

THE TREE-ACHIEVING SET AND NON-SEPARATING INDEPENDENT SET PROBLEM OF SUBCUBIC GRAPHS

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Abstract

The decycling number $\nabla(G)$ (respectively, tree-achieving number $\nabla_T(G)$) of a graph G is the smallest number of vertices whose deletion yields a forest (respectively, tree). Obviously, $\nabla_T(G) \geq \nabla(G)$ for all graphs. A graph is cubic (respectively, subcubic) if every vertex has degree three (respectively, at most three). A non-separating independent set is an independent vertices set whose deletion yields a connected subgraph. The nsis number $\mathcal{Z}(G)$ is the maximum cardinality of a non-separating independent set. In this article, we present a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in cubic graphs. That is $\nabla_T(G) = \nabla(G)$ if and only if there exists a Xuong-tree [J.L. Gross and R.G. Rieper, *Local extrema in genus-stratified graphs*, J. Graph Theory 15 (1991) 153–171] T_X of G such that every odd component of $G - E(T_X)$ contains at least three edges. Further, we give a formula for $\mathcal{Z}(G)$ in subcubic graphs: there is a Xuong-tree T_X of G such that $\alpha_1(T_X) = \mathcal{Z}(G)$, where $\alpha_1(T_X)$ is the independence number of the subgraph of G induced by leaves of T_X .

Keywords: tree-achieving number, decycling number, nsis number, Xuong-tree.

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1. INTRODUCTION

Graphs considered in this paper are finite and connected. A *decycling set* (also known as *feedback vertex set*) of a graph is a subset of the vertices whose deletion yields a forest. It is not hard to see that determining a decycling set is equivalent to finding an induced forest. The smallest size of a decycling set of a graph G , denoted by $\nabla(G)$, is called the *decycling number* of G . For brevity, we call a decycling set containing exactly $\nabla(G)$ vertices a ∇ -set in this paper.

In a network, the decycling set problem consists in finding a node set of minimum size such that excluding these nodes from the network guarantees an acyclic network. This is a critical problem which has numerous applications in parallel systems, combinatorial circuit design and distributed computing. For these reasons, determining the decycling number of graphs has attracted much attention of a large number of researchers. It has been shown that determining the decycling number of graphs is NP-hard [7]. A direction in the study of decycling number problem is computing the exact value or upper bound for sparse graphs.

A closely related to decycling number problem is to study the minimum size of vertices whose deletion yields a tree. Here, we say a vertex set S of a graph G is a *tree-achieving set* if $G - S$ is an induced tree. Analogously, the smallest cardinality of a tree-achieving set of G is said to be the *tree-achieving number*, and denoted by $\nabla_T(G)$. A tree-achieving set of this cardinality is called a ∇_T -set. The tree-achieving set problem was initiated by Erdős, Saks and Sós in 1986 [3]. Historically, this problem finds its motivation in the theory of energy of graphs. In [12], the authors characterized the extremal graphs with respect to matching energy from $T_t(n)$, where $T_t(n)$ is the set of t -apex trees (i.e., graphs with $\nabla_T(G) = t$). Thus, it is an important work to determine the tree-achieving number of graphs. However, in contrast to decycling number case, few works have been done on this topic.

It is apparent that $\nabla_T(G) \geq \nabla(G)$ for all graphs. In some cases, the gap between $\nabla(G)$ and $\nabla_T(G)$ can be arbitrarily large. It is clear from Figure 1 that $\nabla_T(G) = n, \nabla(G) = 1$. As stated above, computing the decycling number in a graph is NP-hard, not to mention the tree-achieving number. It would be interesting to establish a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in some specific graphs.

Another topic related to the decycling set problem is the non-separating independent set. In a graph, an *independent set* is a subset of vertices no two of which are adjacent. A set of vertices S of a connected graph G is a *non-separating independent set* (*nsis*, for short) if S is independent and $G - S$ is connected. The *nsis number* $\mathcal{Z}(G)$ is the maximum cardinality of a nsis. The nsis problem has many important applications in combinatorial optimization operation research and wireless network design [9]. In theory, the nsis number problem is closely

The *maximum genus* $\gamma_M(G)$ of G is defined to be the maximum integer k such that there exists a cellular embedding of G into an orientable surface of genus k . For general background, see Gross and Tucker [4], or Mohar and Thomassen [10].

Given a spanning tree T of a graph G , the subgraph $G - E(T)$ is called a *co-tree* of G . A component of a co-tree $G - E(T)$ is called odd (respectively, even) if it contains odd (respectively, even) number of edges. We use $w(T; G)$ to denote the number of odd components of $G - E(T)$. The *Betti deficiency* $\xi(G)$ of G is defined to be the minimum $w(T; G)$ over all spanning trees. A spanning tree T of G such that $w(T; G) = \xi(G)$ is said to be a *Xuong-tree* of G [5].

The following basic result, due to Xuong, relates the maximum genus to the Betti deficiency.

Theorem 1 [13]. *The maximum genus of a graph G is*

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G)).$$

Here, the $\beta(G)$ is called the *cycle rank* of G , which is the minimum number of edges whose removal results an acyclic graph. The cycle rank has a simple expression: $\beta(G) = |E(G)| - |V(G)| + 1$.

It is worth mentioning that during the procedure of proving Theorem 1, Xuong obtained the following edge-partition.

Lemma 2 [13]. *Let T_X be a Xuong-tree of a graph G . Then there exists an edge-partition of $G - E(T_X)$ as follows:*

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \dots, f_s\},$$

where (1) $m = \gamma_M(G)$, $s = \xi(G)$; (2) for any $i = 1, 2, \dots, m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$ and $\{f_1, f_2, \dots, f_s\}$ is a matching of G .

Figure 2 shows an edge-partition in K_4 .

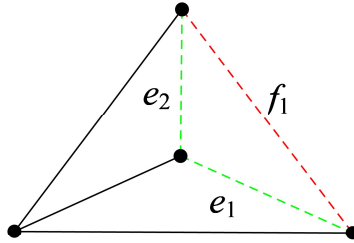


Figure 2. Edge-partition in K_4 .

In the following, we apply Lemma 2 to cubic graphs. Since $e_{2i-1} \cap e_{2i} \neq \emptyset$, they have at least an endvertex in common, say u_i , $i = 1, 2, \dots, m$. Actually,

u_i is a leaf of T_X . Thereby, the set $S_1 = \{u_i: 1 \leq i \leq m\}$ is a subset of leaves in T_X , which implies that $G - S_1$ is connected. In addition, the edges in $\{e_1, e_2, \dots, e_{2m}\}$ are different in pairs. Hence, any two vertices of S_1 are non-adjacent. Consequently, S_1 is a non-separating independent set of G . Together with the result $\mathcal{Z}(G) = \gamma_M(G)$ in cubic graphs [6], S_1 ($|S_1| = m = \gamma_M(G)$) is a maximum non-separating independent set. Let v_j be an endvertex of f_j , $1 \leq j \leq s$ and $S_2 = S_1 \cup \{v_j: 1 \leq j \leq s\}$. Deleting S_2 from G implies deleting all edges of $E(G) - E(T_X)$. So, S_2 is a decycling set. Combining with the equality $\nabla(G) = \gamma_M(G) + \xi(G)$ in cubic graphs [8], it yields that S_2 ($|S_2| = m + s = \gamma_M(G) + \xi(G)$) is a minimum decycling set.

Finally, we introduce a result that will be frequently used in this paper.

Lemma 3 [8]. *Let G' be a subdivision of G . Then*

- (1) $\gamma_M(G) = \gamma_M(G')$;
- (2) $\beta(G) = \beta(G')$;
- (3) $\xi(G) = \xi(G')$;
- (4) $\nabla(G) = \nabla(G')$.

3. SUFFICIENT AND NECESSARY CONDITION

In this section, we shall give a sufficient and necessary condition for $\nabla_T(G) = \nabla(G)$ in cubic graphs. Before going into the details, it is appropriate to say a few words about terminology and notation again. We write $G - e$ or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M . We write $G - v$ or $G - S$ for the induced subgraph obtained by deleting a vertex v or set of vertices S . Adding a set of edges S to a graph G is denoted by $G + S$. Let H be a subgraph of G , u and v be two vertices of G with $u \in V(H)$, $v \notin V(H)$ and the edge $uv \in E(G)$. We write $H + v$ for the subgraph with vertex set $V(H) \cup \{v\}$ and edge set $E(H)$, and simply write $H + v + uv$ as $H + uv$.

First, we give two period basic lemmas.

Lemma 4. *Every subgraph induced by a minimum decycling set in a cubic graph consists of a collection of isolated edges and vertices.*

Lemma 4 provides a structural characterization for the subgraphs induced by minimum decycling sets in a cubic graph. Specifically, it means that each minimum decycling set of a cubic graph contains no paths of length ≥ 2 and parallel edges. Although its proof is trivial, the consequences of this lemma are of major importance.

Lemma 5. *Let G be a cubic graph and S a ∇ -set of G . Then $c + t - 1 = \xi(G)$, where c and t are the numbers of the components of $G - S$ and edges of $G[S]$, respectively.*

Proof. For convenience, let $\nabla = \nabla(G)$ and $E(S) = E(G[S])$ in the proof. Suppose that $|V(G)| = n$. Then $|E(G)| = \frac{3n}{2}$. Set $S = \{x_1, x_2, \dots, x_\nabla\}$. We first assert that $c + t$ is an invariant. It is not hard verify that

$$\begin{aligned} \frac{3n}{2} - \sum_{i=1}^{\nabla} d_G(x_i) &= \frac{3n}{2} - 3\nabla = \frac{3n}{2} - |E(S, G - S)| - 2|E(S)| \\ &= n - \nabla - |E(S)| - c = n - \nabla - t - c. \end{aligned}$$

It follows that $t + c = 2\nabla - \frac{n}{2}$. Thereby, our assertion is true. Hence, it suffices to prove that there exists a ∇ -set S such that $\xi(G) = c + t - 1$.

There are two cases to be treated.

Case 1. G has loops. We complete the proof by applying induction on n . For $n = 2$, there is nothing to prove, so assume $n > 2$. Let v be a vertex incident to a loop and $G_1 = G - v$. Then G_1 is a subdivision of some cubic graph G'_1 . Assume that S'_1 is a ∇ -set of G'_1 . Analogously, $c'_1, t'_1, \xi(G'_1)$ are defined. By the induction hypothesis,

$$(1) \quad c'_1 + t'_1 - 1 = \xi(G'_1).$$

Set $S = S'_1 \cup \{v\}$. Then S is a ∇ -set of G . Now, we let u be the neighbor of v and x, y the other two neighbors of u . If both x and y belong to S'_1 , then $c = c'_1 + 1$ and $t = t'_1$, and $t = t'_1 + 1$ and $c = c'_1$ otherwise. In either case, we have

$$(2) \quad t + c = t'_1 + c'_1 + 1.$$

Let T_1 be a Xuong-tree of G_1 . Then $T_1 + uv$ is a Xuong-tree of G . Thus, $\xi(G) = \xi(G_1) + 1$. Lemma 3 implies that $\xi(G_1) = \xi(G'_1)$. So, we conclude that

$$(3) \quad \xi(G) = \xi(G_1) + 1 = \xi(G'_1) + 1.$$

Combining (1), (2) and (3), we deduce that $c + t - 1 = \xi(G)$.

Case 2. G has no loops. If G has a cut edge, then we can get our statement by a similar argument as Case 1. Hence, it suffices to consider the case of 2-edge connected graphs. Denote

$$N_E(S) = \{e : e = uv \in E(G), u \text{ or } v \in S\}.$$

Then,

$$|N_E(S)| + |E(G - S)| = \frac{3n}{2}, \quad |N_E(S)| = 3|S| - t, \quad |E(G - S)| = n - |S| - c.$$

Thus,

$$|S| = \frac{1}{2} \left(\frac{1}{2}n + c + t \right).$$

By Theorem 1, we deduce that

$$\frac{1}{2} \left(\frac{1}{2}n + c + t \right) = |S| = \gamma_M(G) + \xi(G) = \frac{1}{2} \left\{ \frac{n+2}{2} + \xi(G) \right\}.$$

Thereby,

$$(4) \quad c + t - 1 = \xi(G). \quad \blacksquare$$

Now, we are devoted to tree-achieving number problem in cubic graphs.

Theorem 6. *For a cubic graph G , $\nabla_T(G) = \nabla(G)$ if and only if there exists a Xuong-tree T_X of G such that every odd component of $G - E(T_X)$ contains at least three edges.*

Proof. If $\nabla_T(G) = \nabla(G)$, then there exists a ∇ -set S of G such that $G - S$ is a tree T . Suppose that $G[S]$ contains t edges, say e_1, e_2, \dots, e_t and $|S| - 2t$ isolated vertices, say $v_1, v_2, \dots, v_{|S|-2t}$. By Lemma 5, $c + t - 1 = \xi(G)$. Here, $c = 1$. Therefore, $t = \xi(G)$. Let first $e_i = x_{2i-1}x_{2i}$, $1 \leq i \leq t$. We use g_i^1, g_i^2 and f_i^1, f_i^2 to denote the other two edges incident to x_{2i-1} and x_{2i} , respectively, and use h_j^1, h_j^2, h_j^3 to denote the three edges incident to v_j , $1 \leq j \leq |S| - 2t$.

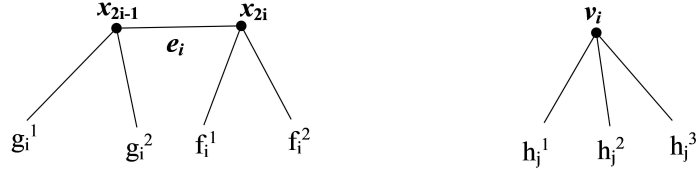


Figure 3. $g_i^1, g_i^2, f_i^1, f_i^2$ and h_j^1, h_j^2, h_j^3 .

Define $T_X = T + \{f_1^1, g_1^1, f_2^1, g_2^1, \dots, f_t^1, g_t^1, h_1^1, h_2^1, \dots, h_{|S|-2t}^1\}$. Then T_X is a spanning tree of G . Notice that $w(T_X; T_X) = 0$. For a given co-tree of a graph, adding a pair of adjacent edges to this graph can not increase the number of odd components of the co-tree. On the other hand, adding an edge to this graph increases at most one by the number of odd components of the co-tree. Let

$$H = T_X + \left\{ f_1^2, e_1, f_2^2, e_2, \dots, f_t^2, e_t, h_1^2, h_1^3, h_2^2, h_2^3, \dots, h_{|S|-2t}^2, h_{|S|-2t}^3 \right\}.$$

Then $w(T_X; H) = 0$. While, $G = H + \{g_1^2, g_2^2, \dots, g_t^2\}$. We obtain that $w(T_X; G) \leq t$. Combining $t = \xi(G)$, we conclude that $t = \xi(G) \leq w(T_X; G) \leq t$, which implies that T_X is a Xuong-tree of G .

From the construction T_X , every odd component of $G - E(T_X)$ contains the edges g_k^2, f_k^2, e_k for some $1 \leq k \leq t$. In other words, every odd component of $G - E(T_X)$ contains at least three edges.

Conversely, assume that G contains a Xuong-tree T_X , such that every odd component of $G - E(T_X)$ contains at least three edges. Bearing Lemma 2 in mind, there is an edge-partition

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \dots, f_s\},$$

where (1) $m = \gamma_M(G)$, $s = \xi(G)$; (2) for any $i = 1, 2, \dots, m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$ and $\{f_1, f_2, \dots, f_s\}$ is a matching of G . Note that each vertex of $G - E(T_X)$ has degree at most two, then every component of $G - E(T_X)$ is a path, a cycle, an edge or a vertex. Since every odd component of $G - E(T_X)$ contains at least three edges, each f_j does not form a component of $G - E(T_X)$, $1 \leq j \leq s$. Therefore, each f_j is adjacent to an edge e_{i_0} , $1 \leq j \leq s$, $1 \leq i_0 \leq 2m$. This implies that f_j is incident to leaf of T_X , say v_j , $1 \leq j \leq s$. Consider the vertex set

$$S_X = \{v_j : 1 \leq j \leq s\} \cup \{u_i : u_i \text{ is a common endvertex of } e_{2i-1} \text{ and } e_{2i}, \\ 1 \leq i \leq m\}.$$

Clearly, S_X is a ∇ -set of G (see Section 2). Since S_X is a subset of leaves of T_X , $G - S_X$ is connected. Therefore, we conclude that S_X is a ∇_T -set of G . The proof is completed. \blacksquare

Figure 4 provides an example of a choice for u_1, u_2, v_1 when one of the components of $G - E(T_X)$ is a 4 (even)-cycle, 5 (odd)-cycle, 4 (even)-path and 5 (even)-path, respectively.

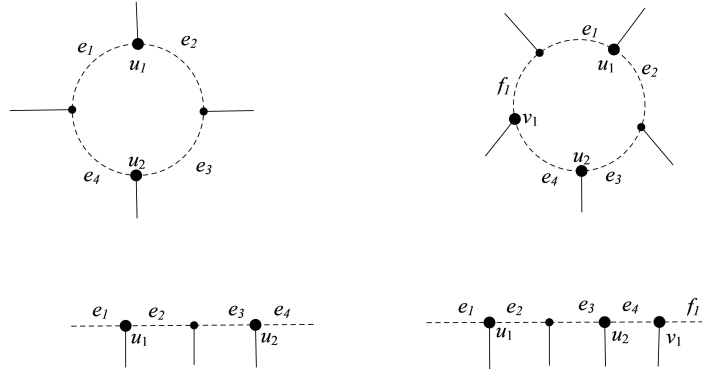


Figure 4. Choice for u_1, u_2, v_1 .

4. NSIS NUMBER

Let T be a spanning tree of a graph G and L be a vertex set of all leaves of T ; we denote by $\alpha_1(T)$ the independence number of $G[L]$. In other words, $\alpha_1(T)$ is the independence number of the subgraph in G induced by the leaves of T . In [2] we have proved that $\mathcal{Z}(G) = \max_T \{\alpha_1(T) : T \text{ is a spanning tree of } G\}$. Hence, $\alpha_1(T_X) \leq \mathcal{Z}(G)$ for each Xuong-tree T_X in a cubic graph. Recall the set S_1 defined in Section 2, S_1 is a maximum non-separating independent set. In addition, S_1 is a subset of leaves of T_X . Thus, $\alpha_1(T_X) \geq |S_1| = \mathcal{Z}(G)$. Consequently, $\alpha_1(T_X) = \mathcal{Z}(G)$ for cubic graphs. As for the case of subcubic graphs, we have the following result.

Theorem 7. *For each subcubic graph G , there is a Xuong-tree T_X of G such that $\alpha_1(T_X) = \mathcal{Z}(G)$.*

To prove Theorem 7, we need two operations.

Operation I. Let u be a vertex of degree 2 and v, w the vertices adjacent to u (v and w may be identical) in G . Let H be the graph obtained from G by deleting u , adding two vertices x, y , connecting them by two parallel edges and adding edges xv, yw (see Figure 6).

The following result shows that this operation does not change the nsis number of G .

Lemma 8. $\mathcal{Z}(G) = \mathcal{Z}(H)$.

The proof of this lemma is simple and straightforward. The next result shows the relation between $\gamma_M(H)$ and $\gamma_M(G)$.

Lemma 9. $\gamma_M(G) \leq \gamma_M(H) \leq \gamma_M(G) + 1$.

Proof. Since G can be obtained by shrinking the vertex set $\{x, y\}$, $\gamma_M(G) \leq \gamma_M(H)$. We now prove that $\gamma_M(H) \leq \gamma_M(G) + 1$. Let T_H be a Xuong-tree of H . We use x_1y_1 and x_2y_2 to denote the parallel edges connecting x and y . There are three case to be treated.

Case 1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T = T_H - \{x, y\} + \{vu, uw\}$. Then T is a spanning tree of G and $w(T; G) = \xi(H) - 1$ (see the left picture below). According to Theorem 1,

$$\begin{aligned} \gamma_M(G) &= \frac{1}{2}(\beta(G) - \xi(G)) = \frac{1}{2}(\beta(H) - 1 - \xi(G)) \geq \frac{1}{2}(\beta(H) - 1 - w(T; G)) \\ &= \frac{1}{2}(\beta(H) - 1 - (\xi(H) - 1)) = \gamma_M(H). \end{aligned}$$

Case 2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. Define $T = T_H - \{x, y\} + uw$. Then T is a spanning tree of G and $w(T; G)$ equals either $\xi(H) + 1$ or $\xi(H) - 1$ (see the right picture below). Therefore,

$$\begin{aligned} \gamma_M(G) &= \frac{1}{2}(\beta(G) - \xi(G)) = \frac{1}{2}(\beta(H) - 1 - \xi(G)) \geq \frac{1}{2}(\beta(H) - 1 - w(T; G)) \\ &= \frac{1}{2}(\beta(H) - 1 - (\xi(H) + 1)) \geq \gamma_M(H) - 1. \end{aligned}$$

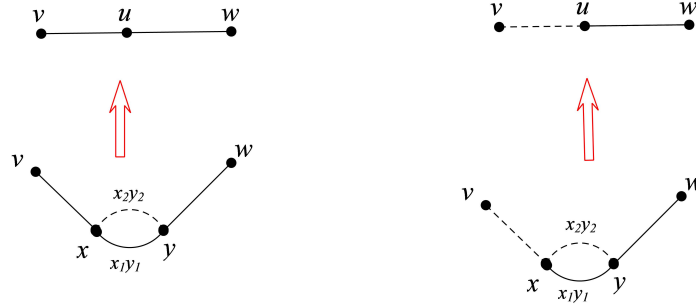


Figure 5. Operation I.

Case 3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. Let $T'_H = T_H - \{x, y\} + \{x_1y_1, yw\}$. Obviously, $w(T'_H; H) = w(T_H; H)$, i.e., T'_H is a Xuong-tree of H . Repeating the discussion as we did in Case 2 leads to the desired inequality. ■

The following lemma is an immediate consequence of Lemma 9.

Lemma 10. (i) $\gamma_M(G) = \gamma_M(H)$ if and only if $\xi(H) = \xi(G) + 1$; (ii) $\gamma_M(H) = \gamma_M(G) + 1$ if and only if $\xi(H) = \xi(G) - 1$.

Using Lemmas 8, 9 and 10, one may obtain the following result.

Lemma 11. If there is a Xuong-tree T_H of H such that $\alpha_1(T_H) = \mathcal{Z}(H)$, then G has a Xuong-tree T_G with $\alpha_1(T_G) = \mathcal{Z}(G)$.

Proof. Considering the relation between $\gamma_M(G)$ and $\gamma_M(H)$, we deal with the following cases.

Case 1. $\gamma_M(G) = \gamma_M(H)$, i.e., $\xi(H) = \xi(G) + 1$.

Subcase 1.1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + \{vu, uw\}$. Then T_G is a spanning tree of G and $w(T_G; G) = \xi(H) - 1$. So, T_G is a Xuong-tree of G .

Subcase 1.2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. $T_G = T_H - \{x, y\} + uw$. Then T_G is a spanning tree of G . We claim that the edge vx belongs to an odd component of $H - E(T_H)$. Otherwise, by the edge-pairing method of Xuong [13] to increase the genus of a graph, $\gamma_M(H) = \gamma_M(H - x_2y_2) + 1 = \gamma_M(G) + 1$, a contradiction. Therefore, $w(T_G; G) = \xi(H) - 1$. As a consequence, T_G is a Xuong-tree of G .

Subcase 1.3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. Combining the transformation approach described in the Lemma 9 and Subcase 1.2, we could find such a Xuong-tree T_G .

Case 2. $\gamma_M(G) + 1 = \gamma_M(H)$, i.e., $\xi(H) = \xi(G) - 1$.

Subcase 2.1. $vx, x_1y_1, yw \in E(T_H)$ and $x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + \{vu, uw\}$. Then T_G is a spanning tree of G and $w(T_G; G) = \xi(H) - 1 < \xi(G)$, a contradiction. So, this subcase fails to happen.

Subcase 2.2. $x_1y_1, yw \in E(T_H)$ and $vx, x_2y_2 \notin E(T_H)$. Let $T_G = T_H - \{x, y\} + uw$. Then T_G is a spanning tree of G . Notice that the edge vx belongs to an even component of $H - E(T_H)$. By way of contradiction, $\gamma_M(H) = \gamma_M(H - xy) = \gamma_M(G)$. This contradicts to the fact $\gamma_M(H) = \gamma_M(G) + 1$. Thereby, $w(T_G; G) = \xi(H) + 1$. As a result, T_G is a Xuong-tree of G .

Subcase 2.3. $vx, yw \in E(T_H)$ and $x_1y_1, x_2y_2 \notin E(T_H)$. This subcase can be solved by an argument similar to Subcase 1.3.

In each case above, $\alpha_1(T_G) = \alpha_1(T_H)$. Together with Lemma 8, it implies that $\alpha_1(T_G) = \mathcal{Z}(G)$. The lemma is builded. ■

In order to replace the vertices of degree 1, we perform another operation.

Operation II. Let u be a vertex of degree 1 of a subcubic graph G and K the graph obtained from G by adding two vertices x and y , connecting them by two parallel edges and adding edges ux and uy .

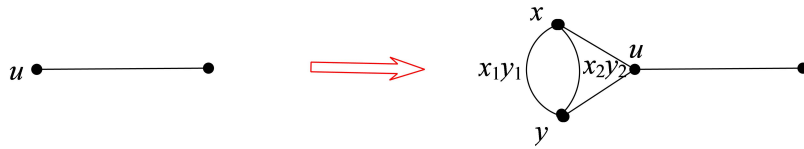


Figure 6. Operation II.

Lemma 12. *If there is a Xuong-tree T_K of K such that $\alpha_1(T_K) = \mathcal{Z}(K)$, then G has a Xuong-tree T_G with $\alpha_1(T_G) = \mathcal{Z}(G)$.*

Proof. Let S_G be a maximum non-separating independent set of G . Since, u is a vertex of degree 1, $u \in S_G$. It is easy check that $S_G \setminus \{u\} \cup \{x\}$ is a maximum non-separating independent set of K . Thus, $\mathcal{Z}(K) = \mathcal{Z}(G)$. We use x_1y_1 and x_2y_2 to denote the parallel edges connecting x and y . There are two case to deal with.

Case 1. $x_1y_1, x_2y_2 \notin E(T_K)$. Under this case, ux and uy belongs to $E(T_K)$. Let $T_G = T_K - \{x, y\}$. Since edges x_1y_1 and x_2y_2 form an even component of the co-tree $K - E(T_K)$, $w(T_K; K) = w(T_G; G)$. Therefore, T_G is a Xuong-tree of G . Let S_1 be a maximum independent set of the subgraph of K induced by the leaves of T_K . Then one of $\{x, y\}$ belongs to S_1 , without loss of generality, say x . Then $S_1 \setminus \{x\} \cup \{u\}$ is a maximum independent set of the subgraph of G induced by the leaves of T_G . We derive that $\mathcal{Z}(G) = \mathcal{Z}(K) = \alpha_1(T_K) = \alpha_1(T_G)$.

Case 2. One of $\{x_1y_1, x_2y_2\}$ belongs in $E(T_K)$. Without loss of generality, suppose that x_1y_1 belongs in $E(T_K)$. Further, suppose that $xu \in E(T_K)$. Let $T_G = T_H - \{x, y\}$. Since edges x_2y_2 and yu form an even component of the co-tree $K - E(T_K)$, $w(T_K; K) = w(T_G; G)$. Therefore, T_G is a Xuong-tree of G . Let S_1 be a maximum independent set of the subgraph of K induced by the leaves of T_K . Then $S_1 \setminus \{y\} \cup \{u\}$ is a maximum independent set of the subgraph of G induced by the leaves of T_G . We obtain that $\mathcal{Z}(G) = \mathcal{Z}(K) = \alpha_1(T_K) = \alpha_1(T_G)$. ■

Depending on the results above, we can get Theorem 7.

Proof of Theorem 7. Given the fact that this result is true for cubic graphs, we transform a subcubic graph G into a cubic graph G' by means of the two operations above. Conversely, a Xuong-tree of G' will result in a corresponding Xuong-tree of G . □

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