq |W| - 2 \leq |W_1| - 2 = \left\lceil \frac{1}{2}|T| \right\rceil - 1$. Suppose first that w is adjacent to a vertex $w_1 \in W$. Then it cannot be adjacent to any other vertex in $W_1 \cup W_2 - \{w_1\}$, since otherwise we are done. As w has degree at least $\left\lceil \frac{1}{2}|T| \right\rceil + 1$, we have, by the pigeonhole principle, that w has at most $\left\lceil \frac{1}{2}|T| \right\rceil - 1$ neighbors in T_2, implying that the degree of w is at most $\left\lceil \frac{1}{2}|T| \right\rceil$, a contradiction.

We may assume, without loss of generality, that w is adjacent to a vertex $w_1 \in W_1 - W$, whence w cannot be adjacent to any vertex in W_2, since otherwise we are done. Hence, w is adjacent to at most $|W_1| - |W| = \left\lceil \frac{1}{2}|T| \right\rceil + 1 - |W|$ vertices in $W_1 \cup W_2$. By the pigeonhole principle, w is adjacent to at most $|W| - 2$ vertices in T_2, implying that w has degree at most $\left\lceil \frac{1}{2}|T| \right\rceil - 1$, a contradiction. ■

In what follows, consider a blue-red edge coloring of $H = K_{2s,2s-1}$, with $s \geq 18$. Let G_R (G_B, respectively) denote the spanning subgraph of H with edge set comprising of red (blue, respectively) edges. In H we will say that a vertex u has a blue (red, respectively) neighbor v if the edge uv is blue (red, respectively). Assume that G_R has a cycle $C = C_{2s-2}$. Let $C : u_1, w_1, u_2, w_2, \ldots, u_i, w_i, \ldots, u_{s-1}, w_{s-1}$ denote the vertices of C, where $u_i \in \mathcal{L}(H)$ and $w_i \in \mathcal{R}(H)$ for all $1 \leq i \leq s-1$. Define $U = \bigcup_{i=1}^{s-1}\{u_i\}$, $W = \bigcup_{i=1}^{s-1}\{w_i\}$, $L = \mathcal{L}(H) - U$ and $Y = \mathcal{R}(H) - W$. To prove Lemma 2, we need to show that $G_B \langle L \cup Y \rangle$ has some specific structural properties. These properties will be proven in Lemmas 14, 19, 21 and 22. Lemmas 9, 10, 11 13 and 17 will assist in proving the structural properties. Lemmas 15, 16 and 20 will be used throughout the paper.

Lemma 9. *If, for some $i > j$, $x' \in L$ is adjacent to w_i and w_j in G_R, and $y' \in Y$ is adjacent to u_i and u_j in G_R, then G_R has a C_{2s}.*

Proof. Assume that for $i > j$, $x' \in L$ is adjacent to w_i and w_j in G_R, and $y' \in Y$ is adjacent to u_i and u_j in G_R. Consider the path segments P and P' on C given by $P : w_i, u_{i+1}, w_{i+1}, \ldots, u_{s-1}, w_{s-1}, u_1, w_1, \ldots, u_{j-1}, w_{j-1}, u_j$ and P' :

$u_i, w_{i-1}, u_{i-1}, \ldots, w_{j+1}, u_{j+1}, w_j$. Observe that the sequence P', x', P, y' forms a cycle in G_R on $2s$ vertices. ∎

Lemma 10. *Let $S \subseteq L$ and $T \subseteq Y$ such that $|S| \geq |T| \geq s-2$. If $G_R \langle S \cup T \rangle = K_{|S|,|T|}$, then $G_B \langle S \cup W \rangle = K_{|S|,|W|}$ or $G_B \langle T \cup U \rangle = K_{|T|,|U|}$, or G_R has a C_{2s}.*

Proof. Let $x_1 \in S$, $y_1 \in T$, $u_i \in U$ and $w_j \in W$. Assume the edges $x_1 w_j$ and $y_1 u_i$ are red. We may assume, without loss of generality, that $j = 1$ as we can relabel the vertices of C if necessary. Note that for all $1 \leq k \leq s-2$, there exists an $x_1 - y_1$ path P in $G_R \langle S \cup T \rangle$ on $2k$ vertices. If $i = 1$, then the vertices $V(C) \cup \{x_1, y_1\}$ form a red C_{2s}, whence $2 \leq i \leq s-1$. In G_R, consider the path $P' : w_1, u_2, w_2, \ldots, w_{i-1}, u_i$ on C. In G_R, let P'' be the path segment on C with vertices $(V(C) - V(P')) \cup \{u_i, w_1\}$. Note that P'' has $2s - 2 - (2i - 2 - 2) = 2s - 2(i-1)$ vertices. Observe that $1 \leq i-1 \leq s-2$. Pick $k = i-1$. The vertices $V(P) \cup V(P'')$ form a red C_{2s}. ∎

Lemma 11. *Let $S' \subseteq L$ ($S' \subseteq Y$, respectively) and $T' \subseteq Y$ ($T' \subseteq L$, respectively) such that $|S'| \geq |T'| \geq s-2$. If, in G_R, every vertex in S' is adjacent to at least $|T'| - 1$ vertices in T', then there exists a vertex z such that, in G_B, exactly one of the following holds.*

1. *The vertex z is in S' and every vertex in $S' - \{z\}$ is adjacent to every vertex in W (U, respectively).*
2. *The vertex z is in W (U, respectively) and every vertex in S' is adjacent to every vertex in $W - \{z\}$ ($U - \{z\}$, respectively).*

Proof. We consider only the case where $S' \subseteq L$ and $T' \subseteq Y$, since the case where $S' \subseteq Y$ and $T' \subseteq L$ is symmetrical. We will prove that every edge in $G_R \langle S' \cup W \rangle$ is incident with a single vertex z. Recall that $w_j, w_i \in W$. Let $x_1 w_j$ and $x_2 w_i$ be red edges in $G_R \langle S' \cup W \rangle$ such that $x_1 \neq x_2$ and $w_i \neq w_j$. We may assume that $j = 1$, since we can relabel the vertices of C if necessary. In addition, $2 \leq i \leq s-1$. Let T be a subset of $4 \leq \ell \leq s-2$ vertices in T'. Set $S = S'$. By Part 1 of Lemma 6, we can set $k = 0$, and deduce that there exists an $x_1 - x_2$ path P, that alternates between S and T, on $2|T| - 1 = 2\ell - 1$ vertices, with $4 \leq \ell \leq s-2$. Note that, in G_R, x_1 and x_2 have at least $s - 4 \geq 14$ common neighbors in T'. Set $S = S'$ and $T = T'$ and so, by Lemma 5, there exists, for $2 \leq \ell \leq 3$, an $x_1 - x_2$ path P that alternates between S and T, on $2\ell - 1$ vertices. We can therefore assume that P has $2\ell - 1$ vertices with $2 \leq \ell \leq s-2$.

Consider the path segment $P' : w_1, u_2, w_2, \ldots, w_{i-1}, u_i, w_i$ on C. Note that P' has $2i-1$ vertices. Consider that path segment P'' on C with vertices $(V(C) - V(P')) \cup \{w_1, w_i\}$. Note that P'' has $2s - 2 - (2i-3) = 2s - 2i + 1$ vertices. If $2 \leq i \leq s-2$, then pick $\ell = i$ and so the vertices $V(P'') \cup V(P)$ form a C_{2s} in G_R. If $i = s-1$, then pick $\ell = 2$ and observe that the vertices $V(P) \cup V(P')$ form a C_{2s} in G_R. Hence, either $x_1 = x_2$ or $w_i = w_1$. It can easily be deduced that all edges in $G_R \langle S' \cup W \rangle$ must be incident with a single vertex z, where $z \in S'$ or $z \in W$, whence $G_B \langle S' \cup W \rangle$ has the required property. ∎

Observation 12. *Let S and T be disjoint vertex sets with $|S| \geq |T| > 0$. If a vertex $w \notin S \cup T$ has two neighbors $u, v \in S$ and there exists a $u - v$ path P that alternates between S and T on $2s-1$ vertices, then the vertices $S \cup T \cup \{w\}$ form a C_{2s}.*

Lemma 13. *If $G_B \langle L \cup Y \rangle$ has two disjoint $K_{1,2}$'s with end vertices in L (Y, respectively), or a P_5 that starts and ends in L (Y, respectively), and every vertex in L (Y, respectively) has at least $s-2$ blue neighbors in W (U respectively), then G_B has a C_{2s}.*

Proof. We consider the case where the end vertices are in L, as the case where the end vertices are in Y is symmetrical. Let us assume first that $G_B \langle L \cup Y \rangle$ has a $P = P_5$ with end vertices in L. Let S be a subset of $L - V(P)$ of cardinality $s-3$. Let Z_2 be the set containing P and set $T = W$. We apply Part 3 of Lemma 6. Observe that $|S| + 1 = s - 3 + 1 \geq |T| - 1$. It follows that G_B has a cycle on $2|T| + 2 = 2(s-1) + 2$ vertices.

Assume now that $G_B \langle L \cup Y \rangle$ has two disjoint $K_{1,2}$'s. Let S be a subset of $L - V(P)$ of cardinality $s-4$. Let Z_1 be the set containing the two disjoint $K_{1,2}$'s and set $T = W$. We apply Part 3 of Lemma 6. Observe that $|S| + 2 = s - 4 + 2 \geq |T| - 1$. It follows that G_B has a cycle on $2|T| + 2 = 2(s-1) + 2$ vertices. ∎

Lemma 14. *The graph $G_B \langle L \cup Y \rangle$ has a $K_{1,2}$ with end (central, respectively) vertices (vertex, respectively) in L (Y, respectively) or G_R has a C_{2s}.*

Proof. Suppose, to the contrary, that each vertex in Y has at most one blue neighbor in L. This implies that each vertex in Y has at least $|L|-1$ red neighbors in L. Let $w \in L$ such that w has two red neighbors in Y. Set $S = Y$ and $T = L - \{w\}$. Note that $|S| = s$ and $|T| = s$. Furthermore, in G_R, every vertex in S has at least $|T| - 1$ neighbors in T. By Part 1 of Lemma 6, we have, for each $u, v \in Y$, that there exists a $u - v$ path in $G_R \langle S \cup T \rangle$ on $2|T| - 1 = 2s - 1$ vertices. By Observation 12, G_R has a C_{2s}. ∎

Lemma 15. *Let $x \in L$ ($y \in Y$, respectively) be a vertex that has exactly t blue neighbors in W (U, respectively), such that t is as small as possible. In G_B, there exists a set $U_1 \subseteq U$ ($W_1 \subseteq W$, respectively) such that $|U_1| = s - 1 - t$ ($|W_1| = s - 1 - t$, respectively) and, if $|U_1| \geq 1$ ($|W_1| \geq 1$, respectively), every vertex in Y (L, respectively) has $|U_1| - 1$ ($|W_1| - 1$, respectively) blue neighbors in U_1 (W_1, respectively) or a C_{2s} exists in G_R.*

Proof. Let $x \in L$ ($y \in Y$, respectively) be a vertex that has exactly t blue neighbors in W (U, respectively) such that t is as small as possible. The case where $y \in Y$ is a vertex with exactly t blue neighbors in U such that t is as small as possible, will be omitted as it follows symmetrically. In G_B, define $U_1 \subseteq U$ such that $U_1 = \{u_i \in U \mid w_i \text{ is not adjacent to } x \text{ in } G_B\}$. Observe that as x has exactly t blue neighbors in W, we have that $|U_1| = s - 1 - t$.

Suppose that $|U_1| \geq 1$. If $|U_1| = 1$, then clearly every vertex in Y has at least $|U_1| - 1 = 0$ neighbors in U_1 and so we are done, whence $|U_1| \geq 2$. Let $y' \in Y$, and suppose, to the contrary, that there are two vertices $u_i, u_j \in U_1$ such that, in G_B, y' is adjacent to neither u_i nor u_j, with $i > j$. Recall that the vertex x is not adjacent to w_i and w_j in G_B. In the graph G_R, the vertex x is adjacent to both w_i and w_j, and y' is adjacent to both u_i and u_j. By Lemma 9, G_R has a C_{2s}. ■

Lemma 16. *Let $x \in L$ ($y \in Y$, respectively) be a vertex that has exactly t blue neighbors in W (U, respectively) such that t is as small as possible. If $|U_1| \geq 1$ ($|W_1| \geq 1$, respectively), then, in G_B, if $y' \in Y$ ($x' \in L$, respectively) is a vertex not adjacent to x (y, respectively), then y' (x', respectively) must be adjacent to all vertices in U_1 (W_1, respectively), or a C_{2s} exists in G_R.*

Proof. Let $x \in L$ be a vertex that has exactly t blue neighbors in W such that t is as small as possible. The case where $y \in Y$, with y having exactly t blue neighbors in U, such that t is as small as possible, will be omitted as it follows symmetrically. Let U_1 be defined as it was in the proof of Lemma 15. Let $y' \in Y$ such that y' is not adjacent to x in G_B. Assume that $|U_1| \geq 1$. By Lemma 15, y' is adjacent to $|U_1| - 1$ vertices of U_1 in G_B. Suppose, for some i, that y' is not adjacent to $u_i \in U_1$ in G_B. Recall from the definition of U_1 that x is not adjacent to w_i in G_B. Consider the path segments on C given by $P : u_i, w_{i-1}, u_{i-1}, \ldots, w_1, u_1$ and $P' : w_{s-1}, u_{s-1}, w_{s-2}, u_{s-2}, \ldots, w_{i+1}, u_{i+1}, w_i$. The sequence P', x, y', P forms a C_{2s} in G_R. ■

Lemma 17. *Let $x_1 \in L$ and $y_1 \in Y$. If the graph $G_R \langle L \cup Y - \{x_1, y_1\} \rangle = K_{s,s-1}$, then either G_R or G_B has a C_{2s}.*

Proof. Let $x_1 \in L$ and $y_1 \in Y$ such that $G_R \langle L \cup Y - \{x_1, y_1\} \rangle = K_{s,s-1}$. If a vertex $w \in \{y_1\} \cup W$ has two red neighbors in $L - \{x_1\}$, then, by Observation 12, a C_{2s} exists in G_R. Hence, each vertex in $\{y_1\} \cup W$ has at least $|L| - 2 \geq 17$ blue neighbors in $L - \{x_1\}$. By setting $S' = L - \{x_1\}$ and $T' = Y - \{y_1\}$ and applying Lemma 11, we have that there exists a vertex z that satisfies Part 1 or Part 2 of Lemma 11. Let $x_1', x_2' \in L - \{x_1, z\}$ be two blue neighbors of y_1.

We assume first that $z \in W$. Recall that, in G_B, every vertex in $L - \{x_1\}$ is adjacent to every vertex in $W - \{z\}$. From the previous paragraph, the vertex z has $|L| - 2 \geq 17$ blue neighbors in $L - \{x_1\}$. This implies that there are $|L| - 2 = s + 1 - 2 \geq 17$ vertices in $L - \{x_1\}$ that, in G_B, are adjacent to every vertex in W. Set $S = L - \{x_1\}$, $T = W$ and let $u', v' \in S - \{x'_1, x'_2\}$ be two vertices that are, in G_B, adjacent to every vertex in W. By Part 4 of Lemma 6, there exists an $x'_1 - x'_2$ path P, in $G_B \langle S \cup T \rangle$, on $2|T| - 1 = 2(s-1) + 1 = 2s - 1$ vertices. Whence, the sequence P, y_1 forms a C_{2s}. We may assume that $z \in L - \{x_1\}$ and so, in G_B, every vertex in $L - \{x_1, z\}$ is adjacent to every vertex in W. Note that if, in G_B, z is adjacent to every vertex in W, then $G_B \langle L \cup W - \{x_1\} \rangle = K_{s,s-1}$, and so $G_B \langle L \cup W \cup \{y_1\} - \{x_1\} \rangle$ has a C_{2s}, whence z has a red neighbor in W.

We claim that z can have at most one blue neighbor in W. Suppose $w, w' \in W$ are two distinct blue neighbors of z. Let $x'_3 \in L - \{x_1, z, x'_1, x'_2\}$ and set $S = L - \{x_1, z, x'_2\}$ and $T = W - \{w, w'\}$. By Part 4 of Lemma 6, $G_B \langle S \cup T \rangle$ has an $x'_1 - x'_3$ path P on $2|T| - 1 = 2(s-3) + 1 = 2s - 5$ vertices. The sequence P, w', z, w, x'_2, y_1 forms a C_{2s}, whence z has at most one blue neighbor in W. This implies, by Lemma 15, that, in G_B, $|U_1| \geq s - 1 - \deg_W(z) \geq s - 2$. Hence, in G_B, every vertex in Y is adjacent to at least $|U_1| - 1 \geq s - 3$ vertices in U. Recall that there exists at least one edge (incident with z) in $G_R \langle L \cup W - \{x_1\} \rangle$. Set $S = L - \{x_1\}$ and $T = Y - \{y_1\}$. By Lemma 10, $G_B \langle Y \cup U - \{y_1\} \rangle = K_{|T|,|U|} = K_{s-1,s-1}$.

We claim that x_1 has at most one blue neighbor in $Y - \{y_1\}$. Suppose $y_2, y_3 \in Y - \{y_1\}$ are blue neighbors of x_1. Let $u, u' \in U$ be two blue neighbors of y_1. Set $S = Y - \{y_1, y_2\}$ and $T = U - \{u, u'\}$, and let $y_4 \in Y - \{y_1, y_2, y_3\}$. By Part 4 of Lemma 6, $G_B \langle S \cup T \rangle$ has a $y_3 - y_4$ path P on $2|T| + 1 = 2(s-3) + 1 = 2s - 5$ vertices. The sequence P, u', y_1, u, y_2, x_1 forms a C_{2s}. It follows that x_1 has at least $s - 2$ red neighbors in $Y - \{y_1\}$. Set $S' = L$ and $T' = Y - \{y_1\}$ and apply Lemma 11. There exists a vertex z' such that either Part 1 or 2 of Lemma 11 hold. Suppose first that $z' \in S'$. Then, in G_B, every vertex in $L - \{z'\}$ is adjacent to every vertex in W. Recall that z has a red neighbor in W. This implies that $z = z'$. Hence $G_B \langle L \cup W - \{z'\} \rangle = K_{s,s-1}$, and, since $x'_1, x'_2 \in L - \{z, x_1\}$, the vertices $L \cup W \cup \{y_1\} - \{z'\}$ form a C_{2s} in G_B. We may assume that $z' \in W$, and that, in G_B, every vertex in L is adjacent to every vertex in $W - \{z'\}$.

If z' has two red neighbors $u, v \in L$, then we can set $S = L$ and $T = Y - \{y_1\}$, and so, by Part 4 of Lemma 6, $G_R \langle S \cup T \rangle$ has a $u - v$ path on $2|T| + 1 = 2(s-1) + 1 = 2s - 1$ vertices. The sequence P, z' forms a C_{2s} in G_R, whence the vertex z' must have at least $|L| - 1$ blue neighbors in L. The blue neighbors of z' in L must be adjacent to every vertex in W. Set $S = L$, $T = W$ and let $u', v' \in S - \{x'_1, x'_2\}$ be two vertices that are, in G_B, adjacent to every vertex in W. By Part 4 of Lemma 6, there exists an $x'_1 - x'_2$ path P, in $G_B \langle S \cup T \rangle$, on $2|T| - 1 = 2(s-1) + 1 = 2s - 1$ vertices. Whence, the sequence P, y_1 forms a C_{2s}. ■

Lemma 18. *The graph $G_B \langle L \cup Y \rangle$ has two disjoint K_2's or either G_R or G_B has a C_{2s}.*

Proof. Note that $G_B\langle L\cup Y\rangle$ must have a K_2 since otherwise $G_R\langle L\cup Y\rangle = K_{s+1,s}$ and so G_R will have a C_{2s}. Let $x_1 \in L$ and $y_1 \in Y$ be the vertices of the K_2 in $G_B\langle L\cup Y\rangle$. If $G_B\langle L\cup Y-\{x_1,y_1\}\rangle$ has a K_2 we are done, whence $G_R\langle L\cup Y-\{x_1,y_1\}\rangle = K_{s,s-1}$. By Lemma 17, we are done. ■

Lemma 19. *The graph $G_B\langle L\cup Y\rangle$ has a $K_{1,2}$ with end (central, respectively) vertices (vertex, respectively) in Y (L, respectively) or G_R has a C_{2s}.*

Proof. Suppose, to the contrary, that each vertex in L has at most one blue neighbor in Y. This implies that each vertex in L has at least $|Y|-1$ red neighbors in Y. Set $S=L$ and $T=Y$. By Part 1 of Lemma 6, we have, for any $u,v\in S$, that there exists a $u-v$ path in $G_R\langle S\cup T\rangle$ on $2|T|-1=2s-1$ vertices. If, in G_R, $w\in W$ is adjacent to say u and v, then, by Observation 12, a red C_{2s} exists, whence each vertex in W has at least $|L|-1$ blue neighbors in L. Set $S'=L$ and $T'=Y$. By Lemma 11, there exists a vertex z that satisfies either Part 1 or 2 of Lemma 11. By Lemma 14, $G_B\langle L\cup Y\rangle$ has a $K_{1,2}$ with central vertex $y_1'\in Y$ and end vertices $x_1',x_2'\in L$.

Let us assume first that $z\in W$. In G_B, every vertex in L is adjacent to every vertex in $W-\{z\}$. Recall that z has $|L|-1$ blue neighbors in L. This implies that there are $|L|-1=s+1-1\geq 18$ vertices in L that have $|W|$ blue neighbors in W. Set $S=L$ and $T=W$ and note that there are two vertices in $L-\{x_1',x_2'\}$ with $|T|$ blue neighbors in T. We apply Part 4 of Lemma 6 and deduce that there exists an $x_1'-x_2'$ path P in $G_B\langle S\cup T\rangle$ on $2|T|-1=2(s-1)+1=2s-1$ vertices. The vertices $V(P)\cup\{y_1'\}$ form a C_{2s} in G_B. We may assume that $z\in L$, and that, in G_B, every vertex in $L-\{z\}$ is adjacent to every vertex in W. Assume first that $z\in L-\{x_1',x_2'\}$. Set $S=L-\{z\}$ and $T=W$. It is clear that there exists an $x_1'-x_2'$ path P in $G_B\langle S\cup T\rangle$ on $2s-1$ vertices. The vertices $V(P)\cup\{y_1'\}$ form a C_{2s}. Hence, without loss of generality $z=x_1'$.

We claim that z has no blue neighbor in W. Suppose it has the neighbor $w\in W$. Set $S=L-\{z\}$, $T=W-\{w\}$ and let $x_3'\in L-\{z,x_2'\}$. Clearly there exists an $x_2'-x_3'$ path P in $G_B\langle S\cup T\rangle$ on $2(s-2)+1=2s-3$ vertices. The sequence P,w,z,y_1' forms a C_{2s}. It follows that z has no blue neighbors in W. Set $t=0$ and so, by Lemma 15, it follows that $|U_1|=s-1-t=s-1=|U|$ and each vertex in Y has at least $|U_1|-1=s-2=|U|-1$ blue neighbors in U_1. By Lemma 18, recall that $G_B\langle L\cup Y\rangle$ has two disjoint K_2's, say x_1y_1 and x_2y_2, with $x_1,x_2\in L$ and $y_1,y_2\in Y$. Set $S=Y$ and $T=U$. By Part 1 of Lemma 6, the graph $G_B\langle S\cup T\rangle$ has a y_2-y_1 path P on $2|T|-1=2(s-1)-1=2s-3$ vertices. Let $w\in W$. If $z\in L-\{x_1,x_2\}$, then the sequence P,x_1,w,x_2 forms a C_{2s}. Hence, without loss of generality, $z=x_2=x_1'$. Since z has only one blue neighbor in Y we have that $y_2=y_1'$. If $x_1=x_2'$ then x_1 has 2 blue neighbors in Y, a contradiction, whence $x_2'\in L-\{x_1,x_2\}$. The sequence P,x_2',w,x_1 forms a blue C_{2s}. ■

Lemma 20. *Let $x \in L$ ($y \in Y$, respectively) be a vertex that has exactly t (t', respectively) blue neighbors in W (U, respectively), such that t (t', respectively) is as small as possible. Then $t \leq s-3$ ($t' \leq s-3$, respectively).*

Proof. Suppose, to the contrary, that $t \geq s-2$ ($t' \geq s-2$, respectively). For t (t', respectively), set, for convenience, $S = L$, $T = W$, $S' = Y$, $T' = U$ and $x = z'$ ($Y = S$, $U = T$, $S' = L$, $T' = W$ and $y = z'$, respectively). Consider the case where $t = s-1$ ($t' = s-1$, respectively). Then $G_B\langle S \cup T\rangle = K_{|S|,s-1}$. By Lemma 14 (Lemma 19, respectively), $G_B\langle S \cup S'\rangle$ has a $K_{1,2}$ with central vertex $y_1' \in S'$ and end vertices $x_1', x_2' \in S$. There exists an $x_1' - x_2'$ path P in $G_B\langle S \cup T\rangle$ on $2s-1$ vertices. The sequence P, y_1' forms a blue C_{2s}, whence $t = s-2$ ($t' = s-2$, respectively), and so Lemma 15 implies the existence of a set $Z_{z'} \subset T'$ with $|Z_{z'}| = s-1-t = 1$ ($|Z_{z'}| = s-1-t' = 1$, respectively). Let $Z_{z'} = \{z\}$. By Lemma 18, $G_B\langle S \cup S'\rangle$ has two disjoint K_2's, say x_1y_1 and x_2y_2, with $x_1, x_2 \in S$ and $y_1, y_2 \in S'$. If, in G_B, both y_1 and y_2 are adjacent to the single vertex z, then by Part 1 of Lemma 6, $G_B\langle S \cup T\rangle$ has an $x_1 - x_2$ path P on $2|T| - 1 = 2(s-1) - 1 = 2s-3$ vertices, whence P, y_1, z, y_2 forms a blue C_{2s}. Thus, at least one vertex of y_1 and y_2, say y_1, is, in G_B, adjacent to exactly $|Z_{z'}| - 1 = 0$ vertices in $Z_{z'}$. It follows, by Lemma 16, that z' is adjacent to y_1.

We claim that S has $|S| - 3$ (≥ 15) vertices with exactly $s-2$ blue neighbors in T. Suppose that $S - \{x_1', x_2'\}$ has two vertices, say u' snd v', such that both have $s-1$ blue neighbors in T. By Part 4 of Lemma 6, $G_B\langle S \cup T\rangle$ has an $x_1' - x_2'$ path P on $2|T| + 1 = 2(s-1) + 1 = 2s-1$ vertices. The sequence P, y_1' forms a blue C_{2s}. Hence, $S - \{x_1', x_2'\}$ has at most one vertex with $s-1$ blue neighbors in T, and so the claim holds. By the pigeonhole principle, the set $S - \{x_1, x_2\}$ has two vertices with exactly $s-2$ blue neighbors in T. Let $X' = \{x', x''\}$ be the set of these two vertices. We can label x' as z' or x'' as z' and deduce that, in G_B, y_1 is adjacent to a vertex in X'. If, in G_B, y_2 is adjacent to a vertex in X', then the vertices x_1, y_1, x', y_2, x_2 and x'' will form either a P_5 with end vertices in S or two disjoint $K_{1,2}$'s with end vertices in S. By Lemma 13, G_B has a C_{2s}. We may assume that, in G_B, y_2 is adjacent to no vertex in X'. By labeling any arbitrary vertex in X' as z', we have, by Lemma 16, that, in G_B, y_2 is adjacent to the single vertex z in $Z_{z'}$.

If, in G_B, $v \in S' - \{y_1, y_2\}$ is adjacent to both x' and x'', then the vertices x_1, y_1, x', v and x'' form a P_5 with end vertices in S. By Lemma 13, G_B has a C_{2s}. We may assume, without loss of generality, that, in G_B, every vertex in $S' - \{y_1, y_2\}$ is not adjacent to some vertex in X'. Thus, for every $v \in S' - \{y_1, y_2\}$, we can, in G_B, label the vertex in X' that v is not adjacent to as z', and apply Lemma 16. Whence, in G_B, v is adjacent to the single vertex z in $Z_{z'}$. Let $v \in S' - \{y_1, y_2\}$. If, in G_B, v is adjacent to a vertex $x_3 \in S - \{x_2\}$, then we can find, using Part 1 of Lemma 6, an $x_3 - x_2$ path P in $G_B\langle S \cup T\rangle$ on $2s-3$ vertices, whence P, v, z, y_2 forms a blue C_{2s}. It follows that $G_R\langle S \cup S' - \{x_2, y_1, y_2\}\rangle =$

$K_{|S|-1,|S'|-2}$. If the vertex y_2 has a blue neighbor in $S - \{x_2\}$, then $G_B \langle S \cup S' \rangle$ has either a P_5 or two disjoint $K_{1,2}$'s, with end vertices in S. By Lemma 13 a blue C_{2s} exists, whence $G_R \langle S \cup S' - \{x_2, y_1\} \rangle = K_{s,s-1}$, and so, by Lemma 17 we are done. ■

Lemma 21. *The graph $G_B \langle L \cup Y \rangle$ has three disjoint K_2's, or either G_R or G_B has a C_{2s}.*

Proof. Let $x \in L$ ($y \in Y$, respectively) have t (t', respectively) blue neighbors in W (U, respectively) such that t (t', respectively) is as small as possible. By Lemma 18, $G_B \langle L \cup Y \rangle$ has two disjoint K_2's, say x_1y_1 and x_2y_2, with $x_1, x_2 \in L$ and $y_1, y_2 \in Y$. We may assume that $G_R \langle L \cup Y - \{x_1, x_2, y_1, y_2\} \rangle = K_{s-1,s-2}$. By Lemma 20 and 15, there exists a set $U_1 \subseteq U$ ($W_1 \subseteq W$, respectively) with $|U_1| = s-1-t \geq 2$ ($|W_1| = s-1-t' \geq 2$, respectively), such that every vertex in Y (L, respectively) has $|U_1|-1$ ($|W_1|-1$, respectively) blue neighbors in U_1 (W_1, respectively). Thus, x_1 and x_2 (y_1 and y_2, respectively) must have at least one blue neighbor in W (U respectively). Let $x'_1 \in W$ and $x'_2 \in W$ be blue neighbors of x_1 and x_2 respectively, and let $y'_1 \in U$ and $y'_2 \in U$ be blue neighbors of y_1 and y_2 respectively.

We will make the following useful observation. For convenience set $D = L$, $F = W$, $x_1 = u$, $x_2 = v$, $x'_1 = u'$ and $x'_2 = v'$ ($D = Y$, $F = U$, $y_1 = u$, $y_2 = v$, $y'_1 = u'$ and $y'_2 = v'$, respectively). Suppose $u' \neq v'$. Let X_D (X_F, respectively) be a subset of $D - \{u, v\}$ ($F - \{u', v'\}$, respectively) consisting of i (j, respectively) vertices, with $i > j > 0$. If, in G_B, every vertex in X_D is adjacent to every vertex in $X_F \cup \{u', v'\}$, then we claim that there exists a $u-v$ path $P_{D,F}(j)$ in $G_B \langle D \cup F \rangle$ on $2j+5$ vertices. Let $x', x'' \in X_D$. Since $|X_D| = i > j = |X_F|$, there exists an $x' - x''$ path P in $G_B \langle X_D \cup X_F \rangle$ on $2j+1$ vertices. The sequence u, u', P, v', v forms the desired path.

We claim that $x'_1 \neq x'_2$. Suppose, to the contrary, that $x'_1 = x'_2$. If x_1 has a blue neighbor $x'_3 \in W - \{x'_2\}$, then we can relabel x'_3 as x'_1 and deduce that $x'_1 \neq x'_2$. We may assume that both x_1 and x_2 have one blue neighbor in W, whence $t \leq 1$, and so $|U_1| = s-1-t \geq s-2$. Lemma 15 implies that every vertex in Y has at least $s-3$ blue neighbors in U, whence $t' \geq s-3$. We apply Lemma 11. Let $S' = Y - \{y_1, y_2\}$, and choose a set $T' \subset L - \{x_1, x_2\}$ of cardinality $s-2$. Recall that $G_R \langle S' \cup T' \rangle = K_{s-2,s-2}$, whence there exists a vertex z which satisfies either Part 1 or 2 of Lemma 11. Suppose z satisfies Part 1 of Lemma 11. Then $z \in S'$ and, in G_B, every vertex in $S' - \{z\}$ is adjacent to every vertex in U. Set $D = Y$ and $F = U$. Define $X_D = Y - \{y_1, y_2, z\}$ and let X_F be a subset of $F - \{y'_1, y'_2\}$ of cardinality $s-4$. By our earlier observation the mentioned path exists, and so the sequence $P_{D,F}(s-4), x_1, x_2, x'_1$ forms a C_{2s}. If z satisfies Part 2 of Lemma 11 then, in G_B, $z \in U$ and every vertex in S' is adjacent to every vertex in $U - \{z\}$. Since $t' \geq s-3$, we can pick $y'_1, y'_2 \in U - \{z\}$. Define

$X_F = U - \{y'_1, y'_2, z\}$ and $X_D = Y - \{y_1, y_2\}$. By our earlier observation, the sequence $P_{D,F}(s-4), x_1, x_2, x'_1$ forms a C_{2s}. We may assume that $x'_1 \neq x'_2$.

Set $S' = L - \{x_1, x_2\}$ and $T' = Y - \{y_1, y_2\}$. Recall that $G_R \langle S' \cup T' \rangle = K_{s-1,s-2}$. There exists a vertex z which satisfies either Part 1 or 2 of Lemma 11. Set $D = L$ and $F = W$. Suppose first that $z \in S'$ or, in G_B, every vertex in S' is adjacent to every vertex in W. In both instances there exists a vertex $z \in S'$ such that, in G_B, every vertex in $S' - \{z\}$ is adjacent to every vertex in W. Assume first that y_1 and y_2 have a common neighbor $y' \in \{z\} \cup U$. Let $X_D = L - \{x_1, x_2, z\}$ ($|X_D| = s-2$) and let X_F be a subset of $W - \{x'_1, x'_2\}$ of cardinality $s-4$. By our earlier observation the mentioned path exists, whence the sequence $P_{D,F}(s-4), y_1, y', y_2$ forms a C_{2s}. We may assume, in G_B, that y_1 and y_2 have no common neighbor in $U \cup \{z\}$, and so since every vertex in Y has $|U_1| - 1$ blue neighbors in U_1, we have that $|U_1| = 2$. Relabeling if necessary, let $U_1 = \{y'_1, y'_2\}$. Note that $x \in \{x_1, x_2, z\}$. By Lemma 16, both y_1 and y_2 must be adjacent to x in G_B, and so, without loss of generality, $x = x_2$. Let X_F be a subset of $W - \{x'_1, x'_2\}$ of cardinality $s-3$. The sequence $P_{D,F}(s-3), y_1$ forms a C_{2s}.

We may assume that $z \in W$ and that $G_R \langle S' \cup W \rangle$ has a red edge. We apply Lemma 10. Set $S' = S$ and $T' = T$. Recall that $G_R \langle S' \cup T' \rangle = K_{s-1,s-2}$. Hence, $G_B \langle T \cup U \rangle = K_{s-2,s-1}$. Suppose first that $z \in W - \{x'_1, x'_2\}$. In G_B, every vertex in S' is adjacent to every vertex in $W - \{z\}$. If y_1 and y_2 have a common blue neighbor $y' \in U$, then set $X_D = S'$ and $X_F = W - \{x'_1, x'_2, z\}$. It follows that the sequence $P_{D,F}(s-4), y_1, y', y_2$ forms a C_{2s}. Again, it follows that $|U_1| = 2$, since otherwise y_1 and y_2 will have a common neighbor in U_1. Without loss of generality, $U_1 = \{y'_1, y'_2\}$. Pick $y_3 \in Y - \{y_1, y_2\}$ and $x'_3 \in W - \{x'_1, x'_2, z\}$. Let $X_D = S'$ and $X_F = W - \{x'_1, x'_2, x'_3, z\}$. The sequence $P_{D,F}(s-5), y_1, y'_1, y_3, y'_2, y_2$ forms a C_{2s}. We may assume, without loss of generality, that $z = x'_1$. If x_1 has a blue neighbor $x'_3 \in W - \{x'_1, x'_2\}$, then we can relabel x'_3 as x'_1 and so $z \in W - \{x'_1, x'_2\}$, which we already considered. Hence, $t \leq 2$ and so, by Lemma 15, $|U_1| = s-1-t \geq s-3$, implying that every vertex in Y is adjacent to at least $s-4$ vertices in U. Set $D = Y$, $F = U$ and $X_D = Y - \{y_1, y_2\}$. Suppose x_1 and x_2 have a common blue neighbor $x' \in W$. Let X_F be a subset of cardinality $s-4$ of $U - \{y'_1, y'_2\}$. The sequence $P_{D,F}(s-4), x_1, x', x_2$ forms a C_{2s}. Thus, x_1 and x_2 have no common blue neighbor in W, implying that $|W_1| = 2$. Without loss of generality, assume $W_1 = \{x'_1, x'_2\}$. Observe that $y \in \{y_1, y_2\}$. By Lemma 16, both x_1 and x_2 must be adjacent to y in G_B, and so, without loss of generality, x_1 has the two blue neighbors y_1 and y_2. Set $X_F = U - \{y'_1, y'_2\}$, whence the sequence $P_{D,F}(s-3), x_1$ forms a C_{2s}. ∎

Lemma 22. *The graph $G' = G_B \langle L \cup Y \rangle$ either has a P_4 and K_2 (called Configuration 1), which are both disjoint, or, two K_2's, and a $K_{1,2}$ with central vertex in Y (L, respectively) (called Configuration 2), all of which are disjoint.*

Proof. By Lemma 19 (14, respectively), the graph $G_B\langle L\cup Y\rangle$ has a $K_{1,2}$, with central vertex y_4 in Y (L, respectively) and two endpoints x_4, x_5 in L (Y, respectively). By Lemma 21, the graph G' has three disjoint copies of of K_2, say x_1y_1, x_2y_2 and x_3y_3, where x_1, x_2 and x_3 are in L (Y, respectively), and y_1, y_2 and y_3 are in Y (L, respectively). Suppose first that $y_4\in\{y_1,y_2,y_3\}$. Without loss of generality, $y_4=y_3$. If $x_4,x_5\in X-\{x_1,x_2\}$, then the vertices x_1,x_2,y_1,y_2 form two K_2's, and the vertices y_4,x_4,x_5 form a $K_{1,2}$. If, without loss of generality, $x_4=x_2$, then the vertices y_2,x_2,y_3,x_3 (x_1,y_1, respectively) form a P_4 (K_2, respectively). We may assume that $y_4\in Y-\{y_1,y_2,y_3\}$.

If $x_4,x_5\in X-\{x_1,x_2,x_3\}$, then the vertices x_1,x_2,y_1,y_2 form two K_2's, and the vertices y_4,x_4,x_5 form a $K_{1,2}$. If, without loss of generality, $x_4=x_3$ and $x_5=x_2$, then the vertices y_2,x_2,y_4,x_3 (x_1,y_1, respectively) form a P_4 (K_2, respectively) and our claim is verified. We may assume that $x_4\in X-\{x_1,x_2,x_3\}$. If $x_5=x_2$ then the vertices y_2,x_2,y_4,x_4 (x_1,y_1, respectively) form a P_4 (K_2, respectively) and our claim is verified. ■

Consider, again, two disjoint vertex sets S and T, such that $|S|\geq|T|\geq 3$, and join each vertex in S to at least $|T|-1$ vertices in T. Let P' (P'', respectively) be two vertex disjoint paths with vertices $P': u_1,u_2,\ldots,u_\ell$ ($P'': v_1,v_2,\ldots,v_k$, respectively), with $\ell\geq 1$ ($k\geq 1$, respectively) being odd. Join the vertex u_i (v_j, respectively) to at least $|T|-1$ vertices in T, for each odd i (j, respectively). Define $S_1=\bigcup_{i=0}^{(\ell-1)/2}\{u_{2i+1}\}$, $S_2=\bigcup_{j=0}^{(k-1)/2}\{v_{2j+1}\}$, $T_1=V(P')-S_1$ and $T_2=V(P'')-S_2$. Let z be a vertex such that the following construction is made. If a vertex $w\in S\cup S_1\cup S_2$ is adjacent to exactly $|T|-1$ vertices in T, then join z to w. The sets S,T,T_1,T_2,S_1,S_2 and $\{z\}$, are all disjoint. From this construction, we derive the following necessary lemma.

Lemma 23. *Let $u,v\in S$ and $|T|\geq 3$. There exists a $u-v$ path that alternates between the sets $S\cup S_1\cup S_2$ and $T\cup T_1\cup T_2\cup\{z\}$, on $2|T|-1+k+\ell$ vertices.*

Proof. If $|T|\geq 4$, then pick $s_1,s_2,\ldots,s_{|T|-3}\in S-\{u,v\}$. Recall that every two vertices in $S\cup S_1\cup S_2$ have $|T|-2$ common neighbors in T. If $|T|=3$, then pick $t_1\in N(u)\cap N(u_1)$. If $|T|\geq 4$, pick a sequence of vertices in T as follows: $t_1\in N(u)\cap N(s_1), t_2\in N(s_1)\cap N(s_2)-\{t_1\}, t_3\in N(s_2)\cap N(s_3)-\{t_1,t_2\},\ldots,t_{|T|-3}\in N(s_{|T|-4})\cap N(s_{|T|-3})-\{t_1,t_2,\ldots,t_{|T|-4}\}, t_{|T|-2}\in N(s_{|T|-3})\cap N(u_1)-\{t_1,t_2,\ldots,t_{|T|-3}\}$. We now consider the adjacency of the vertex z to the end vertices of the paths P' and P''. By relabeling if necessary, we need to consider three cases.

Case 1. The vertex z is not adjacent to any of the three vertices u_ℓ, v_1 and v_k. By our construction, the vertices u_ℓ, v_1 and v_k are adjacent to all vertices in T. Let $t_{|T|}\in N(v)\cap T-\{t_1,t_2,\ldots,t_{|T|-2}\}$. Let $t_{|T|-1}\in T-\{t_1,t_2,\ldots,t_{|T|-2},t_{|T|}\}$.

The path $P : u, t_1, s_1, t_2, s_2, t_3, \ldots, s_{|T|-3}, t_{|T|-2}, u_1, u_2, \ldots, u_\ell, t_{|T|-1}, v_1, v_2, \ldots, v_k, t_{|T|}, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices.

Case 2. The vertex z is adjacent to v_k but not u_ℓ. As u_ℓ is not adjacent to z, it must be adjacent to all vertices in T. Let $t_{|T|-1} \in N(v_1) \cap T - \{t_1, t_2, \ldots, t_{|T|-2}\}$. If v is adjacent to z, then $P : u, t_1, s_1, t_2, s_2, t_3, \ldots, s_{|T|-3}, t_{|T|-2}, u_1, u_2, \ldots, u_\ell, t_{|T|-1}, v_1, v_2, \ldots, v_k, z, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices. Hence, we may assume that v is not adjacent to z and so, by our construction, $N(v) = T$. If u is adjacent to z, then $P : u, z, v_k, \ldots, v_2, v_1, t_{|T|-1}, u_\ell, \ldots, u_1, t_{|T|-2}, s_{|T|-3}, \ldots, s_1, t_1, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices, whence, by our construction, $N(u) = T$.

Let $t_{|T|} \in T - \{t_1, t_2, \ldots, t_{|T|-1}\}$. If v_k is adjacent to $t_{|T|}$, then $P : u, t_1, s_1, t_2, s_2, t_3, \ldots, s_{|T|-3}, t_{|T|-2}, u_1, u_2, \ldots, u_\ell, t_{|T|-1}, v_1, v_2, \ldots, v_k, t_{|T|}, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices, whence $N(v_k) = \{t_1, t_2, \ldots, t_{|T|-1}\}$. It follows immediately that $P : u, t_{|T|}, u_\ell, \ldots, u_2, u_1, t_{|T|-2}, s_{|T|-3}, \ldots, s_1, t_1, v_k, \ldots, v_2, v_1, t_{|T|-1}, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices.

Case 3. The vertex z is adjacent to u_ℓ and v_1. Let $t_{|T|-1} \in N(v_k) \cap T - \{t_1, t_2, \ldots, t_{|T|-2}\}$. If v is adjacent to $t_{|T|-1}$, then $P : u, t_1, s_1, t_2, s_2, t_3, \ldots, s_{|T|-3}, t_{|T|-2}, u_1, u_2, \ldots, u_\ell, z, v_1, v_2, \ldots, v_k, t_{|T|-1}, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices. By the pigeonhole principle, there exists a vertex $t_{|T|} \in T - \{t_1, t_2, \ldots, t_{|T|-2}\}$ such that $N(v) = \{t_1, t_2, \ldots, t_{|T|-2}, t_{|T|}\}$. Pick $t_{|T|} \in N(v) - \{t_1, t_2, \ldots, t_{|T|-2}\}$.

By the same argument, v_k cannot be adjacent to $t_{|T|}$, whence, $N(v_k) \cap T = \{t_1, t_2, \ldots, t_{|T|-2}, t_{|T|-1}\}$. By our construction, v is adjacent to z. If u is adjacent to $t_{|T|-1}$, then $P : u, t_{|T|-1}, v_k, \ldots, v_2, v_1, z, u_\ell, \ldots, u_2, u_1, t_{|T|-2}, s_{|T|-3}, \ldots, s_2, t_2, s_1, t_1, v$ is a path on $2\,|T| - 1 + \ell + k$ vertices. It follows that $N(u) = \{t_1, t_2, \ldots, t_{|T|-2}, t_{|T|}\}$.

Suppose first that u_ℓ is adjacent to $t_{|T|}$. The path $P : u, t_{|T|}, u_\ell, \ldots, u_1, t_{|T|-2}, s_{|T|-3}, \ldots, s_2, t_2, s_1, t_1, v_k, \ldots, v_2, v_1, z, v,$ is a $u - v$ path on $2\,|T| - 1 + \ell + k$ vertices. It follows, by the pigeonhole principle, that u_ℓ is adjacent to every vertex in $T - \{t_{|T|}\}$. The path $P : u, t_1, s_1, t_2, s_2, t_3, \ldots, s_{|T|-3}, t_{|T|-2}, u_1, u_2, \ldots, u_\ell, t_{|T|-1}, v_k, \ldots, v_2, v_1, z, v$ is a $u - v$ path on $2\,|T| - 1 + k + \ell$ vertices. ■

5. Proof of Lemma 2

Recall the statement of Lemma 2. Consider a blue-red coloring of $G = K_{2s,2s-1}$ such that G_R has a C_{2s-2}, say C. We will show that G_B has an C_{2s}. Recall the defined sets $U = \mathcal{L}(G) \cap V(C)$, $W = \mathcal{R}(G) \cap V(C)$, $L = \mathcal{L}(G) - U$ and $Y = \mathcal{R}(G) - W$. Observe that $|U| = |W| = s - 1$, $|L| = s + 1$ and $|Y| = s$. By Lemma 21, $G_B\langle L \cup Y\rangle$ has 3 disjoint K_2's. By Lemma 22, $G_B\langle L \cup Y\rangle$ has either a K_2 and a P_4, both of which are disjoint, or two K_2's, and a $K_{1,2}$, all of which are disjoint. We can also pick the $K_{1,2}$ to either have the central vertex in Y or in L. In what follows, we will only be concerned with the blue graph $G_B\langle L \cup Y \cup U \cup W\rangle$. Pick $x \in L$ ($y \in Y$, respectively) with t (t', respectively) blue neighbors in W (U, respectively) such that t (t', respectively) is as small as possible. Applying Lemma 15, we have that there exists a set $U_1 \subseteq U$ ($W_1 \subseteq W$, respectively) such that each vertex in Y (L, respectively) is adjacent to $|U_1| - 1$ ($|W_1|-1$, respectively) vertices in U_1 (W_1, respectively). Note that $|U_1| = s-1-t$ and $|W_1| = s - 1 - t'$. Define $W_2 = W - W_1$ and $U_2 = U - U_1$.

For convenience, let X be a subset of L with s vertices such that X contains x and $G_B\langle X \cup Y\rangle$ contains the three disjoint K_2's, the $K_{1,2}$ with central vertex in L and the $K_{1,2}$ with central vertex in Y. Without loss of generality, we may assume that $t \geq t'$. (We can relabel the sets X, Y, U and W if necessary). By our choice of X and by Lemma 22, Configuration 1 or Configuration 2 holds for $G_B\langle X \cup Y\rangle$. Let $x_1, x_2, x_3, x_4 \in X$ and $y_1, y_2, y_3 \in Y$ (all distinct vertices). Let x_1y_1, x_2y_2 and x_3y_3 denote the three disjoint K_2's in $G_B\langle X \cup Y\rangle$. By the pigeonhole principle, we may assume, for the rest of the paper, that, without loss of generality, $x \in X - \{x_1, x_2\}$. For Configuration 1, the vertices x_1, y_1, x_2, y_2 (x_3y_3, respectively) will denote the P_4 (K_2, respectively). For Configuration 2, the vertices x_1y_1 and x_2y_2 (x_3, x_4, y_3, respectively) will denote the two K_2's ($K_{1,2}$, respectively). By Lemma 20, $t' \leq t \leq s-3$, implying that $|U_1| \geq 2$ and $|W_1| \geq 2$.

Case 1. There exist integers $i, j \geq 1$ such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor + j$. As $t \geq t'$, we have that $i \geq j$. Furthermore, $\deg_W(x) = t \leq s-3$ ($\deg_U(y) = t' \leq s-3$, respectively) and so $i \leq \lceil s/2 \rceil - 3$ ($j \leq \lceil s/2 \rceil - 3$, respectively).

Claim 24. $|U_1| \geq 3$ (*and* $|W_1| \geq 3$).

Proof. Suppose, to the contrary, that $|U_1| \leq 2$. This implies that $\deg_W(x) \geq |W| - 2$. Note that every vertex in X is adjacent to $|W| - 2$ vertices in W. Observe that $\deg_U(y) = \lfloor s/2 \rfloor + j \geq \lfloor (s-1)/2 + 1/2 \rfloor + 1 = \lfloor |U|/2 + 1/2 \rfloor + 1$. If $|U|$ is even (odd, respectively), then $\deg_U(y) \geq \lceil |U|/2 \rceil + 1$. Note that as $\deg_U(y) \geq \lceil |U|/2 \rceil + 1$, we have that every vertex in Y has at least $\lceil |U|/2 \rceil + 1$ neighbors in U.

Assume first that Configuration 1 holds. Set $S = X - \{x_2\}$, $T = W$ and $Z_1 = \emptyset$. By Part 7 of Lemma 6, there exists an $x_1 - x_3$ path P, that alternates between S and T, on $2|T| - 5 = 2|W| - 5 = 2s - 7$ vertices. Now set $S = Y - \{y_1\}$ and $T = U$. As each vertex in S has at least $\lceil |T|/2 \rceil + 1$ neighbors in T and $|T| \geq 8$, we have, by Lemma 8, that there exists a $y_3 - y_2$ path P', that alternates between S and T, on 5 vertices. The sequence P, P', x_2, y_1 forms a monochromatic C_{2s}.

We may assume that Configuration 2 holds. Let Z_1 be the set containing the $K_{1,2}$ (with vertices x_3, y_3, x_4). Define $S = X - \{x_3, x_4\}$, $T = W$ and set $|Z_1| = k = 1$. By Part 7 of Lemma 6, there exists an $x_1 - x_2$ path P, that alternates between $S \cup \{x_3, x_4\}$ and $T \cup \{y_3\}$, on $2|T| - 5 + 2 = 2|W| - 5 + 2 = 2s - 5$ vertices. Now define $S = Y - \{y_3\}$ and $T = U$. As each vertex in S has at least

$\lceil |T|/2 \rceil + 1$ neighbors in T and $|T| \geq 8$, we have, by Lemma 8, that there exists a $y_2 - y_1$ path P' on 5 vertices. The sequence P, P' forms a monochromatic C_{2s}. It follows that $|U_1| \geq 3$. Hence $\deg_U(y) \leq \deg_W(x) \leq |W| - 3 = |U| - 3$, whence $|W_1| \geq 3$. ∎

Lemma 25. *Suppose that $s/2 - j - 2 > j + 2$ ($s/2 - i - 2 > i + 2$, respectively). Then there exists an $x_1 - x_2$ ($y_1 - y_2$, respectively) path P, that alternates between X (Y, respectively) and W (U, respectively), where P can be chosen to have $2\lfloor s/2 \rfloor + 1$ vertices or chosen to have $2\lfloor s/2 \rfloor - 1$ vertices.*

Proof. Assume that $s/2 - j - 2 > j + 2$. The case where $s/2 - i - 2 > i + 2$ is perfectly symmetrical (we can interchange the i's and j's in the argument below and use the vertex y as reference). We can deduce that $j < s/4 - 2$. Observe that $\deg_{W_2}(x) \geq \deg_W(x) - |W_1| \geq t - (s - t' - 1) = t - s + t' + 1$. Set $S' = X - \{x_1, x_2\}$, $T' = W_2$ and $k' = t - s + t' + 1 - i \geq 0$. It follows that, as $i \geq 1$, x (and every vertex in $X - \{x\}$) has more than k' neighbors in T'. Note that $|S'| - k' = 2s - 2 - t - t' - 1 + i \geq s - 3 - j = s/2 + s/4 + s/4 - 3 - j > s/2 + s/4 - 1 > \lfloor s/2 \rfloor + j = |W_2| = |T'|$. Applying Lemma 7, there exist $k' + 1$ disjoint copies of $K_{1,2}$, all with central vertices in W_2 and with end vertices in $X - \{x_1, x_2\}$. Let X'' (W'', respectively) denote the set consisting of the end (central, respectively) vertices of the $k' + 1$ disjoint copies of $K_{1,2}$, in $X - \{x_1, x_2\}$ (W_2, respectively). Let Z_1 be the collection of $k' + 1$ disjoint copies of $K_{1,2}$.

Now set $S = X - X''$ and $T = W_1$. Our aim is to apply Part 1 of Lemma 6. We need to check that the requirements hold. First note that $|T| = s - t' - 1 \geq \lceil s/2 \rceil - j - 1 \geq s/2 - j - 1 > j + 3 \geq 4$, whence $|T| \geq 5$. We claim that $|S| + k' - 1 \geq |T|$ (and $|S| + (k' + 1) - 1 \geq |T|$). Note that $|S| + k' - 1 = s - 2k' - 2 + k' - 1 \geq s - j - 4$ and $|T| = s - t' - 1 = s - \lfloor s/2 \rfloor - j - 1 \leq s - j - 7$ (as $s > 12$), whence $|S| + k' - 1 \geq |T|$. We now claim that $|T| - 1 \geq (k' + 1) + 1$ (and $|T| - 1 \geq k' + 1$). Note that $|T| - 1 = s - t' - 2 = \lceil s/2 \rceil - j - 2$ and $k' + 2 = 2\lfloor s/2 \rfloor + i + j - s + 1 - i + 2 = 2\lfloor s/2 \rfloor - s + j + 3$. If s is even, then, as $j + 2 < s/2 - j - 2$, we have that $j + 3 \leq s/2 - j - 2$, whence $k' + 2 = j + 3 \leq s/2 - j - 2 = |T| - 1$. If s is odd, then $k' + 2 = j + 2 < s/2 - j - 2 \leq \lceil s/2 \rceil - j - 2 = |T| - 1$.

By Part 1 of Lemma 6, there exists an $x_1 - x_2$ path P, that alternates between X and W, on $2|T| - 1 + 2(k' + 1) = 2\lfloor s/2 \rfloor + 1$ vertices. Alternatively, as $|T| \geq 4$, $|S| + k' - 1 \geq |T|$ and $|T| - 1 \geq k' + 1$, we can modify Z_1, to only include k' copies of $K_{1,2}$ and deduce that there exists an $x_1 - x_2$ path P, that alternates between X and W', on $2|T| - 1 + 2k' = 2\lfloor s/2 \rfloor - 1$ vertices. ∎

Case 1.1. $s/2 - j - 2 > j + 2$.

Case 1.1.1. $s/2 - i - 2 > i + 2$. Suppose s is odd. By Lemma 25, pick the $x_1 - x_2$ ($y_2 - y_1$, respectively) path P (P', respectively), that alternates between X and W (Y and U, respectively), on $2\lfloor s/2 \rfloor + 1$ vertices. It follows that the sequence P, P' forms a monochromatic C_{2s}. If s is even, then pick the $x_1 - x_2$ ($y_2 - y_1$, respectively) path P (P', respectively), that alternates between X and W (Y and U, respectively), on $2\lfloor s/2 \rfloor + 1$ ($2\lfloor s/2 \rfloor - 1$, respectively) vertices. It follows that the sequence P, P' forms a monochromatic C_{2s}. Observe that $j < s/4 - 2$.

Case 1.1.2. $s/2 - i - 2 \leq i + 2$. Observe that $i \geq s/4 - 2$. As $s \geq 18$, we have that $i \geq 3$. Observe that $t - s + t' + 1 \geq 2$. If s is even (odd, respectively), we have, for every $x' \in X - \{x, x_1, x_2\}$, that $\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j$ ($\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = t - s + t' + 1 > i + j - 1$, respectively) and so x' has at least $i + j$ ($i + j - 1$, respectively) neighbors in W_2.

Now set $S'' = Y - \{y_1, y_2\}$ and $T'' = U_2$. Set $k'' = 1$. Observe that since $i \geq s/4 - 2$ and $s \geq 18$, we have, for every vertex $y' \in Y$, that $\deg_{U_2}(y') \geq \deg_{U'}(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j > 2$. This implies that y' has more than one neighbor in T''. Recall that, by Claim 24, $|U_1| = s - 1 - t \geq 3$, which implies that $|S''| - k'' = s - 3 > t = \lfloor s/2 \rfloor + i = |T''|$. By Lemma 7, there exist $k'' + 1 = 2$ copies of $K_{1,2}$, say $P(1)$ and $P(2)$, with respective central vertices in U_2 and respective end vertices in $Y - \{y_1, y_2\}$.

Claim 26. *For the graph $G' = G_B \langle X \cup W_2 - \{x, x_1, x_2\} \rangle$, one of the following holds.*

1. *If s is even, then G' has disjoint paths $P(3)$, $P(4)$ and $P(5)$ that start and end in $X - \{x, x_1, x_2\}$, with $P(3) = P_{2(i+j)-3}$, $P(4) = P_3$ and $P(5) = P_3$.*

2. *If s is odd then G' has disjoint paths $P(3)$, $P(4)$, $P(5)$ and $P(6)$ that all start and end in $X - \{x, x_1, x_2\}$ and $P(3) = P_{2(i+j)-7}$, $P(4) = P_3$, $P(5) = P_3$ and $P(6) = P_3$.*

Proof. Define $S = X - \{x, x_1, x_2\}$ and $T = W_2$. Let $b = |X - \{x, x_1, x_2\}| = s - 3$ and $a = |W_2| = t' = \lfloor s/2 \rfloor + j$. Observe that $b = s - 3 \geq a = \lfloor s/2 \rfloor + j$ since otherwise $\deg_U(y) = \lfloor s/2 \rfloor + j \geq s - 2$, a contradiction. Recall that each vertex in $X - \{x, x_1, x_2\}$ has more than $t - s + t' = 2\lfloor s/2 \rfloor + i + j - s$ neighbors in W_2, whence $m(G') \geq (s-3)(t - s + t') = (s-3)(2\lfloor s/2 \rfloor + i + j - s)$. Since $s/2 - i - 2 \leq i + 2$ and $s \geq 18$, we have that $i \geq 2$ ($i \geq 3$, if s is odd). Note that if s is odd and $i + j = 4$, then $P(3) = P_{2(i+j)-7}$ exists and so we are done, whence $i + j \geq 5$. If s is even (odd, respectively), pick $c = i + j - 2$ ($c = i + j - 4$, respectively) and note that as $j \geq 1$, we have $c > 0$. If s is even (odd, respectively), let us assume that the graph G' has no path on $2(i + j - 1)$ ($2(i + j) - 6$, respectively) vertices.

Suppose first that $c < a < 2c$. Applying the second part of Theorem 1, we have that if s is even (odd, respectively), then $(s-3)(i+j) \leq m(G') \leq bc = (s-3)(i+j-2)$ ($(s-3)(i+j-1) \leq m(G') \leq bc = (s-3)(i+j-4)$, respectively), a contradiction. If $a \leq c$, then, applying the first part of Theorem 1, we have that if s is even (odd, respectively), then $(s-3)(i+j) \leq m(G') \leq ab \leq cb =$

$(s-3)(i+j-2)$ $((s-3)(i+j-1) \le m(G') \le ab \le cb = (s-3)(i+j-4)$, respectively), a contradiction. We may assume that $a \ge 2c$. If s is even (odd, respectively) we have, by the third part of Theorem 1, that $m(G') \le (i+j-2)(s-3+s/2+j-2(i+j-2))$ $(m(G') \le (i+j-4)(s-3+\lfloor s/2 \rfloor+j-2(i+j-4))$, respectively).

Let us first assume that s is even. To obtain a contradiction, we will prove that $m(G') \ge (s-3)(i+j) > (i+j-2)(s-3+s/2+j-2(i+j-2))$, by showing that $(s-3)(i+j)-(i+j-2)(s-3+s/2+j-2(i+j-2)) > 0$. For the proof see the appendix in Section 6. It follows that G' has a path, that starts in $X-\{x, x_1, x_2\}$ and ends in W_2, on $2(i+j)-2$ vertices. Whence the graph G' has a path $P(3) = P_{2(i+j)-3}$ that starts and ends in $X-\{x, x_1, x_2\}$.

Let us assume that s is odd. To obtain a contradiction, we will prove that $m(G') \ge (s-3)(i+j-1) > (i+j-4)(s-3+s/2+j-2(i+j-4))$ by showing that $(s-3)(i+j-1)-(i+j-4)(s-3+s/2+j-2(i+j-4)) > 0$. For the proof see the appendix in Section 6. It follows that G' has a path, that starts in $X-\{x, x_1, x_2\}$ and ends in W_2, on $2(i+j)-6$ vertices. Whence the graph G' has a path $P(3) = P_{2(i+j)-7}$, that starts and ends in $X-\{x, x_1, x_2\}$.

To complete the proof, we apply Lemma 7. Set $S' = X-\{x, x_1, x_2\}-V(P(3))$ and $T' = W_2 - V(P(3))$. If s is even (odd, respectively), set $k'' = 1$ $(k'' = 2$, respectively). If s is even (odd, respectively), every vertex in S' has at least $i+j-(i+j-2) \ge 2 > k''$ $(i+j-1-(i+j-4) \ge 3 > k''$, respectively) neighbors in T'. Assume first that s is odd. Observe that since $j < s/4-2$, we have that $|S'|-k'' = s-3-(i+j-3)-2 = s-(i+j)-2 = \lfloor s/2 \rfloor + j-(i+j-4)+\lceil s/2 \rceil -4-j-2 > |T'|+s/2-6-j > |T'|+s/4-4 > |T'| > 0$ (as $i \le \lceil s/2 \rceil -3$). By Lemma 7, there exist 3 copies of $K_{1,2}$ with end vertices in S' and central vertex in W_2, whence the desired paths $P(4)$, $P(5)$ and $P(6)$ exist. Assume now that s is even. Then $|S'|-k'' = s-3-(i+j-1)-1 = s-(i+j)-2-1 = s/2+j-(i+j-2)+s/2-2-j-3 = |T'|+s/2-5-j > |T'|+s/4-3 > |T'| > 0$ (as $i \le \lceil s/2 \rceil - 3$). By Lemma 7, there exist $k''+1 = 2$ copies of $K_{1,2}$, with end vertices in S' and central vertex in W_2, whence the desired paths $P(4)$ and $P(5)$ exist. ■

Let $u = x_1$ and $v = x_2$. We refer to Claim 26. For s even (odd repectively), set $Z_1 = \{P(4), P(5)\}$ $(Z_1 = \{P(4), P(5), P(6)\}$, respectively) with $k = |Z_1|$. It will now be shown that if s is even (odd, respectively), then there exists a $u-v$ path P, that alternates between X and W, such that P has $s+2i-3$ $(s+2i-4$, respectively) vertices.

For s even (odd, respectively), we set $S = X-V(P(3))-V(P(4))-V(P(5))-\{x\}$ $(S = X-V(P(3))-V(P(4))-V(P(5))-V(P(6))-\{x\}$, respectively) and $T = W_1$. We apply Part 5 of Lemma 6 (or Part 1 of Lemma 6 if s is odd and $i+j = 4$). Observe that if s is even (odd, respectively), then $|T|-1 = s/2-j-2 > s/2-(s/4-2)-2 = s/4 \ge 4 = k+2$ $(|T|-1 = \lceil s/2 \rceil -j-2 > \lceil s/2 \rceil -(s/4-2)-2 =$

$(s+2)/4 \geq 5 = k+2$, respectively). If s is even (odd, respectively), then $|S|+k+1 = s-(i+j-1)-4-1+k+1 = s-i-j-1 > s/2-j-1 = |T|$ ($|S|+k+1 = s-(i+j-3)-6-1+k+1 = s-i-j > \lceil s/2\rceil - j - 1 = |T|$, respectively) (as $i < s/2$).

By Part 5 of Lemma 6 (or Part 1 of Lemma 6 if s is odd and $i+j=4$), we have that if s is even (odd, respectively) there exists an $x_1 - x_2$ path P on $2|T|-1+2k+n(P(3))-1 = 2(s/2-j-1)-1+2.2+2(i+j)-3-1 = s+2i-3$ ($2|T|-1+2k+n(P(3))-1 = 2(\lceil s/2\rceil-j-1)-1+2.3+2(i+j)-7-1 = s+2i-4$, respectively) vertices.

Let $u = y_2$, $v = y_1$, $S_2 = V(P(2)) \cap Y$ and $S_1 = V(P(1)) \cap Y$. Set $S = Y - V(P(1)) - V(P(2))$, $T = U_1$, $z = x$, $k = |V(P(1))|$ and $\ell = |V(P(2))|$. Note that $|S| \geq |T| \geq 3$. Recall that each vertex in Y is adjacent to at least $|T|-1$ vertices in T, and, by Lemma 16, every vertex in Y is adjacent to either $x = z$, or to every vertex in T. It follows that the construction for Lemma 23 is satisfied. If s is even (odd, respectively), there exists a $y_2 - y_1$ path P' that has $2|T|-1+k+\ell = 2(s/2-i-1)-1+3+3 = s-2i+3$ ($2|T|-1+k+\ell = 2(\lceil s/2\rceil - i-1)-1+3+3 = s-2i+4$, respectively) vertices. The sequence of vertices P, P' creates a C_{2s}.

Case 1.2. $s/2-j-2 \leq j+2$. Observe that $j \geq s/4-2$ and, as $i \geq j$, we have that $i \geq s/4-2$. Note that if s is odd, then $i \geq s/4-2+1/4$ (and $j \geq s/4-2+1/4$). This implies that $\deg_W(x) = \lfloor s/2\rfloor + i \geq 3s/4-9/4$ ($\deg_U(y) = \lfloor s/2\rfloor + j \geq 3s/4-9/4$, respectively). Recall that $\deg_W(x) \leq s-3$ and $\deg_U(y) \leq s-3$. To apply Lemma 6, we claim the following.

Claim 27. $|U_1| \geq 5$ *and* $|W_1| \geq 4$, *or* $|U_1| \geq 4$ *and* $|W_1| \geq 5$.

Proof. Suppose, without loss of generality, that $s-1-t = |U_1| \leq 3$. This implies that $t = \deg_W(x) \geq s-4$, whence every vertex in X has at least $s-4$ neighbors in W, and the fact that $\deg_U(y) \geq 3s/4-9/4$, implies that every vertex in Y has at least $3s/4-9/4$ neighbors in U. First set $S = X$ and $T = W$. For any $u, v \in S$, observe that $N_W(u) \cap N_W(v) = W - ((W - N_W(u)) \cup (W - N_W(v)))$ and so $|N_W(u) \cap N_W(v)| \geq s-7$. (Recall that for every $u' \in S$, $|W - N_W(u')| \leq s-1-t \leq s-1-(s-4) = 3$). By Lemma 5, there exists an $x_1 - x_2$ path P, that alternates between X and W, on $2s-13$ vertices.

Now set $S = Y$ and $T = U$. For any $u, v \in S$, observe again that $N_U(u) \cap N_U(v) = U - ((U - N_U(u)) \cup (U - N_U(v)))$, with $|U - N_U(u)| \leq s-1-(3s/4 - 9/4) = s/4+5/4$ and $|U - N_U(v)| \leq s/4+5/4$. Hence, $|N_U(u) \cap N_U(v)| \geq s-1-2(s/4+5/4) = s/2-7/2 \geq 11/2$. Since $|N_U(u) \cap N_U(v)|$ is an integer, we have that $|N_U(u) \cap N_U(v)| \geq 6$. By Lemma 5, there exists a $y_2 - y_1$ path P', that alternates between Y and U, on 13 vertices. The sequence P, P' forms a monochromatic C_{2s}. It follows that $|U_1| \geq 4$ and, by symmetry, $|W_1| \geq 4$.

To complete the proof of our claim, assume that $|U_1| = 4$ and $|W_1| = 4$. This implies that $\deg_W(x) = \deg_U(y) = s - 5$. Set $S = X$ and $T = W_1$. For any $u, v \in S$, observe, again, that $N_W(u) \cap N_W(v) = W - ((W - N_W(u)) \cup (W - N_W(v)))$ and so $|N_W(u) \cap N_W(v)| \geq s - 9$. By Lemma 5, there exists an $x_1 - x_2$ path P, that alternates between X and W, on $2s - 17$ vertices. By symmetry, we have, for any $u, v \in Y$, that $|N_U(u) \cap N_U(v)| \geq s - 9 > 8$. Whence, by Lemma 5, there exists a $y_2 - y_1$ path P', that alternates between Y and U, on 17 vertices. The sequence P, P' forms a monochromatic C_{2s}. ■

Let $i', j' \in \{0, 1\}$. Observe that Claim 27 implies that $i \leq \lceil s/2 \rceil - 5 - i'$, $j \leq \lceil s/2 \rceil - 5 - j'$ and that either $i' \neq 0$ or $j' \neq 0$.

Claim 28. *The graph $G' = G_B \langle X \cup W_2 - \{x_1, x_2\} \rangle$ has a path $P(1)$ that starts and ends in $X - \{x_1, x_2\}$ and alternates between X and W_2 such that $P(1) = P_{2j+3}$.*

Proof. Define $S = X - \{x_1, x_2\}$ and $T = W_2$. Let $b = |X - \{x_1, x_2\}| = s - 2$ and $a = |W_2| = t'$. It follows that $a = t' = j + \lfloor s/2 \rfloor \leq s - 2 = b$. Recall that $\deg_{W_2}(x) \geq \deg_W(x) - |W_1| \geq t - s + t' + 1 \geq i + j$, and so each vertex in X has degree at least $t - s + t' + 1$ in W_2, whence $m(G') \geq (s-2)(t - s + t' + 1) \geq (s-2)(i+j)$. Set $\ell = j + 2$ and $c = j + 1$. As $j \geq s/4 - 2$ ($i \geq s/4 - 2$, respectively) and $s \geq 18$, we have that $c = j + 1 \geq 4 > 0$ ($i \geq 3$, respectively). Assume that the graph G' has no path on $2(j+2)$ vertices. If $a \leq c$, then, by the first part of Theorem 1, $(i+j)(s-2) \leq m(G') \leq ab \leq cb \leq (j+1)(s-2)$, a contradiction. If $c < a \leq 2c$, then, by the second part of Theorem 1, $(i+j)(s-2) \leq m(G') \leq bc \leq (j+1)(s-2)$, a contradiction. Hence, $a \geq 2c$.

Applying the third part of Theorem 1, we have that $m(G') \leq (j+1)(s - 2 + \lfloor s/2 \rfloor + j - 2(j+1))$. To obtain a contradiction, we will show that $m(G') \geq (s-2)(i+j) > (j+1)(s - 2 + \lfloor s/2 \rfloor + j - 2(j+1))$. We will start by showing that $(s-2)(i+j) - (j+1)(s - 2 + s/2 + j - 2(j+1)) > 0$. Consider the expression $(s-2)(i+j) - (j+1)(s - 2 + s/2 + j - 2(j+1))$, which simplifies to $(s-2)i + j^2 - sj/2 - 3s/2 + 3j + 4$. As $i \geq j$, $s \geq 18$ and $j \geq 3$, we have that $(s-2)i + j^2 - sj/2 - 3s/2 + 3j + 4 \geq (s-2)j + j^2 - sj/2 - 3s/2 + 3j + 4 = j^2 + j(s/2 + 1) - 3s/2 + 4 \geq j^2 + 3(s/2 + 1) - 3s/2 + 4 > 0$. It immediately follows that $m(G') \geq (s-2)(i+j) > (j+1)(s - 2 + s/2 + j - 2(j+1)) \geq (j+1)(s - 2 + \lfloor s/2 \rfloor + j - 2(j+1)) \geq m(G')$, a contradiction. Hence, G' has a path on $2(j+2)$ vertices. It follows that a path $P(1)$ exists, that alternates between X and W_2 and starts and ends in $X - \{x_1, x_2\}$, and $P(1) = P_{2(j+2)-1}$. ■

Claim 29. *The graph $G'' = G_B \langle Y \cup U_2 - \{y_1, y_2\} \rangle$ has a path $P(2)$, that starts and ends in $Y - \{y_1, y_2\}$ and alternates between Y and U_2, such that $P(2) = P_{2i-1}$.*

Proof. Define $S = Y - \{y_1, y_2\}$ and $T = U_2$. Let $b = |Y - \{y_1, y_2\}| = s - 2$ and $a = |U_2| = t$. It follows that $a = t = i + \lfloor s/2 \rfloor < s - 2 = b$. Recall that

$\deg_{U_2}(y) \geq \deg_U(y) - |U_1| \geq t' - s + t + 1$, and so each vertex in Y has degree at least $t' - s + t + 1$ in U_2, whence $m(G'') \geq (s-2)(t' - s + t + 1)$. Set $\ell = i$ and $c = i - 1$. As $i \geq s/4 - 2$ and $s \geq 18$, we have that $c = i - 1 > 0$. Assume that the graph G'' has no path on $2i$ vertices. As $i \leq \lceil s/2 \rceil - 3$, we have that $2c = 2(i-1) \leq \lceil s/2 \rceil - 5 + i \leq \lfloor s/2 \rfloor + i - 4 = t - 4 < |U_2| = a$. Applying the third part of Theorem 1, we have that $m(G') \leq (i-1)(s-2+\lfloor s/2 \rfloor + i - 2(i-1))$.

To obtain a contradiction, we will show that $m(G'') \geq (s-2)(t' - s + t + 1) > (i-1)(s - 2 + \lfloor s/2 \rfloor + i - 2(i-1))$. We will start by showing that $(s-2)(i+j) - (i-1)(s - 2 + s/2 + i - 2(i-1)) > 0$. Consider the expression $(s-2)(i+j) - (i-1)(s-2+s/2+i-2(i-1))$, which simplifies to $(s-2)j + i^2 - si/2 + 3s/2 - 3i$. Let $f(i) = i^2 - si/2 + 3s/2 - 3i$. Using calculus, $f(i)$ obtains an absolute minimum at $i = s/4 + 3/2$. Hence, as $s \geq 18$ and $j \geq s/4 - 2$, we have $(s-2)j + i^2 - si/2 + 3s/2 - 3i \geq (s-2)j + f(s/4+3/2) \geq (s-2)(s/4-2) + f(s/4+3/2) = 3s^2/16 - 7s/4 + 7/4 > 0$. It immediately follows that $m(G') \geq (s-2)(t-s+t'+1) \geq (s-2)(i+j) > (i-1)(s-2+s/2+i-2(i-1)) \geq (i-1)(s-2+\lfloor s/2 \rfloor + i - 2(i-1)) \geq m(G')$, a contradiction. It follows that G'' has a path that alternates between Y and U_2, on $2i$ vertices. Whence, G'' has a path $P(2)$, that starts and ends in $Y - \{y_1, y_2\}$, alternates between $Y - \{y_1, y_2\}$ and U_2, with $P(2) = P_{2i-1}$. ■

Claim 30. *The graph $G'' = G_B \langle X \cup W_2 - \{x_1, x_2\} - V(P(1)) \rangle$ ($G''' = G_B \langle Y \cup U_2 - \{y_1, y_2\} - V(P(2)) \rangle$, respectively) has $j'+1$ ($i'+1$, respectively) P_3 path(s) (disjoint), that start and end in $X - \{x_1, x_2\}$ ($Y - \{y_1, y_2\}$, respectively).*

Proof. We will apply Lemma 7. Consider the graph $G'' = G_B \langle X \cup W_2 - \{x_1, x_2\} - V(P(1)) \rangle$ ($G''' = G_B \langle Y \cup U_2 - \{y_1, y_2\} - V(P(2)) \rangle$, respectively). Recall that for each vertex $x' \in X$ ($y' \in Y$, respectively), $\deg_{W_2}(x') \geq \deg_W(x) - |W_1| = \lfloor s/2 \rfloor + i - (\lceil s/2 \rceil - j - 1) \geq i + j$ ($\deg_{U_2}(y') \geq \deg_U(y) - |U_1| = \lfloor s/2 \rfloor + j - (\lceil s/2 \rceil - i - 1) \geq i + j$, respectively). Set $S' = X - \{x_1, x_2\} - V(P(1))$, $T' = W_2 - V(P(1))$, $S'' = Y - \{y_1, y_2\} - V(P(2))$ and $T'' = U_2 - V(P(2))$.

Whence, each $x' \in S'$ has at least $i + j - (j+1) = i - 1 \geq 2 (> j')$ (as $i \geq s/4 - 2 \geq 5/2$) neighbors in T'. Likewise, each $y' \in S''$ has at least $i + j - (i-1) \geq j + 1 \geq 4 (> i')$ (as $j \geq s/4 - 2 \geq 5/2$) neighbors in T''. Set $k' = j'$ and $k'' = i'$. For the graph G'', observe that as $j \leq \lceil s/2 \rceil - 5 - j'$, we have that $|S'| - k' = s - 2 - (j+2) - j' = s - 4 - j - j' = \lfloor s/2 \rfloor + \lceil s/2 \rceil - j - j' - 4 \geq \lfloor s/2 \rfloor + 1 > |T'|$. For the graph G''', we have, as $i \leq \lceil s/2 \rceil - 5 - i'$, that $|S''| - k'' = s - 2 - i - i' = \lfloor s/2 \rfloor + \lceil s/2 \rceil - i - i' - 2 \geq \lfloor s/2 \rfloor + 3 > |T''|$. By Lemma 7, G'' (G''', respectively) has $j'+1$ ($i'+1$, respectively) P_3 path(s) (disjoint). ■

We now apply Part 5 of Lemma 6. Referring to Claim 30, let Z (Z', respectively) denote the set of $j'+1$ ($i'+1$, respectively) P_3's of the graph G'' (G''', respectively). If s is even (odd, respectively) and $i' = j' = 1$, remove one P_3 from

Z (Z and Z', respectively). If s is odd and $i' \neq j'$, then remove one P_3 from Z. Set $S = X - V(P(1)) - V(Z)$ and $T = W_1$. We claim that an $x_1 - x_2$ path P exists, that alternates between X and W, on $2|W_1| - 1 + 2|Z| + |V(P(1))| - 1$ vertices. We first check that the requirements for Part 5 of Lemma 6 hold. Observe that $|S| + |Z| + 1 > |S| \geq s - (j+2) - 2|Z| \geq s - j - 2 - 2j' - 2 \geq \lfloor s/2 \rfloor + \lceil s/2 \rceil - j - 6 > |T|$. Furthermore, as $j \leq \lceil s/2 \rceil - 5 - j'$, we have that $|T| - 1 = \lceil s/2 \rceil - j - 2 \geq \lceil s/2 \rceil - (\lceil s/2 \rceil - j' - 5) - 2 = j' + 3 = j' + 1 + 2 \geq |Z| + 2$. It follows, by Part 5 of Lemma 6, that P exists.

Set $S = Y - V(P(2)) - V(Z')$ and $T = U_1$. We claim that a $y_2 - y_1$ path P' exists, that alternates between Y and U', on $2|U_1| - 1 + 2|Z'| + |V(P(2))| - 1$ vertices. Observe that $|S| + |Z'| + 1 > |S| \geq s - i - 2|Z'| \geq s - i - 2i' - 2 \geq \lfloor s/2 \rfloor + \lceil s/2 \rceil - i - 4 > |T|$. Furthermore, as $i \leq \lceil s/2 \rceil - 5 - i'$, we have that $|T| - 1 = \lceil s/2 \rceil - i - 2 \geq \lceil s/2 \rceil - (\lceil s/2 \rceil - i' - 5) - 2 = i' + 3 = i' + 1 + 2 \geq |Z'| + 2$. It follows, by Part 5 of Lemma 6, that P' exists.

If, in G_B, x_1 is adjacent to y_1 (x_1 is not adjacent to y_1, respectively), then the sequence P, P' forms a cycle D' on $2(|U_1| + |W_1|) + 2(|Z| + |Z'|) + 2j + 2i - 2$ vertices. Suppose first that s is even. If $0 \leq i', j' \leq 1$ and $i' \neq j'$, then D' has $2(s-i-j-2)+2(i'+j'+2)+2i+2j-2 = 2s$ vertices and so $D' = C_{2s}$. If $i' = j' = 1$, then D' has $2(|U_1|+|W_1|)+2(|Z|+|Z'|)+2j+2i-2 = 2(s-i-j-2)+2(i'+j'+1)+2i+2j-2 = 2s$ vertices and so $D' = C_{2s}$. Assume s is odd. If $0 \leq i', j' \leq 1$ and $i' \neq j'$, then D' has $2(|U_1|+|W_1|)+2(|Z|+|Z'|)+2j+2i-2 = 2(s-i-j-1)+2(i'+j'+1)+2i+2j-2 = 2s$ vertices and so $D' = C_{2s}$. If $i' = j' = 1$, then D' has $2(|U_1|+|W_1|)+2(|Z|+|Z'|)+2j+2i-2 = 2(s-i-j-1)+2(i'+j')+2i+2j-2 = 2s$ vertices and so $D' = C_{2s}$.

Case 2. There exist integers $i, j \geq 0$, such that $t = \lfloor s/2 \rfloor + i$ and $t' = \lfloor s/2 \rfloor - j$. Recall that $|U_1| = s - 1 - t = \lceil s/2 \rceil - i - 1$, $|W_1| = s - 1 - t' = \lceil s/2 \rceil + j - 1 > 2$, $|W_2| = \lfloor s/2 \rfloor - j$ and $\deg_{W_2}(x) \geq \deg_W(x) - |W_1| \geq t - s + 1 + t'$. Furthermore, recall that $1 \leq t' \leq t \leq s - 3$, whence $i \leq \lceil s/2 \rceil - 3$.

Claim 31. *Let $u, v \in \{y_1, y_2, y_3\}$ and $|U_1| = 2$. If $j \in \{0, 1, 2\}$, then, for the graph $G' = G_B \langle X \cup Y \cup U \rangle$, we have that for some $1 \leq i', j' \leq 3$ and $i' \neq j'$, there exists a $y_{i'} - y_{j'}$ path P on 5 vertices, such that, in G', P alternates between Y and $\{x\} \cup U$, and $V(P) \cap \{x_{i'}, x_{j'}\} = \emptyset$.*

Proof. Let $U_1 = \{u_1, u_2\}$. As $|U_1| = 2$, we have that $\deg_W(x) = s - 3 = |W| - 2$. If $x \in \{x_1, x_2, x_3\}$, then, without loss of generality, let $x = x_3$.

Case A. y_1 and y_2 have no common neighbors in U_1. Let $w \in Y - \{y_1, y_2, y_3\}$. Then, without loss of generality, $N_{U_1}(y_1) = \{u_1\}$ and $N_{U_1}(y_2) = \{u_2\}$, and w is adjacent to say u_1. If w is adjacent to u_2, then the path $P : y_2, u_2, w, u_1, y_1$ has the desired properties. If w is not adjacent to u_2, then, by Lemma 16, w is adjacent to x. As y_1 and y_2 do not have a common neighbor in U_1, we can apply Lemma

16 and deduce that both y_1 and y_2 are adjacent to x. The path $P: y_2, x, w, u_1, y_1$ has the desired properties.

Case B. y_1 and y_2 are adjacent to say u_1.

Case B1. y_3 is adjacent to u_1. If, without loss of generality, y_3 and y_1 (y_2, respectively) have a common neighbor w', with $w' \in U - \{u_1\}$, then $P: y_1, w', y_3, u_1, y_2$ ($P: y_2, w', y_3, u_1, y_1$, respectively) has the desired properties. If y_2 and y_1 have a common neighbor w', with $w' \in U - \{u_1\}$, then $P: y_1, w', y_2, u_1, y_3$ has the desired properties. It follows that each pair of vertices in $\{y_1, y_2, y_3\}$ has only one common neighbor in U (in G'), which is u_1. This implies that $N_U(y_1) \cup N_U(y_2) \cup N_U(y_3)$ has at least $3(\lfloor s/2 \rfloor - 3) + 1$ vertices. But, as $s \geq 18$, $3(\lfloor s/2 \rfloor - 3) + 1 \geq s > |U| = s - 1$, a contradiction.

Case B2. y_3 is adjacent to u_2, and not u_1. By Lemma 16, y_3 is adjacent to x. If y_1 (y_2, respectively) and y_3 have a common neighbor $w' \in U - \{u_1\}$, then $P: y_3, w', y_1, u_1, y_2$ ($P: y_3, w', y_2, u_1, y_1$, respectively) is a path with the desired properties. Whence, in G', neither y_1 nor y_2 has a common neighbor with y_3 in U. This implies that as y_3 is adjacent to u_2, both y_1 and y_2 are not adjacent to u_2. By Lemma 16, y_1 and y_2 are both adjacent to x. If $x \in X - \{x_1, x_2, x_3\}$, then $P: y_3, x, y_1, u_1, y_2$ is a path with the desired properties. Hence, we may assume that $x \in \{x_1, x_2, x_3\}$ and we can let $x = x_3$. We may therefore assume that each vertex in $X - \{x_1, x_2, x_3\}$ must have more neighbors in W than x, since otherwise some vertex $x' \in X - \{x_1, x_2, x_3\}$ has $\deg_W(x) = \deg_W(x')$, and we can simply relabel x' as x, and so we are done.

If, without loss of generality, $\deg_W(x_1) = \deg_W(x)$, then we can relabel x as x_1. Recall that y_2 and y_3 have no common neighbors in U_1, whence, with an appropriate relabeling, this translates into Case A. We can deduce that there is a path P with the desired properties. It follows that each vertex in $X - \{x_3\}$ must have at least $s - 2$ neighbors in W. Set $S = X - \{x_3\}$ and $T = W$. By Part 1 of Lemma 6, there exists an $x_1 - x_2$ path P on $2|T| - 1 = 2(s-1) - 1 = 2s - 3$ vertices, where $x \in X - \{x_1, x_2\}$. Recall that each vertex in $\{y_1, y_2\}$ is adjacent to x or adjacent to the two vertices in U_1. Hence, y_1 and y_2 have a common neighbor $x' \in U_1 \cup \{x\}$. The sequence P, y_2, x', y_1 forms a monochromatic C_{2s}. ■

Case 2.1. $i - j \geq -1$. Define X' as a subset of X. Let Z_1 be a set (possibly empty) of disjoint $K_{1,2}$'s, where each one of these $K_{1,2}$'s has central vertex in W_2, and end vertices in X'. Let Z_2 be a set of disjoint P_5's, where each one of these P_5's starts and ends in X', and alternates between X' and W_2. Let the P_5's and $K_{1,2}$'s in $Z_1 \cup Z_2$ be disjoint. Let $V(Z_i)$ denote the vertices of all the components in Z_i, for $1 \leq i \leq 2$.

Claim 32. *Let $i' \in \{i+1, i\}$ and suppose that if n is even (odd, respectively), every vertex in X' has at least $i' - j + 1$ ($i' - j$, respectively) neighbors in W_2 and at least*

$\lfloor s/2\rfloor + i'$ neighbors in W. If n is even (odd, respectively) and $|X'|+2j > s/2+i'$ ($|X'| + 2j + 1 > \lfloor s/2\rfloor + i'$, respectively), then $|Z_1| = i' - j + 1$ ($|Z_1| = i' - j + 1$ if $i' - j = -1$ and $|Z_1| = i' - j$ if $i' \geq j$, respectively).

Proof. Suppose that if n is even (odd, respectively), every vertex in X' has at least $i'-j+1$ ($i'-j$, respectively) neighbors in W_2 and at least $\lfloor s/2\rfloor+i'$ neighbors in W. Assume that if n is even (odd, respectively), then $|X'| + 2j > s/2 + i'$ ($|X'| + 2j + 1 > \lfloor s/2\rfloor + i'$, respectively). If s is even (odd, respectively), set $k' = i' - j$ if $i' \geq j$ and $k' = i' - j + 1$ if $i' - j = -1$ ($k' = i' - j$ if $i' \geq j$, and $k' = i' - j + 1$ if $i' - j = -1$, respectively). Set $S' = X'$ and $T' = W_2$. Observe that if $|W_2| = 0$, then $j = \lfloor s/2\rfloor$ and so $r = 0$ which will imply that $|W_1| = s - 1$ and so $\deg_W(x) = \lfloor s/2\rfloor + i \geq s - 2$, contradicting the fact that $i \leq \lceil s/2\rceil - 3$. Whence, $|T'| \geq 1$.

Suppose first that n is even. If $i' - j = -1$ ($i' - j + 1 = 0$) then we are done as there are at least zero copies of $K_{1,2}$ in Z_1. Suppose $i' - j \geq 0$. Now $|S'|-k' = |X'|-i'+j > s/2-j = |T'|$. By Lemma 7, there exist $k'+1 = i'-j+1$ disjoint copies of $K_{1,2}$ with central vertices in T' and end vertices in X'.

Suppose now that n is odd. If $i' - j = -1$ ($i' - j = 0$, respectively), then we are done as there are at least $i'-j+1 = 0$ ($i'-j = 0$) copies of $K_{1,2}$ in Z_1. Assume that $i'-j \geq 1$. Set $k' = i'-j-1$. Now $|S'|-k' = |X'|-i'+j+1 > \lfloor s/2\rfloor-j = |T'|$. By Lemma 7, there exist $k'+1 = i'-j$ disjoint copies of $K_{1,2}$ with central vertices in T' and end vertices in X'. ∎

Claim 33. *Let $i' \in \{i + 1, i\}$ and suppose that if n is even (odd, respectively), every vertex in X' has at least $i' - j + 1$ ($i' - j$, respectively) neighbors in W_2 and at least $\lfloor s/2\rfloor + i'$ neighbors in W. If n is even (odd, respectively) and $|X'| - 2 + 2j > s/2 + i'$ ($|X'| - 1 + 2j > \lfloor s/2\rfloor + i'$, respectively), then one of the following holds.*

1. *If n is even (odd, respectively), then $|Z_1| = i' - j + 2$ ($|Z_1| = i' - j + 1$, respectively).*

2. *If n is even (odd, respectively), then $|Z_1| = i' - j + 1$ ($|Z_1| = i' - j + 1$ if $i' - j = -1$ and $|Z_1| = i' - j$ if $i' \geq j$, respectively) and there are two vertices u' and v' in $X' - V(Z_1)$ that are both adjacent to every vertex in W_1.*

Proof. Assume that if n is even (odd, respectively), every vertex in X' has at least $i'-j+1$ ($i'-j$, respectively) neighbors in W_2 and at least $\lfloor s/2\rfloor+i'$ neighbors in W. If n is even (odd, respectively) we have, by Claim 32, that there exists a set Z_1 of $i' - j + 1$ ($i' - j + 1$ if $i' - j = -1$ and $i' - j$ if $i' \geq j$, respectively) disjoint $K_{1,2}$'s, with end vertices in X' and central vertices in W_2. Let X_1 (W', respectively) denote the set of end (central, respectively) vertices of the $K_{1,2}$'s in Z_1. If two vertices $u', v' \in X' - X_1$ are adjacent to every vertex in W_1, then the second part holds. It follows that at most one vertex, say $x' \in X' - X_1$, is adjacent to all vertices in W_1. Hence, every vertex in $X' - X_1 - \{x'\}$ has exactly $|W_1| - 1$ neighbors in W_1.

Suppose first n is even. For each vertex $u' \in X' - X_1 - \{x'\}$, we have that $\deg_{W_2}(u') \geq \deg_W(u') - (|W_1| - 1) = s/2 + i' - (s/2 + j - 2) = i' - j + 2$. By the pigeonhole principle, every vertex in $X' - X_1 - \{x'\}$ has a neighbor in $W_2 - W'$. Set $S' = X' - X_1 - \{x'\}$ and $T' = W_2 - W'$ ($|T'| \geq 1$, as $i \leq s/2 - 3$). Set $k' = 0$. Then, as $|X'| - 2 + 2j > s/2 + i'$, $|S'| - k' = |X'| - 1 - 2(i' - j + 1) > s/2 - j - (i' - j + 1) = |W_2| - |W'| = |T'|$. By Lemma 7, there exists $k' + 1 = 1$ copy of $K_{1,2}$ with central (end, respectively) vertex (vertices, respectively) in T' (S', respectively). Add the extra copy of $K_{1,2}$ to Z_1. So there are $i' - j + 2$ disjoint copies of $K_{1,2}$.

We may assume that n is odd. If $i' - j = -1$, then $|Z_1| = i' - j + 1 = 0$ and we are done. It follows that $i' - j \geq 0$ ($i' - j = |Z_1|$). For each vertex $u' \in X' - X_1 - \{x'\}$, we have that $\deg_{W_2}(u') \geq \deg_W(u') - (|W_1| - 1) = \lfloor s/2 \rfloor + i' - (\lceil s/2 \rceil + j - 2) = i' - j + 1$. By the pigeonhole principle, every vertex in $X' - X_1 - \{x'\}$ has a neighbor in $W_2 - W'$. Set $S' = X' - X_1 - \{x'\}$ and $T' = W_2 - W'$ ($|T'| \geq 1$, as $i' \leq \lceil s/2 \rceil - 2$). Set $k' = 0$. Then, as $|X'| - 1 + 2j > \lfloor s/2 \rfloor + i'$, $|S'| - k' = |X'| - 1 - 2(i' - j) > \lfloor s/2 \rfloor - j - (i' - j) = |W_2| - |W'| = |T'|$. By Lemma 7, there exists $k' + 1 = 1$ copy of $K_{1,2}$ with central (end, respectively) vertex (vertices, respectively) in T' (S', respectively). Add the extra copy of $K_{1,2}$ to Z_1. So there are $i' - j + 1$ disjoint copies of $K_{1,2}$ and so we are done. ■

Claim 34. *Suppose n is even and $|X'| = s - 3$. If $i = s/2 - 3$ and $j \in \{0, 1, 2\}$, then for the graph $G' = G_B \langle X' \cup W_2 \rangle$, one of the following holds.*

1. *There are two disjoint paths P and P', such that $P = P_{2(i-j)+1}$ and $P' = P_3$, and both P and P' start and end in X', and there are two vertices $u', v' \in X' - V(P) - V(P')$, such that u' and v' are both adjacent to all the vertices in W_1.*

2. *There are three disjoint paths P, P' and P'', such that $P = P_{2(i-j)+1}$, both P'' and P' are paths on three vertices, and P, P' and P'' start and end in X'.*

Proof. We claim that G' does have a path P with the desired property. Suppose G' does not have a path on $2(i - j + 1)$ vertices. Pick $c = i - j > 0$. Define $S = X'$ and $T = W_2$. Let $|X'| = s - 3 = b$ and $|W_2| = s/2 - j = a$. Recall that $\deg_{W_2}(x) \geq \deg_W(x) - |W_1| \geq i - j + 1$, and so every vertex in X' has at least $i - j + 1$ neighbors in W_2. Note that $m(G') \geq (s - 3)(i - j + 1)$. If $c < a < 2c$, then, applying the second part of Theorem 1, we have that $(s - 3)(i - j + 1) \leq m(G') \leq bc = (s - 3)(i - j)$, a contradiction. If $a \leq c$, then, applying the first part of Theorem 1, we have that $(s - 3)(i - j + 1) \leq m(G') \leq ab \leq cb = (s - 3)(i - j)$, a contradiction. We may assume that $a \geq 2c$. By the third part of Theorem 1, $m(G') \leq (i - j)(s - 3 + s/2 - j - 2(i - j))$.

To obtain a contradiction, we will prove that $m(G') \geq (s-3)(i-j+1) > (i-j)(s-3+s/2-j-2(i-j))$. Simplifying, we obtain the inequality $2i^2 - i(s/2+3j) + sj/2 + j^2 + s - 3 > 0$, which, if verified, will prove the desired result. Observe that $f(i) = 2i^2 - i(s/2+3j) + sj/2 + j^2 + s - 3$ is a parabola with minimum value occurring at $i = s/8 + 3j/4$. Since $f(i)$ is increasing on the interval $[s/8+3j/4, +\infty)$ and $i = s/2 - 3 \geq s/8 + 3j/4$, we have that $f(i) \geq f(s/2-3)$. If $j = 0$, then, as $s \geq 18$, $f(s/2-3) = s^2/4 - 7s/2 + 15 > 0$. If $j = 1$, then, as $s \geq 18$, $f(s/2-3) = s^2/4 - 9s/2 + 25 > 0$. If $j = 2$, then, as $s \geq 18$, $f(s/2-3) = s^2/4 - 11s/2 + 37 > 0$. It follows that there exists a path in G' on $2(i-j+1)$ vertices, whence there exists the desired path P in G'. Let $S' = X' - V(P)$ and $T' = W_2 - V(P)$. Then $|S'| = s - 3 - (i-j+1)$ and $|T'| = |W_2| - (i-j) = s/2 - j - (i-j) \geq 3$. Set $k' = 0$. By the pigeonhole principle, every vertex in S' has more than k' neighbors in T'. Clearly, $|S'| - k' > |T'|$. By Lemma 7, a path $P' = P_3$ (a $K_{1,2}$) with the desired property exists.

If there are two vertices $u', v' \in X' - V(P) - V(P')$ such that u' and v' are both adjacent to all the vertices in W_1, then the first property holds. It follows that at most one vertex $u' \in X' - V(P) - V(P')$ has $|W_1|$ neighbors in W_1. Hence, for each vertex $x' \in X' - V(P) - V(P') - \{u'\}$, we have that $\deg_{W_2}(x') \geq \deg_W(x) - |N_{W_1}(x')| \geq s/2 + i - (s/2 + j - 2) = i - j + 2$. It follows that each x' has 1 neighbor in $W_2 - V(P) - V(P')$. Set $S' = X' - V(P) - V(P') - \{u'\}$, $T' = W_2 - V(P) - V(P')$ and set $k' = 0$. Then $|S'| - k' = s - 4 - (i - j + 1 + 2) > s/2 - j - (i - j + 1) = |T'| \geq 2$. By Lemma 7, a path $P'' = P_3$ with the desired property exists. ■

Claim 35. *Suppose n is odd and $|X'| = s-3$. If $i = \lceil s/2 \rceil - 3$ and $j \in \{0,1,2\}$, then, for the graph $G' = G_B\langle X' \cup W_2 \rangle$, we have that $|Z_1| = i-j-1$ and $|Z_2| = 1$.*

Proof. Define $S = X'$ and $T = W_2$. Let $|X'| = s - 3 = b$ and $|W_2| = \lfloor s/2 \rfloor - j = a$. Recall that $\deg_{W_2}(x) \geq \deg_W(x) - |W_1| \geq i - j$, and so every vertex in X' has at least $i - j$ neighbors in W_2. Note that $m(G') \geq (s-3)(i-j) = (\lceil s/2 \rceil - 3 - j)(s-3)$. We claim that G' has a path on 6 vertices. Suppose it does not. Pick $c = 2$. Observe that $a > 5 > 4 = 2c$. By the third part of Theorem 1, $m(G') \leq c(a+b-2c) = 2(s-3+(s-1)/2-j-4)$. Observe that $2(s-3+(s-1)/2-j-4) < (\lceil s/2 \rceil - 3 - j)(s-3) \leq m(G')$, a contradiction. It follows that a path P_6 exists. The end vertex of this path that is in W_2 can be deleted, and so a P_5 will result that starts and ends in X' and alternates between X' and W_2. We can place this path in Z_2 and so we may assume that $|Z_2| = 1$.

Set $S' = X' - V(Z_2)$ and $T' = W_2 - V(Z_2)$. Let $k' = i - j - 2$. By the pigeonhole principle, every vertex in S' has more than $i - j - 2$ (at least $i - j$) neighbors in T'. As $i = \lceil s/2 \rceil - 3$, we have that $|S'| - k' = s - 3 - 3 - (i - j - 2) > \lfloor s/2 \rfloor - j - 2 = |T'| > 0$. By Lemma 7, the graph $G_B\langle S' \cup T' \rangle$ has $i - j - 1$ copies of $K_{1,2}$, whence we can let $|Z_1| = i - j - 1$. ■

Recall that $G_B \langle X \cup Y \rangle$ has 3 disjoint K_2's. By the pigeonhole principle, we may assume that $x \in X - \{x_1, x_2\}$.

Case 2.1.1. $|U_1| \geq 3$. Observe that if $\deg_W(x) = t = \lfloor s/2 \rfloor + i \geq s-3$, then $|U_1| = s - 1 - t \leq 2$, a contradiction. Hence $\deg_W(x) = \lfloor s/2 \rfloor + i < s - 3$, implying that $i < \lceil s/2 \rceil - 3$.

Case 2.1.1.1. $i \geq \lceil s/2 \rceil + 2j - 5$. If $j \geq 1$, then $i \geq \lceil s/2 \rceil - 3$, contradicting the fact that $i < \lceil s/2 \rceil - 3$. It follows that $j = 0$, whence $i \in \{\lceil s/2 \rceil - 4, \lceil s/2 \rceil - 5\}$. Hence, $i - j \geq 1$ and so $i - j + 1 \geq 2$. Observe that $\deg_{U_2}(y) \geq \deg_U(y) - |U_1| \geq \lfloor s/2 \rfloor - j - (\lceil s/2 \rceil - i - 1)$. It follows that if s is even (odd, respectively), then $\deg_{U_2}(y) \geq i - j + 1 \geq 2$ ($\deg_{U_2}(y) \geq i - j \geq 1$, respectively) and so every vertex in Y has at least 2 (1, respectively) neighbors (neighbor, respectively) in U_2. If s is even (odd, respectively), let $S' = Y - \{y_1, y_2\}$, $T' = U_2$ and $k' = 1$ ($k' = 0$, respectively). Then $|S'| - k' \geq s - 3 > \lfloor s/2 \rfloor + i = |U_2| = |T'|$. By Lemma 7, we have, if s is even (odd, respectively), that the graph $G_B \langle Y \cup U_2 - \{y_1, y_2\} \rangle$ has two disjoint copies of $K_{1,2}$ (a copy of $K_{1,2}$ and K_1, both being disjoint, respectively), say P' and P'', such that both P' and P'' start and end in $Y - \{y_1, y_2\}$. Let $X' = X - \{x, x_1, x_2\}$. By Claim 32 (we can check that the conditions hold using $i < \lceil s/2 \rceil - 3$) there exists a set Z_1 of $i - j$ disjoint copies of $K_{1,2}$, where each $K_{1,2}$ has central vertex in W_2 and end vertices in X'. Set $S = X - \{x\} - V(Z_1)$, $T = W_1$ and $k = i - j$. We apply Part 1 of Lemma 6. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 \geq k + 1 = i - j + 1$, since otherwise $i > \lceil s/2 \rceil + 2j - 3 \geq \lceil s/2 \rceil - 3$. By Lemma 6, there exists an $x_1 - x_2$ path P''', that alternates between X and W, on $2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j)$ vertices.

Recall the paths P' and P''. Set $S = Y - V(P') - V(P'')$ and $T = U_1$. Define $S_1 = V(P') \cap Y$, $S_2 = V(P'') \cap Y$, $u = y_2$, $v = y_1$ and $z = x$. By Lemma 16, each vertex in Y is adjacent to z or adjacent to every vertex in T. The construction for the application of Lemma 23 is satisfied. If s is even (odd, respectively), then there exists a $y_2 - y_1$ path P, on $2|T| - 1 + 3 + 3 = 2(s/2 - i - 1) - 1 + 6 = s - 2i + 3$ ($2|T| - 1 + 1 + 3 = 2(\lceil s/2 \rceil - i - 1) - 1 + 4 = s - 2i + 2$, respectively) vertices. The sequence P, P''' forms a monochromatic C_{2s}.

Case 2.1.1.2. $i < \lceil s/2 \rceil + 2j - 5$. Define $X' = X - \{x, x_1, x_2\}$. We will first check that the requirements of Claim 33 hold and then apply it. Recall the definition of the set Z_1 in the statement. Observe that as $i < \lceil s/2 \rceil + 2j - 5$, we have that $i + \lfloor s/2 \rfloor < \lceil s/2 \rceil + \lfloor s/2 \rfloor + 2j - 5 = s - 5 + 2j = |X'| - 2 + 2j$. . Suppose the first part of Claim 33 holds. Note that if s is even (odd, respectively), then $|Z_1| = i - j + 2 = k$ ($|Z_1| = i - j + 1 = k$, respectively). Set $S = X - \{x\} - V(Z_1)$ and $T = W_1$. Note that $|S| + k = s - 1 - 2k + k = s - k - 1 \geq s - i + j - 3 \geq \lceil s/2 \rceil + j - 1 = |T|$. We apply Part 1 of Lemma 6. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 \geq i - j + 3 \geq k + 1$. If s is even (odd, respectively), then, by Lemma 6, there exists an $x_1 - x_2$ path P''', that alternates between X and W, on $2|T| - 1 + 2k = 2(s/2 + j - 1) - 1 + 2(i - j + 2) = s + 1 + 2i$ ($2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 1) = s + 2i$, respectively) vertices. Suppose the second part of Claim 33 holds. Note that if s is even (odd, respectively) then $|Z_1| = i - j + 1 = k$ ($|Z_1| = i - j + 1 = k$ if $i - j = -1$ and $|Z_1| = i - j = k$ if $i \geq j$, respectively), and there are two vertices $u', v' \in X' - V(Z_1)$, that are adjacent to all vertices in W_1. Set $S = X - \{x\} - V(Z_1)$ and $T = W_1$. Note that $|S| + k - 1 = s - 1 - 2k + k - 1 = s - k - 2 \geq s - i + j - 3 \geq \lceil s/2 \rceil + j - 1 = |T|$. We apply Part 4 (or Part 1) of Lemma 6. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 > i - j + 3 \geq k + 2$. Suppose s is even. There exists an $x_1 - x_2$ path P''', that alternates between X and W, on $2|T| + 1 + 2k = 2(s/2 + j - 1) + 1 + 2(i - j + 1) = s + 1 + 2i$ vertices. If s is odd, then if $k = i - j$ ($k = i - j + 1$, respectively), we have, by Part 4 (Part 1, respectively) of Lemma 6, that there exists an $x_1 - x_2$ path P''', that alternates between X and W, on $2|T| + 1 + 2k = 2(\lceil s/2 \rceil + j - 1) + 1 + 2(i - j) = s + 2i$ ($2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 1) = s + 2i$, respectively) vertices.

Set $S = Y$ and $T = U_1$. Choose two disjoint one vertex paths P' and P'' in $Y - \{y_1, y_2\}$. Define $S_1 = V(P') \cap Y$, $S_2 = V(P'') \cap Y$, $u = y_2$, $v = y_1$ and $z = x$. By Lemma 16, each vertex in S is adjacent to z or adjacent to every vertex in T. The construction for the application of Lemma 23 is satisfied. If s is even (odd, respectively), there exists a $y_2 - y_1$ path P, on $2|T| - 1 + 1 + 1 = 2(s/2 - i - 1) - 1 + 2 = s - 2i - 1$ ($2|T| - 1 + 1 + 1 = 2(\lceil s/2 \rceil - i - 1) - 1 + 2 = s - 2i$, respectively) vertices. The sequence P, P''' forms a monochromatic C_{2s}.

Case 2.1.2. $|U_1| \leq 2$. Observe that if $\deg_W(x) = t = \lfloor s/2 \rfloor + i \leq s - 4$, then $|U_1| = s - 1 - t \geq 3$, a contradiction. Hence $\deg_W(x) = \lfloor s/2 \rfloor + i \geq s - 3$, implying that $i = \lceil s/2 \rceil - 3$, and so $|U_1| = 2$.

Case 2.1.2.1. $i < \lceil s/2 \rceil + 2j - 7$. Recall the definition of Configuration 1 and 2. Let us assume first that for Configuration 1 (Configuration 2, respectively), we have that for every $x' \in X - \{x_1, x_2, x_3\}$ ($x' \in X - \{x_1, x_2, x_3, x_4\}$, respectively), $\deg_W(x') \geq \deg_W(x) + 1 = \lfloor s/2 \rfloor + (i+1)$. Note that if s is even (odd, respectively) $\deg_{W_2}(x') \geq \deg_W(x') - |W_1| \geq s/2 + i + 1 - (s/2 + j - 1) = (i + 1) - j + 1$ ($\deg_{W_2}(x') \geq \deg_W(x') - |W_1| \geq \lfloor s/2 \rfloor + i + 1 - (\lceil s/2 \rceil + j - 1) = (i + 1) - j$, respectively).

Consider first the case where Configuration 1 holds. We focus our attention on the three disjoint K_2's with edges x_1y_1, x_2y_2 and x_3y_3. Without loss of generality, $x_3 = x$. We apply Lemma 16 and conclude that every vertex in Y is adjacent to either the two vertices in U_1, or to x. This implies that y_1 and y_2 have a common neighbor $x'' \in \{x\} \cup U_1$. If s is even (odd, respectively), there exists a $y_2 - y_1$ path P on $3 = 2|U_1| - 1 = 2(s/2 - i - 1) - 1 = s - 2i - 3$ ($3 = 2(\lceil s/2 \rceil - i - 1) - 1 = s - 2i - 2$, respectively) vertices.

Consider the case where Configuration 2 holds. It is clear that $x \in \{x_1, x_2, x_3, x_4\}$. If one vertex of y_1 and y_2 is not adjacent to x, then, by Lemma 16, y_1

and y_2 have a common neighbor $u_1 \in U_1$. If s is even (odd, respectively), there exists a $y_2 - y_1$ path P on $3 = 2|U_1| - 1 = 2(s/2 - i - 1) - 1 = s - 2i - 3$ ($3 = 2(\lceil s/2 \rceil - i - 1) - 1 = s - 2i - 2$, respectively) vertices. We may assume that both y_1 and y_2 are adjacent to x. It follows that $G_B\langle X \cup Y \rangle$ has a disjoint P_4 and a K_2. This reverts back to Configuration 1, and this will be treated in what follows.

For Configuration 1 (Configuration 2, respectively), set $X' = X - \{x, x_1, x_2\}$ ($X' = X - \{x_1, x_2, x_3, x_4\}$, respectively). Observe that $|X'| \geq s - 4$ and as $i < \lceil s/2 \rceil + 2j - 7$, we have that $|X'| - 1 + 2j > |X'| - 2 + 2j \geq s - 6 + 2j > \lfloor s/2 \rfloor + (i+1)$. If s is even (odd, respectively), then each vertex in X' has at least $(i+1) - j + 1$ ($(i+1) - j$, respectively) neighbors in W_2. We apply Claim 33. Suppose the first part of Claim 33 holds. If s is even (odd, respectively), then $|Z_1| = (i+1) - j + 2 = i - j + 3$ ($|Z_1| = (i+1) - j + 1 = i - j + 2$, respectively). We apply Part 1 of Lemma 6. For Configuration 1 (Configuration 2, respectively), set $S = X - \{x_3\} - V(Z_1)$ ($S = X - V(Z_1)$, respectively), $T = W_1$ and $k = |Z_1|$. Note that as $i = \lceil s/2 \rceil - 3$, $|S| + k \geq s - 1 - 2k + k \geq \lceil s/2 \rceil + j - 1 = |T|$. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 > i - j + 4 \geq k + 1$. If s is even (odd, respectively), there exists an $x_1 - x_2$ path P' on $2|T| - 1 + 2k = 2(s/2 + j - 1) - 1 + 2(i - j + 3) = s + 2i + 3$ ($2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 2) = s + 2i + 2$, respectively) vertices. The sequence P, P' forms a monochromatic C_{2s}.

Suppose the second part of Claim 33 holds. Then if s is even (odd, respectively) $|Z_1| = (i+1) - j + 1$ ($|Z_1| = (i+1) - j$, respectively) and there are two vertices $u', v' \in X' - V(Z_1)$ that are adjacent to every vertex in W_1. We apply Part 4 of Lemma 6. For Configuration 1 (Configuration 2, respectively), set $S = X - \{x_3\} - V(Z_1)$ ($S = X - V(Z_1)$, respectively), $T = W_1$ and $k = |Z_1|$. Note again that $|S| + k - 1 \geq s - 1 - 2k + k - 1 \geq \lceil s/2 \rceil + j - 1 = |T|$. In addition, as $i < \lceil s/2 \rceil + 2j - 7$, $|T| = \lceil s/2 \rceil + j - 1 > i - j + 6 > k + 3$. If s is even (odd, respectively), then there exists an $x_1 - x_2$ path P' on $2|T| + 1 + 2k = 2(s/2 + j - 1) + 1 + 2(i - j + 2) = s + 2i + 3$ ($2|T| + 1 + 2k = 2(\lceil s/2 \rceil + j - 1) + 1 + 2(i - j + 1) = s + 2i + 2$, respectively) vertices. The sequence P', P forms a monochromatic C_{2s}.

We may assume that for Configuration 1 (Configuration 2, respectively), $x \in X - \{x_1, x_2, x_3\}$ ($x \in X - \{x_1, x_2, x_3, x_4\}$, respectively).

Case A. Configuration 1 holds. We apply Lemma 16 and conclude that every vertex in Y is adjacent to either the two vertices in U_1, or to x. This implies that y_2 and y_3 have a common neighbor $x' \in \{x\} \cup U_1$. If s is even (odd, respectively), there exists a $y_3 - y_2$ path P on $3 = 2|U_1| - 1 = 2(s/2 - i - 1) - 1 = s - 2i - 3$ ($3 = 2(\lceil s/2 \rceil - i - 1) - 1 = s - 2i - 2$, respectively) vertices. Let $X' = X - \{x_1, x_2, x_3, x\}$. Observe that $|X'| = s - 4$ and as $i < \lceil s/2 \rceil + 2j - 7$, we have that $|X'| - 1 + 2j > |X'| - 2 + 2j > \lfloor s/2 \rfloor + i$.

We apply Claim 33. Suppose the first part of Claim 33 holds. If s is even (odd, respectively), then $|Z_1| = i - j + 2$ ($|Z_1| = i - j + 1$, respectively). We apply Part

1 of Lemma 6. Set $S = X - \{x_2, x\} - V(Z_1)$, $T = W_1$ and $k = |Z_1|$. Note that, as $i = \lceil s/2 \rceil - 3$, $|S| + k = s - 2 - 2k + k = s - 2 - k \geq |T| = \lceil s/2 \rceil + j - 1$. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 > i - j + 5 > k + 1$. If s is even (odd, respectively), there exists an $x_1 - x_3$ path P' on $2|T| - 1 + 2k = 2(s/2 + j - 1) - 1 + 2(i - j + 2) = s + 2i + 1$ ($2|T| - 1 + 2k \geq 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 1) = s + 2i$, respectively) vertices. The sequence P', P, x_2, y_1 forms a monochromatic C_{2s}. Suppose the second part of Claim 33 holds. Then if s is even (odd, respectively), $|Z_1| = i - j + 1$ ($|Z_1| = i - j + 1$ if $i - j = -1$ and $|Z_1| = i - j$ if $i \geq j$, respectively) and there are two vertices $u', v' \in X' - V(Z_1)$ that are adjacent to every vertex in W_1. We apply Part 4 (or Part 1) of Lemma 6. Set $S = X - \{x_2, x\} - V(Z_1)$, $T = W_1$ and $k = |Z_1|$. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 > i - j + 5 > k + 2$. Suppose s is even. Note that $|S| + k - 1 = s - 2 - 2k + k - 1 \geq s - 3 - (i - j + 1) \geq \lceil s/2 \rceil + j - 1 = |T|$. By Part 4 of Lemma 6, there exists an $x_1 - x_3$ path P' on $2|T| + 1 + 2k = 2(s/2 + j - 1) + 1 + 2(i - j + 1) = s + 2i + 1$ vertices. Suppose s is odd. If $|Z_1| = i - j$ ($|Z_1| = i - j + 1$, respectively), note that $|S| + k - 1 = s - 2 - 2k + k - 1 \geq s - 3 - (i - j) \geq \lceil s/2 \rceil + j - 1 = |T|$ ($|S| + k = s - 2 - 2k + k \geq s - 2 - (i - j + 1) \geq \lceil s/2 \rceil + j - 1 = |T|$, respectively) and so, by Part 4 (part 1, respectively) of Lemma 6, there exists an $x_1 - x_3$ path P' on $2|T| + 1 + 2k = 2(\lceil s/2 \rceil + j - 1) + 1 + 2(i - j) = s + 2i$ ($2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 1) = s + 2i$, respectively) vertices. The sequence P', P, x_2, y_1 forms a monochromatic C_{2s}.

Case B. Configuration 2 holds. We apply Lemma 16 and conclude that every vertex in Y is adjacent to either the two vertices in U_1, or to x. This implies that y_2 and y_1 have a common neighbor $x' \in \{x\} \cup U_1$. If s is even (odd, respectively), there exists a $y_2 - y_1$ path P on $3 = 2|U_1| - 1 = 2(s/2 - i - 1) - 1 = s - 2i - 3$ ($3 = 2(\lceil s/2 \rceil - i - 1) - 1 = s - 2i - 2$, respectively) vertices. Let $X' = X - \{x_1, x_2, x_3, x_4, x\}$. Observe that $|X'| = s - 5$ and as $i < \lceil s/2 \rceil + 2j - 7$, we have that $|X'| - 1 + 2j > |X'| - 2 + 2j > \lfloor s/2 \rfloor + i$.

We apply Claim 33. Suppose the first part of Claim 33 holds. Add the $K_{1,2}$, with vertices x_3, y_3, x_4, to Z_1. Then if s is even (odd, respectively), $|Z_1| = i - j + 3$ ($|Z_1| = i - j + 2$, respectively). We apply Part 1 of Lemma 6. Set $S = X - \{x\} - V(Z_1)$, $T = W_1$ and $k = |Z_1|$. Note that as $i = \lceil s/2 \rceil - 3$, $|S| + k = s - 1 - 2k + k = s - k - 1 \geq |T| = \lceil s/2 \rceil + j - 1$. Observe that $|T| - 1 = \lceil s/2 \rceil + j - 2 > i - j + 5 > k + 1$. If s is even (odd, respectively), there exists an $x_1 - x_2$ path P' on $2|T| - 1 + 2k = 2(s/2 + j - 1) - 1 + 2(i - j + 3) = s + 2i + 3$ ($2|T| - 1 + 2k = 2(\lceil s/2 \rceil + j - 1) - 1 + 2(i - j + 2) = s + 2i + 2$, respectively) vertices. The sequence P', P forms a monochromatic C_{2s}.

Suppose the second part of Claim 33 holds. If s is even or s is odd and $|Z_1| = i - j$, add the $K_{1,2}$ with vertices x_3, y_3, x_4 to Z_1. Then if s is even (odd, respectively), $|Z_1| = i - j + 2$ ($|Z_1| = i - j + 1$, respectively) and there are two vertices $u', v' \in X' - V(Z_1)$ that are adjacent to every vertex in W_1. We apply Part 4 of Lemma 6. Set $S = X - \{x\} - V(Z_1)$, $T = W_1$ and $k = |Z_1|$. Note that, as $i = \lceil s/2\rceil - 3$, $|S| + k - 1 = s - 1 - 2k + k - 1 = s - k - 2 \geq |T| = \lceil s/2\rceil + j - 1$. If s is even (odd, respectively), there exists an $x_1 - x_2$ path P' on $2|T| + 1 + 2k = 2(s/2 + j - 1) + 1 + 2(i - j + 2) = s + 2i + 3$ ($2|T| + 1 + 2k = 2(\lceil s/2\rceil + j - 1) + 1 + 2(i - j + 1) = s + 2i + 2$, respectively) vertices. The sequence P', P forms a monochromatic C_{2s}.

Case 2.1.2.2. $i \geq \lceil s/2\rceil + 2j - 7$. As $i = \lceil s/2\rceil - 3 \geq \lceil s/2\rceil + 2j - 7$, we have that $j \in \{0, 1, 2\}$. We apply Claims 31, 34 and 35. By Claim 31, the graph $G_B\langle Y \cup U \cup \{x\}\rangle$ has, without loss of generality, a y_2-y_1 path $P : y_2, u, u', x'', y_1$, such that $\{x_2, x_1\} \cap V(P) = \emptyset$ and $y_1, y_2, u' \in Y$ and $u, x'' \in \{x\} \cup U$. Note that if s is even (odd, respectively) the path P has $5 = 2|U_1| + 1 = 2(s/2 - i - 1) + 1 = s - 2i - 1$ ($5 = 2|U_1| + 1 = 2(\lceil s/2\rceil - i - 1) + 1 = s - 2i$, respectively) vertices. Let $x' \in X - \{x_1, x_2\}$ such that if $x \in X - \{x_1, x_2\}$, then $x = x'$. Set $X' = X - \{x_1, x_2, x'\}$.

Suppose that s is even and that the first part of Claim 34 holds. The graph $G_B\langle X' \cup W\rangle$ has paths $P' = P_{2(i-j)+1}$ and $P'' = P_3$, both of which are disjoint, with both starting and ending in X', and with two vertices $u', v' \in X' - (P') - V(P'')$ such that u' and v' are adjacent to all vertices in W_1. We apply Part 6 of Lemma 6. Set $S = X - \{x'\} - V(P') - V(P'')$ and $T = W_1$. Set $Z_1 = \{P''\}$, with $k = |Z_1| = 1$. Observe that as $s \geq 18$, we have that $|T| = s/2 + j - 1 \geq k + 4 = 5$. We have, as $i = \lceil s/2\rceil - 3$, that $|S| + k + 1 = s - 1 - (i - j + 1) - 2 + 2 = s - i + j - 2 \geq s/2 + j - 1 + 1 = |T| + 1$. By Part 6 of Lemma 6, we have that the graph $G_B\langle X \cup W\rangle$ has an $x_1 - x_2$ path P''' on $2|T| + 2k + \ell = 2(s/2 + j - 1) + 2 + 2(i - j) + 1 = s + 2i + 1$ vertices. The sequence P''', P forms a monochromatic C_{2s}. We may assume that if s is even then the second part of Claim 34 holds.

If s is even (odd, respectively), then by Claim 34 (Claim 35, respectively), $G_B\langle X' \cup W\rangle$ has three paths $P' = P_{2(i-j)+1}$, $P'' = P_3$ and $P''' = P_3$ (has a $P' = P_5$ and a set Z_1 of $i - j - 1$ copies of $K_{1,2}$, respectively), all of which are disjoint, and with all ending and starting in X'. If s is even (odd, respectively), set $S = X - \{x'\} - V(P') - V(P'') - V(P''')$ ($S = X - \{x'\} - V(P') - V(Z_1)$, respectively) and $T = W_1$. If s is even, then let $Z_1 = \{P'', P'''\}$ and $|Z_1| = k = 2$. If s is odd then recall that $|Z_1| = i - j - 1 = k$. We apply Part 5 of Lemma 6. Observe that as $s \geq 18$, we have, for s even, that $|T| - 1 = s/2 + j - 2 \geq k + 2 = 4$. If s is odd, then, as $j \in \{0, 1, 2\}$ and $i = \lceil s/2\rceil - 3$, $|T| - 1 = \lceil s/2\rceil + j - 2 \geq i - j - 1 + 2 = k + 2$. In addition, if s is even (odd, respectively), then $|S| + k + 1 \geq s - 1 - (i - j + 1) - 2 - 2 + 2 + 1 = s - i + j - 3 \geq s/2 + j - 1 + 1 = |T| + 1$ ($|S| + k + 1 = s - 1 - 2k - 3 + k + 1 = s - 3 - (i - j - 1) = s - i + j - 2 \geq \lceil s/2\rceil + j - 1 + 1 = |T| + 1$, respectively). If s is even (odd, respectively), then the graph $G_B\langle X \cup W\rangle$ has an $x_1 - x_2$ path P'''' on $2|T| - 1 + 2k + \ell - 1 = 2(s/2 + j - 1) - 1 + 2(i - j) + 4 = s + 2i + 1$ ($2|T| - 1 + 2k + \ell - 1 = 2(\lceil s/2\rceil + j - 1) - 1 + 2(i - j - 1) + 5 - 1 = s + 2i$, respectively) vertices. The sequence P'''', P forms a monochromatic C_{2s}.

Case 2.2. $i - j \leq -2$.

Case 2.2.1. $|U_1| \geq 3$. Observe that $j \geq i + 2$. Recall that we assumed much earlier that $x \in X - \{x_1, x_2\}$. Set $S = X - \{x\}$. If s is even (odd, respectively), let T be a subset of W_1, of cardinality $s/2 + i + 2 - 1$ ($\lceil s/2 \rceil + i + 1 - 1$, respectively). Observe that as every vertex in X has at least $|W_1| - 1$ neighbors in W_1, we have that every vertex in X has $|T| - 1$ neighbors in T. If s is even (odd, respectively), then, by Part 1 of Lemma 6 ($|S| \geq |T|$, as $i \leq \lceil s/2 \rceil - 3$), there exists an $x_1 - x_2$ path P, that alternates between S and W, on $2|T| - 1 = 2(s/2 + i + 2 - 1) - 1 = s + 2i + 1$ ($2|T| - 1 = 2(\lceil s/2 \rceil + i + 1 - 1) - 1 = s + 2i$, respectively) vertices.

We apply Lemma 23. Pick two one vertex paths P' and P'' in Y. Set $S = Y - V(P') - V(P'')$, $T = U_1$, $S_1 = Y \cap V(P')$ and $S_2 = Y \cap V(P'')$. Set $x = z$, $u = y_2$ and $v = y_1$. Observe that every vertex $w \in S \cup S_1 \cup S_2$ is adjacent to $|T| - 1$ vertices in T, and is either adjacent to z, or adjacent to $|T|$ vertices in T. If s is even (odd, respectively), then there exists a $y_2 - y_1$ path P''' on $2|T| - 1 + 1 + 1 = 2(s/2 - i - 1) - 1 + 1 + 1 = s - 2i - 1$ ($2|T| - 1 + 1 + 1 = 2(\lceil s/2 \rceil - i - 1) + 1 = s - 2i$, respectively). The sequence P, P''' forms a monochromatic C_{2s}.

Case 2.2.2. $|U_1| \leq 2$. If $\deg_W(x) = t \leq s - 4$, then $|U_1| = s - 1 - t \geq 3$, a contradiction. Hence $\deg_W(x) \geq s - 3$, and as $\deg_W(x) \leq s - 3$, we have $\deg_W(x) = \lfloor s/2 \rfloor + i = s - 3$, and so $i = \lceil s/2 \rceil - 3$. Recall that $j \geq i + 2$. If s is even, then $|W_1| = s/2 + j - 1 \geq s/2 + i + 2 - 1 = s/2 + s/2 - 3 + 2 - 1 = s - 2$. If $|W_1| = s - 1$, then every vertex in X has at least $|W_1| - 1 = s - 2$ neighbors in W. This contradicts the fact that $\deg_W(x) = s - 3$, whence $|W_1| = s - 2$. If s is odd, then $|W_1| = \lceil s/2 \rceil + j - 1 \geq \lceil s/2 \rceil + i + 2 - 1 = \lceil s/2 \rceil + \lceil s/2 \rceil - 3 + 2 - 1 = s - 1$. This implies that every vertex in X has at least $|W_1| - 1 \geq s - 2$ neighbors in W. This contradicts the fact that $\deg_W(x) = s - 3$, whence s is even.

We claim that for any two vertices $y', y'' \in Y$, there exists, within G_B, a $y' - y''$ path P, that alternates between $Y \cup \{x\} \cup U$, on $s - 2i - 3$ vertices. Now each vertex in Y is adjacent to either x, or to the two vertices in U_1. Then both y', y'' have a common neighbor $x''' \in \{x\} \cup U_1$. Thus, $P : y', x''', y''$ is a path on $3 = 2|U_1| - 1 = 2(s/2 - i - 1) - 1 = s - 2i - 3$ vertices. Let T be a subset of W_1 of cardinality $s/2 + i + 2 - 1$. Let us assume first that for Configuration 1 (Configuration 2, respectively), $x \in X - \{x_1, x_2, x_3\}$ ($x \in X - \{x_1, x_2, x_3, x_4\}$, respectively).

Let us deal with Configuration 1 first. Set $y' = y_3$, $y'' = y_2$ and $S = X - \{x, x_2\}$. By Part 1 of Lemma 6, there exists an $x_1 - x_3$ path P', that alternates between X and W, on $2|T| - 1 = 2(s/2 + i + 2 - 1) - 1 = s + 2i + 1$ vertices. The sequence P', P, x_2, y_1 forms a monochromatic C_{2s}. Let us deal with Configuration 2. Let $y' = y_2$ and $y'' = y_1$. Now set $S = X - \{x, x_3, x_4\}$ and let Z_1 be the set containing the $K_{1,2}$ with vertices x_3, x_4, y_3. Observe that $k = |Z_1| = 1$. Then $|S| + k = s - 3 + 1 = s - 2 \geq s/2 + i + 2 - 1 = s/2 + s/2 - 3 + 2 - 1 = s - 2 = |T|$. By Part 1 of Lemma 6, there exists an $x_1 - x_2$ path P', that alternates between X and W, on $2\,|T| - 1 + 2k = 2(s/2 + i + 2 - 1) - 1 + 2 = s + 2i + 3$ vertices. The sequence P', P forms a monochromatic C_{2s}. Let us assume now that for Configuration 1 (Configuration 2, respectively), $x \in \{x_1, x_2, x_3\}$ ($x \in \{x_1, x_2, x_3, x_4\}$, respectively).

This implies that for every vertex $x' \in X - \{x_1, x_2, x_3\}$ ($x' \in X - \{x_1, x_2, x_3, x_4\}$, respectively), we have that $\deg_W(x') \geq s - 2 > s - 3 = \deg_W(x)$, since otherwise we can relabel x' as x. It follows that X has at least $|X| - 4 = s - 4 \geq 18 - 4 = 14$ vertices that have at least $s - 2$ neighbors in W. Let $y' = y_2$ and $y'' = y_1$. By the pigheonhole principle, the set $X - \{x_1, x_2, x\}$ has at least $14 - 3 = 11$ vertices (let X' be the set of these 11 vertices) that have at least $s - 2$ neighbors in W.

Suppose that there are two vertices $x', x'' \in X'$ that have $s - 2 = |W_1|$ neighbors in W_1. Set $S = X - \{x\}$ and $T = W_1$. By Part 4 of Lemma 6, there exists an $x_1 - x_2$ path P', that alternates between X and W, on $2\,|T| + 1 = 2s - 3$ vertices. The sequence P', P forms a monochromatic C_{2s}. We may assume that at most one vertex in X' has $s - 2 = |W_1|$ neighbors in W_1. It follows that two vertices $x', x'' \in X'$ have $s-3$ neighbors in W_1 and one neighbor in W_2 ($|W_2| = 1$). For $w \in W_2$, the vertices x', w, x'' form a $K_{1,2}$, say K. Set $S = X - \{x\} - V(K)$, $T = W_1$ and let $Z_1 = \{K\}$ (where $k = 1$). By Part 1 of Lemma 6, there exists an $x_1 - x_2$ path P', that alternates between X and W, on $2\,|T| - 1 + 2k = 2s - 3$ vertices. The sequence P', P forms a monochromatic C_{2s}.

Case 3. There exist integers $i, j \geq 1$ such that $t = \lfloor s/2 \rfloor - i$ and $t' = \lfloor s/2 \rfloor - j$. Observe that $|W_1| = s - 1 - t' = \lceil s/2 \rceil + j - 1$ and $|U_1| = s - 1 - t = \lceil s/2 \rceil + i - 1$. Assume first that s is odd. Let $S = X$ and let T be a subset of W_1 of cardinality $\lceil s/2 \rceil$. Observe that every vertex in X has $|T| - 1$ neighbors in T. By Part 1 of Lemma 6, there exists an $x_1 - x_2$ path P, that alternates between S and T, on $2\,|T| - 1 = s$ vertices. Likewise, set $S = Y$ and let T be a subset of U_1 of cardinality $\lceil s/2 \rceil$. By Part 1 of Lemma 6, there exists a $y_2 - y_1$ path P' on $2\,|T| - 1 = s$ vertices. The sequence P, P' forms a monochromatic C_{2s}. We may assume that s is even.

For Configuration 1 (Configuration 2, respectively), let $S = X - \{x_2\}$ ($S = X - \{x_3, x_4\}$, respectively) and let T be a subset of W_1 of cardinality $s/2$, and for Configuration 2 let Z_1 be the set containing the $K_{1,2}$ with vertices x_3, y_3, x_4. For Configuration 1 (Configuration 2, respectively), by Part 1 of Lemma 6, there exists an $x_1 - x_3$ ($x_1 - x_2$, respectively) path P, that alternates between X and W, on $2\,|T| - 1 = s - 1$ ($2\,|T| - 1 + 2k = s - 1 + 2$, respectively) vertices. Now, for Configuration 1 (Configuration 2, respectively), let $S = Y - \{y_1\}$ ($S = Y - \{y_3\}$, respectively) and let T be a subset of U_1 of cardinality $s/2$. By Part 1 of Lemma 6, there exists a $y_3 - y_2$ ($y_2 - y_1$, respectively) path P' on $2\,|T| - 1 = s - 1$ vertices. For Configuration 1 (Configuration 2, respectively), the sequence P, P', x_2, y_1 (P, P', respectively) forms a monochromatic C_{2s}. ∎

6. APPENDIX

Proof of Parts 2 to 7 of Lemma 6.

***Proof of Part* 2.** Pick the vertices $s_1, s_2, \ldots, s_{|T|-3} \in S - \{u, v\}$, and recall that any two vertices in S have $|T|-2$ common neighbors in T. Let $t_1, t_2, \ldots, t_{|T|-2} \in T$ be a sequence of vertices such that $t_1 \in N(u) \cap N(s_1), t_2 \in N(s_1) \cap N(s_2) - \{t_1\}, t_3 \in N(s_2) \cap N(s_3) - \{t_1, t_2\}, \ldots, t_{|T|-2} \in N(s_{|T|-3}) \cap N(v) - \{t_1, t_2, \ldots, t_{|T|-3}\}$. Let $y \in N(u) - \{t_1, t_2, \ldots, t_{|T|-2}\}$ and $x \in N(v) - \{t_1, t_2, \ldots, t_{|T|-2}\}$.

If u is adjacent to x, then $C : u, t_1, s_1, t_2, s_2, \ldots, s_{|T|-3}, t_{|T|-2}, v, x, u$ is a cycle on $2\,|T|-2$ vertices. Hence, u is not adjacent to x and, by symmetry, v is not adjacent to y. By the pigeonhole principle, both u and v are adjacent to t_1 and t_2. If s_1 is adjacent to x then $C : u, t_1, s_1, x, v, t_{|T|-2}, s_{|T|-3}, \ldots, s_2, t_2, u$ is a cycle on $2\,|T|-2$ vertices. If s_1 is adjacent to y then $C : u, y, s_1, t_1, v, t_{|T|-2}, s_{|T|-3}, \ldots, t_3, s_2, t_2, u$ is a cycle on $2\,|T| - 2$ vertices. ■

***Proof of Part* 3.** Let us assume that $|Z_1| = 2$ ($|Z_2| \neq \emptyset$, respectively). Recall that any two vertices in $S \cup \left(\bigcup_{i=1}^{2}\{z_i, z_i'\}\right)$ ($S \cup \{z_1, z_1', z_2'\}$, respectively) have $|T| - 2$ common neighbors in T. If $|Z_1| = 2$, then, as $|T| - 3 \geq 3$, we pick vertices $s_3, s_4, \ldots, s_{|T|-3} \in S - \{u, v\}$. If $Z_2 \neq \emptyset$, then pick vertices $s_2, s_3, \ldots, s_{|T|-3} \in S - \{u, v\}$.

Pick a sequence of vertices $t_1, t_2, \ldots, t_{|T|-2} \in T$ as follows. For the case where we deal with the set Z_1, pick $t_1 \in N(u) \cap N_T(z_1), t_2 \in N_T(z_1') \cap N_T(z_2) - \{t_1\}, t_3 \in N_T(z_2') \cap N(s_3) - \{t_1, t_2\}, \ldots, t_{|T|-2} \in N(s_{|T|-3}) \cap N(v) - \{t_1, t_2, \ldots, t_{|T|-3}\}$. For the set Z_2, pick $t_1 \in N(u) \cap N_T(z_1), t_2 \in N_T(z_2') \cap N(s_2) - \{t_1\}, t_3 \in N(s_2) \cap N(s_3) - \{t_1, t_2\}, \ldots, t_{|T|-2} \in N(s_{|T|-3}) \cap N(v) - \{t_1, t_2, \ldots, t_{|T|-3}\}$. Let $y \in N(u) - \{t_1, t_2, \ldots, t_{|T|-2}\}$ and $x \in N(v) - \{t_1, t_2, \ldots, t_{|T|-2}\}$.

If u is adjacent to x, then $C : u, t_1, z_1, z_1'', z_1', \ldots, s_{|T|-3}, t_{|T|-2}, v, x, u$ is a cycle on $2\,|T| + 2$ vertices. Hence, u is not adjacent to x and, by the same argument, v is not adjacent to y. By the pigeonhole principle, both u and v are adjacent to t_1 and t_2. Assume first that $|Z_1| = 2$. If z_1' is adjacent to x then $C : u, t_1, z_1, z_1'', z_1', x, v, t_{|T|-2}, s_{|T|-3}, \ldots, z_2'', z_2, t_2, u$ is a cycle on $2\,|T|+2$ vertices. If z_1' is adjacent to y then $C : u, y, z_1', z_1'', z_1, t_1, v, t_{|T|-2}, s_{|T|-3}, \ldots, z_2'', z_2, t_2, u$ is a cycle on $2\,|T| + 2$ vertices.

In the case of the set Z_2, we have that if z_2' is adjacent to x, then $C : z_2', x, v, t_{|T|-2}, s_{|T|-3}, t_{|T|-3}, \ldots, s_3, t_3, s_2, t_2, u, t_1, z_1, z_1'', z_1', z_2'', z_2'$ is a cycle on $2\,|T|+2$ vertices. If z_2' is adjacent to y, then $C : z_2', z_2'', z_1', z_1'', z_1, t_1, v, t_{|T|-2}, s_{|T|-3}, t_{|T|-3}, \ldots, s_3, t_3, s_2, t_2, u, y, z_2'$ is a cycle on $2\,|T| + 2$ vertices. ■

***Proof of Part* 4.** Assume that $|T| \geq k + 3$ and that u' and v' both have $|T|$ neighbors in T. Recall that $|S| + k - 1 \geq |T|$, and that any two vertices in $S' = S \cup \left(\bigcup_{i=1}^{k}\{z_i, z_i'\}\right)$ have $|T| - 2$ common neighbors in T. If $|T| > k + 3$, pick vertices $s_{k+1}, s_{k+2}, \ldots, s_{|T|-3}, u', v', v \in S - \{u\}$. If $|T| = k + 3$ label z_k' as $s_{|T|-3}$.

We pick a sequence of vertices $t_1, t_2, \ldots, t_{|T|-2}, t_{|T|-1} \in T$ as follows. If $k \geq 1$, then pick $t_1 \in N(u) \cap N_T(z_1), t_2 \in N_T(z_1') \cap N_T(z_2) - \{t_1\}, t_3 \in N_T(z_2') \cap N_T(z_3) - \{t_1, t_2\}, \ldots, t_{k+1} \in N_T(z_k') \cap N(s_{k+1}) - \{t_1, t_2, \ldots, t_k\}, \ldots, t_{|T|-2} \in N(s_{|T|-3}) \cap N(u') - \{t_1, t_2, \ldots, t_{|T|-3}\}, t_{|T|-1} \in N(u') \cap N(v') - \{t_1, t_2, \ldots, t_{|T|-2}\}$. If $k = 0$, then pick $t_1 \in N(u) \cap N(s_1), t_2 \in N(s_1) \cap N(s_2) - \{t_1\}, t_3 \in N(s_2) \cap N(s_3) - \{t_1, t_2\}, \ldots, t_{|T|-2} \in N(s_{|T|-3}) \cap N(u') - \{t_1, t_2, \ldots, t_{|T|-3}\}, t_{|T|-1} \in N(u') \cap N(v') - \{t_1, t_2, \ldots, t_{|T|-2}\}$. Let $z \in N(v') \cap T - \{t_1, t_2, \ldots, t_{|T|-1}\}$.

If v is adjacent to z then $P : u, t_1, \ldots, t_{|T|-2}, u', t_{|T|-1}, v', z, v$ is a $u - v$ path on $2|T| + 1 + 2k$ vertices, whence $N(v) = \{t_1, t_2, \ldots, t_{|T|-1}\}$. If $k \geq 1$ ($k = 0$, respectively) then if u is adjacent to z, then $P : u, z, v', t_{|T|-1}, u', \ldots, s_{k+1}, t_{k+1}, z_k', \ldots, z_2, t_2, z_1', z_1'', z_1, t_1, v$ ($P : u, z, v', t_{|T|-1}, u', \ldots,$ s_2, t_2, s_1, t_1, v, respectively) is a $u - v$ path on $2|T| + 1 + 2k$ vertices. It follows that $N(u) = \{t_1, t_2, \ldots, t_{|T|-1}\}$. Hence, as u' and v' both are adjacent to z, we have, if $k \geq 1$ ($k = 0$, respectively) that $P : u, t_{|T|-1}, v', z, u', t_{|T|-2}, \ldots, z_k', z_k'', z_k, \ldots, z_2', z_2'', z_2, t_2, z_1', z_1'', z_1, t_1, v$ ($P : u, t_{|T|-1}, v', z, u', t_{|T|-2}, \ldots, s_2, t_2, s_1, t_1, v$, respectively) is a $u - v$ path on $2|T| + 1 + 2k$ vertices. ■

***Proof of Part* 5.** If $\ell > 3$ we modify the path P' by deleting the vertices $w_2, \ldots, w_{\ell-2}$, and joining w_1 to $w_{\ell-1}$. Label $z_{k+1} = w_1$, $z_{k+1}'' = w_{\ell-1}$ and $z_{k+1}' = w_\ell$. Now add the $K_{1,2}$ with vertices z_{k+1}, z_{k+1}'' and z_{k+1}', to the set Z_1. As $|T| - 1 \geq (k+1) + 1$, we can apply Part 1 and deduce that there exists a $u - v$ path P on $2|T| - 1 + 2k + 2$ vertices. Since $\deg(z_{k+1}'') = 2$, the path P must contain the edges $z_{k+1}z_{k+1}''$ and $z_{k+1}''z_{k+1}'$. On the path P, delete the edge $z_{k+1}z_{k+1}''$, and insert the section of P' that comprises of vertices $w_2, w_3, \ldots, w_{\ell-2}$. A new $u - v$ path arises on $2|T| - 1 + 2k + \ell - 1$ vertices. ■

***Proof of Part* 6.** If $\ell > 3$ we modify the path P' by deleting the vertices $w_2, \ldots, w_{\ell-2}$, and joining w_1 to $w_{\ell-1}$. Label $z_{k+1} = w_1$, $z_{k+1}'' = w_{\ell-1}$ and $z_{k+1}' = w_\ell$. Now add the $K_{1,2}$ with vertices z_{k+1}, z_{k+1}'' and z_{k+1}', to the set Z_1. As $|T| - 1 \geq (k+1) + 3$, we can apply Part 4 and deduce that there exists a $u - v$ path P on $2|T| + 1 + 2k + 2$ vertices. Since $\deg(z_{k+1}'') = 2$, the path P must contain the edges $z_{k+1}z_{k+1}''$ and $z_{k+1}''z_{k+1}'$. On the path P, delete the edge $z_{k+1}z_{k+1}''$, and insert the section of P' that comprises of vertices $w_2, w_3, \ldots, w_{\ell-2}$. A new $u - v$ path arises on $2|T| + 2k + \ell$ vertices. ■

***Proof of Part* 7.** Observe that any two vertices in $S \cup \{z_1, z_1'\}$ have at least $|T| - 4$ common neighbors in T. Pick a sequence of vertices $s_{k+1}, s_{k+2}, \ldots, s_{|T|-4} \in S - \{u, v\}$. If $k = 1$ pick vertices in T as follows: $t_1 \in N(u) \cap N_T(z_1), t_2 \in N_T(z_1') \cap N(s_2) - \{t_1\}, \ldots, t_{|T|-4} \in N(s_{|T|-5}) \cap N(s_{|T|-4}) - \{t_1, t_2, \ldots, t_{|T|-5}\}$. For the case where $k = 0$, pick $t_1 \in N(u) \cap N(s_1), t_2 \in N(s_1) \cap N(s_2) - \{t_1\}, \ldots, t_{|T|-4} \in N(s_{|T|-5}) \cap N(s_{|T|-4}) - \{t_1, t_2, \ldots, t_{|T|-5}\}$. Let $x_1, x_2 \in N(s_{|T|-4}) - \{t_1, t_2, \ldots, t_{|T|-4}\}$ and $y_1, y_2 \in N(v) - \{t_1, t_2, \ldots, t_{|T|-4}\}$.

If $x_1 = y_1$ then $P : u, t_1, \dots, s_{|T|-5}, t_{|T|-4}, s_{|T|-4}, x_1, v$ is a path on $2|T| - 5 + 2k$ vertices. Hence, $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. By the pigeonhole principle $N(v) \supseteq \{y_1, y_2, t_1, t_2, \dots, t_{|T|-4}\}$ and $N(s_{|T|-4}) \supseteq \{x_1, x_2, t_1, t_2, \dots, t_{|T|-4}\}$. Suppose u is adjacent to say x_1. If $k = 1$ ($k = 0$, respectively), the path P : $u, x_1, s_{|T|-4}, t_{|T|-4}, \dots, s_2, t_2, z_1', z_1'', z_1, t_1, v$ ($P : u, x_1, s_{|T|-4}, t_{|T|-4}, \dots, s_2, t_2, s_1, t_1, v$, respectively) is a path on $2|T| - 5 + 2k$ vertices. It follows that $N(u) \supseteq \{y_1, y_2, t_1, t_2, \dots, t_{|T|-4}\}$. If $k = 1$ ($k = 0$, respectively), and z_1' (s_1, respectively) is adjacent to say x_1, then $P : u, t_1, z_1, z_1'', z_1', x_1, s_{|T|-4}, t_{|T|-4}, \dots, s_2, t_2, v$ (P : $u, t_1, s_1, x_1, s_{|T|-4}, t_{|T|-4}, \dots, t_3, s_2, t_2, v$, respectively) is a path on $2|T| - 5 + 2k$ vertices, whence $N_T(z_1') \supseteq \{y_1, y_2, t_1, t_2, \dots, t_{|T|-4}\}$ ($N(s_1) \supseteq \{y_1, y_2, t_1, t_2, \dots, t_{|T|-4}\}$, respectively). It follows immediately that for $k = 1$ ($k = 0$, respectively), $P : u, y_1, z_1', z_1'', z_1, t_1, s_{|T|-4}, t_{|T|-4}, \dots, s_2, t_2, v$ ($P : u, y_1, s_1, t_2, s_2, \dots, t_{|T|-4}, s_{|T|-4}, t_1, v$, respectively) is a path with the desired property. ■

Proof of the mentioned inequalities in Claim 26. We first prove the inequality $(s-3)(i+j) - (i+j-2)(s-3+s/2+j-2(i+j-2)) > 0$ when s is even. Let $h(i) = (s-3)(i+j) - (i+j-2)(s-3+s/2+j-2(i+j-2))$. Observe that $h(i) = h(i) + (i+j) - (i+j-2) - 2$. We will show that $g(i) = h(i) + (i+j) - (i+j-2) > 3$, implying that $h(i) > 0$. Note that $g(i) = (s-2)(i+j) - (i+j-2)(s-2+s/2+j-2(i+j-2))$ simplifies to $g(i) = s(-i/2 - j/2 + 3) + 2 + (i+j)^2 + ij + i^2 - 6j - 8i + 2$. Note that $g(i)$ is a parabola with minimum value occurring at $i = s/8 - 3j/4 + 2$.

Assume first that $s/8 - 3j/4 + 2 \le s/4 - 2 \le i$. As $g(i)$ is increasing on the interval $[s/8 - 3j/4 + 2, +\infty)$, we have, as $s \ge 18$, that $g(i) \ge g(s/4 - 2) = j^2 + j(s/4 - 12) + 28 \ge j^2 - 8j + 26 > 3 > 0$, for $j \ge 1$. We may assume that $s/8 - 3j/4 + 2 > s/4 - 2$. This implies that $1 \le j \le 2$. Observe that $g(s/8 - 3j/4 + 2) = s(-s/32 - j/8 + 2) - 4 - j^2/8$. If $j = 1$, then $18 \le s \le 25$, and so $g(i) \ge g(s/8 - 3j/4 + 2) = s(-s/32 - 1/8 + 2) - 4 - 1/8 > 3 > 0$. If $j = 2$, then $18 \le s \le 19$, and so $g(i) \ge g(s/8 - 3j/4 + 2) = s(-s/32 - 2/8 + 2) - 4 - 4/8 > 3 > 0$.

We now prove the inequality $(s-3)(i+j-1) - (i+j-4)(s-3+s/2+j-2(i+j-4)) > 0$ when s is odd. Let $h(i) = (s-3)(i+j-1) - (i+j-4)(s-3+s/2+j-2(i+j-4))$. Simplifying, we obtain $h(i) = s(-i/2 - j/2 + 5) + (i+j)^2 + i^2 + ij - 16i - 12j + 23$. Note that $h(i)$ is a parabola with minimum value occurring at $i = s/8 - 3j/4 + 4$. Assume first that $s/4 - 2 \ge s/8 - 3j/4 + 4$. Since $h(i)$ is increasing on the interval $[s/8 - 3j/4 + 4, +\infty)$, we have, as $j \ge 1$, that $h(i) \ge h(s/4 - 2) = j^2 - 18j + sj/4 + 63 \ge j^2 - 14j + 63 > 0$. We may assume that $s/8 - 3j/4 + 4 > s/4 - 2$, and so, as $s \ge 19$, we have that $1 \le j \le 4$, and $s \le 41$. Observe that $h(s/8 - 3j/4 + 4) = s(-s/32 + 3 - j/8) - j^2/8 - 9 \ge -s^2/32 + 3s - 4s/8 - 16/8 - 9 > 0$ (using calculus). ■

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