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ON LINK-IRREGULAR GRAPHS

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Abstract

The subgraph of a graph G that is induced by the set of neighbors of a vertex v of G is the link of v. If every two distinct vertices of G have non-isomorphic links, then G is link-irregular. It is shown that there exists a link-irregular graph of order n if and only if $n \geq 6$. The degree set $\mathcal{D}(G)$ of G is the set of degrees of the vertices of G. While there is no link-irregular graph G of order n such that $|\mathcal{D}(G)| \in \{n, n-1\}$, it is shown that there exists a link-irregular graph G of order n such that $|\mathcal{D}(G)| = n-2$ if and only if $n \geq 7$. Further, for each pair (d, n) of integers with $3 \leq d \leq 8$ and $n \geq d+4$, there is a link-irregular graph of order n whose degree set consists of $n-d$ elements. The link-irregular ratio $\text{lir}(G)$ of a link-irregular graph G is defined as $|\mathcal{D}(G)|/|V(G)|$. For the set $\mathcal{L}$ of link-irregular graphs, it is shown that $\sup\{\text{lir}(G) : G \in \mathcal{L}\} = 1$ and that $0 \leq \inf\{\text{lir}(G) : G \in \mathcal{L}\} \leq 1/9$. Other results, problems, and conjectures on link-irregular graphs are also presented.

Keywords: degree of a vertex, link of a vertex, link-irregular graph.

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1. Introduction

Of the many classes of graphs that have been popular to study, it is the class of regular graphs that are among the most studied. A graph G is *regular* if every vertex of G has the same degree. If this degree is r, then G is r*-regular*. If $r = 0$, then G consists only of isolated vertices; if $r = 1$, then G is a matching; while if $r = 2$, then each component of G is a cycle. Therefore, the situation when $r \geq 3$ has drawn the most interest. The r-regular graphs of minimum order having a specific girth g (the length of a smallest cycle) are (r, g)*-cages*. These graphs have been studied by many (see [5], for example). A 3-regular graph is often called a *cubic graph*. A *cubic map* is a connected cubic bridgeless plane graph. Due to the work of Tait [8], it was known for decades that if it could be shown that every cubic map had a proper 3-coloring of its edges, then the famous Four Color Problem would have an affirmative solution.

While the vertices of a regular graph have the same degree, this does not mean that these graphs are locally regular in other senses. The *link* $L(v)$ of a vertex v in a graph G is the subgraph induced by the set of neighbors of v in G, that is, $L(v) = G[N(v)]$. When discussing the links of the vertices of a graph G, we always assume that G has no isolated vertices. If every two vertices of a graph G have the same link, then G is said to be *link-regular*. If there exists a graph H such that $L(v) \cong H$ for every vertex v of G, then G is H*-link-regular*. A graph H is a *link graph* if there exists a graph G that is H-link-regular. Clearly, if G is link-regular, then G is regular. The converse is not true, however. For two vertex-disjoint graphs G_1 and G_2, let $G_1 + G_2$ denote the union of G_1 and G_2. For example, the cubic graph G of Figure 1 is not link-regular; for this graph, $L(u) \cong \overline{K}_3$, $L(v) \cong K_2 + K_1$, and $L(w) \cong P_3$.

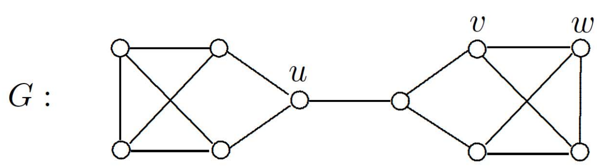

Figure 1. A regular graph that is not link-regular.

This topic was described in the book [1] and the concept was suggested by the Russian mathematician Alexander Zykov [9], author of the first textbook in graph theory written in Russian. At the symposium in Smolenice on the Theory of Graphs and Its Applications, which took place during 17–20 June 1963, Zykov presented the following problem, namely Problem #30, which appeared in the proceedings of this conference.

Problem # 30. *Given a finite graph H, does there exist a nonempty (graph) G with all neighbourhoods of its vertices isomorphic to H?*

Every vertex-transitive graph is not only regular, it is link-regular. However, there are link-regular graphs that are not vertex-transitive. For example, the two cubic graphs G_1 and G_2 shown in Figure 2 are not vertex-transitive but are link-regular, where $L(v) = \overline{K_3}$ for each vertex v of G_1 and $L(v) = K_2 + K_1$ for each vertex v of G_2.

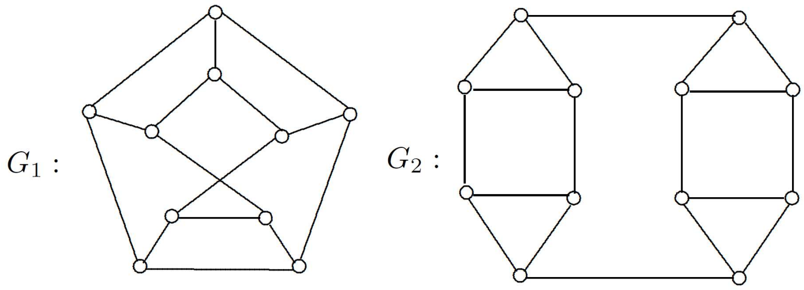

Figure 2. Two link-regular graphs that are not vertex-transitive.

Several familiar classes of graphs are known to be link graphs. For example, every complete graph is a link graph since K_n is the link of every vertex of K_{n+1} for each positive integer n. Also, every empty graph is a link graph since $\overline{K}_n$ is the link of every vertex of the regular complete bipartite graph $K_{n,n}$. Indeed, for each integer $r \geq 2$, every r-regular triangle-free graph is $\overline{K}_r$-link-regular. More generally, every regular complete multipartite graph is a link graph. For example, $K_{r,r,r}$ is the link of every vertex of the graph $K_{r,r,r,r}$ for each positive integer r.

Since there is a K_3-link-regular graph, namely K_4, there is a C_3-link-regular graph. Also, there is a C_4-link-regular graph since $C_4 = K_{2,2}$ and $K_{2,2,2}$ is a $K_{2,2}$-link-regular graph. In fact, it was shown by Brown and Connelly in [4] that there is a C_n-link-regular graph for each integer $n \geq 3$.

Theorem 1.1 [4]**.** *For each integer $n \geq 3$, there is a C_n-link-regular graph.*

Among the link graphs are the friendship graphs. For each positive integer k, the graph $F_k = kK_2 \vee K_1$ (the join of kK_2 and K_1) is called a *friendship graph.* The following result was obtained in [1].

Theorem 1.2 [1]**.** *For each positive integer k, the friendship graph F_k is a link graph.*

In addition to the friendship graphs, another well-known class of graphs is that of the Kneser graphs. For positive integers k and n with $n > 2k$, the

Kneser graph $KG_{n,k}$ is that graph whose vertices are the k-element subsets of $[n] = \{1, 2, \ldots, n\}$ and where two vertices (k-element subsets) A and B are adjacent if and only if A and B are disjoint. Consequently, the Kneser graph $KG_{n,1}$ is the complete graph K_n and the Kneser graph $KG_{5,2}$ is isomorphic to the Petersen graph. Since the Kneser graph $KG_{n+k,k}$ is $KG_{n,k}$-link-regular for every two positive integers k and n with $n > 2k$, it follows that every Kneser graph is a link graph. In particular, the 10-regular Kneser graph $KG_{7,2}$ of order 21 is $KG_{5,2}$-link-regular. Therefore, the Petersen graph P is a link graph and $KG_{7,2}$ is P-link-regular. Hall [6] showed that only two other graphs are P-link-regular.

Theorem 1.3 [6]**.** *For the Petersen graph P, there are exactly three non-isomorphic graphs that are P-link-regular.*

Other graphs that are link graphs have been obtained in [3, 4, 7].

2. Link-Irregular Graphs

The graphs that are opposite to the regular graphs in a sense are the irregular graphs. A nontrivial graph G is *irregular* if no two vertices of G have the same degree. It is well known that no graph is irregular.

Theorem 2.1 [2]**.** *For every integer $n \geq 2$, there is no irregular graph of order n.*

The graphs that are opposite to the link-regular graphs are the link-irregular graphs. A graph G is *link-irregular* if every two vertices of G have distinct links; that is, for every two vertices u and v of G, $L(u) \ncong L(v)$. Contrary to the situation for irregular graphs, there are link-irregular graphs. For example, the graph G_6 of order 6 in Figure 3 is link-irregular. Since it can be ready shown that no graph of order 6 or less other than G_6 is link-irregular, it follows that G_6 is the unique link-irregular graph of smallest order. The links of the vertices of G_6 are also shown in Figure 3. Observe that if u and v are vertices of distinct degrees in a graph G, then $L(u)$ and $L(v)$ have different orders and so $L(u) \ncong L(v)$. Thus, to verify that G is link-irregular, it suffices to show that the links of every two vertices with the same degree are non-isomorphic.

Not only is there a link-irregular graph of order 6, there is a link-irregular graph of order n for every integer $n \geq 6$.

Theorem 2.2. *There exists a link-irregular graph of order n if and only if $n \geq 6$.*

Proof. We have already mentioned that no graph of order less than 6 is link-irregular. It therefore remains to show that there is a link-irregular graph G_n of order n for each integer $n \geq 6$. We saw that the graph G_6 of order 6 in Figure 3 is link-irregular. For each integer $n \geq 7$, we construct a graph G_n recursively as

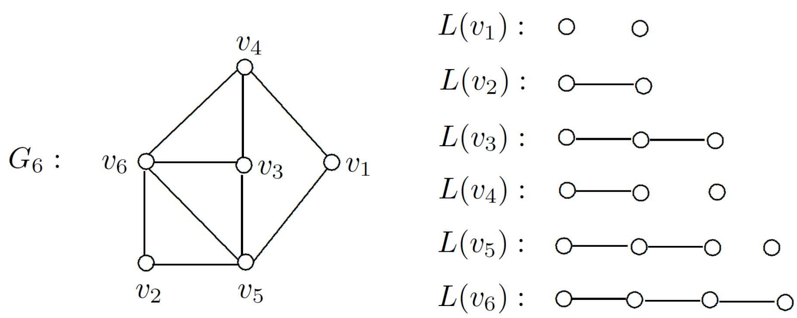

Figure 3. The unique link-irregular graph of order 6.

follows. Let $G_7 = G_6 \vee K_1$ be the join of the graph G_6 of Figure 3 and K_1, and let G_8 be the graph obtained from G_7 by adding a pendant edge at a vertex of minimum degree in G_7. For an integer $n \geq 9$, the graph G_n is constructed from G_{n-1} as follows.

- ⋆ If n is odd, let $G_n = G_{n-1} \vee K_1$ be the join of G_{n-1} and K_1. Thus, $\Delta(G_n) = n - 1$.
- ⋆ If n is even, let G_n be the graph obtained by adding a pendant edge at a vertex of minimum degree in G_{n-1}. Thus, $\Delta(G_n) = \Delta(G_{n-1}) = n - 2$.

First, observe that for each integer $n \geq 7$, the graph G_n is a connected graph of order n. It remains to show that G_n is link-irregular. Before doing this, however, we verify the following two claims.

Claim 1. *If $n \geq 7$ is odd, then the link of every vertex in G_n is a nontrivial connected subgraph of G_n.*

Proof. Recall, for each odd integer $n \geq 7$, that $G_n = G_{n-1} \vee K_1$. For a vertex v in G_{n-1}, let $L_n(v)$ and $L_{n-1}(v)$ denote the links of v in G_n and in G_{n-1}, respectively. Then $L_n(v) = L_{n-1}(v) \vee K_1$ is a connected nontrivial graph. If $v \in V(G_n) \setminus V(G_{n-1})$, then $L_n(v) = G_{n-1}$, which is a connected graph of order $n - 1$. Thus, Claim 1 holds. □

Claim 2. *If $n \geq 7$ is odd, then G_n has a unique vertex of maximum degree $n-1$.*

Proof. Since $\Delta(G_6) = 4$ and $G_7 = G_6 \vee K_1$, it follows that G_7 has a unique vertex of maximum degree 6. Let $n \geq 9$ be an odd integer. Then $G_{n-2} = G_{n-3} \vee K_1$ and so $\Delta(G_{n-2}) = n-3$. Since G_{n-1} is obtained from G_{n-2} by adding a pendant edge at a vertex of minimum degree in G_{n-2}, it follows that $\Delta(G_{n-1}) = \Delta(G_{n-2}) = n - 3$. Therefore, the graph $G_n = G_{n-1} \vee K_1$ has a unique vertex of maximum degree $n - 1$. Thus, Claim 2 holds. □

Next, we proceed by induction to show that G_n is link-irregular for each integer $n \geq 6$. We saw that G_6 is link-irregular and so the base step holds. Assume that G_{n-1} is link-irregular for some integer $n \geq 7$. We show that G_n is link-irregular.

Let $V(G_n) = \{v_1, v_2, \ldots, v_n\}$ where $v_n \notin V(G_{n-1})$. For $1 \leq i \leq n-1$, let $L_{n-1}(v_i)$ denote the link of v_i in G_{n-1}. For $1 \leq i \leq n$, let $L_n(v_i)$ be the link of v_i in G_n. We consider two cases, according to the parity of n.

Case 1. *$n \geq 8$ is even.* Then G_n is constructed from G_{n-1} by adding the vertex v_n and joining v_n to a vertex of minimum degree in G_{n-1}, say v_n is joined to v_{n-1} in G_{n-1}. Observe that $L_n(v_i) = L_{n-1}(v_i)$ for $1 \leq i \leq n-2$, $L_n(v_{n-1}) = L_{n-1}(v_{n-1}) + K_1$, which is a disconnected graph, and $L_n(v_n) \cong K_1$, which is the trivial graph. Since G_{n-1} is link-irregular, $L_{n-1}(v_i) \ncong L_{n-1}(v_j)$ for every pair i, j of integers with $i \neq j$ and $1 \leq i, j \leq n-2$. Thus, $L_n(v_i) \ncong L_n(v_j)$ if $i \neq j$ and $1 \leq i, j \leq n-2$. By Claim 1, for each integer i with $1 \leq i \leq n-2$, the link $L_{n-1}(v_i)$ of v_i in G_{n-1} is a nontrivial connected graph. Hence, $L_n(v_i) \ncong L_n(v_n)$ and $L_n(v_i) \ncong L_n(v_{n-1})$ for $1 \leq i \leq n-2$. Furthermore, $L_n(v_n) \ncong L_n(v_{n-1})$. Therefore, G_n is link-irregular.

Case 2. *$n \geq 7$ is odd.* Then $G_n = G_{n-1} \vee K_1$. Thus, $L_n(v_i) = L_{n-1}(v_i) \vee K_1$ for $1 \leq i \leq n-1$ and $L_n(v_n) = G_{n-1}$. Since G_{n-1} is link-irregular, $L_{n-1}(v_i) \ncong L_{n-1}(v_j)$ for every pair i, j of integers with $i \neq j$ and $1 \leq i, j \leq n-1$. Thus, $L_n(v_i) \ncong L_n(v_j)$ if $i \neq j$ and $1 \leq i, j \leq n-1$. By Claim 2, v_n is the only vertex of maximum degree $n-1$ in G_n and so $\deg_{G_n}(v_i) \leq n-2$ for $1 \leq i \leq n-1$. Hence, $L_n(v_n) \ncong L_n(v_i)$ for each integer i with $1 \leq i \leq n-1$. Therefore, G_n is link-irregular. ∎

A nontrivial graph G has been called *antiregular* if exactly two vertices of G have the same degree. While no nontrivial graph is irregular, there are antiregular graphs of every order $n \geq 2$ (see [2], for example).

Theorem 2.3 [2]**.** *For every integer $n \geq 2$, there are exactly two non-isomorphic antiregular graphs of order n, one of which is connected and the other is its disconnected complement.*

The connected antiregular graph G_n of order $n \geq 2$ referred to in Theorem 2.3 can be defined as the unique graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ for which $v_i v_j \in E(G_n)$ if and only if $i + j \geq n + 1$.

Proposition 1. *No antiregular graph is link-irregular.*

Proof. As we mentioned earlier, no graph of order at most 5 is link-irregular, so only antiregular graphs of order 6 or more need be considered. In the connected antiregular graph G_n of order $n \geq 6$, only the two vertices of degree $\lfloor n/2 \rfloor$ have the same degree. Since the links of these two vertices are both $K_{\lfloor n/2 \rfloor}$, it follows that G_n is not link-irregular. The only other antiregular graph of order n is the complement $\overline{G}_n$ of G_n. The nontrivial component of $\overline{G}_n$ is the connected antiregular graph G_{n-1} of order $n-1$ and so $\overline{G}_n$ is not link-irregular either. ■

For a graph G, let $\mathcal{D}(G)$ denote the degree set of G (the set of degrees of the vertices of G). The following is a consequence of Theorem 2.1 and Proposition 1.

Corollary 2. *For each integer $n \geq 2$, there is no link-irregular graph G of order n such that $|\mathcal{D}(G)| = n$ or $|\mathcal{D}(G)| = n-1$.*

This brings up the question as whether there is a link-irregular graph G of order n such that $|\mathcal{D}(G)| = n-2$. For $n = 7$, the graph H_7 in Figure 4 is a link-irregular graph of order 7 with $|\mathcal{D}(H_7)| = 5$. In order to answer this question in general, we present two lemmas, the first of which is a consequence of the proof of Theorem 2.2.

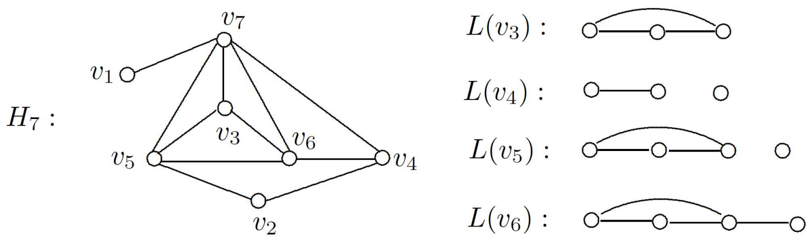

Figure 4. A link-irregular graph H_7 of order 7.

Lemma 2.4. *Let H be a link-irregular graph of order $n \geq 6$. If $\Delta(H) \leq n-2$, then $H \vee K_1$ is also a link-irregular graph.*

Proof. Let H be a link-irregular graph of order $n \geq 6$ with $\Delta(H) \leq n-2$ and let $G = H \vee K_1$. Thus, G has only one vertex w of degree n in G and $L_G(w) = H$. Let u and v be any two vertices of G different from w. Since $L_H(u) \ncong L_H(v)$, it follows that $L_G(u) = L_H(u) \vee K_1 \ncong L_H(v) \vee K_1 = L_G(v)$. Therefore, G is link-irregular. ■

As we saw in the proof of Theorem 2.2, the graph G_6 of order 6 in Figure 3 has $\delta(G_6) = 2$, and $\Delta(G_6) = 4$. By Lemma 2.4, the graph $G_6 \vee K_1$ is a link-irregular graph of order 7 with $\delta(G_6 \vee K_1) = 3$ and $\Delta(G_6 \vee K_1) = 6$.

Lemma 2.5. *If H is a link-irregular graph, then $(H + K_1) \vee K_1$ is also a link-irregular graph.*

Proof. Let $G = (H + K_1) \vee K_1$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $V(H) = \{v_1, v_2, \ldots, v_{n-2}\}$, the vertex v_{n-1} is the isolated vertex in $H+K_1$, and the vertex

v_n is the unique vertex of maximum degree $n-1$ in G_n. For $1 \le i \le n-2$, the link of v_i is $L_G(v_i) = L_H(v_i) \vee K_1$, where $L_H(v_i)$ is the link of v_i in H, $L_G(v_{n-1}) = K_1$, and $L_G(v_n) = H + K_1$. Since H is link-irregular, $L_H(v_i) \ncong L_H(v_j)$ for every pair i, j of integers with $i \neq j$ and $1 \le i, j \le n-2$. Thus, $L_G(v_i) \ncong L_G(v_j)$ if $i \neq j$ and $1 \le i, j \le n-2$. Since $L(v_n)$ is the only link of order $n-1$ in G and each $L_G(v_i)$ is nontrivial for $1 \le i \le n-1$, it follows that $L_G(v_n) \ncong L_G(v_i)$ for each integer i with $1 \le i \le n-1$ and $L_G(v_{n-1}) \ncong L_G(v_i)$ for each integer i with $1 \le i \le n-2$. Therefore, G is link-irregular. ∎

Theorem 2.6. *There exists a link-irregular graph H_n of order n such that $|\mathcal{D}(H_n)| = n-2$ if and only if $n \ge 7$.*

Proof. As we mentioned, no graph of order less than 6 is link-irregular and the graph G_6 of Figure 3 is the only link-irregular graph of order 6. Since $|\mathcal{D}(G_6)| = 3$, there is no link-irregular graph of order 6 whose degree set has cardinality 4. It therefore remains to show that there is a link-irregular graph H_n of order n such that $|\mathcal{D}(H_n)| = n-2$ for each integer $n \ge 7$. We saw that the graph H_7 in Figure 4 is a link-irregular graph of order 7 with $|\mathcal{D}(H_7)| = 5$. For each integer $n \ge 8$, we construct a graph H_n recursively as follows. Let $H_8 = H_7 \vee K_1$ and let $H_9 = (H_7 + K_1) \vee K_1$. For an integer $n \ge 10$, let $H_n = (H_{n-2} + K_1) \vee K_1$. Since H_7 is a link-irregular graph of order 7 with $\Delta(H_7) = 5$, it follows by Lemma 2.4 that H_8 is link-irregular. Furthermore, since H_7 is link-irregular, it follows by Lemma 2.5 that H_9 is link-irregular. Therefore, H_n is link-irregular for each integer $n \ge 10$ by applying Lemma 2.5 repeatedly.

It remains to show for each integer $n \ge 7$ that $|\mathcal{D}(H_n)| = n-2$. We proceed by induction. It is not difficult to see that $|\mathcal{D}(H_n)| = n-2$ for $n = 7, 8, 9$. Suppose that $|\mathcal{D}(H_{n-2})| = n-4$ for some integer n such that $n-2 \ge 9$. Let $a_1, a_2, \ldots, a_{n-2}$ be the degree sequence of H_{n-2}, where then $1 \le a_i \le n-3$ for $1 \le i \le n-2$. Since the degree sequence of $H_n = (H_{n-2} + K_1) \vee K_1$ is

$$1, a_1+1, a_2+1, \ldots, a_{n-2}+1, n-1$$

and $1 < a_i + 1 < n-1$ for $1 \le i \le n-2$, it follows that

$$\mathcal{D}(H_n) = \{1, 2, \ldots, n-1\} - \left\{\lfloor n/2 \rfloor + (-1)^{n+1} \cdot 3\right\},$$

where there are two vertices of degree $\lfloor n/2 \rfloor$, two vertices of degree $\lfloor n/2 \rfloor + (-1)^n$, and one vertex of every other degree. Thus, $|\mathcal{D}(H_n)| = n-2$. ∎

By Theorem 2.6, it follows that for each integer $n \ge 7$, there exists a link-irregular graph H_n of order n such that $|\mathcal{D}(H_n)| = n-2$. In fact, for each integer $d \in \{3, 4, 5, 6, 7, 8\}$, there is a link-irregular graph H_n of order $n \ge d+4$ such that $|\mathcal{D}(H_n)| = n-d$. In order to establish this result, we first present some preliminary results.

Observation 3. *Let G be a graph of order $n \geq 3$.*

- ⋆ *If $\Delta(G) \leq n-2$, then $|\mathcal{D}(G \vee K_1)| = |\mathcal{D}(G)| + 1$.*
- ⋆ *If $\delta(G) \geq 1$, then $|\mathcal{D}((G + K_1) \vee K_1)| = |\mathcal{D}(G)| + 2$.*

With the aid of Lemmas 2.4 and 2.5 and Observation 3, we are now able to present the following result.

Proposition 4. *If there exists a link-irregular graph G of order $p \geq 6$ such that $1 \leq \delta(G) \leq \Delta(G) \leq p-2$, then there exists a link-irregular graph H of order $p+k$ with $|\mathcal{D}(H)| = |\mathcal{D}(G)| + k$ for each positive integer k. Consequently, $|V(H)| - |\mathcal{D}(H)| = |V(G)| - |\mathcal{D}(G)|$.*

Proof. Let G be a link-irregular graph of order $p \geq 6$ such that $1 \leq \delta(G) \leq \Delta(G) \leq p-2$ and let k be a positive integer. We consider two cases, according to the parity of k.

Case 1. *$k \geq 1$ is odd.* For $k = 1$, let $H = G \vee K_1$. Since $\Delta(G) \leq p-2$, it follows by Lemma 2.4 and Observation 3 that H is a link-irregular graph of order $p+1$ with $|\mathcal{D}(H)| = |\mathcal{D}(G)|+1$. Hence, $|V(H)|-|\mathcal{D}(H)| = |V(G)|-|\mathcal{D}(G)|$. Thus, we may assume that $k = 2\ell + 1$ for some integer $\ell \geq 1$. Let $H_1 = ((G \vee K_1) + K_1) \vee K_1$. Since $\delta(G \vee K_1) \geq 1$, it follows by Lemma 2.5 and Observation 3 that H_1 is a link-irregular graph of order $(p+1)+2$ with $|\mathcal{D}(H)| = (|\mathcal{D}(G)|+1)+2$. For each integer $t \geq 2$, let $H_t = (H_{t-1} + K_1) \vee K_1$. Then $\delta(H_t) \geq 1$ for each integer $t \geq 1$. Applying Lemma 2.5 and Observation 3 repeatedly, we see that H_t is a link-irregular graph of order $(p+1)+2t$ with $|\mathcal{D}(H_t)| = |(\mathcal{D}(G)|+1)+2t$ for $t \geq 2$. In particular, H_ℓ is a link-irregular graph of order $(p+1)+2\ell = p+k$ with $|\mathcal{D}(H_\ell)| = (|\mathcal{D}(G)|+1)+2\ell = |\mathcal{D}(G)|+k$.

Case 2. *$k \geq 2$ is even.* Then $k = 2\ell$ for some integer $\ell \geq 1$. Let $H_1 = (G + K_1) \vee K_1$ and let $H_t = (H_{t-1} + K_1) \vee K_1$ for each integer $t \geq 2$. Since $\delta(G) \geq 1$ and $\delta(H_t) \geq 1$ for each integer $t \geq 1$, it follows that H_t is a link-irregular graph of order $p+2t$ with $|\mathcal{D}(H_t)| = |\mathcal{D}(G)| + 2t$ for $t \geq 1$ (again by applying Lemma 2.5 and Observation 3 repeatedly). In particular, H_ℓ is a link-irregular graph of order $p+2\ell = p+k$ with $|\mathcal{D}(H_\ell)| = |\mathcal{D}(G)|+2\ell = |\mathcal{D}(G)|+k$.

We saw for $k = 1$ that $H = G \vee K_1$ and $|V(H)| - |\mathcal{D}(H)| = |V(G)| - |\mathcal{D}(G)|$. Thus, we may assume that $k \geq 2$. Let $\ell = \lfloor k/2 \rfloor$ and let $H = H_\ell$ be defined as in *Case* 1 or *Case* 2 according to the parity of k. Then $|V(H)| - |\mathcal{D}(H)| = (p+k) - (|\mathcal{D}(G)|+k) = p - |\mathcal{D}(G)| = |V(G)| - |\mathcal{D}(G)|$. ■

We are now prepared to present the following result.

Theorem 2.7. *For each pair (d, n) of integers with $d \in \{3, 4, 5, 6, 7, 8\}$ and $n \geq d+4$, there is a link-irregular graph H_n of order n such that $|\mathcal{D}(H_n)| = n - d$.*

Proof. First, we show that for each integer $d \in \{3,4,5,6,7,8\}$, there is a link-irregular graph F_{d+4} of order $d+4$ such that $\mathcal{D}(F_{d+4}) = 4$. We consider these six cases.

Case 1. $d = 3$. We saw that the graph G_6 of Figure 3 is a link-irregular graph of order 6 with $|\mathcal{D}(G_6)| = 3$. Let $F_7 = G_6 \vee K_1$. Since $\Delta(G_6) = 4$, it follows by Lemma 2.4 and Observation 3 that F_7 is a link-irregular graph of order 7 with $\mathcal{D}(G_7) = 4$.

Case 2. $d = 4$. The graph F_8 of Figure 5 is a link-irregular graph of order 8 with degree sequence $2,3,3,3,4,4,4,5$. Thus, $|\mathcal{D}(F_8)| = 4$. The links of the vertices of degree 3 or 4 in F_8 are also shown in Figure 5.

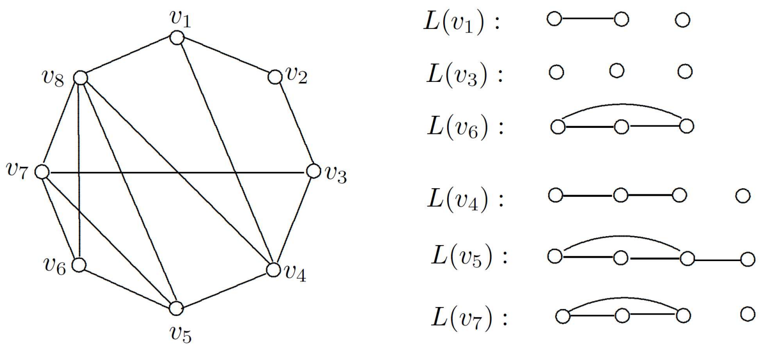

Figure 5. A link-irregular graph F_8 of order 8 with $|\mathcal{D}(F_8)| = 4$.

Case 3. $d = 5$. By adding a new vertex v and joining v to the vertices v_3 and v_7 in the graph F_8 of Figure 5, we obtain a link-irregular graph F_9 of order 9 with degree sequence $2,2,3,3,4,4,4,5,5$. Thus, $|\mathcal{D}(F_9)| = 4$. In this graph, the links of the two vertices of degree 2 are $L(v) \cong K_2$ and $L(v_2) \cong \overline{K}_2$; the links of the two vertices of degree 3 are $L(v_1) \cong K_2 + K_1$ and $L(v_6) \cong K_3$; the links of the three vertices of degree 4 are $L(v_3) \cong P_2 + \overline{K}_2$, $L(v_4) \cong P_3 + K_1$, and $L(v_5) \cong K_3 \star K_1$ (the graph obtained by adding a pendant edge at a vertex of K_3); while the links of the two vertices of degree 5 are $L(v_7) \cong K_3 + K_2$ and $L(v_8)$ which is isomorphic to the graph obtained from the 5-path $(v_7, v_6, v_5, v_4, v_1)$ by adding the edge v_5v_7.

Case 4. $d = 6$. By adding a new vertex u and joining u to the vertices v_4, v_5, and v_6 in the graph F_9 in *Case* 3, we obtain a link-irregular graph F_{10} of order 10 with degree sequence $2,2,3,3,4,4,5,5,5,5$. Thus, $|\mathcal{D}(F_{10})| = 4$. In this graph, the links of the two vertices of degree 2 are $L(v) \cong K_2$ and $L(v_2) \cong \overline{K}_2$; the links of the two vertices of degree 3 are $L(v_1) \cong K_2 + K_1$ and $L(u) \cong P_3$; the links of the two vertices of degree 4 are $L(v_6) \cong K_3 \star K_1$, and $L(v_3) \cong P_2 + \overline{K}_2$; while the links of the four vertices of degree 5 are $L(v_4) \cong P_4 + K_1$, and $L(v_5) \cong C_5 + e$ (the graph obtained from the 5-cycle $(u, v_6, v_7, v_8, v_4, u)$ by adding the edge v_6v_8), $L(v_7) \cong K_3 + K_2$, and $L(v_8)$ which is isomorphic to the graph obtained from the 5-path $(v_7, v_6, v_5, v_4, v_1)$ by adding the edge v_5v_7.

Case 5. $d = 7$. By adding a new vertex w and joining w to the four vertices v_4, v_5, v_7, and v_8 in degree 5 of the graph F_{10} in *Case* 4, we obtain a link-irregular graph F_{11} of order 11 with degree sequence $2, 2, 3, 3, 4, 4, 4, 6, 6, 6, 6$. Thus, $|\mathcal{D}(F_{11})| = 4$. The graph F_{11} is shown in Figure 6 together with the links of all vertices of F_{11}.

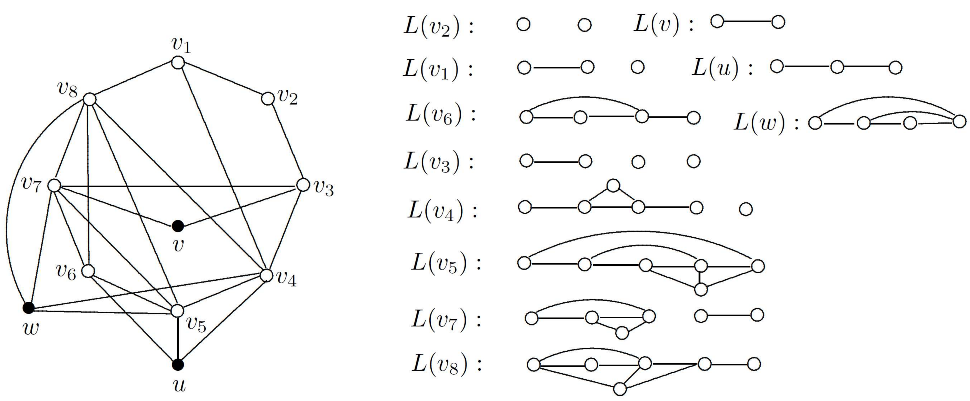

Figure 6. A link-irregular graph F_{11} of order 11 with $|\mathcal{D}(F_{11})| = 4$.

Case 6. $d = 8$. The graph F_{12} of Figure 7 is a link-irregular graph of order 12 with degree sequence $2, 2, 3, 3, 4, 4, 5, 5, 5, 5, 5, 5$. Thus, $|\mathcal{D}(F_{12})| = 4$. In this graph, the links of the two vertices of degree 2 are $L(v_1) \cong K_2$ and $L(v_{11}) \cong \overline{K}_2$; the links of the two vertices of degree 3 are $L(v_5) \cong K_2 + K_1$ and $L(v_9) \cong P_3$; the links of the two vertices of degree 4 are $L(v_2) \cong P_3 + K_1$ and $L(v_3) \cong 2P_2$; while the links of the six vertices of degree 5 are $L(v_4) \cong S_{2,3}$ (the double star whose central vertices have degree 2 and 3), $L(v_6) \cong C_4 \star K_1$ (the graph obtained by adding a pendant edge at a vertex of C_4), $L(v_7) \cong P_5$, $L(v_8) \cong C_4 + K_1$, $L(v_{10}) \cong P_4 + K_1$, and $L(v_{12}) \cong P_2 + \overline{K}_2$.

Next, let d and n be integers with $d \in \{3, 4, 5, 6, 7, 8\}$ and $n \geq d + 4$. Since there is a link-irregular graph F_{d+4} of order $d + 4$ such that $\mathcal{D}(F_{d+4}) = 4$, it follows by Proposition 4 that there is a link-irregular graph H_n of order n such that

$$|V(H_n)| - |\mathcal{D}(H_n)| = |V(F_{d+4})| - |\mathcal{D}(F_{d+4})| = (d + 4) - 4 = d.$$

Consequently, $|\mathcal{D}(H_n)| = n - d$ where $d \in \{3, 4, 5, 6, 7, 8\}$ and $n \geq d + 4$. ■

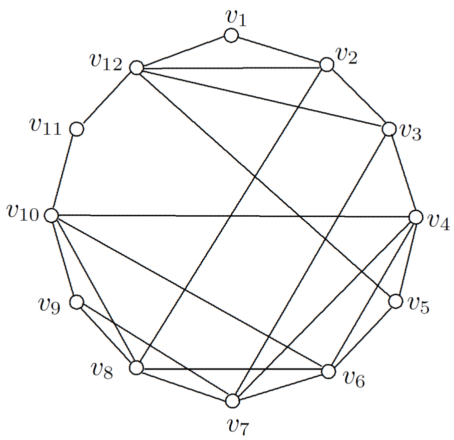

Figure 7. A link-irregular graph G of order 12 with $|\mathcal{D}(G)| = 4$.

Corollary 5. *There exists a link-irregular graph H_n of order n such that $|\mathcal{D}(H_n)| = n - 3$ if and only if $n \geq 6$.*

Proof. We have mentioned that no graph of order less than 6 is link-irregular and that the graph G_6 of Figure 3 is a link-irregular graph of order 6 with $|\mathcal{D}(G_6)| = 3$. By Theorem 2.7, for each integer $n \geq 7$, there is a link-irregular graph G of order n such that $|\mathcal{D}(G)| = n - 3$, giving the desired result. ■

The following problem is suggested by Theorem 2.7.

Problem 2.8. Does there exist a link-irregular graph H_n of order n with $|\mathcal{D}(H_n)| = n - d$ for each pair (d, n) of integers with $n \geq d + 4 \geq 7$?

It was observed in [1] that there is no r-regular link-irregular graph for $r = 2, 3$ and proved that there is no 4-regular link-irregular graph. The following conjecture was stated in [1].

Conjecture 2.9. *There is no regular link-irregular graph.*

If Conjecture 2.9 is true, then for every link-irregular graph G of order n with $|\mathcal{D}(G)| = k$, it follows that $2 \leq k \leq n - 2$. This brings up the following question:

Problem 2.10. For which integers $n \geq 7$, is it true that for every integer k with $2 \leq k \leq n - 2$, there exists a link-irregular graph $G_{n,k}$ of order n such that $|\mathcal{D}(G_{n,k})| = k$?

While we are not aware of a link-irregular graph $G_{8,2}$ of order 8 with $|\mathcal{D}(G_{8,2})| = 2$, it can be shown that there exists no link-irregular graph $G_{7,2}$ of order 7 such that $|\mathcal{D}(G_{7,2})| = 2$. Furthermore, we have the following result on link-irregular graphs of order n for $7 \leq n \leq 10$.

Proposition 6. *Let (n,k) be a pair of integers with $7 \leq n \leq 10$ and $2 \leq k \leq n-2$. If $n \in \{7,8\}$ and $3 \leq k \leq n-2$ or $n \in \{9,10\}$ and $2 \leq k \leq n-2$, then there exists a link-irregular graph $G_{n,k}$ of order n such that $|\mathcal{D}(G_{n,k})| = k$.*

Proof. First, let $(n,k) \in X = \{(7,3),(8,3),(9,2),(9,3),(10,2),(10,3)\}$ and consider the six graphs $G_{n,k}$ of Figure 8, where $\mathcal{D}(G_{7,3}) = \mathcal{D}(G_{8,3}) = \{3,4,5\}$, $\mathcal{D}(G_{9,2}) = \{4,6\}$, $\mathcal{D}(G_{9,3}) = \{4,5,6\}$, $\mathcal{D}(G_{10,2}) = \{4,5\}$, and $\mathcal{D}(G_{10,3}) = \{3,4,5\}$. It is straightforward to verify that for each $(n,k) \in X$, the graph $G_{n,k}$ of order n in Figure 8 is link-irregular with $|\mathcal{D}(G_{n,k})| = k$.

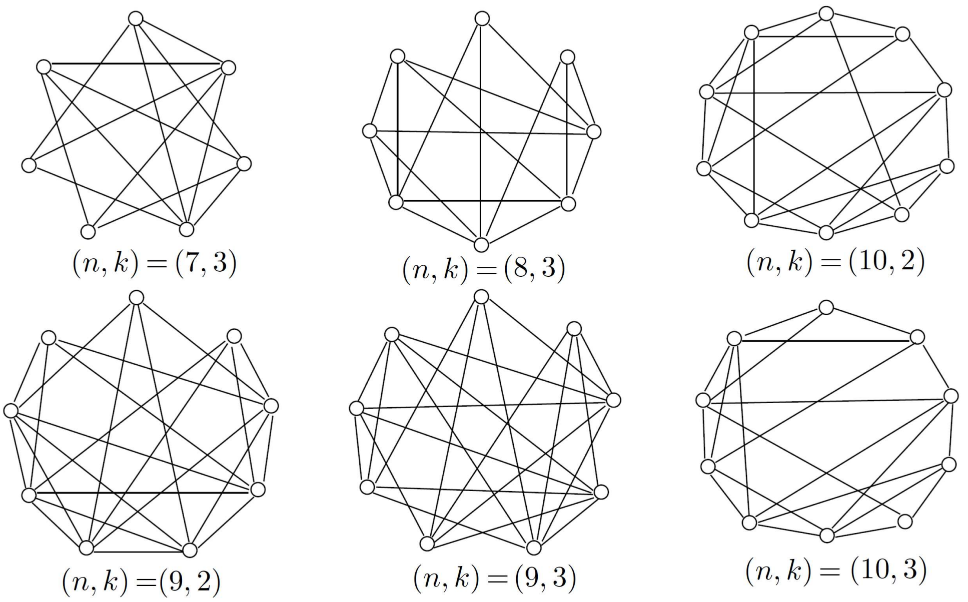

Figure 8. The link-irregular graphs $G_{n,k}$, where $(n,k) \in X$ in the proof of Proposition 6.

Hence, we assume that $7 \leq n \leq 10$ and $4 \leq k \leq n-2$. By Theorems 2.6 and 2.7, it follows that for each integer $d \in \{2,3,\ldots,n-4\}$ (where then $n-4 \leq 6$), there is a link-irregular graph H of order n such that $|\mathcal{D}(H)| = n-d \in \{4,5,\ldots,n-2\}$. Consequently, for each pair (n,k) of integers with $7 \leq n \leq 10$ and $4 \leq k \leq n-2$, there is a link-irregular graph $G_{n,k}$ of order n such that $|\mathcal{D}(G_{n,k})| = k \in \{4,5,\ldots,n-2\}$. ∎

3. LINK-IRREGULAR RATIO

Let $\mathcal{L}$ denote the set of link-irregular graphs. For a graph $G \in \mathcal{L}$ with degree set $\mathcal{D}(G)$, the *link-irregular ratio* $\mathrm{lir}(G)$ of G is defined as

$$\mathrm{lir}(G) = \frac{|\mathcal{D}(G)|}{|V(G)|}.$$

The following is a consequence of Theorem 2.6.

Corollary 7. $\sup\{\mathrm{lir}(G) : G \in \mathcal{L}\} = 1$.

Proof. For every graph $G \in \mathcal{L}$ of order n, we have $|\mathcal{D}(G)| < n$. Thus, $\sup\{\mathrm{lir}(G) : G \in \mathcal{L}\} \le 1$. By Theorem 2.6, for each integer $n \ge 7$, there exists a graph $H_n \in \mathcal{L}$ of order n such that $|\mathcal{D}(G)| = n - 2$. Thus, $\lim_{n\to\infty} \frac{|\mathcal{D}(H_n)|}{|V(H_n)|} = \frac{n-2}{n} = 1$ and so $\sup\{\mathrm{lir}(G) : G \in \mathcal{L}\} \ge 1$, giving the desired result. ■

This brings up the question as to the value of $\inf\{\mathrm{lir}(G) : G \in \mathcal{L}\}$. The graph G of Figure 9 has order 18 with degree set $\mathcal{D}(G) = \{7, 9\}$, where $\deg v_i = 7$ for $1 \le i \le 8$ and $\deg v_i = 9$ for $9 \le i \le 18$. We show that G is link-irregular. For $1 \le i \le 18$, let s_i denote the degree sequence of $L(v_i)$. Observe that

$$\begin{aligned}
s_1 &= (4,3,2,2,2,2,1),\ s_2 = (4,4,3,3,2,2,2),\ s_3 = (4,3,2,2,2,2,1),\\
s_4 &= (3,3,2,2,2,2,2),\ s_5 = (3,2,2,2,1,1,1),\ s_6 = (5,4,4,3,2,2,2),\\
s_7 &= (2,2,2,2,2,1,1),\ s_8 = (6,4,3,3,2,2,2),\ s_9 = (5,4,3,2,2,2,2,2,2),\\
s_{10} &= (5,5,5,5,4,4,3,3,2),\ s_{11} = (5,5,5,5,5,4,3,2,2),\ s_{12} = (7,4,4,2,2,2,1,1,1),\\
s_{13} &= (6,5,4,3,3,3,2,2,2),\ s_{14} = (5,4,4,4,4,3,3,3,2),\ s_{15} = (5,4,4,4,4,3,3,2,1),\\
s_{16} &= (7,4,3,3,2,2,2,2,1),\ s_{17} = (6,5,5,4,3,3,3,3,2),\ s_{18} = (6,5,5,4,4,2,2,1,1).
\end{aligned}$$

Thus, if $i, j \in \{1, 2, \ldots, 18\}$, $i \ne j$, and $\{i, j\} \ne \{1, 3\}$, then $s_i \ne s_j$ and so $L(v_i) \not\cong L(v_j)$. Since $L(v_1)$ contains exactly one triangle and $L(v_3)$ contains exactly two triangles, it follows that $L(v_1) \not\cong L(v_3)$. Hence, G is a link-irregular graph of order 18 with $|\mathcal{D}(G)| = 2$ and so $\mathrm{lir}(G) = 1/9$. Consequently, $0 \le \inf\{\mathrm{lir}(G) : G \in \mathcal{L}\} \le 1/9$.

In this connection, we have the following problem.

Problem 3.1. Does there exist a positive integer constant c for which there is an infinite class of link-irregular graphs such that $|D(G)| \le c$ for each graph G in the class?

If the answer to Problem 3.1 is yes, then this would mean that $\inf\{\mathrm{lir}(G) : G \in \mathcal{L}\} = 0$. In fact, we have the following conjecture.

Conjecture 3.2. $\inf\{\mathrm{lir}(G) : G \in \mathcal{L}\} = 0$.

By Proposition 6, if $r = p/q$ is a rational number such that either (i) $q \in \{7, 8\}$ and $3 \le p \le q - 2$ or (ii) $q \in \{9, 10\}$ and $2 \le p \le q - 2$, then r is realizable as the link-irregular ratio $\mathrm{lir}(G)$ of some link-irregular graph G. This suggests the following question.

Problem 3.3. For which rational numbers $r \in (0, 1)$, does there exist a link-irregular graph G such that $\mathrm{lir}(G) = r$?

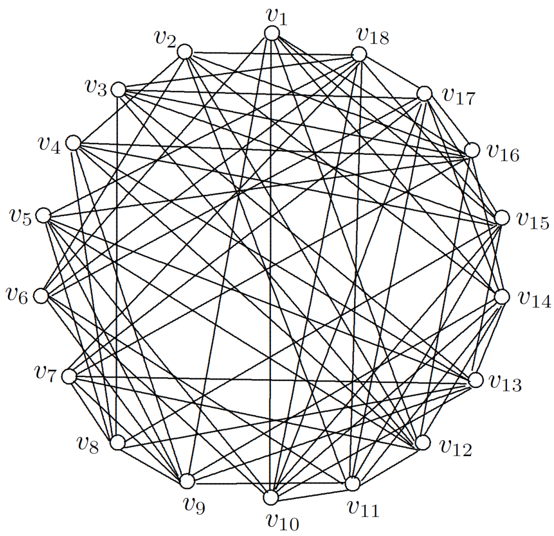

Figure 9. A link-irregular graph G of order 18 with $|\mathcal{D}(G)| = 2$.

For all of the examples of link-irregular graphs of a certain order n that we have seen, their sizes have been relatively close to $\frac{1}{2}\binom{n}{2}$. Consequently, this suggests that there exist real numbers a and b such that if G is a link-irregular graph of order n, then $a \leq \frac{|E(G)|}{\binom{n}{2}} \leq b$. This leads us to the following problem.

Problem 3.4. Determine real numbers $a < 0.5$ and $b > 0.5$ such that if G is a graph of order n such that either

$$\frac{|E(G)|}{\binom{n}{2}} < a \quad \text{or} \quad \frac{|E(G)|}{\binom{n}{2}} > b,$$

then G is not link-irregular.

From the many examples that we have seen, it appears that a and b may be relatively close to 0.3 and 0.7, respectively.

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