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THE TREE-ACHIEVING SET AND NON-SEPARATING INDEPENDENT SET PROBLEM OF SUBCUBIC GRAPHS

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Abstract

The decycling number $\nabla(G)$ (respectively, tree-achieving number $\nabla_T(G)$) of a graph G is the smallest number of vertices whose deletion yields a forest (respectively, tree). Obviously, $\nabla_T(G) \geq \nabla(G)$ for all graphs. A graph is cubic (respectively, subcubic) if every vertex has degree three (respectively, at most three). A non-separating independent set is an independent vertices set whose deletion yields a connected subgraph. The nsis number $\mathcal{Z}(G)$ is the maximum cardinality of a non-separating independent set. In this article, we present a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in cubic graphs. That is $\nabla_T(G) = \nabla(G)$ if and only if there exists a Xuong-tree [J.L. Gross and R.G. Rieper, *Local extrema in genus-stratified graphs*, J. Graph Theory 15 (1991) 153–171] T_X of G such that every odd component of $G - E(T_X)$ contains at least three edges. Further, we give a formula for $\mathcal{Z}(G)$ in subcubic graphs: there is a Xuong-tree T_X of G such that $\alpha_1(T_X) = \mathcal{Z}(G)$, where $\alpha_1(T_X)$ is the independence number of the subgraph of G induced by leaves of T_X.

Keywords: tree-achieving number, decycling number, nsis number, Xuong-tree.

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1. Introduction

Graphs considered in this paper are finite and connected. A *decycling set* (also known as *feedback vertex set*) of a graph is a subset of the vertices whose deletion yields a forest. It is not hard to see that determining a decycling set is equivalent to finding an induced forest. The smallest size of a decycling set of a graph G, denoted by $\nabla(G)$, is called the *decycling number* of G. For brevity, we call a decycling set containing exactly $\nabla(G)$ vertices a ∇-set in this paper.

In a network, the decycling set problem consists in finding a node set of minimum size such that excluding these nodes from the network guarantees an acyclic network. This is a critical problem which has numerous applications in parallel systems, combinatorial circuit design and distributed computing. For these reasons, determining the decycling number of graphs has attracted much attention of a large number of researchers. It has been shown that determining the decycling number of graphs is NP-hard [7]. A direction in the study of decycling number problem is computing the exact value or upper bound for sparse graphs.

A closely related to decycling number problem is to study the minimum size of vertices whose deletion yields a tree. Here, we say a vertex set S of a graph G is a *tree-achieving set* if $G - S$ is an induced tree. Analogously, the smallest cardinality of a tree-achieving set of G is said to be the *tree-achieving number*, and denoted by $\nabla_T(G)$. A tree-achieving set of this cardinality is called a ∇_T-set. The tree-achieving set problem was initiated by Erdös, Sakes and Sós in 1986 [3]. Historically, this problem finds its motivation in the theory of energy of graphs. In [12], the authors characterized the extremal graphs with respect to matching energy from $T_t(n)$, where $T_t(n)$ is the set of t-apex trees (i.e., graphs with $\nabla_T(G) = t$). Thus, it is an important work to determine the tree-achieving number of graphs. However, in contrast to decycling number case, few works have been done on this topic.

It is apparent that $\nabla_T(G) \geq \nabla(G)$ for all graphs. In some cases, the gap between $\nabla(G)$ and $\nabla_T(G)$ can be arbitrarily large. It is clear from Figure 1 that $\nabla_T(G) = n, \nabla(G) = 1$. As stated above, computing the decycling number in a graph is NP-hard, not to mention the tree-achieving number. It would be interesting to establish a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in some specific graphs.

Another topic related to the decycling set problem is the non-separating independent set. In a graph, an *independent set* is a subset of vertices no two of which are adjacent. A set of vertices S of a connected graph G is a *non-separating independent set* (*nsis*, for short) if S is independent and $G - S$ is connected. The *nsis number* $\mathcal{Z}(G)$ is the maximum cardinality of a nsis. The nsis problem has many important applications in combinatorial optimization operation research and wireless network design [9]. In theory, the nsis number problem is closely related with decycling number problem. For example, Bondy, Hopkins and Staton provided upper bounds for decycling number in cubic graphs in terms of its nsis number [1]. Speckenmeyer proved that $\nabla(G)+\mathcal{Z}(G) = \frac{|V(G)|}{2}+1$ for cubic graphs [11]. According to the decycling number in Cartesian product of two cycles, Cao and Ren gave its nsis number [2].

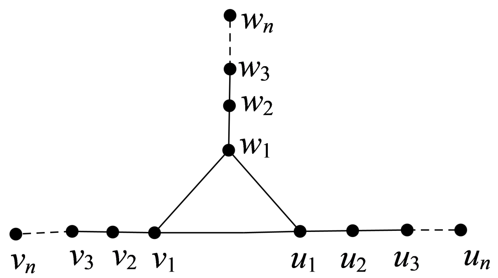

Figure 1. $\nabla_T(G) = n, \nabla(G) = 1$.

In this paper, we first restrict our attention to cubic graphs and give a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ by using graph embedding method. It is $\nabla_T(G) = \nabla(G)$ if and only if there exists a Xuong-tree T_X of G such that every odd component of $G - E(T_X)$ contains at least three edges.

Let T be a spanning tree of G, we denote by $\alpha_1(T)$ the independence number of the subgraph of G induced by leaves in T. Huang *et al.* proved $\mathcal{Z}(G) = \gamma_M(G)$ for cubic graphs [6]. This work derives that $\alpha_1(T_X) = \mathcal{Z}(G)$ for each Xuong-tree T_X in cubic graphs (see Section 4). In this paper, we extend their result to subcubic graphs: there is a Xuong-tree T_X of G such that $\alpha_1(T_X) = \mathcal{Z}(G)$ in subcubic graphs.

2. PRELIMINARIES

In this section, we shall provide some elementary notions of topological graph theory and give some basic but important results.

The orientable surface S_g can be obtained from the sphere with $2g$ pairwise disjoint holes attached with g tubes such that each tube welds two holes. The nonorientable surface N_k ($k \geq 1$) can be obtained from the sphere with k pairwise disjoint discs replaced by k Möbius bands. Recall that g and k are called the *genus* of S_g and N_k, respectively. A graph is said to be *embeddable* on a surface if it can be drawn on that surface in such a way that no two edges cross. Such a drawing is called an *embedding*. An embedding Π of G in a surface S is called a 2-*cell embedding* if each component of $S - \Pi$ is homeomorphic to an open disc.

The *maximum genus* $\gamma_M(G)$ of G is defined to be the maximum integer k such that there exists a cellular embedding of G into an orientable surface of genus k. For general background, see Gross and Tucker [4], or Mohar and Thomassen [10].

Given a spanning tree T of a graph G, the subgraph $G - E(T)$ is called a *co-tree* of G. A component of a co-tree $G - E(T)$ is called odd (respectively, even) if it contains odd (respectively, even) number of edges. We use $w(T; G)$ to denote the number of odd components of $G - E(T)$. The *Betti deficiency* $\xi(G)$ of G is defined to be the minimum $w(T; G)$ over all spanning trees. A spanning tree T of G such that $w(T; G) = \xi(G)$ is said to be a *Xuong-tree* of G [5].

The following basic result, due to Xuong, relates the maximum genus to the Betti deficiency.

Theorem 1 [13]**.** *The maximum genus of a graph G is*

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G)).$$

Here, the $\beta(G)$ is called the *cycle rank* of G, which is the minimum number of edges whose removal results an acyclic graph. The cycle rank has a simple expression: $\beta(G) = |E(G)| - |V(G)| + 1$.

It is worth mentioning that during the procedure of proving Theorem 1, Xuong obtained the following edge-partition.

Lemma 2 [13]**.** *Let T_X be a Xuong-tree of a graph G. Then there exists an edge-partition of $G - E(T_X)$ as follows:*

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \ldots, f_s\},$$

where (1) $m = \gamma_M(G)$, $s = \xi(G)$*;* (2) *for any* $i = 1, 2, \ldots, m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$ *and* $\{f_1, f_2, \ldots, f_s\}$ *is a matching of* G.

Figure 2 shows an edge-partition in K_4.

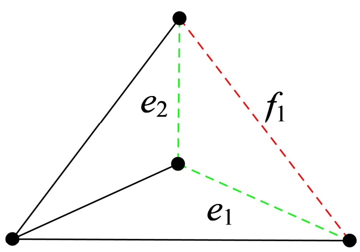

Figure 2. Edge-partition in K_4.

In the following, we apply Lemma 2 to cubic graphs. Since $e_{2i-1} \cap e_{2i} \neq \emptyset$, they have at least an endvertex in common, say u_i, $i = 1, 2, \ldots, m$. Actually,

u_i is a leaf of T_X. Thereby, the set $S_1 = \{u_i \colon 1 \leq i \leq m\}$ is a subset of leaves in T_X, which implies that $G - S_1$ is connected. In addition, the edges in $\{e_1, e_2, \ldots, e_{2m}\}$ are different in pairs. Hence, any two vertices of S_1 are non-adjacent. Consequently, S_1 is a non-separating independent set of G. Together with the result $\mathcal{Z}(G) = \gamma_M(G)$ in cubic graphs [6], S_1 ($|S_1| = m = \gamma_M(G)$) is a maximum non-separating independent set. Let v_j be an endvertex of f_j, $1 \leq j \leq s$ and $S_2 = S_1 \cup \{v_j \colon 1 \leq j \leq s\}$. Deleting S_2 from G implies deleting all edges of $E(G) - E(T_X)$. So, S_2 is a decycling set. Combining with the equality $\nabla(G) = \gamma_M(G) + \xi(G)$ in cubic graphs [8], it yields that S_2 ($|S_2| = m + s = \gamma_M(G) + \xi(G)$) is a minimum decycling set.

Finally, we introduce a result that will be frequently used in this paper.

Lemma 3 [8]**.** *Let G' be a subdivision of G. Then*

(1) $\gamma_M(G) = \gamma_M(G')$;

(2) $\beta(G) = \beta(G')$;

(3) $\xi(G) = \xi(G')$;

(4) $\nabla(G) = \nabla(G')$.

3. Sufficient and Necessary Condition

In this section, we shall give a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in cubic graphs. Before going into the details, it is appropriate to say a few words about terminology and notation again. We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G - v$ or $G - S$ for the induced subgraph obtained by deleting a vertex v or set of vertices S. Adding a set of edges S to a graph G is denoted by $G + S$. Let H be a subgraph of G, u and v be two vertices of G with $u \in V(H)$, $v \notin V(H)$ and the edge $uv \in E(G)$. We write $H + v$ for the subgraph with vertex set $V(H) \cup \{v\}$ and edge set $E(H)$, and simple write $H + v + uv$ as $H + uv$.

First, we give two period basic lemmas.

Lemma 4. *Every subgraph induced by a minimum decycling set in a cubic graph consists of a collection of isolated edges and vertices.*

Lemma 4 provides a structural characterization for the subgraphs induced by minimum decycling sets in a cubic graph. Specifically, it means that each minimum decycling set of a cubic graph contains no paths of length ≥ 2 and parallel edges. Although its proof is trivial, the consequences of this lemma are of major importance.

Lemma 5. *Let G be a cubic graph and S a ∇-set of G. Then $c+t-1=\xi(G)$, where c and t are the numbers of the components of $G-S$ and edges of $G[S]$, respectively.*

Proof. For convenience, let $\nabla=\nabla(G)$ and $E(S)=E(G[S])$ in the proof. Suppose that $|V(G)|=n$. Then $|E(G)|=\frac{3n}{2}$. Set $S=\{x_1,x_2,\ldots,x_\nabla\}$. We first assert that $c+t$ is an invariant. It is not hard verify that

$$\begin{aligned}\frac{3n}{2}-\sum_{i=1}^{\nabla}d_G(x_i)&=\frac{3n}{2}-3\nabla=\frac{3n}{2}-|E(S,G-S)|-2|E(S)|\\&=n-\nabla-|E(S)|-c=n-\nabla-t-c.\end{aligned}$$

It follows that $t+c=2\nabla-\frac{n}{2}$. Thereby, our assertion is true. Hence, it suffices to prove that there exists a ∇-set S such that $\xi(G)=c+t-1$.

There are two cases to be treated.

Case 1. G has loops. We complete the proof by applying induction on n. For $n=2$, there is nothing to prove, so assume $n>2$. Let v be a vertex incident to a loop and $G_1=G-v$. Then G_1 is a subdivision of some cubic graph G_1'. Assume that S_1' is a ∇-set of G_1'. Analogously, c_1', t_1', $\xi(G_1')$ are defined. By the induction hypothesis,

$$c_1'+t_1'-1=\xi\big(G_1'\big). \tag{1}$$

Set $S=S_1'\cup\{v\}$. Then S is a ∇-set of G. Now, we let u be the neighbor of v and x,y the other two neighbors of u. If both x and y belong to S_1', then $c=c_1'+1$ and $t=t_1'$, and $t=t_1'+1$ and $c=c_1'$ otherwise. In either case, we have

$$t+c=t_1'+c_1'+1. \tag{2}$$

Let T_1 be a Xuong-tree of G_1. Then T_1+uv is a Xuong-tree of G. Thus, $\xi(G)=\xi(G_1)+1$. Lemma 3 implies that $\xi(G_1)=\xi(G_1')$. So, we conclude that

$$\xi(G)=\xi(G_1)+1=\xi(G_1')+1. \tag{3}$$

Combining (1), (2) and (3), we deduce that $c+t-1=\xi(G)$.

Case 2. G has no loops. If G has a cut edge, then we can get our statement by a similar argument as Case 1. Hence, it suffices to consider the case of 2-edge connected graphs. Denote

$$N_E(S)=\big\{e:e=uv\in E(G),u\text{ or }v\in S\big\}.$$

Then,

$$|N_E(S)|+|E(G-S)|=\frac{3n}{2},\ |N_E(S)|=3|S|-t,\ |E(G-S)|=n-|S|-c.$$

Thus,

$$|S| = \frac{1}{2}\left(\frac{1}{2}n + c + t\right).$$

By Theorem 1, we deduce that

$$\frac{1}{2}\left(\frac{1}{2}n + c + t\right) = |S| = \gamma_M(G) + \xi(G) = \frac{1}{2}\left\{\frac{n+2}{2} + \xi(G)\right\}.$$

Thereby,

$$c + t - 1 = \xi(G). \tag{4}$$

■

Now, we are devoted to tree-achieving number problem in cubic graphs.

Theorem 6. *For a cubic graph G, $\nabla_T(G) = \nabla(G)$ if and only if there exists a Xuong-tree T_X of G such that every odd component of $G - E(T_X)$ contains at least three edges.*

Proof. If $\nabla_T(G) = \nabla(G)$, then there exists a ∇-set S of G such that $G - S$ is a tree T. Suppose that $G[S]$ contains t edges, say $e_1, e_2, \ldots, e_t$ and $|S| - 2t$ isolated vertices, say $v_1, v_2, \ldots, v_{|S|-2t}$. By Lemma 5, $c + t - 1 = \xi(G)$. Here, $c = 1$. Therefore, $t = \xi(G)$. Let first $e_i = x_{2i-1}x_{2i}$, $1 \le i \le t$. We use g_i^1, g_i^2 and f_i^1, f_i^2 to denote the other two edges incident to x_{2i-1} and x_{2i}, respectively, and use h_j^1, h_j^2, h_j^3 to denote the three edges incident to v_j, $1 \le j \le |S| - 2t$.

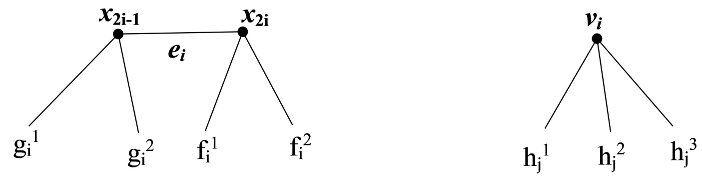

Figure 3. g_i^1, g_i^2, f_i^1, f_i^2 and h_j^1, h_j^2, h_j^3.

Define $T_X = T + \{f_1^1, g_1^1, f_2^1, g_2^1, \ldots, f_t^1, g_t^1, h_1^1, h_2^1, \ldots, h_{|S|-2t}^1\}$. Then T_X is a spanning tree of G. Notice that $w(T_X; T_X) = 0$. For a given co-tree of a graph, adding a pair of adjacent edges to this graph can not increase the number of odd components of the co-tree. On the other hand, adding an edge to this graph increases at most one by the number of odd components of the co-tree. Let

$$H = T_X + \left\{f_1^2, e_1, f_2^2, e_2, \ldots, f_t^2, e_t, h_1^2, h_1^3, h_2^2, h_2^3, \ldots, h_{|S|-2t}^2, h_{|S|-2t}^3\right\}.$$

Then $w(T_X; H)$=0. While, $G = H + \{g_1^2, g_2^2, \ldots, g_t^2\}$. We obtain that $w(T_X; G) \le t$. Combing $t = \xi(G)$, we conclude that $t = \xi(G) \le w(T_X; G) \le t$, which implies that T_X is a Xuong-tree of G.

From the construction T_X, every odd component of $G-E(T_X)$ contains the edges g_k^2, f_k^2, e_k for some $1 \leq k \leq t$. In other words, every odd component of $G-E(T_X)$ contains at least three edges.

Conversely, assume that G contains a Xuong-tree T_X, such that every odd component of $G-E(T_X)$ contains at least three edges. Bearing Lemma 2 in mind, there is an edge-partition

$$E(G)-E(T_X)=\{e_1,e_2\}\cup\{e_3,e_4\}\cup\cdots\cup\{e_{2m-1},e_{2m}\}\cup\{f_1,f_2,\ldots,f_s\},$$

where (1) $m=\gamma_M(G)$, $s=\xi(G)$; (2) for any $i=1,2,\ldots,m, e_{2i-1}\cap e_{2i}\neq\emptyset$ and $\{f_1,f_2,\ldots,f_s\}$ is a matching of G. Note that each vertex of $G-E(T_X)$ has degree at most two, then every component of $G-E(T_X)$ is a path, a cycle, an edge or a vertex. Since every odd component of $G-E(T_X)$ contains at least three edges, each f_j does not form a component of $G-E(T_X)$, $1\leq j\leq s$. Therefore, each f_j is adjacent to an edge e_{i_0}, $1\leq j\leq s$, $1\leq i_0\leq 2m$. This implies that f_j is incident to leaf of T_X, say v_j, $1\leq j\leq s$. Consider the vertex set

$$S_X=\big\{v_j : 1\leq j\leq s\big\}\cup\big\{u_i : u_i \text{ is a common endvertex of } e_{2i-1}\text{and } e_{2i},\\ 1\leq i\leq m\big\}.$$

Clearly, S_X is a ∇-set of G (see Section 2). Since S_X is a subset of leaves of T_X, $G-S_X$ is connected. Therefore, we conclude that S_X is a ∇_T-set of G. The proof is completed. ■

Figure 4 provides an example of a choice for u_1, u_2, v_1 when one of the components of $G-E(T_X)$ is a 4 (even)-cycle, 5 (odd)-cycle, 4 (even)-path and 5 (even)-path, respectively.

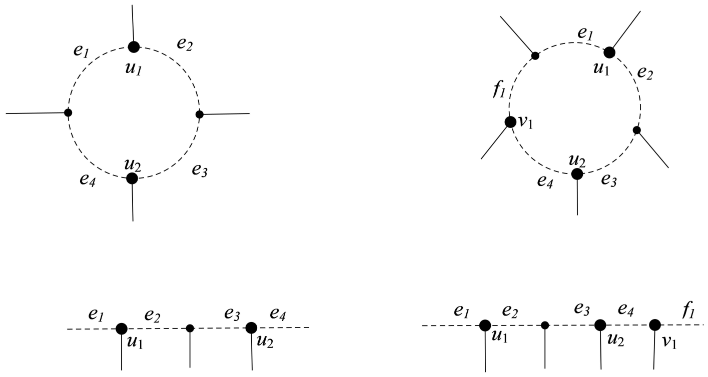

Figure 4. Choice for u_1, u_2, v_1.

4. Nsis Number

Let T be a spanning tree of a graph G and L be a vertex set of all leaves of T; we denote by $\alpha_1(T)$ the independence number of $G[L]$. In other words, $\alpha_1(T)$ is the independence number of the subgraph in G induced by the leaves of T. In [2] we have proved that $\mathcal{Z}(G) = \max_T\{\alpha_1(T) \colon T \text{ is a spanning tree of } G\}$. Hence, $\alpha_1(T_X) \leq \mathcal{Z}(G)$ for each Xuong-tree T_X in a cubic graph. Recall the set S_1 defined in Section 2, S_1 is a maximum non-separating independent set. In addition, S_1 is a subset of leaves of T_X. Thus, $\alpha_1(T_X) \geq |S_1| = \mathcal{Z}(G)$. Consequently, $\alpha_1(T_X) = \mathcal{Z}(G)$ for cubic graphs. As for the case of subcubic graphs, we have the following result.

Theorem 7. *For each subcubic graph G, there is a Xuong-tree T_X of G such that $\alpha_1(T_X) = \mathcal{Z}(G)$.*

To prove Theorem 7, we need two operations.

Operation I. Let u be a vertex of degree 2 and v, w the vertices adjacent to u (v and w may be identical) in G. Let H be the graph obtained from G by deleting u, adding two vertices x, y, connecting them by two parallel edges and adding edges xv, yw (see Figure 6).

The following result shows that this operation does not change the nsis number of G.

Lemma 8. $\mathcal{Z}(G) = \mathcal{Z}(H)$.

The proof of this lemma is simple and straightforward. The next result shows the relation between $\gamma_M(H)$ and $\gamma_M(G)$.

Lemma 9. $\gamma_M(G) \leq \gamma_M(H) \leq \gamma_M(G) + 1$.

Proof. Since G can be obtained by shranking the vertex set $\{x, y\}$, $\gamma_M(G) \leq \gamma_M(H)$. We now prove that $\gamma_M(H) \leq \gamma_M(G) + 1$. Let T_H be a Xuong-tree of H. We use x_1y_1 and x_2y_2 to denote the parallel edges connecting x and y. There are three case to be treated.

Case 1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T = T_H - \{x, y\} + \{vu, uw\}$. Then T is a spanning tree of G and $w(T; G) = \xi(H) - 1$ (see the left picture below). According to Theorem 1,

$$\begin{aligned}\gamma_M(G) &= \frac{1}{2}(\beta(G) - \xi(G)) = \frac{1}{2}(\beta(H) - 1 - \xi(G)) \geq \frac{1}{2}(\beta(H) - 1 - w(T; G)) \\ &= \frac{1}{2}(\beta(H) - 1 - (\xi(H) - 1)) = \gamma_M(H).\end{aligned}$$

Case 2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. Define $T = T_H - \{x, y\} + uw$. Then T is a spanning tree of G and $w(T; G)$ equals either $\xi(H)+1$ or $\xi(H)-1$ (see the right picture below). Therefore,

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G)) = \frac{1}{2}(\beta(H) - 1 - \xi(G)) \geq \frac{1}{2}(\beta(H) - 1 - w(T; G))$$
$$= \frac{1}{2}(\beta(H) - 1 - (\xi(H) + 1)) \geq \gamma_M(H) - 1.$$

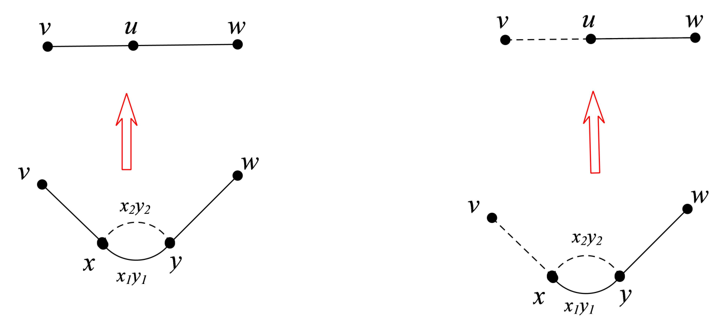

Figure 5. Operation I.

Case 3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. Let $T'_H = T_H - \{x, y\} + \{x_1y_1, yw\}$. Obviously, $w(T'_H; H) = w(T_H; H)$, i.e., T'_H is a Xuong-tree of H. Repeating the discussion as we did in Case 2 leads to the desired inequality. ■

The following lemma is an immediate consequence of Lemma 9.

Lemma 10. (i) $\gamma_M(G) = \gamma_M(H)$ *if and only if* $\xi(H) = \xi(G) + 1$; (ii) $\gamma_M(H) = \gamma_M(G) + 1$ *if and only if* $\xi(H) = \xi(G) - 1$.

Using Lemmas 8, 9 and 10, one may obtain the following result.

Lemma 11. *If there is a Xuong-tree* T_H *of* H *such that* $\alpha_1(T_H) = \mathcal{Z}(H)$, *then* G *has a Xuong-tree* T_G *with* $\alpha_1(T_G) = \mathcal{Z}(G)$.

Proof. Considering the relation between $\gamma_M(G)$ and $\gamma_M(H)$, we deal with the following cases.

Case 1. $\gamma_M(G) = \gamma_M(H)$, i.e., $\xi(H) = \xi(G) + 1$.

Subcase 1.1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + \{vu, uw\}$. Then T_G is a spanning tree of G and $w(T_G; G) = \xi(H) - 1$. So, T_G is a Xuong-tree of G.

Subcase 1.2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. $T_G = T_H - \{x, y\} + uw$. Then T_G is a spanning tree of G. We claim that the edge vx belongs to an odd component of $H - E(T_H)$. Otherwise, by the edge-pairing method of Xuong [13] to increase the genus of a graph, $\gamma_M(H) = \gamma_M(H - x_2y_2) + 1 = \gamma_M(G) + 1$, a contradiction. Therefore, $w(T_G; G) = \xi(H) - 1$. As a consequence, T_G is a Xuong-tree of G.

Subcase 1.3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. Combining the transformation approach described in the Lemma 9 and Subcase 1.2, we could find such a Xuong-tree T_G.

Case 2. $\gamma_M(G) + 1 = \gamma_M(H)$, i.e., $\xi(H) = \xi(G) - 1$.

Subcase 2.1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + \{vu, uw\}$. Then T_G is a spanning tree of G and $w(T_G; G) = \xi(H) - 1 < \xi(G)$, a contradiction. So, this subcase fails to happen.

Subcase 2.2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + uw$. Then T_G is a spanning tree of G. Notice that the edge vx belongs to an even component of $H - E(T_H)$. By way of contradiction, $\gamma_M(H) = \gamma_M(H - xy) = \gamma_M(G)$. This contradicts to the fact $\gamma_M(H) = \gamma_M(G) + 1$. Thereby, $w(T_G; G) = \xi(H) + 1$. As a result, T_G is a Xuong-tree of G.

Subcase 2.3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. This subcase can be solved by an argument similar to Subcase 1.3.

In each case above, $\alpha_1(T_G) = \alpha_1(T_H)$. Together with Lemma 8, it implies that $\alpha_1(T_G) = \mathcal{Z}(G)$. The lemma is builded. ∎

In order to replace the vertices of degree 1, we perform another operation.

Operation II. Let u be a vertex of degree 1 of a subcubic graph G and K the graph obtained from G by adding two vertices x and y, connecting them by two parallel edges and adding edges ux and uy.

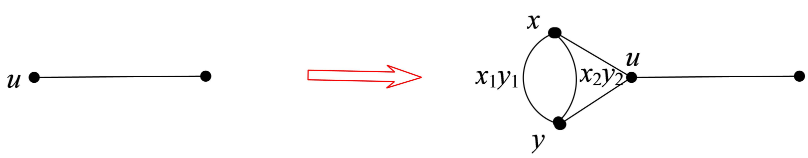

Figure 6. Operation II.

Lemma 12. *If there is a Xuong-tree T_K of K such that $\alpha_1(T_K) = \mathcal{Z}(K)$, then G has a Xuong-tree T_G with $\alpha_1(T_G) = \mathcal{Z}(G)$.*

Proof. Let S_G be a maximum non-separating independent set of G. Since, u is a vertex of degree 1, $u \in S_G$. It is easy check that $S_G \backslash \{u\} \cup \{x\}$ is a maximum non-separating independent set of K. Thus, $\mathcal{Z}(K) = \mathcal{Z}(G)$. We use x_1y_1 and x_2y_2 to denote the parallel edges connecting x and y. There are two case to deal with.

Case 1. $x_1y_1, x_2y_2 \notin E(T_K)$. Under this case, ux and uy belongs to $E(T_K)$. Let $T_G = T_K - \{x, y\}$. Since edges x_1y_1 and x_2y_2 form an even component of the co-tree $K - E(T_K)$, $w(T_K; K) = w(T_G; G)$. Therefore, T_G is a Xuong-tree of G. Let S_1 be a maximum independent set of the subgraph of K induced by the leaves of T_K. Then one of $\{x, y\}$ belongs to S_1, without loss of generality, say x. Then $S_1 \setminus \{x\} \cup \{u\}$ is a maximum independent set of the subgraph of G induced by the leaves of T_G. We derive that $\mathcal{Z}(G) = \mathcal{Z}(K) = \alpha_1(T_K) = \alpha_1(T_G)$.

Case 2. One of $\{x_1y_1, x_2y_2\}$ belongs in $E(T_K)$. Without loss of generality, suppose that x_1y_1 belongs in $E(T_K)$. Further, suppose that $xu \in E(T_K)$. Let $T_G = T_H - \{x, y\}$. Since edges x_2y_2 and yu form an even component of the co-tree $K - E(T_K)$, $w(T_K; K) = w(T_G; G)$. Therefore, T_G is a Xuong-tree of G. Let S_1 be a maximum independent set of the subgraph of K induced by the leaves of T_K. Then $S_1 \setminus \{y\} \cup \{u\}$ is a maximum independent set of the subgraph of G induced by the leaves of T_G. We obtain that $\mathcal{Z}(G) = \mathcal{Z}(K) = \alpha_1(T_K) = \alpha_1(T_G)$. ■

Depending on the results above, we can get Theorem 7.

Proof of Theorem 7. Given the fact that this result is true for cubic graphs, we transform a subcubic graph G into a cubic graph G' by means of the two operations above. Conversely, a Xuong-tree of G' will result in a corresponding Xuong-tree of G. □

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