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ON THE DISTRIBUTION OF DISTANCE SIGNLESS LAPLACIAN EIGENVALUES WITH GIVEN INDEPENDENCE AND CHROMATIC NUMBER

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Abstract

For a connected graph G of order n, let $\mathcal{D}(G)$ be the distance matrix and $Tr(G)$ be the diagonal matrix of vertex transmissions of G. The distance signless Laplacian (dsL, for short) matrix of G is defined as $\mathcal{D}^Q(G) = Tr(G)+\mathcal{D}(G)$, and the corresponding eigenvalues are the dsL eigenvalues of G. For an interval I, let $m_{\mathcal{D}^Q(G)}I$ denote the number of dsL eigenvalues of G lying in the interval I. In this paper, for some prescribed interval I, we obtain bounds for $m_{\mathcal{D}^Q(G)}I$ in terms of the independence number α and the chromatic number χ of G. Furthermore, we provide lower bounds of $\partial_1^Q(G)$, the dsL spectral radius, for certain families of graphs in terms of the order n and the independence number α, or the chromatic number χ.

Keywords: distance matrix, distance signless Laplacian matrix, spectral radius, independence number, chromatic number.

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1. Introduction

Throughout this article, we assume that G is a simple connected graph. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *order* and *size* of G are $|V(G)| = n$ and $|E(G)| = m$, respectively. The *degree* of a vertex v, denoted by $d_G(v)$, is the number of edges incident to the vertex v. Further, $N_G(v)$ is the set of all vertices which are adjacent to v in G. Also, $\overline{G}$ denotes the complement of the graph G. If $e \in E(G)$ is an edge between vertices u and v, then $G - e$ denotes the graph obtained from G by deleting the edge e in G.

The adjacency matrix $A(G) = (a_{ij})$ of G is a $(0,1)$-square matrix of order n whose ij-th entry equals 1 whenever the corresponding vertices v_i and v_j are adjacent, and equals 0 otherwise. Let $\mathrm{diag}(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i)$, where $i = 1, 2, \ldots, n$ associated to G. The matrices $L(G) = \mathrm{diag}(G) - A(G)$ and $Q(G) = \mathrm{diag}(G) + A(G)$ are the Laplacian and the signless Laplacian matrices, respectively, and their spectra are the Laplacian spectrum and signless Laplacian spectrum of the graph G, respectively.

For $v_i, v_j \in V(G)$, the *distance* between v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path between v_i and v_j.

The *diameter* d (or $d(G)$) of a graph G is the maximum distance between any two vertices of G. The *distance matrix* of G defined only for connected graphs, denoted by $\mathcal{D}(G)$, is defined as $\mathcal{D}(G) = (d_G(v_i, v_j))_{v_i, v_j \in V(G)}$. The *transmission* $Tr_G(v_i)$ of a vertex v_i is defined to be the sum of the distances from v_i to all other vertices in G, that is, $Tr_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j)$. For the sake of readability, the subscript or argument G might not be used if the graph is understood.

Let $Tr(G) = \mathrm{diag}(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$ be the diagonal matrix of vertex transmissions of G. For a connected graph G, Aouchiche and Hansen [2] defined the *distance signless Laplacian matrix* (or simply dsL matrix) only for connected graph G as $\mathcal{D}^Q(G) = Tr(G) + \mathcal{D}(G)$. Clearly, $\mathcal{D}^Q(G)$ is a nonnegative positive semidefinite real symmetric matrix, therefore its eigenvalues are real nonnegative numbers. We denote the dsL eigenvalues of G by $\partial_1^Q(G) \geq \partial_2^Q(G) \geq \cdots \geq \partial_{n-1}^Q(G) \geq \partial_n^Q(G)$; in particular, the largest eigenvalue $\partial_1^Q(G)$ is called the dsL spectral radius of G. For some literature on the dsL matrix, we refer the readers to [5, 6, 8, 12, 13].

Let $m_{\mathcal{D}^Q(G)}I$ be the number of dsL eigenvalues of G that lie in the interval I. In particular, $m_{\mathcal{D}^Q(G)}(\partial_i^Q(G))$ denotes the multiplicity of $\partial_i^Q(G)$. We denote dsL spectrum of G by $DSLS(G) = \{\partial_1^Q(G), \partial_2^Q(G), \ldots, \partial_{n-1}^Q(G), \partial_n^Q(G)\}$. If we need to emphasize the k distinct eigenvalues of G, then we write the dsL spectrum of G in the matrix form as

$$DSLS(G) = \begin{pmatrix} \partial_1^Q & \partial_2^Q & \cdots & \partial_k^Q \\ m_{\mathcal{D}^Q(G)}(\partial_1^Q(G)) & m_{\mathcal{D}^Q(G)}(\partial_2^Q(G)) & \cdots & m_{\mathcal{D}^Q(G)}(\partial_k^Q(G)) \end{pmatrix},$$

where $m_{\mathcal{D}^Q(G)}(\partial_1^Q(G)), m_{\mathcal{D}^Q(G)}(\partial_2^Q(G)), \ldots, m_{\mathcal{D}^Q(G)}(\partial_k^Q(G))$ are the corresponding multiplicities.

We denote the complete graph of order n by K_n. A graph G is said to be bipartite if its vertex set can be partitioned into two different (independent) sets U and W with $V = U \cup W$ such that $uv \in E(G)$ if and only if $u \in U$ and $v \in W$. If $|U| = |W|$, then G is called a balanced bipartite graph. The complete multipartite graph with order of parts $t_1, \ldots, t_k$ is denoted by $K_{t_1,\ldots,t_k}$. The star graph of order n is denoted by $K_{1,n-1}$. Finally, $CS(n, \alpha)$ denotes the complete split graph, that is, the join of a clique $K_{n-\alpha}$ and a coclique αK_1 (see Figure 1).

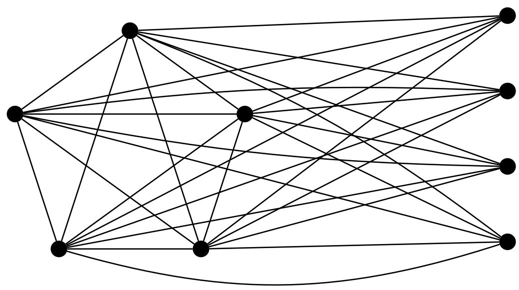

Figure 1. A complete split graph $CS(9, 4)$.

In a graph G, the subset $M \subseteq V(G)$ is called an *independent set* if no two vertices of M are adjacent. The cardinality of the largest independent set of G is the *independence number* of G and is denoted by α. The *chromatic number* of a graph G is the minimum number of colors required to color the vertices of G such that no two adjacent vertices get the same color, and it is denoted by χ. The set of all vertices with the same color is a *color class*. It is well-known that computing the independence number or the chromatic number is in general an NP-hard problem (see [7]). Since the eigenvalues of a graph can be computed in polynomial time, the spectral techniques have been used extensively to provide sharp bounds for such (and other) invariants. For other standard definitions not given here, we refer to [4, 10].

In this paper we study the distribution of the distance signless Laplacian eigenvalues of G in terms of the independence and chromatic numbers. Also, we derive some lower bounds for the dsL spectral radius in terms of the latter mentioned invariants. The remainder of the paper is organized as follows. In Section 2, we study the distribution of dsL eigenvalues of G in relation to the independence number α. In particular, we prove that $m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-4)$

$\leq n - \alpha$, for $\alpha \geq 3$. Also, if G has independence number $\alpha \geq \frac{n+4}{3}$, then $\partial_1^Q(G) \geq 2n + \alpha - 2$, and we characterize the corresponding extremal graph. If G has independence number $\alpha \geq 2$, we show that $n-2$ is a dsL eigenvalue with multiplicity at most $n-\alpha-1$. In Section 3, we find the distribution of dsL eigenvalues of G in relation to the chromatic number χ. If G has at least 5 vertices and has chromatic number χ with $n_1 \geq \cdots \geq n_\chi$ as the cardinalities of its color classes satisfying $n_1 \geq 3$ and $2n_\chi \geq n_1$, then $m_{\mathcal{D}^Q(G)}\left[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil -4\right) \leq n-\chi-\left\lceil \frac{n}{\chi} \right\rceil +1$. We conclude with Section 4, in which we propose some open problems derived from the research exposed in this paper.

2. Distribution of the dsL Eigenvalues and the Independence Number

For a graph G with n vertices, let $Tr_{max}(G) = \max\{Tr(v_i) : v_i \in V(G)\}$. Recall that we omit the graph argument G if clear from the context. Let us consider the following important result from matrix theory.

Lemma 1 [9]**.** *Let $M = (m_{ij})$ be an $n \times n$ complex matrix having $l_1, l_2, \ldots, l_k$ as its distinct eigenvalues. Then*

$$\{l_1, l_2, \ldots, l_k\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}| \right\}.$$

For the dsL matrix of a graph G with n vertices, by using Lemma 1, we get

$$\partial_1^Q(G) \leq 2Tr_{\max}(G). \tag{2.1}$$

The following lemma will be used in sequel.

Lemma 2 [2]**.** *Let G be a connected graph with n vertices and m edges, where $m \geq n$. Let G^* be the connected graph obtained from G by deleting an edge. Let $\partial_1^Q \geq \partial_2^Q \geq \cdots \geq \partial_n^Q$ and $\partial_1^{*Q} \geq \partial_2^{*Q} \geq \cdots \geq \partial_n^{*Q}$ be the distance signless Laplacian spectrum of G and G^*, respectively. Then $\partial_i^{*Q} \geq \partial_i^Q$ for all $i = 1, \ldots, n$.*

An immediate consequence of Lemma 2 is the following.

Lemma 3. *Let G be a connected graph on $n \geq 3$ vertices. Then, ${\partial_1}^Q \geq 2n-2$ and ${\partial_i}^Q \geq n-2$, for all $2 \leq i \leq n$.*

Proof. As

$$DSLS(K_n) = \begin{pmatrix} n-2 & 2n-2 \\ n-1 & 1 \end{pmatrix},$$

by using Lemma 2, the proof follows immediately. ∎

The following lemma can be seen in [11].

Lemma 4 [11]**.** *Let n and α be positive integers with $n-\alpha \geq 1$. Then*

$$DSLS(CS(n,\alpha)) = \begin{pmatrix} n-2 & n+\alpha-4 & \frac{3n+2\alpha-6-\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} & \frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} \\ n-\alpha-1 & \alpha-1 & 1 & 1 \end{pmatrix}.$$

In the following result we provide the upper bound for the quantity $m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-4)$ in terms of order n and independence number α.

Theorem 5. *Let G be a connected graph with n vertices having independence number $\alpha \geq 3$. Then $m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-4) \leq n-\alpha$. For $\alpha = n-1$ equality holds if and only if $G \cong K_{1,n-1}$. If $\frac{n+4}{3} < \alpha \leq n-2$, then the inequality is sharp as can be seen in the complete split graph $CS(n,\alpha)$.*

Proof. As G is connected, so for $\alpha = n$, the graph G reduces to an isolated vertex. Thus we will consider only $\alpha \leq n-1$. Since $\alpha \geq 3$, the interval $[n-2, n+\alpha-4)$ is well defined. Without loss of generality, assume that $U = \{v_1, v_2, \ldots, v_\alpha\}$ be the independent set of G corresponding to the independence number α. Let R be the graph obtained by adding the edges between all non-adjacent vertices in $V(G) \setminus U$ and joining each vertex of U with every vertex of $V(G) \setminus U$. With this construction, it is easily seen that $R \cong CS(n,\alpha)$. Obviously

$$\frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} > n+\alpha-4. \tag{2.2}$$

Therefore, using Lemma 4, we get

$$m_{\mathcal{D}^Q(R)}[n-2, n+\alpha-4) \leq n-\alpha.$$

By Lemma 2 and Lemma 3, we have

$$m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-4) \leq m_{\mathcal{D}^Q(R)}[n-2, n+\alpha-4),$$

which proves the inequality.

We know that $K_{1,n-1} = CS(n, n-1)$ is the only connected graph having independence number $\alpha = n-1$. We will show that the equality holds for $K_{1,n-1}$. When $\alpha = n-1$, we have $n+\alpha-4 = 2n-5$ and $n-\alpha = 1$. Thus, we will show that $m_{\mathcal{D}^Q(K_{1,n-1})}[n-2, 2n-5) = 1$. By Lemma 4, we get

$$DSLS(K_{1,n-1}) = \begin{pmatrix} 2n-5 & \frac{5n-8-\sqrt{9(n-2)^2+4(n-1)}}{2} & \frac{5n-8+\sqrt{9(n-2)^2+4(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

We have

$$2n-5 > \frac{5n-8-\sqrt{9(n-2)^2+4(n-1)}}{2}$$

if $$\sqrt{9(n-2)^2+4(n-1)} > n+2,$$

or if $$(2n-7)(n-1) > 0,$$

or if $$n > \tfrac{7}{2},$$

which is true as $\alpha \geq 3$ and $n \geq \alpha+1$. Using the above observations and Inequality (2.2), we get

$$m_{\mathcal{D}^{Q}(K_{1,n-1})}[n-2, 2n-5) = 1.$$

Lastly, we will show that the inequality is sharp for the graph $CS(n,\alpha)$ whenever $\frac{n+4}{3} < \alpha \leq n-2$. Using the first part of the theorem and Lemma 4, we only need to show that

$$n+\alpha-4 > \frac{3n+2\alpha-6-\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2}.$$

We have

$$n+\alpha-4 > \frac{3n+2\alpha-6-\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2}$$

if $$\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2} > n+2,$$

or if $$12\alpha^2-16\alpha-4\alpha n > 0,$$

or if $$\alpha > \tfrac{4+n}{3},$$

which is true. This completes the proof. ∎

Now, we have the following consequence of Theorem 5.

Corollary 6. *Let G be a connected graph with n vertices and independence number $\alpha \geq 3$. Then $m_{\mathcal{D}^{Q}(G)}[n+\alpha-4, 2Tr_{\max}] \geq \alpha$. For $\alpha = n-1$ equality holds if and only if $G \cong K_{1,n-1}$. If $\frac{n+4}{3} < \alpha \leq n-2$, then the inequality is sharp as shown by $CS(n,\alpha)$.*

Proof. Using Inequality (2.1) and Lemma 3, we have the following equality

$$m_{\mathcal{D}^{Q}(G)}[n-2, n+\alpha-4) + m_{\mathcal{D}^{Q}(G)}[n+\alpha-4, 2Tr_{\max}] = n.$$

Now, applying Theorem 5, we get $m_{\mathcal{D}^{Q}(G)}[n+\alpha-4, 2Tr_{\max}] \geq \alpha$, which proves the inequality. The remaining part of the proof follows from Theorem 5. ∎

The following result shows that K_n is the unique graph having all its dsL eigenvalues except one lying in the interval $[n-2, n-1)$.

Theorem 7. *Let G be a connected graph on $n \geq 6$ vertices. Then $m_{\mathcal{D}^Q(G)}[n-2, n-1) \leq n-1$ with equality holding if and only if $G \cong K_n$.*

Proof. Let α be the independence number of G. When $\alpha = 1$, $G \cong K_n$. Also, we know that

$$DSLS(K_n) = \begin{pmatrix} n-2 & 2n-2 \\ n-1 & 1 \end{pmatrix},$$

which shows that the equality holds for K_n. When $\alpha = n$, the graph G reduces to an isolated vertex which is not the case. Thus, we have to consider the following two cases.

Case 1. Let $3 \leq \alpha \leq n-1$. Using the same construction and reasoning as in Theorem 5, we see that

$$m_{\mathcal{D}^Q(G)}[n-2, n-1) \leq m_{\mathcal{D}^Q(CS(n,\alpha))}[n-2, n-1) \tag{2.3}$$

Since $\alpha \geq 3$, so $n + \alpha - 4 \geq n - 1$. From Lemma 4, we see that $n + \alpha - 4$ is a dsL eigenvalue of $CS(n, \alpha)$ with multiplicity at least two, as $\alpha \geq 3$. Using these observations in Inequality (2.3), we have

$$m_{\mathcal{D}^Q(G)}[n-2, n-1) \leq m_{\mathcal{D}^Q(CS(n,\alpha))}[n-2, n-1) \leq n-2,$$

which proves the result in this case.

Case 2. Let $\alpha = 2$. Clearly, in this case, G can be considered as a spanning subgraph of $CS(n, 2)$. Using Lemma 2, we have the following inequality

$$m_{\mathcal{D}^Q(G)}[n-2, n-1) \leq m_{\mathcal{D}^Q(CS(n,2))}[n-2, n-1). \tag{2.4}$$

By Lemma 4, we have

$$DSLS(CS(n,2)) = \begin{pmatrix} n-2 & \frac{3n-2-\sqrt{n^2-4n+20}}{2} & \frac{3n-2+\sqrt{n^2-4n+20}}{2} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Clearly,

$$\frac{3n-2+\sqrt{n^2-4n+20}}{2} > n-1.$$

Also,

$$\frac{3n-2-\sqrt{n^2-4n+20}}{2} > n-1$$

if

$$n > \sqrt{n^2-4n+20},$$

or if $$n > 5,$$
which is true.

Using the above observations in Inequality (2.4), we have

$$m_{\mathcal{D}^Q(G)}[n-2, n-1) \leq n-2,$$

which proves the result in this case and also completes the proof. ■

Theorem 8. *Let G be a connected graph with n vertices and independence number α such that $6 < \alpha < \sqrt{n(\alpha-3)-3(\alpha-4)}$. Then*

$$m_{\mathcal{D}^Q(G)}[n-2, n+2] \leq n-\alpha-1.$$

The bound is best possible and can be seen holding in the complete split graph $CS(n,\alpha)$.

Proof. Using the same construction technique as in Theorem 5, we observe that G can be considered as a spanning subgraph of $CS(n,\alpha)$. By Lemma 2, we have

$$m_{\mathcal{D}^Q(G)}[n-2, n+2] \leq m_{\mathcal{D}^Q(CS(n,\alpha))}[n-2, n+2]. \tag{2.5}$$

Since $\alpha > 6$,

$$n+\alpha-4 > n+2. \tag{2.6}$$

Also, it is easy to see that

$$\frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} > n+2. \tag{2.7}$$

We claim that

$$\frac{3n+2\alpha-6-\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} > n+2. \tag{2.8}$$

Now

$$\frac{3n+2\alpha-6-\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} > n+2$$

if $$(n+2\alpha-10) > \sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2},$$
or if $$8\alpha n-24(n+\alpha)-8\alpha^2+96 > 0,$$
or if $$\alpha^2 < n(\alpha-3)-3(\alpha-4),$$
or if $$\alpha < \sqrt{n(\alpha-3)-3(\alpha-4)},$$

which is exactly the condition given in the statement of the theorem. This proves the claim.

Using Inequalities (2.6), (2.7), (2.8) and Lemma 4 in Inequality (2.5), we get

$$m_{\mathcal{D}^Q(G)}[n-2, n+2] \leq n - \alpha - 1,$$

which proves the required inequality. Also, it is clear from the proof that the equality holds for $CS(n, \alpha)$. ■

The following lemma will be useful in proving our next result.

Lemma 9 [3]. *Let G be a connected graph with n vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is an independent set of G such that $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial - 2$ is an eigenvalue of $\mathcal{D}^Q(G)$ with multiplicity at least $p - 1$.*

In the following result we provide a lower bound for the quantity $m_{\mathcal{D}^Q(G)}[n+|S|-3, 2Tr_{\max}]$ in terms of $|S|$ only, where S is an independent set of graph G.

Theorem 10. *Let G be a connected graph with n vertices. Let $S = \{v_1, v_2, \ldots, v_p\}$ be an independent set of G such that $|S| \geq 2$ and $N(S) = N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$. If $|S| + |N(S)| \leq n - 1$, then $m_{\mathcal{D}^Q(G)}[n + |S| - 3, 2Tr_{\max}] \geq |S| - 1$.*

Proof. Since $N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, we have

$$T = Tr(v_1) = Tr(v_2) = \cdots = Tr(v_p). \tag{2.9}$$

We observe that $d_G(v_i, v_j) = 2$ for all $i \neq j \in \{1, 2, \ldots, p\}$. Also, $d_G(v_i, v_q) = 1$, for all $v_i \in S$, $v_q \in N(S)$. As $|S| + |N(S)| \leq n - 1$, there is at least one vertex, say u, such that $u \in V(G) \setminus S \cup N(S)$ and $d_G(v_i, u) \geq 2$ for all $i, j \in \{1, 2, \ldots, p\}$.

Using these observations in Equation (2.9), we get

$$T \geq 2(|S| - 1) + 2 + (n - |S| - 1) = n + |S| - 1. \tag{2.10}$$

Using Lemma 9, we see that there are at least $|S| - 1$ distance signless Laplacian eigenvalues of G which are equal to $T - 2$. From Inequality (2.10), we see that all those eigenvalues are greater than or equal to $n + |S| - 3$. Thus, using Inequality (2.1), we get

$$m_{\mathcal{D}^Q(G)}[n + |S| - 3, 2Tr_{\max}] \geq |S| - 1.$$ ■

Now, we have the following observation.

Corollary 11. *Let G be a connected graph with n vertices. Let $S = \{v_1, v_2, \ldots, v_p\}$ be an independent set of G such that $|S| \geq 2$ and $N(S) = N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$. If $|S| + |N(S)| \leq n - 1$, then $m_{\mathcal{D}^Q(G)}[n - 2, n + |S| - 3) \leq n - |S| + 1$.*

Proof. Using Inequality (2.1) and Proposition 3, we have the following equality

$$m_{\mathcal{D}^Q(G)}[n-2, n+|S|-3) + m_{\mathcal{D}^Q(G)}[n+|S|-3, 2Tr_{\max}] = n.$$

Thus, applying Theorem 10, we get

$$m_{\mathcal{D}^Q(G)}[n-2, n+|S|-3) \leq n-|S|+1. \qquad \blacksquare$$

When S happens to be a maximum independent set corresponding to independence number α in Theorem 10, then we get the following result.

Theorem 12. *Let G be a connected graph on n vertices having independence number $\alpha \geq 2$. Let $S = \{v_1, v_2, \ldots, v_\alpha\}$ be an independent set corresponding to independence number α such that $N(S) = N(v_i) = N(v_j)$ for all $i, j \in \{1, 2, \ldots, \alpha\}$. If $|N(S)| \leq n-\alpha-1$, then*

$$m_{\mathcal{D}^Q(G)}[n+\alpha-3, 2Tr_{\max}] \geq \alpha - 1$$

and

$$m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-3) \leq n-\alpha+1.$$

Proof. Since all the conditions in Theorem 10 are met in the hypothesis, therefore, the proof is completed after we replace $|S|$ by α in Theorem 10. $\blacksquare$

Following lemma will be useful in proving our next result.

Lemma 13 [3]**.** *Let G be a connected graph with n vertices. If $\partial^Q = n-2$ is a dsL eigenvalue of G with multiplicity μ, then the complement $\overline{G}$ of G contains at least μ components, each of which is bipartite or an isolated vertex.*

In the following result, we provide an upper bound for the multiplicity of $n-2$ as a dsL eigenvalue of G in terms of order n and independence number α. Also, we give some conditions which are sufficient for the upper bound to be strict.

Theorem 14. *Let G be a connected graph with n vertices and m edges having independence number $\alpha \geq 2$. Then*

$$m_{\mathcal{D}^Q(G)}(n-2) \leq n-\alpha-1, \tag{2.11}$$

and the inequality is sharp as seen in $CS(n,\alpha)$. Let S be the independent set of G corresponding to independence number α. If $\alpha \geq 3$, $m \leq \frac{n(n-1)-\alpha(\alpha-1)}{2} - 2$ and for any $v \in V(G) \setminus S$, $|N_S(v)| \geq \alpha - 1$, where $N_S(v) = \{u : u \in S$ and $uv \in E(G)\}$, then the inequality is always strict.

Proof. We observe that G is a spanning subgraph of $CS(n,\alpha)$. By Lemmas 3 and 2, we get

$$m_{\mathcal{D}^Q(G)}(n-2) \leq m_{\mathcal{D}^Q(CS(n,\alpha))}(n-2). \tag{2.12}$$

From Lemma 4, we have $m_{\mathcal{D}^Q(CS(n,\alpha))}(n-2) = n-\alpha-1$. Combining this fact with Inequality (2.12) proves Inequality (2.11) and also shows that equality holds for $CS(n,\alpha)$.

Now, given the conditions in the statement of the theorem, we will show that Inequality (2.11) is strict. It is clear that $|E(CS(n,\alpha))| = \frac{n(n-1)-\alpha(\alpha-1)}{2}$. Since $m \leq \frac{n(n-1)-\alpha(\alpha-1)}{2} - 2$, therefore, G is a proper spanning subgraph of $CS(n,\alpha)$ with at least two edges deleted from $CS(n,\alpha)$. If $\alpha = n-1$, then $G \cong CS(n,n-1)$, a contradiction. Thus, $\alpha \leq n-2$ and $|V(G)\setminus S| = n-\alpha \geq 2$. Using Lemmas 2 and 3, we only need to show that inequality is strict for G when $m = \frac{n(n-1)-\alpha(\alpha-1)}{2} - 2$, that is, when we delete exactly two edges, say e_1 and e_2, from $CS(n,\alpha)$. Let $S = \{v_1, v_2, \ldots, v_\alpha\}$ and $N = V(G)\setminus S = \{v_{\alpha+1}, v_{\alpha+2}, \ldots, v_n\}$. Given the conditions in the hypothesis, we can delete two edges from $CS(n,\alpha)$ in seven different ways. Without loss of generality we take all the cases one by one as follows.

Case 1. Let $e_1 = \{v_{\alpha+1}, v_1\}$ and $e_2 = \{v_{\alpha+1}, v_2\}$. Then, $|N_S(v_{\alpha+1})| = \alpha-2$, which is a contradiction, as for any $v \in V(G)\setminus S$, $|N_S(v)| \geq \alpha-1$.

Case 2. Let $e_1 = \{v_1, v_{\alpha+1}\}$ and $e_2 = \{v_2, v_{\alpha+2}\}$. Then, $\overline{G}$ contains $n-\alpha-2$ isolated vertices and a connected component with the vertex set $\{v_1, v_2, \ldots, v_\alpha, v_{\alpha+1}, v_{\alpha+2}\}$ which is not bipartite as it contains a triangle, as $\alpha \geq 3$. By Lemma 13, $m_{\mathcal{D}^Q(G)}(n-2) < n-\alpha-1$.

Case 3. Let $e_1 = \{v_1, v_{\alpha+1}\}$ and $e_2 = \{v_1, v_{\alpha+2}\}$. As in Case 2, $\overline{G}$ has exactly $n-\alpha-1$ components which include one connected component that is non-bipartite and $n-\alpha-2$ isolated vertices. Thus, by Lemma 13, $m_{\mathcal{D}^Q(G)}(n-2) < n-\alpha-1$.

Case 4. Let $e_1 = \{v_1, v_{\alpha+1}\}$ and $e_2 = \{v_{\alpha+2}, v_{\alpha+1}\}$. Proceeding as in Case 3, we get the required inequality.

Case 5. Let $e_1 = \{v_{\alpha+1}, v_{\alpha+2}\}$ and $e_2 = \{v_{\alpha+3}, v_1\}$, if $n-\alpha \geq 3$. Then, $\overline{G}$ contains $n-\alpha-3$ isolated vertices, one copy of K_2 and a non-bipartite component on the vertex set $\{v_1, v_2, \ldots, v_\alpha, v_{\alpha+3}\}$. This shows that $\overline{G}$ has exactly $n-\alpha-2$ components which are either isolated vertices or bipartite. Using Lemma 13, proves the result in this case.

Case 6. Let $e_1 = \{v_{\alpha+1}, v_{\alpha+2}\}$ and $e_2 = \{v_{\alpha+1}, v_{\alpha+3}\}$, if $n-\alpha \geq 3$. In this case $\overline{G}$ contains K_α, $K_{1,2}$ and $n-\alpha-3$ isolated vertices. Using the same reasoning as in the above cases proves the result.

Case 7. Let $e_1 = \{v_{\alpha+1}, v_{\alpha+2}\}$ and $e_2 = \{v_{\alpha+3}, v_{\alpha+4}\}$, if $n-\alpha \geq 4$. Then, $\overline{G}$ contains K_α, two copies of K_2 and $n-\alpha-4$ isolated vertices. This shows that $\overline{G}$

has exactly $n-\alpha-2$ components which are either isolated vertices or bipartite. Using Lemma 13, proves the result in this case. ■

Now, we obtain the bounds for the dsL spectral radius of a graph G in terms of the order n and the independence number α. We also characterize the corresponding extremal graphs.

Theorem 15. *Let G be a connected graph with n vertices having independence number α. Then*

$$\partial_1^Q \geq \frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2}. \tag{2.13}$$

Equality holds if and only if $G \cong CS(n,\alpha)$.

Proof. First, let $\alpha=1$. Then $G \cong K_n$ and it is easy to see that $K_n = CS(n,1)$. We know that $\partial_1^Q(K_n) = 2n-2$. When we put $\alpha = 1$ in the right hand side of Inequality (2.13), it comes out to be $2n-2$ which shows that equality holds for K_n. If $\alpha = n$, then G reduces to an isolated vertex because G is connected. So, let $2 \leq \alpha \leq n-1$. As the graph G has independence number α, it can be considered as a spanning subgraph of $CS(n,\alpha)$. Lemma 4 shows that equality always holds whenever $G \cong CS(n,\alpha)$. Thus, to prove the result, we need to show that Inequality (2.13) is strict when G is a proper spanning subgraph of $CS(n,\alpha)$. Using Lemma 2, we only need to prove that Inequality (2.13) is strict when $G = CS(n,\alpha)-e$, where e is any edge of $CS(n,\alpha)$. We know that the dsL matrix corresponding to any connected graph H is symmetric, positive and irreducible. Therefore, by the Perron-Frobenius Theorem, $\partial_1^Q(H-uv) > \partial_1^Q(H)$ whenever $uv \in E(H)$ and $H-uv$ is connected. Using this observation, we get

$$\partial_1^Q(CS(n,\alpha)-e) > \partial_1^Q(CS(n,\alpha)) = \frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2}$$

which proves the result. ■

Theorem 16. *If G be a connected graph with n vertices having independence number $\alpha \geq \frac{n+4}{3}$. Then*

$$\partial_1^Q(G) \geq 2n+\alpha-2 \tag{2.14}$$

with equality if and only if $n+4-3\alpha = 0$ and $G \cong CS(n,\alpha)$.

Proof. We claim that

$$\frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} \geq 2n+\alpha-2. \tag{2.15}$$

If possible, suppose that

$$\frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} < 2n+\alpha-2,$$

which implies that $\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2} < n+2$, or $12\alpha^2-16\alpha-4\alpha n < 0$, or $\alpha < \frac{n+4}{3}$, which is a contradiction as $\alpha \geq \frac{n+4}{3}$. This proves the claim.

Now, from Theorem 15 and Inequality (2.15), we get

$$\partial_1^Q(G) \geq \frac{3n+2\alpha-6+\sqrt{8\alpha^2-8\alpha+(n-2\alpha+2)^2}}{2} \geq 2n+\alpha-2,$$

which proves the required inequality.

We observe that equality holds in the Inequality (2.14) whenever equality holds in both Theorem 15 and Inequality (2.15). It can be easily seen that equality holds in the Inequality (2.15) if and only if $n+4-3\alpha=0$. This fact and the Theorem 15 shows that equality holds in the Inequality (2.14) if and only if $n+4-3\alpha=0$ and $G \cong CS(n,\alpha)$. This completes the proof. ■

3. Distribution of the dsL Eigenvalues and the Chromatic Number

The following lemmas which will be used in sequel.

Lemma 17 [1]**.** *Let $H = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph with $n_1 = \cdots = n_k$ and $n = kn_1$. Then the dsL spectrum of H is given as*

$$DSLS(H) = \begin{pmatrix} 2n+2n_1-4 & n+2n_1-4 & n+n_1-4 \\ 1 & k-1 & n-k \end{pmatrix}.$$

Lemma 18 [3]**.** *Let G ba a connected graph on $n \geq 3$ vertices with diameter $d \geq 2$. Let $\partial_n^Q(G) \leq \partial_{n-1}^Q(G) \leq \cdots \leq \partial_1^Q(G)$ and $\overline{q}_n \leq \overline{q}_{n-1} \leq \cdots \leq \overline{q}_1$ be the dsL eigenvalues of G and the signless Laplacian eigenvalues of the complement $\overline{G}$ of G, respectively. Then*

$$\partial_i^Q(G) \geq n-2+\overline{q}_i \quad \text{for every } 1 \leq i \leq n.$$

Lemma 19 [14]**.** *Let $G = K_{n_1,n_2,\ldots,n_t}$ be a complete t-partite graph with $n = n_1+n_2+\cdots+n_t$ vertices for $t \geq 2$. Let $P_{\mathcal{D}^Q(G)}(\lambda)$ be the characteristic polynomial of $\mathcal{D}^Q(G)$. Then*

$$P_{\mathcal{D}^Q(G)}(\lambda) = \prod_{i=1}^{t}(\lambda-n-n_i+4)^{n_i-1}\left[\prod_{i=1}^{t}(\lambda-n-2n_i+4)-\sum_{i=1}^{t} n_i \prod_{j=1, j\neq i}^{t}(\lambda-n-2n_j+4)\right].$$

Theorem 20. *Let G be a connected graph of order n and chromatic number $\chi \leq \sqrt{n}$. If the color classes are of the same cardinality, then*

$$\partial_1^Q(G) \geq 2n + 2\chi - 4 \tag{3.16}$$

with equality if and only if $G \cong \underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}$.

Proof. For $n = 1$, the result is trivial. So, let $n \geq 2$. So $G \ncong K_n$ as $\chi \leq \sqrt{n}$. Let b be the cardinality of each color class so that $b = \frac{n}{\chi}$. Since $G \ncong K_n$, therefore, $b \geq 2$. Now, let H be the graph obtained from G by joining every two non-adjacent vertices, if there are any, which fall in different color classes. Clearly, $H = \underbrace{K_{b,b,\dots,b}}_{\chi}$ and we can consider G as a spanning subgraph of H. Using Lemma 2 and Lemma 17, we get

$$\partial_1^Q(G) \geq \partial_1^Q(H) = 2n + 2b - 4. \tag{3.17}$$

As $\chi \leq \sqrt{n}$, we see that $b = \frac{n}{\chi} \geq \chi$. Using this observation in Inequality (3.17), we get

$$\partial_1^Q(G) \geq 2n + 2\chi - 4,$$

which proves Inequality (3.16).

From Lemma 17, we see that $\partial_1^Q(\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}) = 2n + 2\chi - 4$ which shows that equality holds for $\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}$. To complete the proof, we will show that Inequality (3.16) is strict when $G \ncong \underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}$. Since $b = \frac{n}{\chi} \geq \chi$, therefore, we have the following two cases.

Case 1. Let $b = \frac{n}{\chi} > \chi$. Using the fact that G is a spanning subgraph of H and Lemmas 2 and 17, we get

$$\partial_1^Q(G) \geq \partial_1^Q(H) = 2n + 2b - 4 > 2n + 2\chi - 4,$$

which proves the result in this case.

Case 2. Let $b = \frac{n}{\chi} = \chi$. In this case we see that G is a proper spanning subgraph of $\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}$. Using Lemma 2, we only need to show that Inequality (3.16) is strict when $G = \underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi} - e$, where e is any edge in $\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}$. Since

$\mathcal{D}^Q(G)$ is positive, symmetric and irreducible, therefore, by the Perron-Frobenius theorem, we get

$$\partial_1^Q(\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi} - e) > \partial_1^Q(\underbrace{K_{\chi,\chi,\dots,\chi}}_{\chi}) = 2n + 2\chi - 4,$$

which proves the result in this case. This completes the proof. ■

As a consequence of Theorem 20, we have the following result for the class of balanced bipartite graphs.

Corollary 21. *Let G be a balanced bipartite graph on $n \geq 4$ vertices. Then $\partial_1^Q(G) \geq 2n$ with equality if and only if $G \cong K_{2,2}$.*

Theorem 22. *Let G be a connected graph with $n \geq 5$ vertices having chromatic number χ. Let $n_1 \geq \cdots \geq n_\chi$ be the cardinalities of the color classes of G with $n_1 \geq 3$ and $2n_\chi \geq n_1$. Then*

$$m_{\mathcal{D}^Q(G)}\Big[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil - 4\Big) \leq n - \chi - \left\lceil \frac{n}{\chi} \right\rceil + 1. \tag{3.18}$$

Proof. First, we show that the interval $\Big[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil - 4\Big)$ is well defined. As $n_1 \geq 3$ and $2n_\chi \geq n_1$, it is easy to see that $\chi < \frac{n}{2}$, which shows that $\frac{n}{\chi} > 2$. Thus, $\left\lceil \frac{n}{\chi} \right\rceil \geq 3$ so that $n + \left\lceil \frac{n}{\chi} \right\rceil - 4 \geq n - 1$. This shows that the interval in Inequality (3.18) is well defined. Since G has chromatic number χ with cardinalities of the color classes given by $n_1 \geq \cdots \geq n_\chi$, therefore, G can be considered as a spanning subgraph of $H = K_{n_1,n_2,\dots,n_\chi}$. Using Lemma 2 and Lemma 3, we get

$$m_{\mathcal{D}^Q(G)}\Big[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil - 4\Big) \leq m_{\mathcal{D}^Q(H)}\Big[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil - 4\Big). \tag{3.19}$$

From Lemma 19, exactly $n-\chi$ dsL eigenvalues of H are known. In particular, the eigenvalue $n + n_1 - 4$ is of multiplicity $n_1 - 1$. Using Lemma 18, we see that the rest of the χ eigenvalues of H are all greater than or equal to $n + 2n_\chi - 4$. Also, it is easy to see that $n_1 \geq \left\lceil \frac{n}{\chi} \right\rceil$, so that $n + 2n_\chi - 4 \geq n + n_1 - 4 \geq n + \left\lceil \frac{n}{\chi} \right\rceil - 4$. By applying these observations and Lemma 19, we get

$$m_{\mathcal{D}^Q(H)}\Big[n-2, n+\left\lceil \frac{n}{\chi} \right\rceil - 4\Big) \leq n - \chi - n_1 + 1 \leq n - \chi - \left\lceil \frac{n}{\chi} \right\rceil + 1.$$

Using the above inequality in Inequality (3.19) completes the proof. ■

After using Lemma 3 and Inequality (2.1), we have the following consequence of Theorem 22.

Corollary 23. *Let G be a connected graph with $n \geq 5$ vertices having chromatic number χ. Let $n_1 \geq \cdots \geq n_\chi$ be the cardinalities of the color classes of G with $n_1 \geq 3$ and $2n_\chi \geq n_1$. Then*

$$m_{\mathcal{D}^Q(G)}\left[n + \left\lceil \frac{n}{\chi} \right\rceil - 4, 2Tr_{\max}\right] \geq \chi + \left\lceil \frac{n}{\chi} \right\rceil - 1.$$

We have the following lemma.

Lemma 24. *Let G be a connected graph with $n \geq 6$ vertices having chromatic number χ. Let $n_1 \geq \cdots \geq n_\chi$ be the cardinalities of the color classes of G. If $n_i \geq 3$ for all $1 \leq i \leq \chi$, then $n-2$ cannot be a dsL eigenvalue of G.*

Proof. If possible, suppose that $n-2$ is a dsL eigenvalue of G with multiplicity $t \geq 1$. By Lemma 13, $\overline{G}$ must contain at least t components, each of which is bipartite or an isolated vertex. Since $n_i \geq 3$ for all $1 \leq i \leq \chi$, therefore, each component of $\overline{G}$ contains at least three mutually adjacent vertices, that is, a triangle. Thus, no component of $\overline{G}$ is bipartite or an isolated vertex, a contradiction. This proves the result. ■

The following result shows that we can improve the Inequality (3.18) in Theorem 22 whenever and $n_i \geq 3$ for all $1 \leq i \leq \chi$.

Theorem 25. *Let G be a connected graph with $n \geq 6$ vertices and having chromatic number χ. Let $n_1 \geq \cdots \geq n_\chi \geq 3$ be the cardinalities of the color classes of G with $2n_\chi \geq n_1$. Then*

$$m_{\mathcal{D}^Q(G)}\left(n-2, n + \left\lceil \frac{n}{\chi} \right\rceil - 4\right) \leq n - \chi - \left\lceil \frac{n}{\chi} \right\rceil + 1.$$

Proof. It is given that $n_i \geq 3$ for all $1 \leq i \leq \chi$. Using Lemma 24, we see that $n-2$ cannot be a dsL eigenvalue of G. The rest of the proof follows by Theorem 22. ■

4. Concluding Remarks

Although we were able to show that the bound in Theorem 5 as well as in Theorem 14 is best possible but still all the graphs satisfying the respective bounds have not been characterized. So in this direction, we propose the following research problems.

Problem 1. Determine the families of graphs ϑ for which $m_{\mathcal{D}^Q(G)}[n-2, n+\alpha-4) = n - \alpha$, for any $G \in \vartheta$.

Problem 2. Determine the families of graphs ϑ for which $m_{\mathcal{D}^Q(G)}(n-2) = n-\alpha-1$, for any $G \in \vartheta$.

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