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(I, F)-PARTITION OF PLANAR GRAPHS WITHOUT CYCLES OF LENGTH 4, 6, OR 9

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Abstract

A graph G is (I, F)-partitionable if its vertex set can be partitioned into two parts such that one part is an independent set, and the other induces a forest. A k-cycle is a cycle of length k. A 9-cycle $[v_1v_2 \cdots v_9]$ of a plane graph is called special if its interior contains either an edge v_1v_4 or a common neighbor of v_1, v_4, and v_7. In this paper, we prove that every plane graph with neither 4- or 6-cycles nor special 9-cycles is (I, F)-partitionable. As corollaries, for each $k \in \{8, 9\}$, every planar graph without cycles of length from $\{4, 6, k\}$ is (I, F)-partitionable and consequently, they are also signed 3-colorable.

Keywords: planar graph, (I, F)-partition, super-extension, bad cycle, discharging.

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1. Introduction

Graphs considered in this paper are finite and simple. A graph G is *k-degenerate* if every subgraph H of G contains a vertex of degree at most k in H. Clearly, every k-degenerate graph is $(k+1)$-colorable. Let p and q be two nonnegative integers. A graph G is *(p,q)-partitionable* if $V(G)$ can be partitioned into two subsets which induce a p-degenerate subgraph and a q-degenerate subgraph of G, respectively. Thomassen [15, 16] proved that planar graphs are both $(1,2)$-partitionable and $(0,3)$-partitionable.

A graph G is *(I,F)-partitionable* (also called *near-bipartitionable*) if its vertex set can be partitioned into two parts such that one part is an independent set and the other induces a forest. By definition, (I,F)-partition is exactly $(0,1)$-partition, and every (I,F)-partitionable graph is 3-colorable. Hence, it is of interest to see which 3-color theorem can be strengthened in the context of (I,F)-partition.

Borodin and Glebov [1] confirmed that every planar graph of girth at least 5 is (I,F)-partitionable. Kawarabayashi and Thomassen [10] proved an extension of this result and guessed it might be true that every triangle-free planar graph is (I,F)-partitionable.

Conjecture 1 [10]**.** *Every triangle-free planar graph is (I,F)-partitionable.*

The famous Steinberg conjecture, proposed in 1976 (open Problem 2.9 in [6]) and disproved in 2016 [4], states that every planar graph without cycles of length 4 or 5 is 3-colorable. It has motivated a lot of research on 3-coloring of planar graphs with restriction on short cycles. It can be concluded from literature that for integers $4 < i < j < k < 10$, planar graphs without cycles of length from $\{4, i, j, k\}$ are 3-colorable. Further studies give partial results to the following question.

Problem 2. For which pair of integers (i, j) with $4 < i < j < 10$, every planar graph without cycles of length from $\{4, i, j\}$ is 3-colorable?

This question was answered in the affirmative for pairs $(i, j) \in \{(5,7), (5,8), (6,7), (6,8), (6,9), (7,9)\}$ [2, 3, 7, 8, 13, 17, 18], and the question for the remaining cases of (i, j) is still open.

This paper is interested in the following generalized form of Problem 2 and proves a partial result on it.

Problem 3. For which pair of integers (i, j) with $4 < i < j < 10$, every planar graph without cycles of length from $\{4, i, j\}$ is (I,F)-partitionable?

Consider a plane graph G. A vertex is *external* if it lies on the boundary of the unbounded face; *internal* otherwise. For a cycle C, let $\mathrm{int}(C)$ and $\mathrm{ext}(C)$

denote the set of vertices in the interior and exterior of C, respectively. A cycle C is *separating* if both $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ are nonempty. Denote by $\mathrm{int}[C]$ (respectively, $\mathrm{ext}[C]$) the subgraph of G consisting of C and its interior (respectively, C and its exterior).

Denote by $G[S]$ the subgraph of a graph G induced by a set S with $S \subseteq V(G)$ or $S \subseteq E(G)$. Given two disjoint subgraphs H_1 and H_2 of a graph G, denote by $E_G(H_1, H_2)$ the set of edges of G connecting a vertex of H_1 to a vertex of H_2.

Definition. Let C be a cycle of a plane graph G. An edge of $\mathrm{int}[C]$ connecting two non-consecutive vertices of C is called a *chord* of C. If a vertex $v \in \mathrm{int}(C)$ has three neighbors v_1, v_2, v_3 on C, then $G[\{vv_1, vv_2, vv_3\}]$ is called a *claw* of C. If $u \in \mathrm{int}(C)$ has two neighbors u_1 and u_2 on C, $v \in \mathrm{int}(C)$ has two neighbors v_1 and v_2 on C, and $uv \in E(G)$, then $G[\{uv, uu_1, uu_2, vv_1, vv_2\}]$ is called a *biclaw* of C. If each of three pairwise adjacent vertices $u, v, w \in \mathrm{int}(C)$ has a neighbor on C, say u', v', w' respectively, then $G[\{uv, vw, uw, uu', vv', ww'\}]$ is called a *triclaw* of C. The cycles into which a chord, a claw, a biclaw, or a triclaw divides C are called *cells*. A cell of length c_i is called a *c_i-cell*. We further call a (c_1, c_2)-chord, a (c_1, c_2, c_3)-*claw*, a (c_1, c_2, c_3, c_4)-*biclaw*, or a (c_1, c_2, c_3, c_4)-*triclaw*, as depicted in Figure 1.

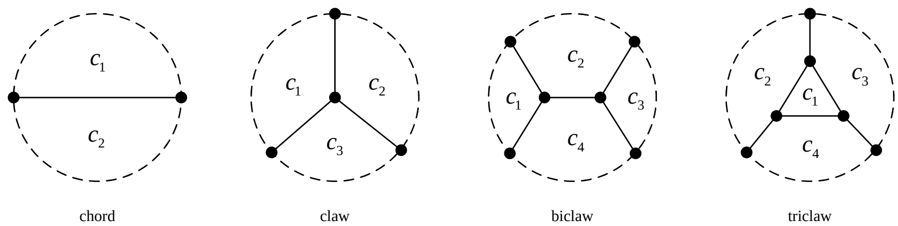

Figure 1. A cycle C in dotted line and a chord, a claw, a biclaw, and a triclaw of C in solid line.

A *k-cycle* is a cycle of length k. A 9-cycle of a plane graph is *special* if it has a $(3, 8)$-chord or a $(5, 5, 5)$-claw. Let $\mathcal{G}$ denote the class of connected plane graphs with neither 4- or 6-cycles nor special 9-cycles.

The following theorem is the main result of this paper.

Theorem 4. *Every graph of $\mathcal{G}$ is (I, F)-partitionable.*

Liu and Yu [11] proved that planar graphs without cycles of length 4, 6, or 8 are (I, F)-partitionable, which is the only known partial result to Problem 3. Lu *et al.* [12] proved an extension of the result of Liu and Yu. Theorem 4 not only extends the result of Liu and Yu, but also implies a new partial result to Problem 3 as follows.

Corollary 5 [11]**.** *Every planar graph without cycles of length* $4, 6,$ *or* 8 *is* (I, F)*-partitionable.*

Corollary 6. *Every planar graph without cycles of length* $4, 6,$ *or* 9 *is* (I, F)*-partitionable.*

A *signed graph* is a pair (G, σ), where G is a graph and $\sigma\colon\ E(G) \to \{1, -1\}$ is a signature of G. The study on coloring of signed graph was initiated by Zaslavsky in the 1980's and has attracted some recent attention. For a positive integer k, let Z_k be the cyclic group of order k, and let $M_k = \{\pm 1, \ldots, \pm p\}$ if $k = 2p$ is even and $M_k = \{0, \pm 1, \ldots, \pm p\}$ if $k = 2p + 1$ is odd. A *k-coloring* of (G, σ) is a mapping $f\colon\ V(G) \to M_k$ such that $f(u) \neq \sigma(e)f(v)$ for each edge $e = uv$. A *Z_k-coloring* of (G, σ) is a mapping $f\colon\ V(G) \to Z_k$ such that $f(u) \neq \sigma(e)f(v)$ for each edge $e = uv$. These two definitions were introduced respectively by Máčajová, Raspaud, and Škoviera [14] and by Kang and Steffen [9]. These two definitions are differ for any even k but equivalent for any odd k. A graph G is *signed k-colorable* if (G, σ) has a k-coloring for any signature σ of G.

We remark that every (I, F)-partitionable graph is signed 3-colorable. This is because no matter what the signature σ of an (I, F)-partitionable graph G is, assigning the independent set part with the color 0 and properly coloring the forest part by color set $\{1, -1\}$ yields a proper 3-coloring of the signed graph (G, σ).

Some 3-color problems were asked in the context of signed 3-coloring. The following question stands in the middle of Problems 2 and 3.

Problem 7. For which pair of integers (i, j) with $4 < i < j < 10$, every planar graph without cycles of length from $\{4, i, j\}$ is signed 3-colorable?

Hu and Li [5] proved that planar graphs without cycles of length from 4 to 8 are signed 3-colorable. Notice that the result of Liu and Yu [11] implies that planar graphs without cycles of length 4, 6, or 8 are signed 3-colorable, which extends the result of Hu and Li. This is the only known partial result to Problem 7.

The following corollary is a direct consequence of Corollary 6, which provides a new partial result to Problem 7.

Corollary 8. *Every planar graph without cycles of length* 4, 6, *or* 9 *is signed* 3*-colorable.*

The structure of the remaining part of the paper is as follows. In Section 2, both the method of super-extended theorem and the technique of bad cycle, which were usually used for solving 3-color problem, are extended to the context of (I, F)-partition. We address the statement of the super-extended theorem, which strengthens Theorem 4. In Section 3, the proof of the super-extended theorem is given by using discharging method. The proof follows a similar way as in [8]. More precisely, for the minimal counterexample to the super-extended theorem, we prove all the necessary reducible configurations proposed in [8] and consequently, the final contradiction can be derived by exactly the same argument of the discharging part of [8]. For the seek of completeness, we provide the discharging part in the section of Appendix.

2. Super-Extended Theorem and Terminology

Denote by $d(v)$ the degree of a vertex v, $|C|$ the length of a cycle C, $|f|$ the size of a face f, and $|P|$ the number of edges a path P contains. Let k be a positive integer. A *k-vertex* (respectively, *k^+-vertex*, and *k^--vertex*) is a vertex v with $d(v) = k$ (respectively, $d(v) \geq k$, and $d(v) \leq k$). Similar definition is applied for cycle, face, and path by constitution $|C|$, $|f|$, and $|P|$ for $d(v)$, respectively.

An *(I, F)-coloring* of a graph G is a mapping from $V(G)$ to the color set $\{I, F\}$ such that vertices of the color I is an independent set and vertices of the color F induce a forest. A vertex of color F is called an *F-vertex*. A path or cycle on only F-vertices is called an *F-path* or *F-cycle*, respectively. An *I-edge* is an edge whose ends are both I-vertices. Let H be a subgraph of a graph G and ϕ be an (I, F)-coloring of H. A *super-extension* of ϕ to G is an (I, F)-coloring of G whose restriction on H is ϕ such that $G - E(H)$ contains no F-path connecting two vertices of H.

Remark 9. Let H_2 be a subgraph of a graph H_3, H_1 be a subgraph of H_2, and ϕ_1 be an (I, F)-coloring of H_1. If ϕ_i is a super-extension of ϕ_{i-1} to H_i for each $i \in \{2, 3\}$, then ϕ_3 is a super-extension of ϕ_1 to H_3.

Proof. By assumption, ϕ_3 is an (I, F)-coloring of H_3, and the restriction of ϕ_3 in H_1 is exactly ϕ_1. So, it suffices to show that $H_3 - E(H_1)$ contains no F-path connecting two vertices of H_1. Otherwise, let P be such an F-path. Since ϕ_2 is a super-extension of ϕ_1 to H_1, P is not a subgraph of H_2. Then $P - E(H_2)$ is an F-path of $H_3 - E(H_2)$ connecting two vertices of H_2, contradicting that ϕ_3 is a super-extension of ϕ_2 to H_3. ■

Given a plane graph G, denote by $D(G)$ the boundary of the unbounded face of G. A *good cycle* is a cycle of length at most 12 which has none of claws, biclaws and triclaws. A *bad cycle* is a cycle of length at most 12 which is not good.

We will prove the following theorem, which strengthens Theorem 4.

Theorem 10 (Super-extended theorem)**.** *Let $G \in \mathcal{G}$. If $D(G)$ is a good cycle, then every (I, F)-coloring of $G[V(D(G))]$ can super-extend to G.*

To see that Theorem 4 follows from Theorem 10, take any graph $G \in \mathcal{G}$. If G has no triangles, then it has girth at least 5 and is known to be (I,F)-partitionable [1]. So, let T be a triangle of G. If there is a 10^--cycle containing T inside, then let C be the outermost one, that is, the one which is contained in the interior of no other 10^--cycles; otherwise, let $C = T$. Take any (I,F)-coloring ϕ of $G[V(C)]$. Denote by H the plane graph obtained from $\text{ext}[C]$ by re-embedding it so that C is the boundary of the unbounded face of H. Suppose that $H \notin \mathcal{G}$. Since $\text{ext}[C] \in \mathcal{G}$, it is only possible that the re-embedding makes a non-special 9-cycle (say C') of $\text{ext}[C]$ be special in H. It follows that C' contains C inside, a contradiction to the choice of C. Therefore, $H \in \mathcal{G}$. Moreover, since $|C| \leq 10$, it is easy to check by definition that C is a good cycle in both $\text{int}[C]$ and H. By Theorem 10, ϕ can super-extend to both $\text{int}[C]$ and H. This results in an (I,F)-coloring of G.

The remainder of this section is devoted to some necessary definitions and terminology.

Consider a plane graph G. A path or a cycle C is *triangular* if it has an edge as the common part between C and some triangle. A cycle C is *ext-triangular* if it has an edge as the common part between C and some triangle of $\text{ext}[C]$. A path is a *splitting path* of a cycle C if its two end-vertices locate on C and all other vertices locate inside C. A *directed path* $\vec{P} = v_1v_2\cdots v_k$ is the path on vertices $v_1, v_2, \ldots, v_k$ with direction $v_1 \to v_2 \to \cdots \to v_k$, and P denotes the undirected path associated with $\vec{P}$. Given an (I,F)-coloring of G, a *defective segment* means an I-edge, an F-cycle, or a splitting F-path of $D(G)$.

Remark 11. Given a plane graph G, a subgraph H of G which contains $D(G)$, and an (I,F)-coloring ϕ of H which is a super-extension from $D(G)$, for any vertex-induced subgraph U of G with $V(U) \cap V(H) = \emptyset$, if assigning each vertex of U with a color from $\{I,F\}$ brings no defective segments, that is, each vertex of U is contained in no defective segments, then the resulting coloring of $H + U + E_G(H,U)$ is a super-extension from $D(G)$.

Consider a plane graph G, a subgraph H of G which contains $D(G)$, and an (I,F)-coloring ϕ of H. Let u be an uncolored vertex which has at most two neighbors locating in H. *Nicely coloring* u means assigning u with the color I if u has no neighbors of color I, and assigning u with the color F otherwise. Let $\vec{P} = v_1v_2\cdots v_k$ $(k \geq 2)$ be a vertex-induced directed path of $G - V(H)$ such that v_i has precisely one neighbor (say t_i) locating in H for each $i \in \{1, 2, \ldots, k\}$. *$I$-nicely coloring* (respectively, *F-nicely coloring*) $\vec{P}$ means assigning v_i with the color F for each i with $\phi(t_i) = I$ and then assigning all the remaining vertices of P with I and F alternately (respectively, with F and I alternately) along $\vec{P}$. It is easy to deduce the following two properties, which will be used often for the proof of reducible configurations in Section 3.1.

(1) Each of nicely coloring u, I-nicely coloring $\vec{P}$, and F-nicely coloring $\vec{P}$ brings no defective segments and therefore, the resulting coloring is a super-extension of ϕ by Remark 11.

(2) For the case of I-nicely coloring $\vec{P}$, let x be an uncolored vertex adjacent to v_1, and let $t = k$ if x has no other neighbors on P; otherwise, let $t \in \{2, 3, \ldots, k\}$ be the minimum such that $v_t x \in E(G)$. If $v_1 v_2 \cdots v_t$ is not an F-path, then assigning x with F brings no defective segment which contains the edge $v_1 x$.

Given a plane graph G and an (I, F)-coloring of G, a pair of vertices (u, v) is *F-linked* if at least one of the following holds.

(1) There exists an F-path between u and v.

(2) There exist two vertex-disjoint F-paths, one connects u with an external vertex, and the other connects v with another external vertex.

3. The Proof of Theorem 10

We shall prove Theorem 10 by contradiction. Let G be a counterexample to Theorem 10 with minimum $|V(G)| + |E(G)|$. Thus, the boundary D of the unbounded face f_0 of G is a good cycle, and there exists an (I, F)-coloring ϕ_0 of $G[V(D)]$ which cannot super-extend to G.

3.1. Reducible configurations

Lemma 12. *D has no chords.*

Proof. Otherwise, let e be a chord of D, which divides D into two cycles, say D_1 and D_2. By the minimality of G, the restriction of ϕ_0 in $D \cap D_i$ can super-extend to $\mathrm{int}[D_i]$ for $i \in \{1, 2\}$. It is easy to verify by definition that the resulting coloring of G is a super-extension of ϕ_0, a contradiction. ■

Lemma 13. *Every internal vertex of G has degree at least* 3.

Proof. Otherwise, let v be an internal vertex with $d(v) \leq 2$. The pre-coloring ϕ_0 can super-extend to $G - v$ by the minimality of G, and further to G by nicely coloring v. ■

Lemma 14. *G has no separating good cycles.*

Proof. Suppose to the contrary that C is a separating good cycle of G. Let $H_1 = G - \mathrm{int}(C)$ and $H_2 = \mathrm{int}[C]$. By the minimality of G, ϕ_0 can super-extend to H_1, and the resulting coloring of C can super-extend to H_2, which can restate by planarity that the resulting coloring of H_1 can super-extend to H_2. By Remark 9, the resulting coloring of G is a super-extension of ϕ_0, a contradiction. ■

The following three lemmas can be concluded easily.

Lemma 15. *Every 9^--cycle of G is facial except that an 8-cycle of G might have a $(3,7)$- or $(5,5)$-chord.*

Lemma 16. *Let $H \in \mathcal{G}$. If C is a bad cycle of H, then C has length either 11 or 12. Furthermore, if $|C| = 11$, then C has a $(3,7,7)$- or $(5,5,7)$-claw; if $|C| = 12$, then C has a $(5,5,8)$-claw, a $(3,7,5,7)$- or $(5,5,5,7)$-biclaw, or a $(3,7,7,7)$-triclaw.*

Lemma 17. *Every bad cycle C of G is adjacent to at most one triangle. Furthermore, if C is ext-triangular, then C has a $(5,5,7)$-claw or $(5,5,5,7)$-biclaw.*

Lemma 18. *G is 2-connected.*

Proof. Otherwise, we may assume that G has a pendant block B with a cut vertex v such that $B - v$ does not intersect with D. By the minimality of G, ϕ_0 can super-extend to $G - (B - v)$. Consider only B. We distinguish two cases as follows. If v is contained in a 10^--cycle, then take the outermost one, that is, the one which is contained in the interior of no other 10^--cycles, denoted by C. Lemma 16 implies that C is good and therefore, the coloring of v can extend to an (I, F)-coloring of $B[V(C)]$, which can further super-extend to both the interior and exterior (if not empty) of C in B. This results in an (I, F)-coloring of B. It remains to assume that v is contained in no 10^--cycles. Insert into the unbounded face f of B an edge e between the two neighbors of v on f, creating a 3-face, say T. Note that the embedding of $B + e$ in the plane which takes T as the unbounded face belongs to $\mathcal{G}$. Similarly, the coloring of v can extend to an (I, F)-coloring of T and can further super-extend to $B + e$. In either case, the resulting coloring of G is a super-extension of ϕ_0, a contradiction. ■

Lemma 19. *Let P be a splitting path of D, which divides D into two cycles D' and D''. If $2 \le |P| \le 5$, then at least one of D' and D'' has length $|P| + 1$ to $2|P| - 1$. More precisely, since $G \in \mathcal{G}$,*

(1) *if $|P| = 2$, then at least one of D' and D'' is a triangle;*

(2) *if $|P| = 3$, then at least one of D' and D'' is a 5-cycle;*

(3) *if $|P| = 4$, then at least one of D' and D'' is a 5- or 7-cycle;*

(4) *if $|P| = 5$, then at least one of D' and D'' is a 7-, 8-, or 9-cycle.*

Proof. Suppose to the contrary that $|D'|, |D''| \ge 2|P|$. Since D has length at most 12, $|D'| + |D''| = |D| + 2|P| \le 12 + 2|P|$. It follows that $2|P| \le |D'|, |D''| \le 12$.

(1) Let $P = xyz$. By Lemma 13, y has a neighbor y' other than x and z. If y' is external, then D has a claw, a contradiction. So, y' lies inside D' or D'',

w.l.o.g., say D'. By Lemma 14, D' is a bad cycle. Moreover, since G has no 4-cycles, $5 \leq |D'|, |D''| \leq 11$. Hence by Lemma 16, D' has a claw, which yields that D has a biclaw, a contradiction.

(2) Let $P = wxyz$. We may let x' and y' be neighbors of x and y with $\{xx', yy'\} \cap E(P) = \emptyset$, respectively. If both x' and y' are external, then D has a biclaw, a contradiction. So, without loss of generality, let x' lie inside D'. Moreover, since G has no 6-cycles, $7 \leq |D'|, |D''| \leq 11$. Hence by Lemmas 14 and 16, D' is a bad 11-cycle with a claw and D'' is a 7-face. So, y' has no choices but coincides with x'. Now, D has a triclaw, a contradiction.

(3) Let $P = vwxyz$. In this case, $8 \leq |D'|, |D''| \leq 12$. We claim that G has no edge connecting two non-consecutive vertices on P. Otherwise, such an edge e together with P forms a triangle as well as a splitting 3-path of D. By the statement (2), we can deduce that e is a (3,5)-chord of D', a contradiction.

Let w', x', and y' be neighbors of w, x, and y with $\{ww', xx', yy'\} \cap E(P) = \emptyset$, respectively. Clearly, x' lies in $\mathrm{int}[D']$ or $\mathrm{int}[D'']$, without loss of generality, say $\mathrm{int}[D']$. If x' is external, then both the paths $vwxx'$ and $x'xyz$ are splitting 3-paths of D. By the statement (2), D' is an 8-cycle with a (5,5)-chord xx'. Hence, y' has no choice for its location but to lie inside D'', and so does w'. So, D'' is a bad cycle and by Lemma 16, either $w' = y'$ which yields a 4-cycle or $w'y' \in E(G)$ which yields a special 9-cycle with a $(5, 5, 5)$-claw, a contradiction. It remains to assume that $x' \in \mathrm{int}(D')$. Thus, D' is a bad cycle, which implies that D'' has length 8 or 9. For $|D''| = 9$, D'' is facial and D' is a bad 11-cycle with a claw, which is impossible because of the locations of w', x' and y'. For $|D''| = 8$, at least one of w' and y' lies in $\mathrm{int}[D']$, which together with x' yields either a 4-cycle or a special 9-cycle with a $(3, 8)$-chord, a contradiction.

(4) Let $P = uvwxyz$. In this case, $10 \leq |D'|, |D''| \leq 12$. By a similar argument as in the case (3), one can conclude that G has no edge connecting two nonconsecutive vertices on P. Let v', w', x', y' be neighbors of v, w, x, y not on P, respectively.

We claim that both w' and x' are internal. Otherwise, let $w' \in V(D')$. Since both $uvww'$ and $w'wxyz$ are splitting paths of D, D' is a 10-cycle with a (5,7)-chord ww'. If $x' \in V(D'')$, then similarly, D'' is a 10-cycle with a (5,7)-chord xx', which yields no locations for v' and y'. Hence, $x' \in \mathrm{int}(D'')$. Moreover, $v' \in \mathrm{int}(D'')$ since otherwise, uvv' is a splitting 2-path of D which yields a triangle adjacent to a 5-cycle. Therefore, $v'x' \in E(G)$ and D'' is a bad 12-cycle with a biclaw, which yields no location for y'.

If one of w' and x' lies inside D' and the other lies inside D'', then both D' and D'' are bad 11-cycles with a claw, yielding $v' = w'$ and $y' = x'$. Now, G has a special 9-cycle with a (3,8)-chord. Otherwise, let $w', x' \in \mathrm{int}(D')$. Since G has no 4-cycles, $x' = w'$ and hence, D' is a bad cycle with either a $(3, 7, 7)$-claw or a (3,7,5,7)-biclaw. If $v' \in V(D'')$, then uvv' is a splitting 2-path of D, forming a

(3,8)-chord uv. Hence, $v' \in \operatorname{int}(D'')$ and similarly, $y' \in \operatorname{int}(D'')$. It follows that either $v' = y'$ or $v'y' \in E(G)$, yielding a 6-cycle in both cases. ■

Lemma 20. *If G' is a plane graph obtained from G by deleting a nonempty set of internal vertices and either identifying two vertices without identifying edges or adding an edge, which satisfies the following two conditions:*

(a) *identify no two vertices on D and create no edge connecting two vertices on D, and*

(b) *create neither 6^--cycles nor ext-triangular 7- or 8-cycles,*

then ϕ_0 can super-extend to G'.

Proof. The item (a) guarantees that D is unchanged and bounds G' and that ϕ_0 is an (I, F)-coloring of $G'[V(D)]$. By the item (b), G' is simple and G' contains no 4- or 6-cycles. Hence, to super-extend ϕ_0 to G' by the minimality of G, it suffices to show both that D is a good cycle in G' and that G' contains no special 9-cycles.

Suppose to the contrary that D is a bad cycle of G', i.e., D has a claw, biclaw, or triclaw, say H. For the case of identifying two vertices, the resulting vertex is incident with k ($k \le 2$) cells of H that are created by the operation. If $k = 0$, then D has H also in G, a contradiction. Moreover, since the operation does not identify edges, $k \ne 1$. Therefore, $k = 2$. It follows by Lemma 16 that there is a 5^--cycle or an ext-triangular 7-cycle created, contradicting the item (b). For the case of inserting a new edge, say e, we can similarly deduce that both cells of H incident with e are created, yielding a similar contradiction as above.

Suppose to the contrary that G' contains a special 9-cycle C. By a similar argument on C as on D above, we can deduce that there is a 5^--cycle or an ext-triangular 8-cycle created, contradicting the item (b). ■

Lemma 21. *Let G' be a plane graph obtained from G by the following operation T: deleting a nonempty set S of internal vertices and then identifying two edges u_1u_2 and v_1v_2 so that u_1 is identified with v_1. For $i \in \{1, 2\}$, let T_i denote the operation on G that consists of deleting all the vertices of S and identifying u_i and v_i. If at least one of u_1u_2 and v_1v_2 is contained in no 8^--cycle of $G - S$, and the conditions* (a) *and* (b) *of Lemma* 20 *hold for both T_1 and T_2, then ϕ_0 can super-extend to G'.*

Proof. For $i \in \{1, 2\}$, denote by w_i the vertex resulting from u_i and v_i by T. Since the condition (a) holds for both T_1 and T_2, D bounds G' and ϕ_0 is an (I, F)-coloring of $G'[V(D)]$.

Suppose that T creates a 6^--cycle or a special 9-cycle or a bad D, denoted by C. Since the two conditions (a) and (b) hold for both T_1 and T_2, by the proof of Lemma 20, each T_i does not create C. Hence, w_1w_2 must be either a common edge of some two cells of C or a chord of some cell of C. This implies that both u_1u_2 and v_1v_2 are contained in a 8^--cycle of $G-S$, contradicting the assumption.

Therefore, ϕ_0 can super-extend to G' by the minimality of G. ■

Given a plane graph, a *good path* is a path $P = v_1v_2v_3v_4$ of the boundary of some face such that the edge v_1v_2 is triangular and all the vertices of P are internal 3-vertices, see Figure 2.

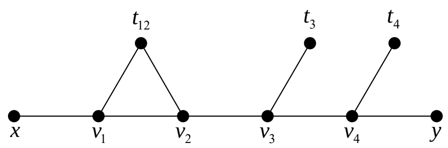

Figure 2. Good path.

Lemma 22. *G has no good paths.*

Proof. Suppose to the contrary that G has a good path $P = v_1v_2v_3v_4$, using the same label for vertices as in Figure 2. Since $G \in \mathcal{G}$, all the vertices in Figure 2 are pairwise distinct except that t_3 and t_4 might coincide. Apply on G the following operation T: remove all the vertice of P and identify x with t_3, obtaining a smaller plane graph G'.

Suppose that T creates a 6^--cycle or an ext-triangular 7- or 8-cycle. Thus, $G - v_4$ has a 12^--cycle C containing $xv_1v_2v_3t_3$ and additionally, if $|C| \in \{11, 12\}$ then the path $C - \{v_1, v_2, v_3\}$ is triangular. By planarity, $t_{12} \in V(C)$ or $t_{12} \in \operatorname{int}(C)$ or $v_4 \in \operatorname{int}(C)$. For the first case, between the two cycles formed by paths $C - v_1v_2$ and $v_1t_{12}v_2$, at least one is a triangular 6^--cycle, contradicting that $G \in \mathcal{G}$. For the last two cases, C is a bad cycle by Lemma 14. But now C is adjacent to two triangles, contradicting Lemma 17. So, the item (b) of Lemma 20 holds for T.

Suppose that T identifies two external vertices or create an edge connecting two external vertices. Thus, $xv_1v_2v_3t_3$ is contained in a splitting 4- or 5-path of D, which together with D forms a 9^--cycle by Lemma 19. Thus, T creates a 5^--cycle, a contradiction. Therefore, the item (a) of Lemma 20 holds for T.

Hence, ϕ_0 can super-extend to G' by Lemma 20 and further to G as follows. Nicely color v_4 and v_3 in turn, which for sure brings no defective segments. Clearly, x and t_3 receive the same color, say α. Denote by β and γ the colors of t_{12} and v_3, respectively. We distinguish the following four cases.

(i) If $\alpha = I$, then color v_1 by F and color v_2 different from t_{12}, which brings no defective segments, we are done by Remark 11.

(ii) If $\alpha = F$ and $\beta = I$, then color both v_1 and v_2 by F, we are done. Notice that $xv_1v_2v_3t_3$ might be an F-path, which however brings neither F-cycle nor splitting F-path of D since otherwise, identifying x with t_3 yields an F-cycle or a splitting F-path of D in G'.

(iii) If $\alpha = \beta = F$ and $\gamma = I$, then color v_1 by I and v_2 by F, we are done.

(iv) Let $\alpha = \beta = F$ and $\gamma = F$. Since identifying x with t_3 yields neither F-cycle nor splitting F-path of D in G', either (x, t_{12}) or (t_{12}, t_3) is not F-linked, for which case we color v_1 by F or by I respectively and color v_2 different from v_1. ■

Lemma 23. *For $k \in \{5, 7\}$, the graph G has no k-face that contains k internal 3-vertices.*

Proof. Suppose to the contrary that G has such a k-face $f = [v_1 \cdots v_k]$. Let t_i be the remaining neighbor of v_i for $i \in \{1, 2, \ldots, k\}$. Since $G \in \mathcal{G}$ and Lemma 22, these vertices $t_1, \ldots, t_k$ are pairwise distinct.

Case 1. Let $k = 5$. Since $G \in \mathcal{G}$, f contains a vertex incident with two 7^+-faces, without loss of generality, say v_2. Apply on G the following operation T: remove $V(f)$ and insert an edge between t_1 and t_3, obtaining a smaller plane graph G'.

Suppose that T creates a 6^--cycle or an ext-triangle 7- or 8-cycle. Then $G - \{v_4, v_5\}$ has an 11^--cycle C containing the path $P = t_1 v_1 v_2 v_3 t_3$ and additionally, $\text{ext}[C]$ has a triangle sharing an edge with $C - E(P)$ when $|C| \in \{10, 11\}$. If C is a good cycle, then $t_2 \in V(C)$ and thus, $v_2 t_2$ is a $(7^+, 7^+)$-chord of a 11^--cycle C, a contradiction. So, C is a bad 11-cycle. By Lemma 16, C must contain t_2 inside and have a $(3, 7, 7)$-claw. Now, C is adjacent to two triangles in G, contradicting Lemma 17. Therefore, the item (b) of Lemma 20 holds for T.

If both t_1 and t_3 are external, then P is a splitting 4-path of D, which together with D forms a 5- or 7-cycle C by Lemma 19. Then T creates a 2- or 4-cycle, contradicting the truth of the item (b). Hence, the item (a) of Lemma 20 holds for T.

Hence, ϕ_0 can super-extend to G' by Lemma 20 and further to G as follows. Firstly, assume that all the vertices of $\{t_1, t_2, \ldots, t_5\}$ are of color F. If both the pairs (t_1, t_2) and (t_2, t_3) are F-linked, then $t_1 t_3$ is contained in an F-cycle or a splitting F-path of D in G', a contradiction. Hence, at least one of the pairs (t_1, t_2) and (t_2, t_3) is not F-linked, without loss of generality, say (t_1, t_2). Assign $v_1, v_2, \ldots, v_5$ with F, F, I, F, I, respectively. Note that the coloring of $V(f)$ brings no defective segments, we are done by Remark 11. It remains to assume that there is a vertex from $\{t_1, t_2, \ldots, t_5\}$ of color I, say t_q. I-nicely color the path $f - v_q$ with any direction. Since not both t_1 and t_3 are of color I, the path $f - v_q$ is not an F-path. So, assigning v_q with color F brings no defective segments, we are done by Remark 11.

Case 2. Let $k = 7$. Apply on G the following operation T: remove all the vertice of f and insert an edge between t_1 and t_4, obtaining a smaller plane graph G'.

Suppose that T creates a 6^--cycle or an ext-triangle 7- or 8-cycle. Then $G - \{v_5, v_6, v_7\}$ has a 12^--cycle C containing the path $P = t_1v_1v_2v_3v_4t_4$ and additionally, ext$[C]$ has a triangle sharing an edge with $C - E(P)$ when $|C| \in \{11, 12\}$. If C is a good cycle, then $t_2, t_3 \in V(C)$. Since $|C| \leq 12$, each edge of $v_1v_2v_3v_4$ is incident with a 5-face. Now $|C| = 11$, which implies that one of those 5-faces is adjacent to a triangle, a contradiction. So, C is a bad cycle. On one hand, C has a (5,5,7)-claw or (5,5,5,7)-biclaw by Lemma 17. On the other hand, either $v_5, v_6, v_7 \in \text{int}(C)$ or C contains t_2t_3 inside by planarity. A contradiction follows. So, the item (b) of Lemma 20 holds for T.

If both t_1 and t_4 are external vertices, then P is a splitting 5-path of D, which together with D forms a 9^--cycle by Lemma 19. Then T creates a 5^--cycle, contradicting the truth of the item (b). So, the item (a) of Lemma 20 holds for T.

Hence, ϕ_0 can super-extend to G' by Lemma 20 and further to G in a similar way as for *Case* 1. ■

A 3-7-*face* H consists of a 3-face $[xzy]$ and a 7-face $[xzv_1 \cdots v_5]$ such that their common part is the edge xz, z is an internal 4-vertex, and all other vertices of H are internal 3-vertices, see Figure 3.

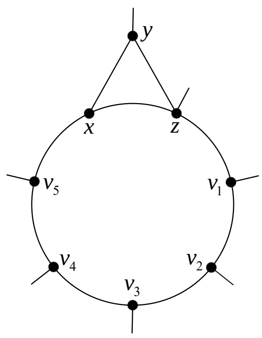

Figure 3. 3-7-face.

Lemma 24. *G has no* 3-7-*faces.*

Proof. Suppose to the contrary that G has a 3-7-face H, using the same label for vertices as in Figure 3. The pre-coloring ϕ_0 can super-extend to $G - V(H)$ by the minimality of G and further to G as follows.

I-nicely color the directed path $\vec{P} = v_5v_4 \cdots v_1zy$. If at least one of y and z is of color I, then assign x with F, which brings no defective segments except that $[xzv_1v_2 \cdots v_5]$ might be an F-cycle. For this exceptional case, the remaining neighbor of each vertex from $\{z, v_1, v_2, \ldots, v_5\}$ is of color I. Reassign x with I

and y with F, which obviously brings no defective segments, we are done. Hence, we may next assume that both y and z are of color F.

If v_5 is of color F, then assign x with I, we are done. So, let v_5 be of color I. Denote by y' the remaining neighbor of y. If y' is of color F, then reassign y with I and assign x with F, we are done. So, let y' be of color I. F-nicely recolor $\vec{P}$, which yields that both v_5 and y are of color F, but the color of z might be changed. Finally, color x different from z, which brings no defective segments, we are done. ∎

A 7-7-*face* H consists of two 7-faces $[xu_6 \cdots u_1]$ and $[xv_1 \cdots v_6]$ such that their common part is the vertex x, u_1 is adjacent to v_1, both x and u_1 are internal 4-vertices, and all other vertices of H are internal 3-vertices, see Figure 4.

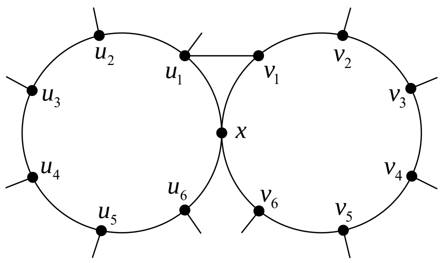

Figure 4. 7-7-face.

Lemma 25. *G has no 7-7-faces.*

Proof. Suppose to the contrary that G has a 7-7-face H, using the same label for vertices as in Figure 4. The pre-coloring ϕ_0 can super-extend to $G - V(H)$ by the minimality of G and further to G as follows. Let $\vec{P_1} = u_6u_5 \cdots u_1$ and $\vec{P_2} = v_6v_5 \cdots v_1$.

I-nicely color the directed path $\vec{P_1}$. If P_1 is an F-path, then F-nicely color $\vec{P_2}$. Note that v_6 must be of color F. Reassign v_1 with F if its color is not F and finally, assign x with I. Note that the coloring of $\{v_1, x\}$ brings no defective segments, we are done by Remark 11. Hence, we may next assume that P_1 is not an F-path.

I-nicely color the directed path $\vec{P_2}$. If P_2 is an F-path, then u_1 must be of color I. F-nicely recolor the path $\vec{P_1}$ regardless of the edge u_1v_1, yielding both u_1 and u_6 of color F. So, we can assign x with I. It is easy to see that the edge u_1v_1 has both ends of color F but is not contained in any F-cycle or splitting F-path of D, we are done. Hence, we may next assume that P_2 is not an F-path.

If not both u_1 and v_1 are of color F, then assigning x with F brings no defective segments, we are done. So, let both u_1 and v_1 be of color F. If v_2 is of color F, then reassign v_1 with I and assign x with F, we are done. So, let v_2

be of color I. Denote by t_1 the neighbor of u_1 not in H. If t_1 is of color F, then F-nicely recolor the path $\vec{P_1}$ regardless of the edge u_1v_1, yielding u_1 of color I. So, the edge u_1v_1 is contained in no defective segments, and assigning x with F brings no defective segments, we are done. Hence, let t_1 be of color I. F-nicely recolor $\vec{P_2}$, yielding v_6, v_2, v_1 of color F, F, I, respectively. Assign x with F, which might make $u_2u_1xv_6$ be contained in an F-cycle or a splitting F-path of D. For this case, remove the colors of x and v_1 and F-nicely recolor $\vec{P_1}$, yielding that u_2u_1 would be contained in no defective segments no matter what colors x and v_1 will receive. Assign x with I and v_1 with F, we are done. ■

An *M-9-face* is a 9-face $[v_1 \cdots v_9]$ such that the edges $v_1v_2, v_3v_4, v_4v_5, v_6v_7$ are triangular, $v_1, v_2, v_3, v_5, v_6, v_7$ are internal 3-vertices, and v_4 is an internal 4-vertex, see Figure 5.

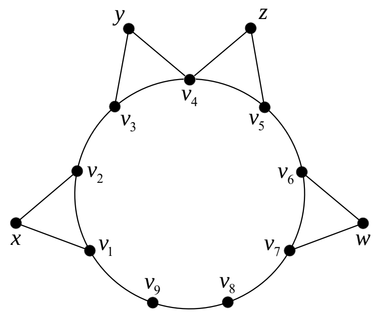

Figure 5. M-9-face.

Lemma 26. *G has no M-9-faces.*

Proof. Suppose to the contrary that G has an M-9-face f, using the same label for vertices as in Figure 5. Let $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_5, v_6, v_7\}$, and $S = S_1 \cup S_2$. Apply on G the operation T as follows: remove all the vertices of S and identify the edges zv_4 with v_8v_9 so that z is identified with v_8, obtaining a smaller plane graph G'. Denote by T_1 (respectively, T_2) the operation on G consisting of removing all the vertices of S and identifying z with v_8 (respectively, v_4 with v_9). Similarly as the proof of Lemma 22, we can deduce that both the items (a) and (b) hold for T_1 as well as T_2. Moreover, notice that v_4z is contained in no 8^--cycle of $G - S$.

By Lemma 21, the pre-coloring ϕ_0 can super-extend to G' and further to G as follows. Color the vertices of S_1 as well as S_2 in the same way as we did for good path in the proof of Lemma 22. Clearly, the coloring of S brings no I-edges. Hence, it remains to show that the coloring of S brings neither F-cycle nor splitting F-path of D. Otherwise, denote by H such a new F-cycle or splitting

F-path of D in G. The way we color S_1 and S_2 implies that $V(H) \cap S_1 \neq \emptyset$ and $V(H) \cap S_2 \neq \emptyset$, and the coloring of S_1 as well as S_2 belongs to case (ii) or (iv) of the proof of Lemma 22. Thus, all the four vertices we identified are of color F and so, v_5 is of color I. It follows that the coloring of S_2 belongs to case (ii), for which the coloring of S_2 brings neither F-cycle nor splitting F-path of D, contradicting that $V(H) \cap S_2 \neq \emptyset$. ∎

3.2. Incompatibility of reducible configurations

By exactly the same discharging procedure as in the article [8], we can derive the incompatibility of reducible configurations as depicted in Lemmas 12 up to 26, which completes the proof of Theorem 10. More precisely, in Section 2.1 of [8], the authors prove reducible configurations for the minimal counterexample $H \in \mathcal{G}$, which are exactly the same as Lemmas 13 up to 26 of this paper. Subsection 2.2 of [8] are discharging procedure, which shows that these reducible configurations are incompatible for a graph of $\mathcal{G}$. For the seek of completeness, we provide the discharging part as appendix.

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Appendix

A vertex incident with a triangle is called a *triangular vertex*. We say a vertex is *bad* if it is an internal triangular 3-vertex; *good* otherwise. A triangular 7-face is *light* if it contains no external vertices and every incident nontriangular vertex has degree 3.

Recall that G is a minimal counterexample to Theorem 10, f_0 is the unbounded face of G, and D is the boundary of f_0. Let $V = V(G)$, $E = E(G)$, and F be the set of faces of G. Give initial charge $ch(x)$ to each element x of $V \cup F$ as $ch(f_0) = |f_0| + 4$, $ch(v) = d(v) - 4$ for $v \in V$, and $ch(f) = |f| - 4$ for $f \in F \setminus \{f_0\}$. Discharge the elements of $V \cup F$ according to the following rules:

$R1$. Every 3-face receives $\frac{1}{3}$ from each incident vertex.

$R2$. Let v be an internal 3-vertex and f be a face containing v.

(1) v receives $\frac{1}{4}$ from f if $|f| = 5$.

(2) Suppose $|f| \geq 7$. Let a and b denote the lengths of other two faces containing v with $a \leq b$. The vertex v receives from f charge $\frac{2}{3}$ if $a = 3$, charge $\frac{1}{2}$ if $a = b = 5$, charge $\frac{3}{8}$ if $a = 5$ and $b \geq 7$, and charge $\frac{1}{3}$ if $a \geq 7$.

$R3$. Let v be an internal 4-vertex and f be a 7^+-face containing v.

(1) If v is incident with precisely two 3-faces, then v receives $\frac{1}{3}$ from f.

(2) If v is incident with precisely one 3-face that is adjacent to f, then v receives $\frac{1}{6}$ from f.

$R4$. Let f be a light 7-face adjacent to a 3-face T on edge xy, z be the vertex on T other than x and y, and h be the face containing edge yz other than T.

(1) If $d(x) = 3$ and $d(y) \geq 5$, then y sends $\frac{1}{24}$ to f.

(2) If $z \in V(D)$, then z sends $\frac{5}{24}$ to f through T.

(3) If $d(x) = 3, d(y) = 4, z \notin V(D)$, and $d(z) \geq 4$, then h sends $\frac{5}{24}$ to f through y.

$R5$. The face f_0 sends $\frac{4}{3}$ to each incident vertex.

$R6$. Let v be an external vertex and f be a 5^+-face containing v other than f_0.

(1) If $d(v) = 2$, then v receives $\frac{2}{3}$ from f.

(2) Suppose $d(v) = 3$. If v is triangular, then v receives $\frac{1}{12}$ from f; otherwise, v sends $\frac{1}{12}$ to f.

(3) If $d(v) \geq 4$, then v sends $\frac{1}{3}$ to f.

Let $ch^*(x)$ denote the final charge of each element x of $V \cup F$ after discharging. On one hand, by Euler's formula $|V| - |E| + |F| = 2$, we can deduce that $\sum_{x \in V \cup F} ch(x) = 0$. Since charges are only moved around over $V \cup F$ in the discharging procedure, we have $\sum_{x \in V \cup F} ch^*(x) = 0$. On the other hand, we will show that $ch^*(x) \geq 0$ for each $x \in V \cup F$ and $ch^*(x_0) > 0$ for some vertex x_0. Hence, this obvious contradiction completes the proof of Theorem 10.

Claim 27. *$ch^*(f) \geq 0$ for $f \in F$.*

Proof. Denote by $V(f)$ the set of vertices of f.

First suppose that f contains no external vertices.

Let $|f| = 3$. By $R1$, we have $ch^*(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$, we are done.

Let $|f| = 5$. Lemma 23 implies that f contains at most four 3-vertices. Hence, $ch^*(f) \geq |f| - 4 - 4 \times \frac{1}{4} = 0$ by $R2(1)$.

Let $|f| = 7$. If G has no 3-face adjacent to f, then f sends at most $\frac{1}{2}$ to each incident 3-vertex by $R2(2)$. Since Lemma 23 implies that f contains at most six 3-vertices, we have $ch^*(f) \geq |f| - 4 - 6 \times \frac{1}{2} = 0$. Hence, we may next assume that f is adjacent to a 3-face $[xyz]$ on the edge xy with $d(x) \leq d(y)$. Since G has no special 9-cycles, f is adjacent to no other 3-faces. Notice that now only rules $R2(2)$, $R3(2)$, and $R4(3)$ might make f send charge out.

Suppose $d(y) = 3$. In this case, f sends $\frac{2}{3}$ to both x and y, and at most $\frac{1}{2}$ to each of other incident 3-vertices. Moreover, it follows from Lemma 22 that f contains at least two 4^+-vertices. Hence, we have $ch^*(f) \geq |f| - 4 - 2 \times \frac{2}{3} - 3 \times \frac{1}{2} > 0$.

Suppose $d(x) = 3$ and $d(y) = 4$. In this case, f sends $\frac{2}{3}$ to x, $\frac{1}{6}$ to y, and at most $\frac{3}{8}$ to the neighbor of x on f other than y. If z is not an internal 3-vertex, then f receives charge $\frac{5}{24}$ either from z by $R4(2)$ or from the face containing yz other than T by $R4(3)$, yielding $ch^*(f) \geq |f| - 4 - \frac{2}{3} - \frac{1}{6} - \frac{3}{8} - 4 \times \frac{1}{2} + \frac{5}{24} = 0$. Hence, we may next assume that z is an internal 3-vertex. Since G has no 3-7-faces by Lemma 24, f is not light. So, $ch^*(f) \geq |f| - 4 - \frac{2}{3} - \frac{1}{6} - 4 \times \frac{1}{2} > 0$.

Suppose $d(x) = 3$ and $d(y) \geq 5$. In this case, f sends $\frac{2}{3}$ to x and at most $\frac{3}{8}$ to the neighbor of x on f other than y. By R4(1), f receives $\frac{1}{24}$ from y. Thus, we have $ch^*(f) \geq |f| - 4 - \frac{2}{3} - \frac{3}{8} + \frac{1}{24} - 4 \times \frac{1}{2} = 0$.

It remains to suppose $d(x) \geq 4$. In this case, f might send charge out through x and y by $R4(3)$. If f is not light, then $ch^*(f) \geq |f| - 4 - 2(\frac{1}{6} + \frac{5}{24}) - 4 \times \frac{1}{2} > 0$. Moreover, if $d(y) \geq 5$, then f sends nothing to y or through y, yielding $ch^*(f) \geq |f| - 4 - (\frac{1}{6} + \frac{5}{24}) - 5 \times \frac{1}{2} > 0$. Hence, we may next assume that f is light and

$d(x) = d(y) = 4$. Since G has no 7-7-faces by Lemma 25, f sends nothing out through x or y. It follows that $ch^*(f) \geq |f| - 4 - 2 \times \frac{1}{6} - 5 \times \frac{1}{2} > 0$.

Let $|f| = 8$. Since f sends at most $\frac{1}{2}$ to each incident vertex by $R2(2)$, we have $ch^*(f) \geq |f| - 4 - 8 \times \frac{1}{2} = 0$.

Let $|f| \geq 9$. We define

$$\begin{aligned}
A(f) &= \{v\colon\ uvw \text{ is a path on } f, \text{ both } u \text{ and } w \text{ are bad, and } v \text{ is good}\},\\
B(f) &= \{v\colon\ uvw \text{ is a path on } f,\ u \text{ is bad, and both } v \text{ and } w \text{ are good}\},\\
C(f) &= \{v\colon\ uvw \text{ is a path on } f, \text{ and all of } u, v, w \text{ are good}\},\\
D(f) &= \{v\colon\ v \text{ is a bad vertex on } f\}.
\end{aligned}$$

Clearly, $A(f)$, $B(f)$, $C(f)$, and $D(f)$ are pairwise disjoint sets whose union is $V(f)$. By our rules, f sends at most $\frac{1}{3}$ to each vertex in $A(f)$, at most $\frac{3}{8}$ in total to and through each vertex in $B(f)$, at most $\frac{1}{2}$ in total to and through each vertex in $C(f)$, and $\frac{2}{3}$ to each vertex in $D(f)$. Hence, we have

$$\begin{aligned}
ch^*(f) &\geq |f| - 4 - \frac{1}{3}|A(f)| - \frac{3}{8}|B(f)| - \frac{1}{2}|C(f)| - \frac{2}{3}|D(f)|\\
&= |f| - 4 - \frac{1}{3}|A(f)| - \frac{3}{8}|B(f)| - \frac{1}{2}|C(f)| - \frac{2}{3}(|f| - |A(f)| - |B(f)|\\
&\quad - |C(f)|)\\
(*) \qquad &= \frac{1}{3}|A(f)| + \frac{7}{24}|B(f)| + \frac{1}{6}|C(f)| + \frac{1}{3}|f| - 4.
\end{aligned}$$

Clearly, $|B(f)|$ is always even, and if $B(f) = \emptyset$ then either $C(f) = \emptyset$ or $C(f) = V(f)$.

Suppose $|f| = 9$. By the inequality $(*)$, it suffices to consider the following three cases.

Case 1. Let $|A(f)| \leq 2$ and $|B(f)| = |C(f)| = 0$. By Lemma 22, one can deduce that $|A(f)| = 2$ (say $A(f) = \{u, v\}$), $D(f)$ is divided by u and v as 3+4 on the boundary of f, and $d(u), d(v) \geq 4$. Furthermore, by the drawing of 3-faces adjacent to f, we can apply Lemma 26 to get that $\max\{d(u), d(v)\} \geq 5$. Hence, $ch^*(f) \geq |f| - 4 - 7 \times \frac{2}{3} - \frac{1}{3} = 0$.

Case 2. Let $|A(f)| = 1$, $|B(f)| = 2$, and $|C(f)| = 0$. By Lemma 22, $D(f)$ is divided by $B(f) \cup A(f)$ as 3+3 or 2+4 on the boundary of f.

For the case 3+3, let $A(f) = \{u\}$. By Lemma 22, $d(u) \geq 4$. Moreover, u is not a 4-vertex incident with two 3-faces by Lemma 26. Hence, u receives at most $\frac{1}{6}$ from f, which yields $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 2 \times \frac{3}{8} - \frac{1}{6} > 0$.

For the case 2+4, let $f = [u_1 \cdots u_9]$ with $A(f) = \{u_1\}$ and $B(f) = \{u_4, u_5\}$. Lemma 22 implies that $d(u_1), d(u_5) \geq 4$. If u_1 is not a 4-vertex incident with two 3-faces, then f sends at most $\frac{1}{6}$ to u_1, which yields $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 2 \times$

$\frac{3}{8} - \frac{1}{6} > 0$; otherwise, the drawing of 3-faces adjacent to f shows that $d(u_4) \geq 4$ and f sends nothing through u_4 or u_5 and at most $\frac{1}{3}$ to each of them, yielding $ch^*(f) \geq |f| - 4 - 6 \times \frac{2}{3} - 3 \times \frac{1}{3} = 0$.

Case 3. Let $|A(f)| = 0$, $|B(f)| = 2$, and $|C(f)| \leq 2$. It follows that f contains five consecutive bad vertices, which form a good path, contradicting Lemma 22.

Suppose $|f| \geq 10$. By the inequality $(*)$, it suffices to consider two cases: (1) $|B(f)| = 0$ and $2|A(f)| + |C(f)| < 4$; (2) $|B(f)| = 2$ and $|A(f)| = |C(f)| = 0$. For either case, f contains five consecutive bad vertices, contradicting Lemma 22.

Next suppose that f contains external vertices.

Since $|f_0| \leq 12$, if $f = f_0$ then by $R5$ we have $ch^*(f) = |f_0| + 4 - |f_0| \times \frac{4}{3} \geq 0$. Hence, we may assume $f \neq f_0$. By our rules, f sends at most $\frac{2}{3}$ to each incident vertex. Lemma 19 implies that if $|f| \leq 8$, then the external vertices on f are consecutive one by one. Furthermore, f has at most one 2-vertex if $|f| = 5$, and has at most two 2-vertices if $|f| \in \{7, 8\}$.

Let $|f| = 3$. We have $ch^*(f) = |f| - 4 + 3 \times \frac{1}{3} = 0$ by $R1$.

Let $|f| = 5$. If f has no 2-vertices, then f sends at most $\frac{1}{4}$ to each vertex, yielding $ch^*(f) \geq |f| - 4 - 4 \times \frac{1}{4} = 0$. Hence, we may assume f has precisely one 2-vertex. It follows that f has two external 3-vertices, both of which send at least $\frac{1}{12}$ to f by $R6$. Hence, we have $ch^*(f) \geq |f| - 4 - \frac{2}{3} + 2 \times \frac{1}{12} - 2 \times \frac{1}{4} = 0$.

Let $|f| = 7$. Note that f contains at most two bad vertices. First assume that f has precisely one external vertex, say u. Then u is a 4^+-vertex, which sends $\frac{1}{3}$ to f by $R6(3)$, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{3} - 2 \times \frac{2}{3} - 4 \times \frac{1}{2} = 0$. It remains to assume that f has at least two external vertices. Then f has at least two external 3^+-vertices, say u and v. If both u and v are not triangular, then they send $2 \times \frac{1}{12}$ in total to f, yielding $ch^*(f) \geq |f| - 4 + 2 \times \frac{1}{12} - 4 \times \frac{2}{3} - \frac{1}{2} = 0$; otherwise, one of u and v is triangular and the other is not, and f has at most one bad vertex, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{12} - \frac{1}{12} - 3 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

Let $|f| = 8$. Clearly, f contains no bad vertices. If f has no 2-vertices, then f sends at most $\frac{1}{2}$ to each incident vertex, yielding $ch^*(f) \geq |f| - 4 - 8 \times \frac{1}{2} = 0$. Hence, we may assume that f has precisely one or two 2-vertices. It follows that f has two external 3^+-vertices, each of which sends at least $\frac{1}{12}$ to f. Thus, $ch^*(f) \geq |f| - 4 - 2 \times \frac{2}{3} + 2 \times \frac{1}{12} - 4 \times \frac{1}{2} > 0$.

It remains to suppose $|f| \geq 9$. If f has an external 4^+-vertex, then f receives $\frac{1}{3}$ from this vertex by $R6(3)$, yielding $ch^*(f) \geq |f| - 4 + \frac{1}{3} - (|f| - 1) \times \frac{2}{3} \geq 0$. Hence, we may assume that f has no external 4^+-vertex, which implies f has at least two external 3-vertices. By $R6$, we have $ch^*(f) \geq |f| - 4 - 2 \times \frac{1}{12} - (|f| - 2) \times \frac{2}{3} > 0$. ■

Claim 28. *$ch^*(v) \geq 0$ for $v \in V$.*

Proof. First suppose that v is internal. We have $d(v) \geq 3$ by Lemma 13.

Let $d(v) = 3$. Since $G \in \mathcal{G}$, the list of lengths of the faces containing v is one of the followings: $\{3, 7^+, 7^+\}$, $\{5, 5, 7^+\}$, $\{5, 7^+, 7^+\}$, and $\{7^+, 7^+, 7^+\}$. We are done for each case by $R1$ and $R2$.

If $d(v) = 4$, then the charge v sends out equals to what v receives by $R1$ and $R3$, yielding that $ch^*(v) = d(v) - 4 = 0$.

It remains to suppose $d(v) \geq 5$. By $R1$ and $R4(1)$, v sends $\frac{1}{3}$ to each incident 3-face and at most $\frac{1}{24}$ to each other incident face, which gives $ch^*(v) \geq d(v) - 4 - \frac{d(v)}{2} \times \frac{1}{3} - \frac{d(v)}{2} \times \frac{1}{24} > 0$.

Next suppose that v is external. Clearly, $d(v) \geq 2$.

By $R1$, $R5$ and $R6$, we have $ch^*(v) = d(v) - 4 + \frac{4}{3} + \frac{2}{3} = 0$ if $d(v) = 2$, $ch^*(v) = d(v) - 4 + \frac{4}{3} - \frac{1}{3} + \frac{1}{12} > 0$ if $d(v) = 3$ and v is triangular, and $ch^*(v) = d(v) - 4 + \frac{4}{3} - \frac{1}{12} - \frac{1}{12} > 0$ if $d(v) = 3$ and v is not triangular.

It remains to suppose $d(v) \geq 4$. The vertex v receives $\frac{4}{3}$ from f_0 by $R5$, sends $\frac{1}{3}$ to each other incident face by $R1$ and $R6(3)$, and might send $\frac{5}{24}$ out through each incident 3-face whose other two vertices are internal. It follows that $ch^*(v) \geq d(v) - 4 + \frac{4}{3} - (d(v) - 1) \times \frac{1}{3} - \frac{d(v)-2}{2} \times \frac{5}{24} > 0$. ■

Claim 29. *D contains a vertex x_0 such that $ch^*(x_0) > 0$.*

Proof. Let x_0 be any 3^+-vertex on D, as desired. ■

The proof of Theorem 10 is completed by Claims 27, 28 and 29.

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