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THE TRIANGLE-FREE GRAPHS THAT ARE COMPETITION GRAPHS OF MULTIPARTITE TOURNAMENTS

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Abstract

In this paper, we discover all the triangle-free graphs that are competition graphs of multipartite tournaments.

Keywords: competition graph, triangle-free graph, multipartite tournament.

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1. Introduction

Given a digraph D, $N^+_D(x)$ and $N^-_D(x)$ denote the sets of out-neighbors and in-neighbors, respectively, of a vertex x in D. The nonnegative integers $|N^+_D(x)|$ and $|N^-_D(x)|$ are called the *outdegree* and the *indegree*, respectively, of x and denoted by $d^+_D(x)$ and $d^-_D(x)$, respectively. When no confusion is likely, we omit D in $N^+_D(x)$, $N^-_D(x)$, $d^+_D(x)$, and $d^-_D(x)$, to just write $N^+(x)$, $N^-(x)$, $d^+(x)$, and $d^-(x)$, respectively.

The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph G defined by $V(G) = V(D)$ and $E(G) = \{uv \mid u, v \in V(D), u \neq v, N^+_D(u) \cap N^+_D(v) \neq \emptyset\}$. Competition graphs arose in connection with an application in ecology (see [4]) and also have applications in coding, radio transmission, and modeling of complex economic systems. Early literature of the study on competition graphs is summarized in the survey papers by Kim [11] and Lundgren [13].

For a digraph D, the *underlying graph* of D is the graph G such that $V(G) = V(D)$ and $E(G) = \{uv \mid (u,v) \in A(D)\}$. An *orientation* of a graph G is a digraph having no directed 2-cycles, no loops, and no multiple arcs whose underlying graph is G. A *tournament* is an orientation of a complete graph. A *k-partite tournament* is an orientation of a complete k-partite graph for some positive integer $k \geq 2$.

The competition graphs of tournaments and those of bipartite tournaments have been actively studied (see [1,2,5–9], and [12] for papers related to this topic).

Recently, the authors of this paper began to study competition graphs of k-partite tournaments for $k \geq 2$ and figured out the sizes of partite sets of multipartite tournaments whose competition graphs are complete [3].

In this paper, following up those results, we study triangle-free graphs which are competition graphs of multipartite tournaments. We show that a connected triangle-free graph is the competition graph of a k-partite tournament if and only if $k \in \{3,4,5\}$, and list all the connected triangle-free graphs which are competition graphs of multipartite tournaments (Theorem 2.19).

We also show that a disconnected triangle-free graph is the competition graph of a k-partite tournament if and only if $k \in \{2,3,4\}$, and list all the disconnected triangle-free graphs which are competition graphs of multipartite tournaments (Theorems 3.4, 3.9, and 3.10).

2. The Connected Triangle-Free Competition Graphs of Multipartite Tournaments

Lemma 2.1. *Let D be an orientation of K_{n_1,n_2,n_3} whose competition graph has no isolated vertex for some positive integers n_1, n_2, and n_3. Then at least two of n_1, n_2, and n_3 are greater than* 1.

Proof. Suppose, to the contrary, that at most one of n_1, n_2, and n_3 is greater than 1, that is, at least two of n_1, n_2, and n_3 equal 1. Without loss of generality, we may assume that $n_1 = n_2 = 1$. Let $\{u\}, \{v\}$, and V be the partite sets of D with $|V| = n_3$. Since D is an orientation of K_{n_1,n_2,n_3}, either $(u,v) \in A(D)$ or $(v,u) \in A(D)$. By symmetry, we may assume that $(u,v) \in A(D)$. Since $C(D)$ has no isolated vertex, v is adjacent to some vertex. Since $(u,v) \in A(D)$, v is not adjacent to any vertex in V. Thus u and v are adjacent in $C(D)$ and so u and v have a common out-neighbor w in V. Then neither u nor v is an out-neighbor of w and so $N^+(w) = \emptyset$. Therefore w is isolated in $C(D)$, which is a contradiction. ■

Lemma 2.2. *Let D be a digraph with n vertices for a positive integer n. If the competition graph $C(D)$ of D is triangle-free, then $|E(C(D))| \leq |A(D)|/2 \leq |V(D)|$.*

Proof. Suppose that the competition graph $C(D)$ of D is triangle-free. Then $d^-(v) \leq 2$ for each vertex v in D. Therefore

$$|E(C(D))| \leq |\{v \in V(D) \mid d^-(v) = 2\}| \leq \frac{|A(D)|}{2} \leq |V(D)|. \qquad \blacksquare$$

Lemma 2.3. *There is no orientation of $K_{4,2,1}$ whose competition graph is triangle-free.*

Proof. Suppose, to the contrary, that there exists an orientation D of $K_{4,2,1}$ whose competition graph is triangle-free. Then $|A(D)| = 14$. Since $C(D)$ is triangle-free, each vertex has indegree at most 2. Then, since $|V(D)| = 7$ and $|A(D)| = 14$,

$$d^-(v) = 2 \tag{1}$$

for each vertex v in D. Let $V_1 = \{x_1, x_2, x_3, x_4\}$, $V_2 = \{y_1, y_2\}$, and $V_3 = \{z\}$ be the partite sets of D. By (1), each vertex in V_1 is a common out-neighbor of two vertices in $V_2 \cup V_3$ and so has outdegree 1. We note that if a vertex a in $V_2 \cup V_3$ is an out-neighbor of a vertex b in V_1, then b is a common out-neighbor of the two vertices in $V_2 \cup V_3 \setminus \{a\}$ and so they are adjacent in $C(D)$. Therefore there must be a vertex in $V_2 \cup V_3$ which is not an out-neighbor of any vertex in V_1 to prevent from creating a triangle y_1y_2z in $C(D)$. By (1), such a vertex in $V_2 \cup V_3$ must be z and

$$N^-(z) = \{y_1, y_2\}.$$

Then $N^+(z) = V_1$. By (1) again, each of y_1 and y_2 is a common out-neighbor of two vertices in V_1. Since each vertex in V_1 has outdegree 1, $N^-(y_1) \cap N^-(y_2) = \emptyset$. Without loss of generality, we may assume $N^-(y_1) = \{x_1, x_2\}$ and $N^-(y_2) = \{x_3, x_4\}$. Then $N^+(x_1) = N^+(x_2) = \{y_1\}$ and $N^+(x_3) = N^+(x_4) = \{y_2\}$, so

$$N^-(x_1) = \{y_2, z\} \text{ and } N^-(x_3) = \{y_1, z\}.$$

Hence $\{y_1, y_2, z\}$ forms a triangle in $C(D)$, which is a contradiction. $\blacksquare$

Lemma 2.4. *Let n_1, n_2, and n_3 be positive integers such that $n_1 \geq n_2 \geq n_3$. Suppose that there exists an orientation D of K_{n_1,n_2,n_3} whose competition graph $C(D)$ is triangle-free. Then one of the following holds.*

(a) *$n_1 = n_2 = n_3 = 2$;*

(b) *$n_1 \leq 3$, $n_2 = 2$, and $n_3 = 1$;*

(c) $n_2 = n_3 = 1$.

In particular, if $C(D)$ is connected, then the case (c) *does not occur.*

Proof. It is easy to check that $|A(D)| = n_1n_2 + n_2n_3 + n_3n_1$. Then, by Lemma 2.2,

$$n_1n_2 + n_2n_3 + n_3n_1 \leq 2(n_1 + n_2 + n_3).$$

Thus

$$n_1(n_2 - 2) + n_2(n_3 - 2) + n_3(n_1 - 2) \leq 0 \tag{2}$$

and so at least one of n_1-2, n_2-2, and n_3-2 is nonpositive. Since $n_1 \geq n_2 \geq n_3$, $n_3-2 \leq 0$. Suppose $n_3 = 2$. Then, by (2), $n_1n_2 \leq 4$ and so $(n_1, n_2, n_3) = (2, 2, 2)$. Now we suppose $n_3 = 1$. Then, by (2), $(n_1 - 1)(n_2 - 1) \leq 3$. Since $n_1 \geq n_2$, $n_2 \leq 2$. Suppose $n_2 = 2$. Then $n_1 \leq 4$. If $n_1 = 4$, then $(n_1, n_2, n_3) = (4, 2, 1)$, which contradicts Lemma 2.3. Therefore $n_1 \leq 3$ and so (b) holds. If $n_2 = 1$, then $n_3 = 1$ and so (c) holds. If $C(D)$ is connected, then $C(D)$ has no isolated vertices and so none of n_1 and n_2 equals 1 by Lemma 2.1 and so the "in particular" part is true. ■

The following lemma is an immediate consequence of Lemma 2.4.

Lemma 2.5. *For a connected triangle-free graph G of order n, if G is the competition graph of a tripartite tournament, then $n \in \{5, 6\}$.*

Lemma 2.6. *For a positive integer $n \geq 3$, a cycle C_n of length n is the competition graph of a tripartite tournament if and only if $n = 6$.*

Proof. Let D be the digraph in Figure 1 which is an orientation of $K_{2,2,2}$. It is easy to check that $C(D) \cong C_6$. Therefore the "if" part is true.

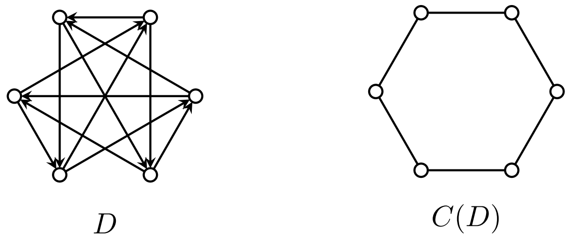


Figure 1. A digraph D which is an orientation of $K_{2,2,2}$ and whose competition graph is isomorphic to C_6.

Now suppose that a cycle C_n is the competition graph of a tripartite tournament T for a positive integer $n \geq 3$. If $n = 3$, then the only possible size of partite sets is $(1,1,1)$ which is impossible by Lemma 2.1. Therefore $n \geq 4$. Thus $n = 5$ or $n = 6$ by Lemma 2.5. Suppose, to the contrary, that $n = 5$. Then T is an orientation of $K_{2,2,1}$ by Lemma 2.4. Since $C(T)$ is triangle-free, $d^-(v) \leq 2$ for each $v \in V(T)$. Moreover, since each edge is a maximal clique and there are five edges in $C(T)$, each vertex has indegree 2 in T. Therefore

$$8 = |A(T)| = \sum_{v \in V(T)} d^-(v) = 10$$

and we reach a contradiction. Thus $n = 6$. ■

Lemma 2.7. *For a positive integer $n \geq 3$, a path P_n of length $n - 1$ is the competition graph of a tripartite tournament if and only if $n = 6$.*

Proof. Let D be the digraph in Figure 2 which is an orientation of $K_{3,2,1}$. It is easy to check that $C(D) \cong P_6$. Therefore the "if" part is true.

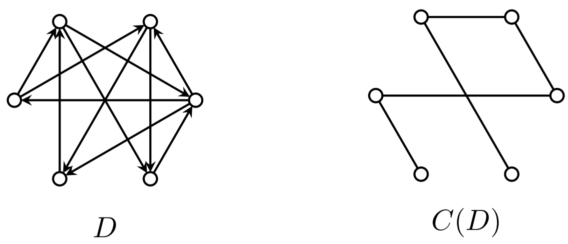


Figure 2. A digraph D which is an orientation of $K_{3,2,1}$ and whose competition graph is isomorphic to P_6.

Now suppose that a path P_n is the competition graph of a tripartite tournament T for a positive integer $n \geq 3$. Since P_n is connected and triangle-free, by Lemma 2.5, $n \in \{5, 6\}$. Suppose, to the contrary, that $n = 5$. Then T is an orientation of $K_{2,2,1}$ by Lemma 2.4. Let V_1, V_2, and V_3 be the partite sets with $|V_1| = |V_2| = 2$ and $|V_3| = 1$. Since $C(T)$ is triangle-free, $d^-(v) \leq 2$ for each $v \in V(T)$. Since there are four edges in $C(T)$, there are four vertices of indegree 2 in T. Since $|A(T)| = 8$ and $n = 5$, there exists exactly one vertex of indegree 0 in T. Let u be the vertex of indegree 0 in T. If $V_3 = \{u\}$, then $N^+(u) = V_1 \cup V_2$. Otherwise, either $N^+(u) = V_2 \cup V_3$ or $N^+(u) = V_1 \cup V_3$. Therefore $d^+(u) = 3$ or 4. Thus u is incident to at least three edges in $C(T)$, which is impossible on a path. Hence $n = 6$. ■

Lemma 2.8. *Let D be an orientation of $K_{3,2,1}$ whose competition graph is connected and triangle-free. Then the following are true.*

(1) *There is no vertex of indegree 0 in D;*

(2) *There are exactly five vertices with indegree* 2 *in* D *no two of which have the same in-neighborhood.*

Proof. Since $C(D)$ is triangle-free, $d^-(v) \leq 2$ for each $v \in V(D)$. If there is a vertex of indegree 0 in D, then $11 = \sum_{v \in V(D)} d^-(v) \leq 10$ and we reach a contradiction. Therefore the statement (1) is true and so the indegree sequence of D is $(2,2,2,2,2,1)$. Meanwhile, since $C(D)$ is connected, the number of edges of $C(D)$ is at least 5. Thus there are exactly five vertices with indegree 2 in D no two of which have the same in-neighborhood. ■

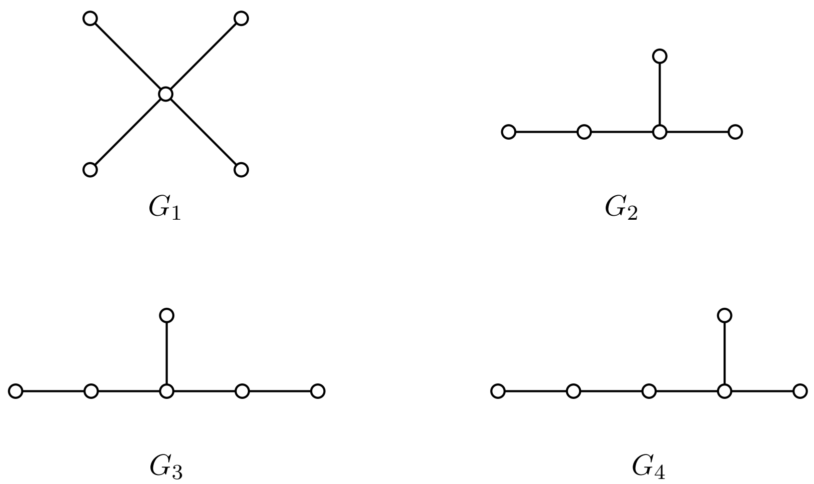


Figure 3. Connected triangle-free graphs mentioned in Lemma 2.9.

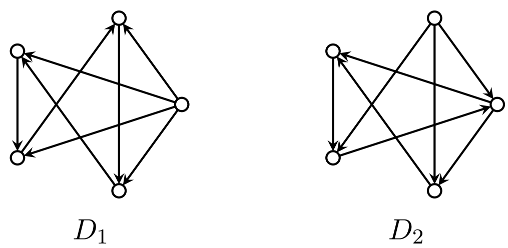


Figure 4. Two digraphs D_1 and D_2 which are orientations of $K_{2,2,1}$ and whose competition graphs are isomorphic to G_1 and G_2, respectively.

Lemma 2.9. *Let* G *be a connected and triangle-free graph with* n *vertices. Then* G *is the competition graph of a tripartite tournament if and only if* G *is isomorphic to a graph belonging to the following set*

$$\begin{cases} \{G_1, G_2\}, & \text{if } n = 5, \\ \{G_3, G_4, P_6, C_6\}, & \text{if } n = 6, \end{cases}$$

where G_i is the graph given in Figure 3 *for each* $1 \leq i \leq 4$.

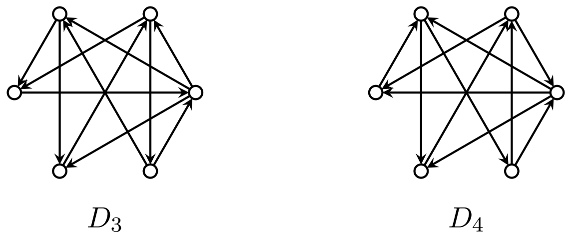


Figure 5. Two digraphs D_3 and D_4 which are orientations of $K_{3,2,1}$ and whose competition graphs are isomorphic to G_3 and G_4, respectively.

Proof. Let D be a tripartite tournament whose competition graph is G. Since G is triangle-free,

$$d^-(v) \leq 2 \tag{3}$$

for each $v \in V(D)$ and $n \in \{5, 6\}$ by Lemma 2.5. If G is a path or a cycle, then, by Lemmas 2.6 and 2.7, G is isomorphic to P_6 or C_6. Now we suppose that G is neither a path nor a cycle. Then, there exists a vertex of degree at least three in $C(D)$.

Case 1. $n = 5$. Then, by Lemma 2.4, D is an orientation of $K_{2,2,1}$. Since $|A(D)| = 8$ and $C(D)$ is connected, by (3), there are exactly four edges in $C(D)$. Therefore $C(D)$ is isomorphic to G_1 or G_2 in Figure 3. Thus the "only if" part is true in this case. To show the "if" part, let D_1 and D_2 be the digraphs in Figure 4 which are some orientations of $K_{2,2,1}$. It is easy to check that $C(D_1) \cong G_1$ and $C(D_2) \cong G_2$. Hence the "if" part is true.

Case 2. $n = 6$. Then, by Lemma 2.4, D is an orientation of $K_{3,2,1}$ or $K_{2,2,2}$. Suppose that D is an orientation of $K_{2,2,2}$. Since $\sum_{v \in V(D)} d^-(v) = 12$, by (3), $d^-(v) = 2$ for each $v \in V(D)$ and so $d^+(v) = 2$ for each $v \in V(D)$. Therefore every vertex has degree at most 2 in $C(D)$, which is a contradiction to the assumption that G is neither a path nor cycle. Thus D is an orientation of $K_{3,2,1}$. By Lemma 2.8, there are exactly five edges in $C(D)$. Let V_1, V_2, and V_3 be the partite sets of D with $|V_i| = i$ for each $i = 1, 2$, and 3.

Suppose that there is a vertex w of degree at least 4 in $C(D)$. Then, by (3), w has outdegree at least 4 in D. Thus w belongs to V_1 or V_2. If w belongs to V_2, then the indegree of w is 0, which contradicts Lemma 2.8(1). Therefore $w \in V_1$ and so $V_1 = \{w\}$. Moreover, the outdegree of w in D is 4 by Lemma 2.8(1). Then the indegree of each vertex in D except w is exactly 2 by Lemma 2.8(2). If three out-neighbors of w belong to the same partite set, then two of them share the same in-neighborhood, which contradicts Lemma 2.8(2). Therefore two of the out-neighbors of w belong to V_2 and the remaining out-neighbors belong to V_3. Since the indegree of each vertex in D except w is exactly 2 by Lemma 2.8(2), each vertex in V_2 has exactly one in-neighbor in V_3. Thus there is one vertex in V_3 which is not an in-neighbor of any vertex in V_2. Then w is the only its out-neighbor and so it is isolated in $C(D)$. Hence we have reached a contradiction and so the degree of each vertex of $C(D)$ is at most 3.

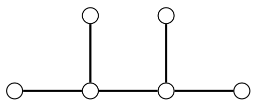


Figure 6. A graph considered in the proof of Lemma 2.9.

Now suppose that there are at least two vertices x and y of degree 3 in $C(D)$. Then, since the number of edges in $C(D)$ is exactly 5, $C(D)$ is isomorphic to the tree given in Figure 6. By (3), $d^+(x) \geq 3$ and $d^+(y) \geq 3$. If x or y belongs to V_3, then $d^-(x) = 0$ or $d^-(y) = 0$, which contradicts Lemma 2.8(1). Therefore x and y belong to V_1 or V_2. Suppose that $V_2 = \{x, y\}$. Then, since $|V_1 \cup V_3| = 4$, there are at least two vertices of indegree 2 in $V_1 \cup V_3$ which have the same in-neighborhood $\{x, y\}$, which contradicts Lemma 2.8(2). Thus one of x and y belongs to V_1 and the other belongs to V_2. Without loss of generality, we may assume that $x \in V_1$ and $y \in V_2$. Then $V_1 = \{x\}$ and, by Lemma 2.8(1), $d^-(y) \neq 0$, so $d^+(y) = 3$. If $N^+(y) = V_3$, then y is adjacent to at most two vertices in $C(D)$, which is a contradiction. Therefore $N^+(y) \cap V_3 = \{z_1, z_2\}$ for some vertices z_1 and z_2 in V_3 and $(y, x) \in A(D)$. Since $C(D)$ is isomorphic to the tree given in Figure 6, x and y have a common out-neighbor in V_3. By Lemma 2.8(2), exactly one of z_1 and z_2 can be a common out-neighbor of x and y in D. By symmetry, we may assume that z_1 is a common out-neighbor of x and y and $(z_2, x) \in A(D)$. Then $N^+(x) = \{y', z_1, z_3\}$ for the vertices y' other than y in V_2 and z_3 other than z_1 and z_2 in V_3. Therefore, by (3), $(z_1, y') \in A(D)$ and so, by (3) again, $N^+(y') = \{z_2, z_3\}$. Then y' is adjacent to y and x in $C(D)$ and so $\{x, y, y'\}$ forms a triangle in $C(D)$, which is a contradiction. Thus we have shown that there is the only one vertex of degree 3 in $C(D)$ and so $C(D)$ is isomorphic to G_3 or G_4 in Figure 3. Hence the "only if" part is true. To show the "if" part is true, let D_3 and D_4 be two digraphs given in Figure 5 which are isomorphic to some orientations of $K_{3,2,1}$. It is easy to check that $C(D_3) \cong G_3$ and $C(D_4) \cong G_4$. Hence the "if" part is true. ■

The following lemma is immediately true by the definition of the competition graph.

Lemma 2.10. *Let D be a digraph and D' be a subdigraph of D. Then the competition graph of D' is a subgraph of the competition graph of D.*

Lemma 2.11. *For a positive integer $k \geq 6$, each competition graph of a k-partite tournament contains a triangle.*

Proof. Suppose that D is a k-partite tournament for a positive integer $k \geq 6$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of D. Then we take a vertex v_i in V_i for each $1 \leq i \leq 6$. Then $\{v_1, \ldots, v_6\}$ forms a 6-tournament T. Since T has 15 arcs, there exists a vertex in T whose indegree is at least 3. Therefore $C(T)$ has a triangle and so, by Lemma 2.10, $C(D)$ contains a triangle. ■

Proposition 2.12 (Fisher *et al.* [9])**.** *For $n \geq 2$, the minimum possible number of edges in the competition graph of an n-tournament is $\binom{n}{2} - n$.*

An n-tournament is *regular* if n is odd and every vertex has outdegree $(n-1)/2$. Fisher *et al.* [10] and Cho *et al.* [1] showed that a path on four or more vertices is not the domination graph of a tournament and that the domination graph of a regular n-tournament ($n \geq 3$) is either an odd cycle or a forest of two or more paths, respectively. Here, the *domination graph* of a tournament T is the complement of the competition graph of the tournament formed by reversing the arcs of T. Accordingly, their results can be restated as follows.

Proposition 2.13 (Fisher *et al.* [10])**.** *A path on four or more vertices is not the complement of the competition graph of a tournament.*

Proposition 2.14 (Cho *et al.* [1])**.** *For a regular n-tournament ($n \geq 3$) T, the complement of the competition graph of T is either an odd cycle or a forest of two or more paths.*

Lemma 2.15. *If the competition graph $C(D)$ of a 5-partite tournament D is triangle-free, then D is a regualr 5-tournament and $C(D)$ is isomorphic to a cycle of length 5.*

Proof. Suppose that D is a 5-partite tournament whose competition graph is triangle-free. Let $V_1, \ldots, V_5$ be the partite sets of D. To show $|V(D)| = 5$ by contrary, suppose $|V(D)| \geq 6$. Then there exists a partite set whose size is at least 2. Without loss of generality, we may assume $|V_1| \geq 2$. We take v_i in V_i for each $1 \leq i \leq 5$. Then we may take a vertex v_1' distinct from v_1 in V_1 so that the subdigraph T induced by $\{v_1, v_1', v_2, \ldots, v_5\}$ is a 5-partite tournament. Since T has 14 arcs and $|V(T)| = 6$, there exists a vertex of indegree at least 3 in T, which is a contradiction. Thus $V_i = \{v_i\}$ for each $1 \leq i \leq 5$. Then D is a tournament.

Since $|V(D)| = 5$, $|E(C(D))| \geq 5$ by Proposition 2.12 and so, by Lemma 2.2, we have $|E(C(D))| = 5$. Then, since $|V(D)| = 5$, each vertex has indegree exactly 2 and so each vertex has outdegree 2. Thus D is a regular 5-tournament. Since it is easy to check that a regular 5-tournament is unique up to isomorphism as shown in Figure 7, $C(D)$ is isomorphic to a cycle of length 5.

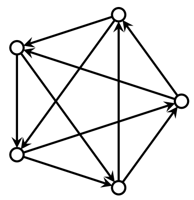


Figure 7. A regular 5-tournament.

■

Lemma 2.16. *Let D be a multipartite tournament whose competition graph is triangle-free. If two vertices u and v with outdegree at least one have the same out-neighborhood or in-neighborhood, then u and v belong to the same partite set of D and form a component in $C(D)$.*

Proof. Suppose, to the contrary, that there are two vertices u and v with outdegree at least one such that $N^+(u) = N^+(v)$ or $N^-(u) = N^-(v)$ but u and v belong to the distinct partite sets. Then $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Without loss of generality, we may assume $(u, v) \in A(D)$. Then $u \in N^-(v)$ but $u \notin N^-(u)$. Therefore $N^-(u) \neq N^-(v)$ and $N^+(u) \neq N^+(v)$, which is a contradiction. Thus u and v belong to the same partite set. Then, since D is a multipartite tournament, $N^+(u) = N^+(v)$ if and only if $N^-(u) = N^-(v)$. Therefore $N^+(u) = N^+(v) \neq \emptyset$ by the hypothesis. Since $C(D)$ is triangle-free, u and v are the only in-neighbors of each vertex in $N^+(u)$ and so they form a component in $C(D)$. ■

Lemma 2.17. *Let n_1, n_2, n_3, n_4 be positive integers such that $n_1 \geq \cdots \geq n_4$. If D is an orientation of K_{n_1,n_2,n_3,n_4} whose competition graph is triangle-free, then $n_1 \leq 2$ and $n_2 = n_3 = n_4 = 1$.*

Proof. Suppose that there exists an orientation D of K_{n_1,n_2,n_3,n_4} whose competition graph is triangle-free. Let $V_1, \ldots, V_4$ be the partite sets of D with $|V_i| = n_i$ for each $1 \leq i \leq 4$. We take a vertex v_i in V_i for each $1 \leq i \leq 4$. Suppose, to the contrary, that $n_2 \geq 2$. Then $n_1 \geq 2$. We may take a vertex v_1' (respectively, v_2') distinct from v_1 (respectively, v_2) in V_1 (respectively, V_2) so that the subdigraph induced by $\{v_1, v_1', v_2, v_2', v_3, v_4\}$ is a 4-partite tournament T. Then T has 13 arcs. Since $|V(T)| = 6$, at least one vertex of T has indegree at least 3, which is a contradiction. Thus at most one partite set of D has size at least 2. Hence $n_2 = n_3 = n_4 = 1$. Therefore $|A(D)| = 3n_1 + 3$. By Lemma 2.2, $|A(D)| \leq 2(|V(D)|) = 2(n_1 + 3)$. Thus $3n_1 + 3 \leq 2(n_1 + 3)$, so $n_1 \leq 3$.

To reach a contradiction, suppose $n_1 = 3$. Then D is an orientation of $K_{3,1,1,1}$. Since $|V(D)| = 6$ and $|A(D)| = 12$,

$$d^-(v) = 2 \tag{4}$$

for each vertex v in D. We note that $|E(C(D))| \leq 6$ by Lemma 2.2. Suppose $|E(C(D))| = 6$. Then, since $|V(D)| = 6$, each pair of vertices shares at most one common out-neighbor in D. Since $n_1 = 3$ and each vertex in V_1 has indegree 2 by (4), each vertex in V_1 is a common out-neighbor of v_i and v_j for some $i, j \in \{2, 3, 4\}$. Therefore v_i and v_j have a common out-neighbor in V_1 for each $2 \leq i \neq j \leq 4$. Thus $\{v_2, v_3, v_4\}$ forms a triangle in $C(D)$, which is a contradiction. Hence $|E(C(D))| \neq 6$ and so $|E(C(D))| \leq 5$. Then, there exists at least one pair of vertices which has two distinct common out-neighbors by (4). Since each vertex in V_1 has outdegree 1 by (4), such a pair of vertices belongs to $\{v_2, v_3, v_4\}$. Without loss of generality, we may assume $\{v_2, v_3\}$ is such a pair. Let x and y be their distinct common out-neighbors of v_2 and v_3. Then

$$N^-(x) = N^-(y) = \{v_2, v_3\}$$

by (4). Since each vertex in D has outdegree at least 1 by (4), x and y belong to the same partite set by Lemma 2.16 and so $\{x, y\} \subset V_1$. Thus $N^+(x) = N^+(y) = \{v_4\}$. Hence $\{x, y\} \subseteq N^-(v_4)$ and so, by (4), $N^-(v_4) = \{x, y\}$. Therefore $N^+(v_4) = \{v_2, v_3, z\}$ where z is a vertex in D distinct from x and y in V_1. Without loss of generality, we may assume

$$(v_3, v_2) \in A(D).$$

Then v_2 is a common out-neighbor of v_3 and v_4. Therefore by (4), $(v_2, z) \in A(D)$. Hence z is a common out-neighbor of v_2 and v_4. Thus $\{v_2, v_3, v_4\}$ forms a triangle in $C(D)$, which is a contradiction. Therefore $|V_1| \neq 3$ and so $|V_1| \leq 2$. ■

Proposition 2.18 (Kim *et al.* [12]). *Let D be an orientation of a bipartite graph with bipartition (V_1, V_2). Then the competition graph of D has no edges between the vertices in V_1 and the vertices in V_2.*

Theorem 2.19. *Let G be a connected and triangle-free graph. Then G is the competition graph of a k-partite tournament for some $k \geq 2$ if and only if $k \in \{3, 4, 5\}$ and G is isomorphic to a graph belonging to the following set*

$$\begin{cases} \{G_1, G_2, G_3, G_4, P_6, C_6\}, & \text{if } k = 3, \\ \{P_5, K_{1,3}, G_2\}, & \text{if } k = 4, \\ \{C_5\}, & \text{if } k = 5, \end{cases}$$

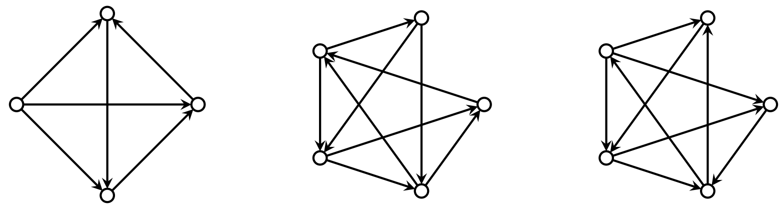
where $K_{1,3}$ is a star graph with four vertices and G_1, G_2, G_3, and G_4 are the graphs given in Figure 3.

Proof. Let D be a k-partite tournament whose competition graph is connected and triangle-free for some $k \geq 2$. Then, $k \in \{3, 4, 5\}$ by Proposition 2.18 and Lemma 2.11. If $k = 3$, then $C(D)$ is isomorphic to a graph in $\{G_1, G_2, G_3, G_4, P_6, C_6\}$ by Lemma 2.9. If $k = 5$, then $C(D)$ is isomorphic to C_5 by Lemma 2.15.

Suppose $k = 4$. Let V_1, V_2, V_3, and V_4 be the partite sets of D. Without loss of generality, we may assume $n_1 \geq n_2 \geq n_3 \geq n_4$ where $|V_i| = n_i$ for each $1 \leq i \leq 4$. Then $n_1 \leq 2$ and $n_2 = n_3 = n_4 = 1$ by Lemma 2.17.

Case 1. $n_1 = 2$. Then $|V(D)| = 5$ and $|A(D)| = 9$. Therefore $|E(C(D))| \leq 4$ by Lemma 2.2. Since $C(D)$ is connected, $|E(C(D))| \geq 4$ and so $|E(C(D))| = 4$. Therefore $C(D)$ is a tree. Thus $C(D)$ is isomorphic to a path graph, G_2, or a star graph. Suppose, to the contrary, that $C(D)$ is a star graph. Then there exists a center v in $C(D)$. Since v has degree 4 in $C(D)$, $d^+(v) \geq 4$. Then $v \in V_2 \cup V_3 \cup V_4$ and so $d^+(v) = 4$ and $d^-(v) = 0$. Since $C(D)$ is triangle-free, each vertex in D has indegree at most 2. Therefore $|A(D)| \leq 8$ and so we have reached a contradiction. Thus $C(D)$ is isomorphic to P_5 or G_2.

Case 2. $n_1 = 1$. Then $|A(D)| = 6$ and so, by Lemma 2.2, $|E(C(D))| \leq 3$. Since $C(D)$ is connected, $|E(C(D))| \geq 3$ and so $|E(C(D))| = 3$. Therefore $C(D)$ is a path graph or a star graph. If $C(D)$ is a path graph, then the complement of $C(D)$ is a path graph, which contradicts Proposition 2.13. Therefore $C(D)$ is a star graph $K_{1,3}$.



D_5 D_6 D_7

Figure 8. Three digraphs D_5, D_6, and D_7 which are orientations of $K_{1,1,1,1}$, $K_{2,1,1,1}$, and $K_{2,1,1,1}$, respectively, and whose competition graphs are isomorphic to $K_{1,3}$, P_5, and G_2, respectively.

Now we show the "if" part. The competition graph of the 5-tournament given in Figure 7 is C_5. For the 4-partite tournaments D_5, D_6, and D_7 given in Figure 8, it is easy to check that $C(D_5) \cong K_{1,3}$, $C(D_6) \cong P_5$, and $C(D_7) \cong G_2$. Each graph in $\{G_1, G_2, G_3, G_4, P_6, C_6\}$ is the competition graph of a tripartite tournament by Lemma 2.9. Hence we have shown that the "if" part is true. ■

3. The Disconnected Triangle-Free Competition Graphs of Multipartite Tournaments

3.1. Bipartite tournaments

Lemma 3.1. *Let n_1 and n_2 be positive integers such that $n_1 \geq n_2$. Suppose that there exists an orientation D of K_{n_1,n_2} whose competition graph is triangle-free. Then one of the following holds:* (a) $n_2 = 1$; (b) $n_2 = 2$; (c) $n_1 \leq 6$ *and* $n_2 = 3$; (d) $n_1 = 4$ *and* $n_2 = 4$.

Proof. It is easy to check that $|A(D)| = n_1 n_2$. Then, by Lemma 2.2, $n_1 n_2 \leq 2(n_1 + n_2)$. Thus

$$(n_1 - 2)(n_2 - 2) \leq 4. \tag{5}$$

Then it is easy to check that $n_2 \leq 4$. If $n_2 = 1$ or $n_2 = 2$, then n_1 can be any positive number satisfying the inequality $n_1 \geq n_2$. If $n_2 = 3$, then $n_1 \leq 6$. If $n_2 = 4$, then $n_1 = 4$. ■

Proposition 3.2 (Kim *et al.* [12]). *Let m and n be positive integers such that $m \geq n$. Then $P_m \cup P_n$ is the competition graph of a bipartite tournament if and only if (m, n) is one of $(1, 1)$, $(2, 1)$, $(3, 3)$, and $(4, 3)$.*

Proposition 3.3 (Kim *et al.* [12]). *Let m and n be positive integers greater than or equal to 3. Then $C_m \cup C_n$ is the competition graph of a bipartite tournament if and only if $(m, n) = (4, 4)$.*

We give a complete characterization for a triangle-free graph which is a competition graph of a bipartite tournament. We denote the set of k isolated vertices in a graph by I_k.

Theorem 3.4. *Let G be a triangle-free graph. Then G is the competition graph of a bipartite tournament if and only if G is isomorphic to one of the followings.*

(a) *An empty graph of order at least 2;*

(b) *P_2 with at least one isolated vertex;*

(c) *$P_2 \cup P_2$ with at least one isolated vertex;*

(d) *$P_3 \cup P_2$ with at least one isolated vertex;*

(e) *$P_2 \cup P_2 \cup P_2$ with at least one isolated vertex;*

(f) *$P_3 \cup I_2$;*

(g) *$P_3 \cup P_3$;*

(h) *$P_4 \cup P_3$;*

(i) *$P_3 \cup P_2 \cup P_2$;*

(j) $C_4 \cup C_4$;

(k) $P_2 \cup P_2 \cup P_2 \cup P_2$.

Proof. We first show the "only if" part. Let D be an orientation of K_{n_1,n_2} with $n_1 \geq n_2$ whose competition graph is G. Let $V_1 = \{u_1, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, \ldots, v_{n_2}\}$ be the partite sets of D. By Proposition 2.18, G is disconnected and there is no edge between the vertices in V_1 and the vertices in V_2. Since G is triangle-free, by Lemma 3.1, there are four cases to consider: $n_2 = 1$; $n_2 = 2$; $n_1 \leq 6$ and $n_2 = 3$; $n_1 = 4$ and $n_2 = 4$.

Case 1. $n_2 = 1$. Then, since each vertex in D has indegree at most 2, G is an empty graph of order at least 2 or P_2 with at least one isolated vertex.

Case 2. $n_2 = 2$. Then G has at most two edges among the vertices in V_1. We denote by H the subgraph obtained from the subgraph $G[V_1]$ induced by V_1 by removing isolated vertices in it, if any. Then H is isomorphic to P_2 or P_3 or $P_2 \cup P_2$. Suppose $n_1 \geq 5$. Then, since each vertex in V_2 has indegree at most 2, each vertex in V_2 has outdegree at least $n_1 - 2$. Therefore v_1 and v_2 have a common out-neighbor u_i in D for some $i \in \{1, \ldots, n_1\}$. Thus v_1 and v_2 are adjacent and u_i is isolated in G. Hence G is isomorphic to $P_2 \cup I_{n-2}$ or $P_2 \cup P_2 \cup I_{n-4}$ or $P_3 \cup P_2 \cup I_{n-5}$ or $P_2 \cup P_2 \cup P_2 \cup I_{n-6}$.

Now we suppose $n_1 \leq 4$. If $H \cong P_2 \cup P_2$, then $G[V_1] \cong P_2 \cup P_2$ and so $G \cong P_2 \cup P_2 \cup I_2$ since the two vertices in V_2 has no common out-neighbor. Suppose $H \cong P_2$. If $G[V_1]$ has two isolated vertices, then at least one of them is a common out-neighbor of v_1 and v_2 and so $G \cong P_2 \cup P_2 \cup I_2$. If $G[V_1]$ has exactly one isolated vertex, then $G \cong P_2 \cup P_2 \cup I_1$ or $G \cong P_2 \cup I_3$. If $H \cong G[V_1]$, then $G \cong P_2 \cup I_2$. Suppose $H \cong P_3$. If $G[V_1]$ has an isolated vertex, then it must be a common out-neighbor of v_1 and v_2 and so $G \cong P_3 \cup P_2 \cup I_1$. If $H \cong G[V_1]$, then $G \cong P_3 \cup I_2$.

Case 3. $n_1 \leq 6$ and $n_2 = 3$. Suppose, to the contrary, that $n_1 \geq 5$. Since each vertex in D has indegree at most 2, each vertex in V_2 has outdegree at least $n_1 - 2$. Since $n_1 - 2 > n_1/2$, any pair of vertices in V_2 has a common out-neighbor in V_1. Therefore the vertices in V_2 form a triangle, which is a contradiction. Thus $n_1 = 3$ or $n_1 = 4$.

Subcase 3.1. $n_1 = 3$. Then, since each vertex in D has indegree at most 2,

$$d^+(v) \geq 1 \tag{6}$$

for each vertex v in D. To reach a contradiction, we suppose that G has at least three isolated vertices. Then at least two isolated vertices belong to the same partite set in D. Without loss of generality, we may assume that V_1 has two isolated vertices u_1 and u_2. Since $|V_1| = 3$, u_3 is also isolated in G. Then, since

$|V_2| = 3$, each vertex in V_1 has exactly one out-neighbor by (6) and the out-neighbors of the vertices in V_1 are distinct. Therefore any pair of the vertices in V_2 has a common out-neighbor in V_1, which implies that the vertices in V_2 form a triangle in G. Thus G has at most two isolated vertices. Hence G is isomorphic to $P_3 \cup P_3$ or $P_3 \cup P_2 \cup I_1$ or $P_2 \cup P_2 \cup I_2$.

Subcase 3.2. $n_1 = 4$. Then, since each vertex in D has indegree at most 2,

$$d^+(v) \geq 2 \tag{7}$$

for each vertex v in V_2. We first suppose that there exists a vertex in V_2 which is isolated in G. Without loss of generality, we may assume v_1 is an isolated vertex in G. Then, since $n_1 = 4$, $d^+(v_1) = 2$ by (7). Without loss of generality, we may assume $N^+(v_1) = \{u_1, u_2\}$. Then, since v_1 is isolated in G, $N^+(v_2) = N^+(v_3) = \{u_3, u_4\}$ by (7). Therefore $G[V_2]$ is isomorphic to $I_1 \cup P_2$ and $G[V_1]$ is isomorphic to $P_2 \cup P_2$. Thus G is isomorphic to $I_1 \cup P_2 \cup P_2 \cup P_2$.

Now we suppose that each vertex in V_2 is not isolated in G. Then $G[V_2]$ is isomorphic to P_3. Without loss of generality, we may assume that $G[V_2]$ is the path $v_1v_2v_3$. Then D contains a subdigraph isomorphic to D' given in Figure 9. We may assume that D' itself is a subdigraph of D. Then, by (7), $N^+(v_1) \cap \{u_3, u_4\} \neq \emptyset$ and $N^+(v_3) \cap \{u_3, u_4\} \neq \emptyset$. Since v_1 and v_3 are not adjacent in G, those intersections are disjoint. We may assume that $N^+(v_1) \cap \{u_3, u_4\} = \{u_3\}$ and $N^+(v_3) \cap \{u_3, u_4\} = \{u_4\}$. Then D contains the subdigraph D'' given in Figure 9. Then v_1 (respectively, v_3) is a common out-neighbor of u_2 and u_4 (respectively, u_1 and u_3). If v_2 is a common out-neighbor of u_3 and u_4, then $G[V_1]$ is the path $u_1u_3u_4u_2$ and so G is isomorphic to $P_4 \cup P_3$. If v_2 is not a common out-neighbor of u_3 and u_4, then $G[V_1]$ is the union of two paths u_1u_3 and u_2u_4, and so G is isomorphic to $P_3 \cup P_2 \cup P_2$.

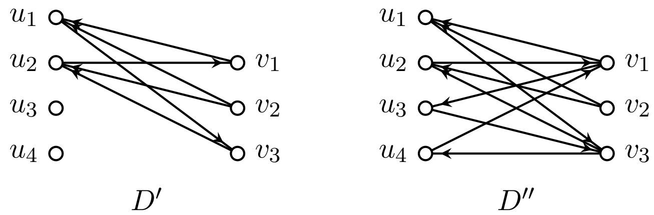


Figure 9. Digraphs D' and D'' in the proof of Theorem 3.4.

Case 4. $n_1 = 4$ and $n_2 = 4$. Then $|A(D)| = 16$. Noting that $|V(D)| = 8$ and each vertex has indegree at most 2,we have

$$d^-(v) = 2 \tag{8}$$

for each vertex v in D. Then, for each vertex v in D,

$$d^+(v) = 2 \tag{9}$$

since v is adjacent to four vertices in D.

Subcase 4.1. $|E(G[V_1])| \geq 4$. Then $|E(G[V_1])| = 4$ by (8) and $G[V_1]$ is isomorphic to C_4 since G has no triangle. Without loss of generality, we may assume $G[V_1] = u_1u_2u_3u_4u_1$. Without loss of generality, we may assume that $N^-(v_1) = \{u_1, u_2\}$, $N^-(v_2) = \{u_2, u_3\}$, $N^-(v_3) = \{u_3, u_4\}$, and $N^-(v_4) = \{u_4, u_1\}$ by (8). Therefore all arcs in D are determined and so $G[V_2]$ is a 4-cycle $v_1v_2v_3v_4v_1$. Thus G is isomorphic to $C_4 \cup C_4$.

Subcase 4.2. $|E(G[V_1])| \leq 3$. Since $|V_2| = 4$, there exists a pair of vertices in V_2 which shares the same in-neighborhood by (8). Without loss of generality, we may assume $N^-(v_1) = N^-(v_2) = \{u_1, u_2\}$. Then $N^+(u_1) = N^+(u_2) = \{v_1, v_2\}$ by (9). Therefore $N^+(u_3) = N^+(u_4) = \{v_3, v_4\}$ by (8) and (9). Then $N^-(u_3) = N^-(u_4) = \{v_1, v_2\}$. Thus it is easy to check that G is isomorphic to $P_2 \cup P_2 \cup P_2 \cup P_2$. Hence we have shown that the "only if" part is true.

To show the "if" part, we fix a positive integer k. Let D_8 be a bipartite tournament with the partite sets $\{u_1, \ldots, u_k\}$ and $\{v\}$, and the arc set

$$A(D_8) = \{(v, u_i) \mid 1 \leq i \leq k\}$$

(see the digraph D_8 given in Figure 10 for an illustration). Then $C(D_8)$ is an empty graph of order $k + 1$.

Let D_9 be a bipartite tournament with the partite sets $\{u_1, \ldots, u_{k+1}\}$ and $\{v\}$, and the arc set

$$A(D_9) = \{(u_1, v), (u_2, v)\} \cup \{(v, u_i) \mid 2 < i \leq k + 1\}$$

(see the digraph D_9 given in Figure 10 for an illustration). Then $C(D_9)$ is the path u_1u_2 with k isolated vertices.

Let D_{10} be a bipartite tournament with the partite sets $\{u_1, \ldots, u_{k+2}\}$ and $\{v_1, v_2\}$, and the arc set

$$A(D_{10}) = \{(u_i, v_j) \mid 1 \leq i, j \leq 2\} \cup \{(v_i, u_j) \mid 1 \leq i \leq 2, 3 \leq j \leq k + 2\}$$

(see the digraph D_{10} given in Figure 10 for an illustration). Then $C(D_{10})$ is the paths u_1u_2 and v_1v_2 with k isolated vertices.

Let D_{11} be a bipartite tournament with the partite sets $\{u_1, \ldots, u_{k+3}\}$ and $\{v_1, v_2\}$, and the arc set

$$\begin{aligned} A(D_{11}) = \{&(u_1, v_1), (u_2, v_1), (u_2, v_2), (u_3, v_2), (v_1, u_3), (v_2, u_1)\} \\ &\cup \{(v_i, u_j) \mid 1 \leq i \leq 2, 4 \leq j \leq k + 3\} \end{aligned}$$

(see the digraph D_{11} given in Figure 10 for an illustration). Then $C(D_{11})$ is the paths $u_1u_2u_3$ and v_1v_2 with k isolated vertices.

Let D_{12} be a bipartite tournament with the partite sets $\{u_1, \ldots, u_{k+4}\}$ and $\{v_1, v_2\}$, and the arc set

$$\begin{aligned} A(D_{12}) = \{(u_i, v_1), (v_2, u_i) \mid i = 1, 2\} \cup \{(u_i, v_2), (v_1, u_i) \mid i = 3, 4\} \\ \cup \{(v_i, u_j) \mid 1 \le i \le 2, 5 \le j \le k+4\} \end{aligned}$$

(see the digraph D_{12} given in Figure 10 for an illustration). Then $C(D_{12})$ is the paths u_1u_2, u_3u_4, and v_1v_2 with k isolated vertices.

The competition graph of the digraph D_{13} given in Figure 10 is isomorphic to $P_3 \cup I_2$. By Proposition 3.2, there exists a bipartite tournament whose competition graph is isomorphic to $P_3 \cup P_3$. By the way, bipartite tournaments whose competition graphs are isomorphic to $P_4 \cup P_3$ and $P_3 \cup P_2 \cup P_2$, respectively, were constructed in the Subcase 3.2. By Proposition 3.3, there exists a bipartite tournament whose competition graph is isomorphic to $C_4 \cup C_4$. It is easy to check that the competition graph of the bipartite tournament D_{14} given in Figure 10 is the disjoint union of the paths u_1u_2, u_3u_4, v_1v_2, and v_3v_4. Hence we have shown that the "if" part is true. ■

3.2. k-partite tournaments for $k \ge 3$

By Lemmas 2.11 and 2.15, the following lemma is true.

Lemma 3.5. *If the competition graph of a k-partite tournament is triangle-free and disconnected for some positive integer $k \ge 3$, then $k = 3$ or $k = 4$.*

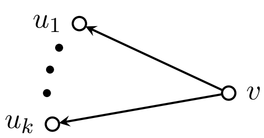
By Lemma 3.5, it is sufficient to consider tripartite tournaments and 4-partite tournaments for studying disconnected triangle-free competition graphs of multipartite tournaments.

Lemma 3.6. *Let D be a multipartite tournament whose competition graph is triangle-free. Suppose that a vertex v is contained in a partite set X of D. Then $|V(D)| - |X| - 2 \le d^+(v)$.*

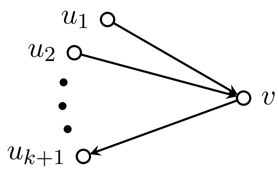
Proof. Since $C(D)$ is triangle-free, $d^-(v) \le 2$. Then, since D is a multipartite tournament, $d^-(v) = |V(D)| - |X| - d^+(v)$ and so $|V(D)| - |X| - 2 \le d^+(v)$. ■

The following is immediately true by Lemma 3.6.

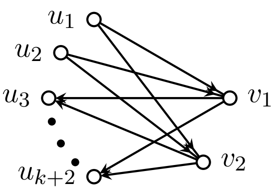
Corollary 3.7. *If the competition graph of a 4-partite tournament D is triangle-free, then each vertex has outdegree at least 1 in D.*



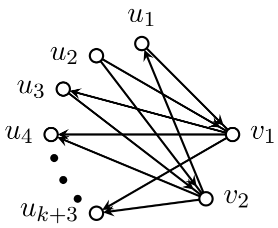
D_8



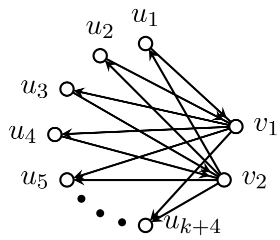
D_9



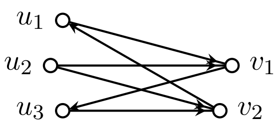
D_{10}



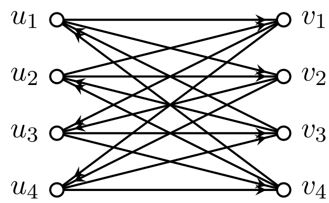
D_{11}



D_{12}



D_{13}



D_{14}

Figure 10. Bipartite tournaments in the proof of Theorem 3.4.

Lemma 3.8. *Let D be a multipartite tournament whose competition graph is triangle-free. If m is the number of vertices of indegree 1 in D, then $2|V(D)| - |A(D)| \geq m$.*

Proof. Let m be the number of vertices of indegree 1 in D. Since $C(D)$ is triangle-free, each vertex has indegree at most 2. Therefore

$$|A(D)| = \sum_{v \in V(D)} d^-(v) \leq 2(|V(D)| - m) + m = 2|V(D)| - m.$$

■

Now we are ready to introduce one of our main theorems.

Theorem 3.9. *Let G be a disconnected and triangle-free graph. Then G is the competition graph of a 4-partite tournament if and only if G is isomorphic to $P_3 \cup P_2$ or $P_3 \cup I_1$.*

Proof. We first show the "only if" part. Let D be an orientation of K_{n_1,n_2,n_3,n_4} with $n_1 \geq \cdots \geq n_4$ whose competition graph $C(D)$ is disconnected and triangle-free. Let $V_1, \ldots, V_4$ be the partite sets of D with $|V_i| = n_i$ for each $1 \leq i \leq 4$. Then $n_1 \leq 2$ and $n_2 = n_3 = n_4 = 1$ by Lemma 2.17.

Case 1. $n_1 = 2$. Then $|A(D)| = 9$. Therefore $|E(C(D))| \leq 4$ by Lemma 2.2. Let l and m be the number of isolated vertices in $C(D)$ and the number of vertices of indegree 1 in D, respectively. By Corollary 3.7, each vertex has outdegree at least 1, so each isolated vertex in $C(D)$ has an out-neighbor in D. Yet, since each out-neighbor of an isolated vertex has indegree 1, $l \leq m$. By Lemma 3.8, $m \leq 1$. Therefore $l \leq 1$.

Suppose, to the contrary, that $l = 1$. Then $m = 1$. Let w be the isolated vertex in $C(D)$. Since each vertex in $N^+(w)$ has indegree 1, $d^+(w) \leq 1$. Since each vertex has outdegree at least 1, $d^+(w) = 1$. Since $C(D)$ is triangle-free, $d^-(w) \leq 2$ and so $w \in V_1$. Let $V_1 = \{v_1, w\}$, $V_2 = \{v_2\}$, $V_3 = \{v_3\}$ and $V_4 = \{v_4\}$. Without loss of generality, we may assume $N^+(w) = \{v_2\}$. Then

$$N^-(w) = \{v_3, v_4\}.$$

Since w is an isolated vertex in $C(D)$, $N^-(v_2) = \{w\}$ and so $N^+(v_2) = \{v_1, v_3, v_4\}$. Without loss of generality, we may assume $(v_3, v_4) \in A(D)$. Then, since $d^-(v_4) \leq 2$,

$$N^-(v_4) = \{v_2, v_3\},$$

and so $(v_4, v_1) \in A(D)$. Therefore

$$N^-(v_1) = \{v_2, v_4\}.$$

Thus $\{v_2, v_3, v_4\}$ forms a triangle in $C(D)$, which is a contradiction. Hence $l = 0$. Since $C(D)$ is disconnected and $|V(D)| = 5$, $C(D)$ has two components each of which has 2 and 3 vertices, respectively. Then, one of the components must be P_2. On the other hand, since $C(D)$ is triangle-free, the other component is isomorphic to P_3. Therefore $C(D)$ is isomorphic to $P_3 \cup P_2$.

Case 2. $n_1 = 1$. Then D is an orientation of $K_{1,1,1,1}$, which is a tournament. By Proposition 2.12, $|E(C(D))| \geq 2$. By the way, since $|A(D)| = 6$, $|E(C(D))| \leq 3$ by Lemma 2.2. Therefore $|E(C(D))| = 2$ or 3. Thus $C(D)$ has exactly two components and so is isomorphic to $I_1 \cup P_3$ or $P_2 \cup P_2$. If $C(D)$ is isomorphic to $P_2 \cup P_2$, then D has two vertices a and b such that $d^-(a) = d^-(b) = 2$ and $N^-(a) \cap N^-(b) = \emptyset$, which is impossible for a digraph of order four. Therefore $C(D)$ is isomorphic to $I_1 \cup P_3$.

To show the "if" part, we consider the 4-partite tournaments D_{15} and D_{16} given in Figure 11. In D_{15}, $\{v_1, v_2\}$, $\{x\}$, $\{y\}$, and $\{z\}$ are the partite sets. Further,

$$N^-(v_1) = N^-(v_2) = \{y, z\},\ N^-(x) = \{v_1, v_2\},\ N^-(y) = \{x, z\},\ \text{and}\ N^-(z) = \{x\}.$$

Thus xyz and v_1v_2 are path components in $C(D_{15})$ and so $C(D_{15}) \cong P_3 \cup P_2$. Now, in D_{16}, every vertex is a partite set and

$$N^-(w) = \{x, y\}, \quad N^-(x) = \{z\}, \quad N^-(y) = \{x\}, \quad \text{and} \quad N^-(z) = \{w, y\}.$$

Thus wyx is a path component and z is isolated in $C(D_{16})$. Hence $C(D_{16}) \cong P_3 \cup I_1$. Therefore we have shown that the "if" part is true. ■

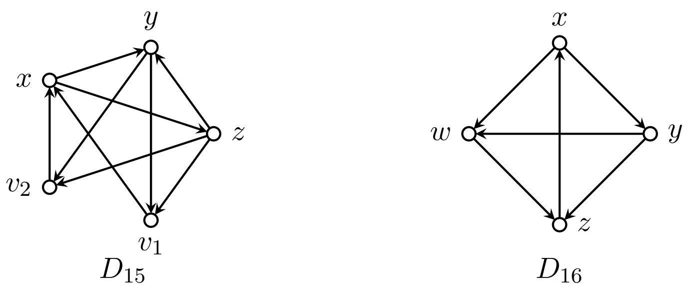


Figure 11. The digraphs D_{15} and D_{16} in the proof of Theorem 3.9.

By Lemma 3.5, it only remains to characterize disconnected and triangle-free competition graphs of tripartite tournaments. The following theorem lists all the disconnected and triangle-free competition graphs of tripartite tournaments

Theorem 3.10. *Let G be a disconnected and triangle-free graph. Then G is the competition graph of a tripartite tournament if and only if G is isomorphic to one of the followings.*

(a) *An empty graph of order* 3*;*

(b) *P_2 with at least one isolated vertex;*

(c) *P_3 with at least one isolated vertex;*

(d) P_4 *with at least one isolated vertex;*

(e) $K_{1,3} \cup I_1$*;*

(f) $K_{1,3} \cup P_2$*;*

(g) $P_2 \cup P_4$*;*

(h) $P_2 \cup P_2$ *with at least one isolated vertex;*

(i) $P_2 \cup P_3$ *with or without isolated vertices;*

(j) $P_2 \cup P_2 \cup P_2$.

Proof. To show the "only if" part, suppose that D is an orientation of K_{n_1,n_2,n_3} whose competition graph is disconnected and triangle-free where n_1, n_2, and n_3 are positive integers such that $n_1 \geq n_2 \geq n_3$. Then, by Lemma 2.4, $(n_1, n_2, n_3) \in A \cup \{(2,2,1), (2,2,2), (3,2,1)\}$ where $A = \{(m,1,1) \mid m \text{ is a positive integer}\}$.

Case 1. $(n_1, n_2, n_3) \in A$. Let $V_1 = \{x_1, \ldots, x_{n_1}\}$, $V_2 = \{y\}$, and $V_3 = \{z\}$ be the partite sets of D. Without loss of generality, we may assume

$$(y, z) \in A(D).$$

Suppose that $C(D)$ is an empty graph. Then y is an isolated vertex in $C(D)$. If $d^+(y) \geq 2$, then a vertex in V_1 should be an out-neighbor of y and so y is adjacent to one of the vertex and z in $C(D)$, which is a contradiction. Therefore $d^+(y) = 1$ and so $N^+(y) = \{z\}$. If $n_1 \geq 2$, then the in-neighbors of y are adjacent in $C(D)$, which is a contradiction. Therefore $n_1 = 1$. Thus $C(D)$ is isomorphic to three isolated vertices.

Now we suppose that $C(D)$ is not an empty graph. Then y is the only possible neighbor of z in $C(D)$. Moreover, z and at most one vertex in V_1 are the only possible neighbors of y in $C(D)$. Since z has indegree at most 2 in D, y is the only possible common out-neighbor of two vertices in V_1. Therefore only one pair of vertices in V_1 is possibly adjacent in $C(D)$. Hence the following are the only possible graphs isomorphic to $C(D)$.

- An empty graph of order 3;
- P_4 with at least one isolated vertex;
- $P_3 \cup P_2$ with at least one isolated vertex;
- P_3 with at least one isolated vertex;
- $P_2 \cup P_2$ with at least one isolated vertex;
- P_2 with at least one isolated vertex.

Case 2. $(n_1, n_2, n_3) = (2,2,1)$. Suppose, to the contrary, that $C(D)$ has l isolated vertices for some positive integer $l \geq 2$. By Lemma 3.6, each vertex in D has outdegree at least 1. By the way, since $2|V(D)| - |A(D)| = 2$, D has at most two vertices of indegree 1 by Lemma 3.8. Then, since each out-neighbor of isolated vertices has indegree 1, $l \leq 2$ and so $l = 2$. Let u_1 and u_2 be the isolated vertices in $C(D)$. Then, $d^+(u_1) = d^+(u_2) = 1$. Suppose that u_1 and u_2 are contained in distinct partite sets in D. Without loss of generality, we may assume $(u_1, u_2) \in A(D)$. Then $d^-(u_2) = 1$. However, since $d^+(u_2) = 1$, $d^+(u_2) + d^-(u_2) = 2 \neq |V(D) \setminus X|$ where X is a partite set containing u_2, which is impossible. Therefore u_1 and u_2 belong to the same partite set of D. Then $\{u_1, u_2\} \subseteq V_1$ or $\{u_1, u_2\} \subseteq V_2$. Without loss of generality, we may assume $\{u_1, u_2\} \subseteq V_1$. Then $V_1 = \{u_1, u_2\}$. Let v_1 and v_2 be the out-neighbors of u_1 and u_2, respectively. Then $N^-(v_1) = \{u_1\}$ and $N^-(v_2) = \{u_2\}$. Therefore there is no arc between v_1 and v_2, and so v_1 and v_2 belong to the same partite set V_2. Then $V_2 = \{v_1, v_2\}$. Since $d^-(v_1) = d^-(v_2) = 1$, the vertex z, in the remaining partite set of D, is a common out-neighbor of v_1 and v_2. Then $N^-(z) = \{v_1, v_2\}$ and so $N^+(z) = \{u_1, u_2\}$. Therefore u_1 (respectively, u_2) is a common out-neighbor of v_2 (respectively, v_1) and z. Thus $\{v_1, v_2, z\}$ forms a triangle in $C(D)$, which is a contradiction. Hence $C(D)$ has at most one isolated vertex. Therefore $C(D)$ has at most three components.

Since $|A(D)| = 8$, $|E(C(D))| \leq 4$ by Lemma 2.2. If $|E(C(D))| \leq 1$, then $C(D)$ has at least 2 isolated vertices, which is a contradiction. Therefore $2 \leq |E(C(D))| \leq 4$. If $|E(C(D))| = 2$, then, $C(D)$ is isomorphic to $P_2 \cup P_2 \cup I_1$. If $|E(C(D))| = 3$, then $C(D)$ is isomorphic to $P_4 \cup I_1$ or $K_{1,3} \cup I_1$ or $P_3 \cup P_2$. Suppose $|E(C(D))| = 4$. Since $|A(D)| = 8$ and $|E(C(D))| = 4$, there exists a vertex w of indegree 0 in D and each vertex in $V(D) \setminus \{w\}$ has indegree 2. Moreover, for distinct vertices a and b of indegree 2, $N^-(a) \neq N^-(b)$. Then, since w has outdegree at least 3, w has degree at least 3 in $C(D)$. Since $C(D)$ is disconnected and triangle-free, $K_{1,3} \cup I_1$ is the only possible graph isomorphic to $C(D)$, which contradicts the assumption that $|E(C(D))| = 4$. Therefore $C(D)$ is isomorphic to one of $P_2 \cup P_2 \cup I_1$ or $P_4 \cup I_1$ or $K_{1,3} \cup I_1$ or $P_3 \cup P_2$.

Case 3. $(n_1, n_2, n_3) = (2, 2, 2)$. Then $|A(D)| = 12$. Since $|V(D)| = 6$ and each vertex of D has indegree at most 2,

$$d^-(v) = 2 \tag{10}$$

and so

$$d^+(v) = 2 \tag{11}$$

for each vertex v in D. Therefore $C(D)$ has no isolated vertex. Thus each component of $C(D)$ contains at least two vertices. Let t be the number of the components of $C(D)$. Then $t \leq 3$. Since $C(D)$ is disconnected, $t = 2$ or $t = 3$. Suppose, to the contrary, that $t = 2$. Then, since $C(D)$ is triangle-free, it is easy to check that $|E(C(D))| \leq 5$. Since $|V(D)| = 6$, there exist at least two vertices a_1

and a_2 sharing the same in-neighborhood by (10). Then a_1 and a_2 are contained in the same partite set and form a component in $C(D)$ by Lemma 2.16. Then the other component must contain four vertices. Without loss of generality, we may assume $V_1 = \{a_1, a_2\}$ is a partite set of D. Let $\{b_1, b_2\} = N^-(a_1) = N^-(a_2)$ for some vertices b_1 and b_2 in D. Then $N^+(b_1) = N^+(b_2) = \{a_1, a_2\}$ by (11). Therefore b_1 and b_2 are contained in the same partite sets and $\{b_1, b_2\}$ forms a component in $C(D)$ by Lemma 2.16, which is a contradiction. Therefore $t \neq 2$ and so $t = 3$. Then, since each component of $C(D)$ contains at least two vertices, $C(D)$ must be isomorphic to $P_2 \cup P_2 \cup P_2$.

Case 4. $(n_1, n_2, n_3) = (3, 2, 1)$. Then $|A(D)| = 11$. Since each vertex has indegree at most 2 in D, one vertex has indegree 1 and the other vertices have indegree 2. Let V_1, V_2, and V_3 be the partite sets of D satisfying $|V_1| = 3$, $|V_2| = 2$, and $|V_3| = 1$ and v^* be the vertex of indegree 1 in D. Then, since $(n_1, n_2, n_3) = (3, 2, 1)$, each vertex in V_1 has outdegree at least 1 and each vertex in $V_2 \cup V_3$ has outdegree at least 2.

Suppose, to the contrary, that $C(D)$ has an isolated vertex u. Since v^* is the only vertex of indegree 1, $N^+(u) = \{v^*\}$. Therefore $u \in V_1$. Then $v^* \in V_2 \cup V_3$. Suppose $v^* \in V_3$. Since $d^-(v^*) = 1$, $N^+(v^*) = (V_1 \cup V_2) \setminus \{u\}$. Moreover, since $V_2 \subset N^+(v^*)$ and each vertex in D other than v^* has indegree 2, each vertex in V_2 has an out-neighbor in $V_1 \setminus \{u\}$. Therefore each vertex in V_2 is adjacent to v^* in $C(D)$. By the way, since $N^+(u) = \{v^*\}$, u is a common out-neighbor of the two vertices in V_2. Therefore $V_2 \cup \{v^*\}$ forms a triangle in $C(D)$, which is a contradiction. Thus $v^* \in V_2$. Let $V_1 = \{u, x_1, x_2\}$, $V_2 = \{v^*, y\}$, and $V_3 = \{z\}$. Then $N^+(v^*) = \{x_1, x_2, z\}$ and

$$N^-(u) = \{y, z\},$$

so y and z are adjacent in $C(D)$. Since $d^-(y) = 2$, $d^+(y) = 2$. Thus

$$N^+(v^*) \cap N^+(y) \neq \emptyset$$

and so v^* and y are adjacent in $C(D)$. Since $d^-(z) = 2$ and $v^* \in N^-(z)$, $\{x_1, x_2\} \not\subset N^-(z)$. Therefore $N^+(z) \cap \{x_1, x_2\} \neq \emptyset$ and so

$$N^+(v^*) \cap N^+(z) \neq \emptyset.$$

Thus v^* and z are adjacent in $C(D)$. Hence $\{v^*, y, z\}$ forms a triangle in $C(D)$, which is a contradiction. Consequently, we have shown that $C(D)$ has no isolated vertex, so each component in $C(D)$ has size at least two. Then, since $|V(D)| = 6$, $C(D)$ has two or three components. If $C(D)$ has three components, then $C(D)$ must be isomorphic to $P_2 \cup P_2 \cup P_2$.

Now we suppose that $C(D)$ has two components. Then it is easy to check that $4 \leq |E(C(D))| \leq 5$ since $C(D)$ is triangle-free and has no isolated vertices.

Suppose, to the contrary, that $|E(C(D))| = 5$. Then, since the vertices in $C(D)$ except the five vertices of indegree 2 have indegree less than 2, no pair of adjacent vertices in $C(D)$ have two distinct common out-neighbors in D. Moreover, $C(D)$ must be isomorphic to $P_2 \cup C_4$ where C_4 is a cycle of length 4. Since only one vertex has indegree 1 and the other vertices have indegree 2 in D, there exist two vertices a and b in V_1 which have outdegree 1 in D. Then a and b have at most degree 1 in $C(D)$, so $\{a, b\}$ is a path component in $C(D)$. Therefore a and b have a common out-neighbor c, in D. Then a and b are common out-neighbors of the two vertices in $V_2 \cup V_3 \setminus \{c\}$, which is a contradiction. Hence $|E(C(D))| \neq 5$ and so $|E(C(D))| = 4$. Since five vertices have indegree 2 in D, there exists a pair of adjacent vertices which have two common out-neighbors. Therefore there exist two vertices whose in-neighbors are the same. Then, since each vertex has outdegree at least 1 by Lemma 3.6, the two vertices form a component in $C(D)$ by Lemma 2.16. Thus $C(D)$ must be isomorphic to $P_2 \cup K_{1,3}$ or $P_2 \cup P_4$. Hence we have shown that the "only if" part is true.

Now we show the "if" part. The competition graph of the digraph D_{17} given in Figure 12 is an empty graph of order 3.

Now we fix a positive integer k. Let D_{18} be a tripartite tournament with the partite sets $\{w_1, \ldots, w_k\}$, $\{x\}$, $\{y\}$ and the arc set

$$A(D_{18}) = \{(x, y)\} \cup \{(x, w_i), (y, w_i) \mid 1 \leq i \leq k\}$$

(see the digraph D_{18} given in Figure 12 for an illustration). Then $C(D_{18})$ is isomorphic to P_2 with k isolated vertices.

Let D_{19} be a tripartite tournament with the partite sets $\{v, w_1, \ldots, w_k\}$, $\{x\}$, $\{y\}$ and the arc set

$$A(D_{19}) = \{(v, x), (v, y), (x, y)\} \cup \{(x, w_i), (y, w_i) \mid 1 \leq i \leq k\}$$

(see the digraph D_{19} given in Figure 12 for an illustration). Then $C(D_{19})$ is the path vxy together with k isolated vertices.

Let D_{20} be a tripartite tournament with the partite sets $\{v_1, v_2, w_1, \ldots, w_k\}$, $\{x\}$, $\{y\}$ and the arc set

$$A(D_{20}) = \{(v_1, x), (v_2, x), (v_2, y), (x, y), (y, v_1)\} \cup \{(x, w_i), (y, w_i) \mid 1 \leq i \leq k\}$$

(see the digraph D_{20} given in Figure 12 for an illustration). Then $C(D_{20})$ is the path v_1v_2xy with k isolated vertices.

The competition graphs of the digraphs D_{21}, D_{22} and D_{23} given in Figure 12 are isomorphic to $K_{1,3} \cup I_1$, $K_{1,3} \cup P_2$, and $P_2 \cup P_4$, respectively.

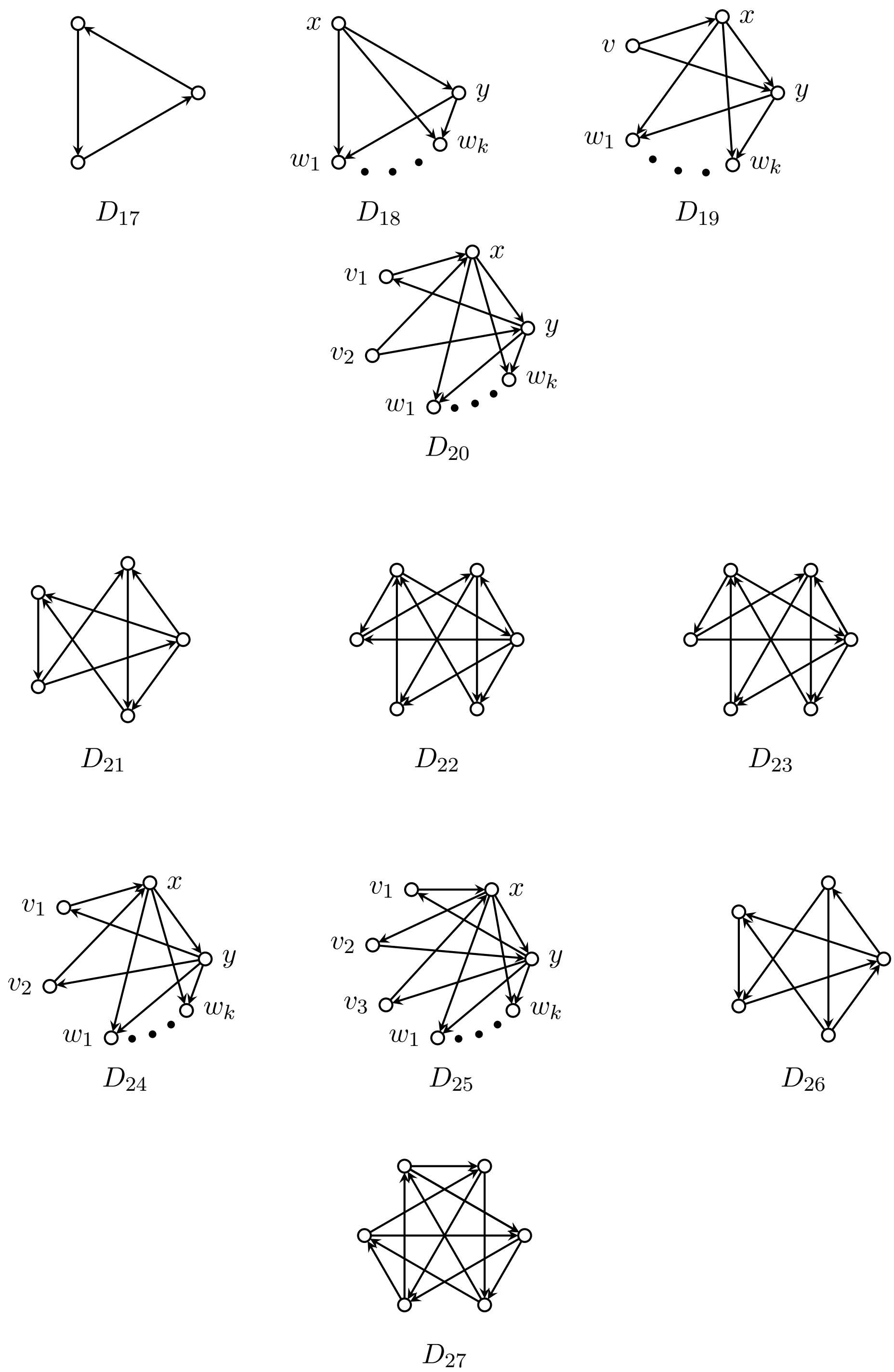


Figure 12. The digraphs in the proof of Theorem 3.10.

Let D_{24} be a tripartite tournament with the partite sets $\{v_1, v_2, w_1, \ldots, w_k\}$, $\{x\}$, $\{y\}$ and the arc set

$$A(D_{24}) = \{(v_1, x), (v_2, x), (x, y), (y, v_1), (y, v_2)\} \cup \{(x, w_i), (y, w_i) \mid 1 \le i \le k\}$$

(see the digraph D_{24} given in Figure 12 for an illustration). Then $C(D_{24})$ is isomorphic to $P_2 \cup P_2$ with k isolated vertices.

Let D_{25} be a tripartite tournament with the partite sets $\{v_1, v_2, v_3, w_1, \ldots, w_k\}$, $\{x\}$, $\{y\}$ and the arc set

$$A(D_{25}) = \big\{(v_1, x), (v_2, y), (v_3, x), (x, v_2), (x, y), (y, v_1), (y, v_3)\} \cup \{(x, w_i), (y, w_i) \mid 1 \le i \le k\big\}$$

(see the digraph D_{25} given in Figure 12 for an illustration). Then $C(D_{25})$ is isomorphic to $P_2 \cup P_3$ with k isolated vertices.

The competition graphs of the digraphs D_{26} and D_{27} given in Figure 12 are isomorphic to $P_2 \cup P_3$ and $P_2 \cup P_2 \cup P_2$, respectively. Hence we have shown that the "if" part is true. ■

4. Closing Remarks

In this paper, we completely identified the triangle-free graphs which are the competition graphs of k-partite tournaments for $k \ge 2$, following up the previous paper [3] in which all the complete graphs which are the competition graphs of k-partite tournaments for $k \ge 2$ are found.

Taking into account the fact that a cycle is a 2-regular graph and a complete graph of order n is an $(n-1)$-regular graph, characterizing the cubic graphs which are the competition graphs of k-partite tournament for $k \ge 2$ seems to be an interesting research problem to be resolved in the next step.

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