Discussiones Mathematicae
Graph Theory 45 (2025) 67–80
https://doi.org/10.7151/dmgt.2520

CHARACTERIZATION OF α-EXCELLENT 2-TREES

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Abstract

A graph is α-excellent if every vertex of the graph is contained in some maximum independent set of the graph. In this paper, we present two characterizations of the α-excellent 2-trees.

Keywords: independence number, excellent graph, k-tree.

2020 Mathematics Subject Classification: 05C69, 05C85.

[1]M.A. Henning has received research support from the University of Johannesburg and from the South African National Research Foundation under grant number 132588.

1. Introduction

For notation and graph theory terminology we, in general, follow the recent books [9–11]. Specifically, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the *open neighborhood* of S is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the *closed neighborhood* $N_G[S] = N_G(S) \cup S$. For a positive integer k, we let $[k] = \{1, \ldots, k\}$.

A subset D of the vertex set $V(G)$ of a graph G is called a *dominating set* of G if every vertex belonging to $V(G) \setminus D$ is adjacent to at least one vertex in D. A subset I of $V(G)$ is *independent* if no two vertices belonging to I are adjacent in G. The cardinality of a largest (i.e., maximum) independent set of G, denoted by $\alpha(G)$, is called the *independence number* of G. Every largest independent set of a graph is called an α-set of the graph. A dominating set D of a graph G is called an *independent dominating set* of G if D is also independent. The *independent domination number* of G, denoted by $i(G)$, is the cardinality of the smallest independent dominating set of G (or equivalently, the cardinality of a minimum maximal independent set of vertices in G). The *common independence number* of a graph G, denoted by $\alpha_c(G)$, is the greatest integer r such that every vertex of G belongs to some independent subset X of $V(G)$ with $|X| \geq r$. It follows immediately from the above definitions that the common independence number is bounded below by the independent domination number and above by the independence number. Formally, for any graph G,

$$i(G) \leq \alpha_c(G) \leq \alpha(G). \tag{1}$$

The study of independent sets in graphs was begun by Berge [1,2] (see also [3]) and Ore [12]. We refer the reader to the book [11] and to the survey [8] of results on independent domination in graphs published in 2013 by Goddard and Henning. A graph G is said to be *well-covered* if $i(G) = \alpha(G)$. Equivalently, G is well-covered if every maximal independent set of G is a maximum independent set of G. The concept of well-covered graphs was introduced by Plummer [13] in 1970. Since then the well-covered graphs were very extensively investigating in many papers. We refer the reader to the excellent (but already older) survey on well-covered graphs by Plummer [14]. We are interested in characterization of *α-excellent graphs*, that is, graphs G for which $\alpha_c(G) = \alpha(G)$. Thus, if G is an α-excellent graph, then every vertex belongs to some α-set of G. It follows from Inequalities (1) that every well-covered graph is an α-excellent graph. The example of the cycle C_6 shows that the set of well-covered graphs is properly contained in the set of α-excellent graphs. The α-excellent graphs have been studied in [4–7,15,16] and in [1,2] as B-graphs. In this paper, we begin the study of α-excellent k-trees.

2. Preliminary results

A vertex v of a graph G is a *simplicial vertex* if every two vertices belonging to $N_G(v)$ are adjacent in G. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique of a graph, where a *clique* of a graph G is a maximal complete subgraph of G. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G. Note that if v is a simplicial vertex of G, then $G[N_G[v]]$ is the unique simplex of G containing v. We begin with a simple proposition.

Proposition 1. *No α-excellent graph contains a vertex belonging to at least two its simplexes.*

Proof. Assume that a vertex v of a graph G belongs to two simplexes of G, say to $G[N_G[u]]$ and $G[N_G[w]]$. If I is a maximum independent set that contains v, then $(I \setminus \{v\}) \cup \{u, w\}$ is an independent set of greater cardinality. Thus, $\alpha(G) \geq |(I \setminus \{v\}) \cup \{u, w\}| > |I|$, implying that v does not belong to any α-set of G, and proving that G is not an α-excellent graph. ■

For a positive integer k, a graph G is called a *k-tree* if it can be obtained from the complete graph K_k by a finite number of applications of the following operation: add a new vertex and join it to k mutually adjacent vertices of the existing graph. Certainly, every 1-tree is a tree and vice versa. In [17], Rose proved that a graph G is a k-tree if and only if the following conditions are fulfilled: (i) G is connected, (ii) G contains K_k as a subgraph and does not contain K_{k+2} as a subgraph, (iii) if v and u are nonadjacent vertices of G, then the subgraph induced by the smallest $v-u$ separator is a complete graph on k vertices. Recall that a $v-u$ separator in a connected graph G is a subset S of $V(G)$ such that u and v are in distinct components of $G[V(G) \setminus S]$. Note that K_k and K_{k+1} are the only k-trees of order k and $k+1$, respectively.

It was proved in [4] that a bipartite graph (and, in particular, a tree) is an α-excellent graph if and only if it has a perfect matching. On the other hand, it was observed in [18] that a k-tree G is a well-covered graph if and only if every vertex of G belongs to exactly one simplex of G. In this paper, we are interested in possible extensions of that characterization to a characterization of α-excellent k-trees for every positive integer k. We begin with the following definition.

A set $\mathcal{P}$ of complete subgraphs of a graph G is said to be a *perfect* $(k+1)$*-cover* of G if each subgraph belonging to $\mathcal{P}$ is of order $k+1$ and every vertex of G belongs to exactly one subgraph in $\mathcal{P}$. It is obvious that for $k = 1$ there exists a one-to-one correspondence between perfect 2-covers of a graph and perfect matchings of the graph. Here we are interested in the existence of perfect $(k+1)$-covers in k-trees. First of all, one can prove that every k-tree has at most one perfect $(k+1)$-cover. In the following proposition, we present the first relationship between k-trees having perfect $(k+1)$-covers and α-excellent graphs.

Proposition 2. *If a k-tree G has a perfect $(k+1)$-cover, then G is an α-excellent graph.*

Proof. Let G be a connected k-tree of order $n \geq k+1$. Then G is a $(k+1)$-partite graph, say $A_1, A_2, \ldots, A_{k+1}$ are partite sets of G and assume that $|A_1| \geq |A_2| \geq \cdots \geq |A_{k+1}| \geq 1$. In addition, since $A_1, A_2, \ldots, A_{k+1}$ are independent sets of vertices and $|A_1| + |A_2| + \cdots + |A_{k+1}| = n$, it follows that $|A_1| \geq n/(k+1)$, and, therefore, $\alpha(G) \geq |A_1| \geq n/(k+1)$. Let I be an α-set of G. Assume now that $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ is a perfect $(k+1)$-cover of G. Then, $\ell = n/(k+1)$, $|I \cap V(P_i)| \leq 1$ for each $i \in [\ell]$, and

$$\alpha(G) = |I| = \left| I \cap \bigcup_{i=1}^{\ell} V(P_i) \right| = \sum_{i=1}^{\ell} |I \cap V(P_i)| \leq \ell = \frac{n}{k+1}.$$

Consequently, $|A_1| = |A_2| = \cdots = |A_{k+1}| = n/(k+1) = \alpha(G)$, and each of the sets $A_1, A_2, \ldots, A_{k+1}$ is an α-set of G. This implies that every vertex of G belongs to an α-set of G and, therefore, G is an α-excellent graph. ■

3. α-Excellent 2-Trees

Proposition 2 shows that a k-tree having a perfect $(k+1)$-cover is an α-excellent k-tree. It is not clear to us whether the converse of this statement is true. That is, we do not know if every α-excellent k-tree of order at least $k+1$ has a perfect $(k+1)$-cover if $k \geq 3$. However, when $k = 2$, we provide in this paper a characterization of α-excellent k-trees. For notational simplicity, in what follows if three vertices a, b, and c are mutually adjacent in a graph G, then the induced subgraph $G[\{a,b,c\}]$ of G is isomorphic to K_3 and is called a *triangle* in G, and we simply write abc rather than $G[\{a,b,c\}]$. To every triangle in a graph G, we assign label R or B (as red or blue, respectively), and by $R(G)$ and $B(G)$ we denote the set of all triangles in G that have label R and B, respectively. We also say that $R(G)$ and $B(G)$ are the sets of all "red" and "blue" triangles in G, respectively.

We are now in position to present a constructive characterization of α-excellent 2-trees. For this purpose, let $\mathcal{E}$ be the family of labeled 2-trees defined recursively as follows.

(1) The family $\mathcal{E}$ contains the 2-tree of order 3 in which the only triangle is red, that is, it has label R.

(2) The family $\mathcal{E}$ is closed under the operations $\mathcal{O}_1$ and $\mathcal{O}_2$ defined below.

Operation $\mathcal{O}_1$. If a graph G' belongs to $\mathcal{E}$ and v_1v_2 is an edge of G', then $\mathcal{O}_1 = \mathcal{O}_1(v_1, v_2)$ forms a graph G by adding three new vertices u_1, u_2, u_3 to

G' in such a way that $v_1v_2u_1$, $v_2u_1u_2$ and $u_1u_2u_3$ are three new triangles, while $R(G) = R(G') \cup \{u_1u_2u_3\}$ and $B(G) = B(G') \cup \{v_1v_2u_1, v_2u_1u_2\}$. In this case we apply the operation $\mathcal{O}_1$ to the edge v_1v_2 of G'.

Operation $\mathcal{O}_2$. If a graph G' belongs to $\mathcal{E}$, $v_1v_2v_3$ is a red triangle in G' (that is, $v_1v_2v_3 \in R(G')$), and v_4 is a neighbor of v_3 (it is possible that $v_4 \in \{v_1, v_2\} \subseteq N_{G'}(v_3)$), then $\mathcal{O}_2 = \mathcal{O}_2(v_1v_2, v_3v_4)$ forms a graph G by adding to G' three new vertices u_0, u_1 and u_2 in such a way that $u_0v_1v_2$, $v_3v_4u_1$, and $v_3u_1u_2$ are new triangles, while $R(G) = (R(G') \setminus \{v_1v_2v_3\}) \cup \{u_0v_1v_2, v_3u_1u_2\}$ and $B(G) = B(G') \cup \{v_1v_2v_3, v_3v_4u_1\}$. In this case we apply the operation $\mathcal{O}_2$ to the edge v_1v_2 of the triangle $v_1v_2v_3$ and to the edge v_3v_4.

The operations $\mathcal{O}_1$ and $\mathcal{O}_2$ are illustrated in Figure 1. Note that the operation $\mathcal{O}_2$ changes "colors" of certain triangles, and the red triangle $v_1v_2v_3$ in G' is recolored blue in G.

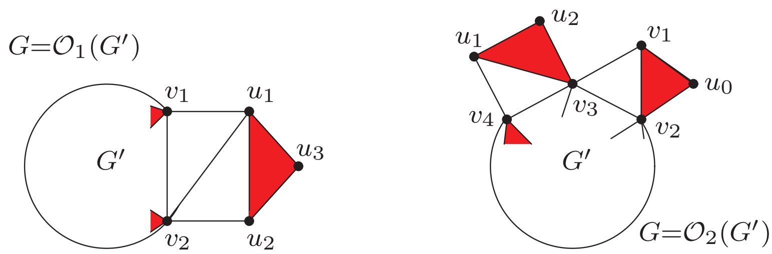

Figure 1. The operations $\mathcal{O}_1$ and $\mathcal{O}_2$.

From the recursive definition of the graphs belonging to the family $\mathcal{E}$ it follows readily that if a 2-tree G belongs to $\mathcal{E}$, then the set $R(G)$ of red triangles in G is a perfect 3-cover of G. From this and from Proposition 2 it follows that G is an α-excellent graph. Thus we have the following proposition that we will need when proving our main theorem.

Proposition 3. *Every 2-tree belonging to the family $\mathcal{E}$ has a perfect 3-cover and it is an α-excellent graph.*

The following theorem is the main result of this paper, and it presents two characterizations of the α-excellent 2-trees: a constructive characterization, and a characterization in terms of perfect 3-covers.

Theorem 4. *If G is a 2-tree of order $n \geq 3$, then the following statements are equivalent.*

(a) *G has a perfect 3-cover.*

(b) *G belongs to the family $\mathcal{E}$.*

(c) *G is an α-excellent graph.*

Proof. The implications (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (a), and (b) $\Rightarrow$ (c) are obvious from Propositions 2 and 3. Thus it suffices to prove the implication (c) $\Rightarrow$ (b) (but we prove the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) at the same time).

Thus assume that G is an α-excellent 2-tree of order at least 3. By induction on the order of G we shall prove that G has a perfect 3-cover and that G belongs to the family $\mathcal{E}$. It is straightforward to observe that the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are true if G is a 2-tree of order $n \leq 6$. Now let G be an α-excellent 2-tree of order greater than 6 and assume that the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are true for smaller α-excellent 2-trees. Let $(T_1, T_2, \ldots, T_p)$ be a longest 3-path in G, that is, a longest sequence $T_1, T_2, \ldots, T_p$ of triangles in G, where $|V(T_i) \cap V(T_j)| = 2$ if $|i-j| = 1$, and $|V(T_i) \cap V(T_j)| \leq 1$ if $|i-j| \geq 2$ $(i, j \in [p])$. From the fact that $(T_1, T_2, \ldots, T_p)$ is a longest 3-path in G (which is an α-excellent 2-tree of order at least 7) it follows that $p \geq 4$. Assume that a, b, c, d, and e are vertices of G for which $V(T_{p-3}) \cap V(T_{p-2}) = \{a, b\}$, $V(T_{p-2}) \setminus V(T_{p-3}) = \{c\}$, $V(T_{p-1}) \setminus V(T_{p-2}) = \{d\}$, and $V(T_p) \setminus V(T_{p-1}) = \{e\}$. From the choice of $(T_1, \ldots, T_p)$, the vertex e is of degree 2 and T_p is a simplex in G. Let $G'_0, G'_1, \ldots, G'_\ell$ be the components of $G - \{a, b\}$, where G'_0 is that component which contains at least one vertex of T_1. It is clear that ℓ is positive integer.

We now let G_i denote the subgraph of G induced by $V(G'_i) \cup \{a, b\}$ for $i \in \{0\} \cup [\ell]$. Among the graphs $G_1, \ldots, G_\ell$, let H be the graph that contains the triangles of the 3-path $\mathcal{P}_0 = (T_{p-2}, T^0_{p-1}, T^0_p)$, where $T^0_{p-1} = T_{p-1}$ and $T^0_p = T_p$. It is obvious that if T_{p-2}, T_{p-1}, and T_p are the only triangles of H, then it is possible that H is one of the graphs H_1, H'_1, H_2, and H'_2 shown in Figure 2. If H contains the triangle T_{p-2}, T_{p-1}, T_p and a simplex T'_{p-1} that shares an edge with the triangle T_{p-2} in H, then it follows readily from Proposition 1 that H is one of the graphs H_3 or H'_3 in Figure 2.

Thus assume that no simplex of H shares an edge with the triangle T_{p-2}. In this case, H is a subgraph induced by the triangles belonging to $\mathcal{P}_0$ and to some additional 3-paths $\mathcal{P}_i = (T_{p-2}, T^i_{p-1}, T^i_p)$, where $i \in [n]$ and n is a positive integer. It follows from Proposition 1 that if $n = 1$, then H is isomorphic to one of the graphs H_4, H_5 or H_6 in Figure 3. Similarly, if $n = 2$, then, as can easily be verified, H is isomorphic to the graph H_7 shown in Figure 3. Finally, we claim that the case $n \geq 3$ is impossible. Suppose, to the contrary, that $n \geq 3$. Then let us first observe that if one of the edges ac and bc of the triangle T_{p-2} belongs to at least 3 of the triangles $T^0_{p-1} = T_{p-1}, T^1_{p-1}, \ldots, T^n_{p-1}$, say to $T^i_{p-1}, T^j_{p-1}, T^k_{p-1}$ (where $0 \leq i < j < k \leq n$), then at least two of the simplexes T^i_p, T^j_p, T^k_p of H (and of G) have a common vertex, which is impossible in an α-excellent graph G. Thus assume that neither ac nor bc belongs to three of the triangles $T^0_{p-1} = T_{p-1}$,

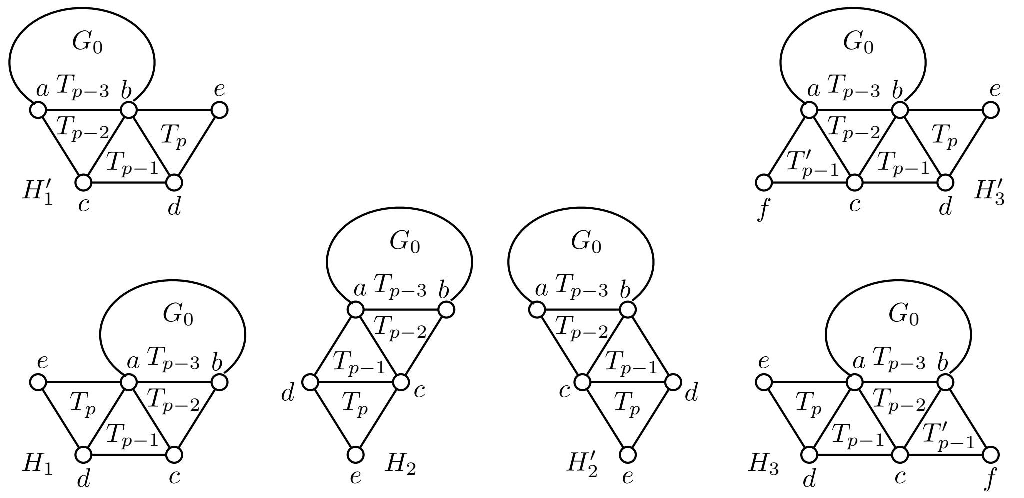

Figure 2. Graphs H_1, H'_1, H_2, H'_2, H_3, and H'_3.

$T^1_{p-1}, \ldots, T^n_{p-1}$. Then necessarily $n = 3$ and each of the edges ac and bc belongs to exactly two of the triangles $T^0_{p-1}, T^1_{p-1}, T^2_{p-1}, T^3_{p-1}$, say ac belongs to T^0_{p-1} and T^1_{p-1}, while bc belongs to T^2_{p-1} and T^3_{p-1}. Therefore, as it is easy to verify, if neither the simplexes T^0_p and T^1_p have a common vertex nor the simplexes T^2_p and T^3_p have a common vertex, then one of the simplexes T^0_p and T^1_p has a common vertex with one of the simplexes T^2_p and T^3_p, which again by Proposition 1 is impossible as G is an α-excellent graph. This proves that the case $n \geq 3$ is impossible.

There are now several cases to consider depending on the structure of H. We begin showing that H cannot be any of the graphs H_1, H'_1, H_4. Suppose, to the contrary, that $H = H_1$ (where H_1 is as illustrated in Figure 2). Then, since G is an α-excellent graph, there exists an α-set I in G that contains a. However in this case, $(I\setminus\{a\})\cup\{c,e\}$ is an independent set in G, and so $\alpha(G) \geq |(I\setminus\{a\})\cup\{c,e\}| > |I| = \alpha(G)$, a contradiction which proves that $H \neq H_1$. Analogously, $H \neq H'_1$. Similarly, suppose that $H = H_4$ (where H_4 is shown in Figure 3). Let I_d be an α-set that contains d in G. We note that $|I_d \cap \{b,f,g\}| = 1$. If $b \in I_d$, then let $I'_d = (I_d \setminus \{b,d\}) \cup \{c,e,g\}$. If $f \in I_d$, then let $I'_d = (I_d \setminus \{d,f\}) \cup \{c,e,g\}$. If $g \in I_d$, then let $I'_d = (I_d \setminus \{d\}) \cup \{c,e\}$. In all cases, the resulting set I'_d is an independent set in G, and so $\alpha(G) \geq |I'_d| > |I_d| = \alpha(G)$, a contradiction. Hence, $H \neq H_4$.

In each of the next five cases (corresponding to the possible graphs H, that is, to the graphs H_2 (and H'_2), H_3, H_5, H_6, and H_7) we prove that G belongs to the family $\mathcal{E}$ and has a perfect 3-cover.

Case 1. $H = H_2$ *or* $H = H'_2$. Without loss of generality, assume that

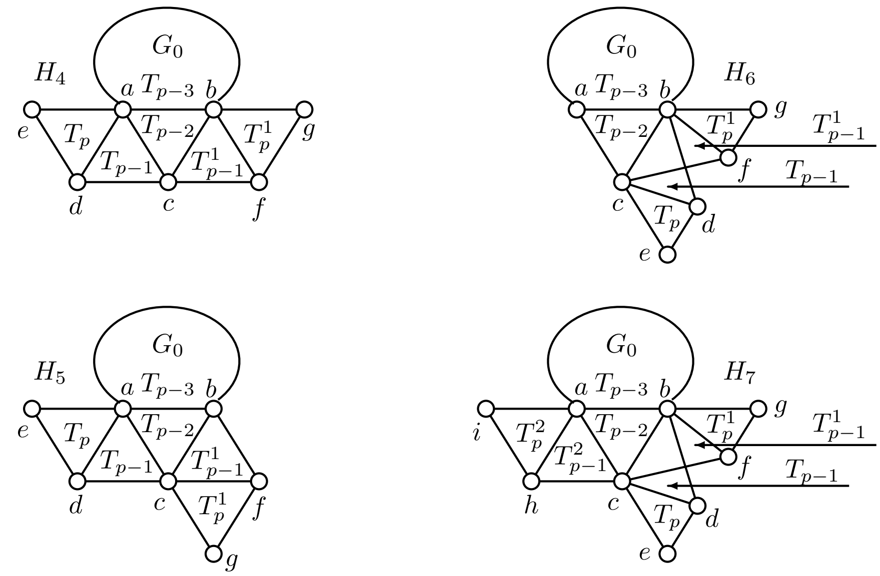

Figure 3. Graphs H_4, H_5, H_6, and H_7.

$H = H_2$ (see Figure 2). In this case, let G' denote the subgraph $G - \{c, d, e\}$ of G. In the following three claims we explain the main relations between properties of the graphs G and G', that is, we prove that: (1) $\alpha(G') = \alpha(G) - 1$; (2) G' is an α-excellent graph; (3) G has a perfect 3-cover and belongs to the family $\mathcal{E}$.

Claim 1.1. $\alpha(G') = \alpha(G) - 1$.

If I' is an α-set of G', then $I' \cup \{e\}$ is an independent set in G (as $N_G[e] \cap I' = \{c, d, e\} \cap I' = \emptyset$) and therefore $\alpha(G) \geq |I' \cup \{e\}| = \alpha(G') + 1$. On the other hand, let I be an α-set of G. In this case, $|I \cap \{c, d, e\}| = 1$ and $I \setminus \{c, d, e\}$ is independent in G'. Thus, $\alpha(G') \geq |I \setminus \{c, d, e\}| = |I| - 1 = \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 1.2. *G' is an α-excellent graph.*

Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G'. Since $\alpha(G') = \alpha(G) - 1$ (by Claim 1.1), it suffices to show that v belongs to an independent set of cardinality $\alpha(G) - 1 = \alpha(G')$ in G'. Let I_v be an α-set of G that contains v, and let I'_v denote the set $I_v \setminus \{c, d, e\}$. Thus, $v \in I'_v$ and the set I'_v is independent in G'. Furthermore, I'_v is an α-set of G' noting that $|I_v \cap \{c, d, e\}| = 1$ and $|I'_v| = |I_v \setminus \{c, d, e\}| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$. This proves that G' is an α-excellent graph.

Claim 1.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

By Claim 1.2, G' is an α-excellent graph. Since the order of G' is less than the order of G, applying the induction hypothesis we infer that G' has a perfect 3-cover and G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $\mathcal{P}' \cup \{cde\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_1 = \mathcal{O}_1(a, b)$ (that is, by $\mathcal{O}_1$ applied to the edge ab of G'), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 2. *$H = H_3$ or $H = H_3'$.* Without loss of generality, assume that $H = H_3$ (see Figure 2). This time, let G' denote the subgraph $G - \{d, e, f\}$ of G. As in Case 1, we study desired relations between properties of the graphs G and G'.

Claim 2.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{a, b, c\}| = 1$ and either $a \in I'$ or $\{b, c\} \cap I' \neq \emptyset$. Consequently, either $I' \cup \{f\}$ or $I' \cup \{e\}$ is an independent set in G, respectively, and therefore $\alpha(G) \geq |I' \cup \{f\}| = |I' \cup \{e\}| = \alpha(G') + 1$. This proves that $\alpha(G) \geq \alpha(G') + 1$. Now, let I be an α-set of G. Then $|I \cap \{a, b, c\}| \leq 1$ and we consider four cases. If $I \cap \{a, b, c\} = \emptyset$, then $|I \cap \{d, e, f\}| = 2$ and $(I \setminus \{d, e, f\}) \cup \{c\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $a \in I$, then $f \in I$, and $I \setminus \{f\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $b \in I$, then $|I \cap \{d, e\}| = 1$, $f \notin I$, and $I \setminus \{d, e\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $c \in I$, then $e \in I$, $f \notin I$, and $I \setminus \{e\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. Consequently, $\alpha(G') \geq \alpha(G) - 1$. This proves that $\alpha(G') = \alpha(G) - 1$.

Claim 2.2. *G' is an α-excellent graph.*

Let v be an arbitrary vertex of G'. As in the proof of Claim 1.2, it suffices to show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. Since G is α-excellent, every vertex of G belongs to some α-set in G. Let I_x be an α-set of G that contains x, where $x \in V(G)$. We note that $I_a \cap \{d, e, f\} = \{f\}$ and, consequently, $I_a \setminus \{f\}$ is an α-set of G' that contains the vertex a. Similarly, from the fact that $|I_b \cap \{d, e\}| = 1$ and $I_c \cap \{d, e, f\} = \{e\}$ it follows that $I_b \setminus \{d, e\}$ and $I_c \setminus \{e\}$ are α-sets of G' and they contain b and c, respectively. If $v \in V(G') \setminus \{a, b, c\}$ and $I_v \cap \{a, b, c\} = \emptyset$, then $|I_v \cap \{d, e, f\}| = 2$ and $(I_v \setminus \{d, e, f\}) \cup \{c\}$ is an α-set of G' and it contains v. Finally, if $v \in V(G') \setminus \{a, b, c\}$ and $I_v \cap \{a, b, c\} \neq \emptyset$, then $|I_v \cap \{a, b, c\}| = 1$, $|I_v \cap \{d, e, f\}| = 1$, and, therefore, $I_v \setminus \{d, e, f\}$ is an α-set of G' and it contains v. Consequently, G' is an α-excellent graph.

Claim 2.3. *G has a perfect* 3*-cover and G belongs to the family $\mathcal{E}$.*

Now, similarly as in the proof of Claim 1.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G' has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Certainly, if $\mathcal{P}'$ is a perfect 3-cover of G', then $abc \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{abc\}) \cup \{ade, bcf\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. From this and from the obvious fact that G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(bc, ac)$ (that is, by $\mathcal{O}_2$ applied to the edge bc of the simplex abc of G' and to the edge ac incident with the third vertex c of the triangle abc), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 3. $H = H_5$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G and, as before, we study desired relations between properties of the graphs G and G'.

Claim 3.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{a, c, d\}| = 1$. If $a \in I'$ or $d \in I'$, then the set $I' \cup \{g\}$ is an independent set of G, while if $c \in I'$, then the set $I' \cup \{e\}$ is an independent set of G, implying that $\alpha(G) \geq \alpha(G') + 1$. It remains to prove that $\alpha(G') \geq \alpha(G) - 1$. To prove this, let I be an α-set of G. Thus, $|I \cap \{a, d, e\}| = 1$ and $|I \cap \{c, f, g\}| = 1$. If $a \in I$ or $d \in I$, then $|I \cap \{f, g\}| = 1$ and we let $I' = I \setminus \{f, g\}$. If $e \in I$ and $c \in I$, then we let $I' = I \setminus \{e\}$. If $e \in I$ and $b \in I$, then $g \in I$ and we let $I' = (I \setminus \{e, g\}) \cup \{d\}$. If $e \in I$ and $I \cap \{b, c\} = \emptyset$, then $|I \cap \{f, g\}| = 1$ and we let $I' = (I \setminus \{e, f, g\}) \cup \{c\}$. In all the above cases, the set I' is an independent set in G' and $|I'| = |I| - 1$, implying that $\alpha(G') \geq |I'| = |I| - 1 = \alpha(G) - 1$. As observed earlier, $\alpha(G') \leq \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 3.2. *G' is an α-excellent graph.*

Since G is an α-excellent graph, every vertex of G belongs to an α-set of G. Let I_x denote an α-set of G that contains the vertex x of G. Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. We note that $|I_v \cap \{a, d, e\}| = 1$, $|I_v \cap \{c, f, g\}| = 1$, and $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$. If $|I_v \cap \{a, b, c, d\}| = 2$, then $I_v \cap \{a, b, c, d\} = \{b, d\}$, $g \in I_v$ and we let $I'_v = I_v \setminus \{g\}$. If $I_v \cap \{a, b, c, d\} = \emptyset$, then $e \in I_v$ and $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = (I_v \setminus \{e, f, g\}) \cup \{c\}$. If $I_v \cap \{a, b, c, d\} = \{a\}$ or $I_v \cap \{a, b, c, d\} = \{d\}$, then $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = I_v \setminus \{f, g\}$. If $I_v \cap \{a, b, c, d\} = \{b\}$, then $e \in I_v$, $g \in I_v$, and we let $I'_v = (I_v \setminus \{e, g\}) \cup \{d\}$. If $I_v \cap \{a, b, c, d\} = \{c\}$, then $e \in I_v$ and we let $I'_v = I_v \setminus \{e\}$. In all cases, the set I'_v is an independent set in G' that contains the vertex v and $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$, implying that v belongs to an α-set of G', as desired.

Claim 3.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

Similarly as in the proofs of Claims 1.3 and 2.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G' has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $acd \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{acd\}) \cup \{ade, cfg\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(ad, cb)$ (that is, by $\mathcal{O}_2$ applied to the edge ad of the simplex acd of G' and to the edge cb incident with the third vertex c of the triangle acd), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 4. $H = H_6$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G and, as before, we study desired relations between properties of the graphs G and G'.

Claim 4.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{b, c, d\}| = 1$. If $c \in I'$ or $d \in I'$, then the set $I' \cup \{g\}$ is an independent set of G, while if $b \in I'$, then the set $I' \cup \{e\}$ is an independent set of G, implying that $\alpha(G) \geq \alpha(G') + 1$. It remains to prove that $\alpha(G') \geq \alpha(G) - 1$. By supposition, G is an α-excellent graph, and so every vertex of G belongs to some α-set of G. Let I be an α-set of G that contains the vertex c. Necessarily, $g \in I$ and $\{b, e, f\} \cap I = \emptyset$. Thus, $I \setminus \{g\}$ is an independent set in G', implying that $\alpha(G') \geq |I| - 1 = \alpha(G) - 1$. As observed earlier, $\alpha(G') \leq \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 4.2. *G' is an α-excellent graph.*

Since G is an α-excellent graph, every vertex of G belongs to an α-set of G. Let I_x denote an α-set of G that contains the vertex x of G. Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. This time we note that $|I_v \cap \{c, d, e\}| = 1$, $|I_v \cap \{b, f, g\}| = 1$, and $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$. If $|I_v \cap \{a, b, c, d\}| = 2$ (that is, if $I_v \cap \{a, b, c, d\} = \{a, d\}$) or $I_v \cap \{a, b, c, d\} = \{d\}$, then $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = I_v \setminus \{f, g\}$. If $I_v \cap \{a, b, c, d\} = \emptyset$ or $I_v \cap \{a, b, c, d\} = \{a\}$, then $e \in I_v$, $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = (I_v \setminus \{e, f, g\}) \cup \{d\}$. If $I_v \cap \{a, b, c, d\} = \{b\}$, then $e \in I_v$ and we let $I'_v = I_v \setminus \{e\}$. If $I_v \cap \{a, b, c, d\} = \{c\}$, then $g \in I_v$ and we let $I'_v = I_v \setminus \{g\}$. In all cases, the set I'_v is an independent set in G' that contains the vertex v and $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$, implying that v belongs to an α-set of G', as desired.

Claim 4.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

Similarly as in the proofs of Claims 1.3, 2.3 and 3.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G'

has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $bcd \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{bcd\}) \cup \{cde, bfg\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(cd, bc)$ (that is, by $\mathcal{O}_2$ applied to the edge cd of the simplex bcd of G' and to the edge bc incident with the third vertex b of the triangle bcd), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 5. $H = H_7$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G. Using identical arguments as in the proof of Case 4, we prove that G has a perfect 3-cover and it belongs to the family $\mathcal{E}$. We omit the details.

Thus all possible cases have been considered. This completes the proof of Theorem 4. ∎

It is easy to observe that the corona graph $H \circ K_1$ (the graph formed from H by adding for each vertex v in H a new vertex v' and the edge vv') is an α-excellent graph for every graph H. This implies that every graph is an induced subgraph of an α-excellent graph. As a consequence of Theorem 4, we have the following property of 2-trees.

Corollary 5. *Every 2-tree is an induced subgraph of an α-excellent 2-tree.*

Proof. Let G be a 2-tree. The statement is obvious if G has order 2. Thus assume that G is a 2-tree of order at least 3. Let $\mathcal{P}$ be a maximal family of vertex-disjoint triangles of G, and let Q be the set of all vertices in G that do not belong to any triangle in $\mathcal{P}$. If the set Q is empty, then, by Theorem 4, G is an α-excellent 2-tree itself (and, therefore, it has the desired property). Thus assume that Q is nonempty. For each vertex $v \in Q$, we do the following. Let v' be an arbitrary neighbor of v in $G - Q$. We now add two new vertices $x_{vv'}$ and $y_{vv'}$, and four edges $x_{vv'}v$, $x_{vv'}v'$, $y_{vv'}v$ and $y_{vv'}x_{vv'}$. We note that the newly constructed triangle $vx_{vv'}y_{vv'}$ covers the vertex v and the two new vertices $x_{vv'}$ and $y_{vv'}$. Let G' be a resulting 2-tree obtained from G by performing this operation for all vertices $v \in Q$. Then the set $\mathcal{P} \cup \bigcup_{v \in Q}\{vx_{vv'}y_{vv'}\}$ is a perfect 3-cover of G'. Thus, by Theorem 4, the graph G' is an α-excellent 2-tree. By construction, the 2-tree G is an induced subgraph of the α-excellent 2-tree G'. This completes the proof of Corollary 5. ∎

We close this paper with the following open question that we have yet to settle. Is it true that if $k \geq 3$ is an integer and G is an α-excellent k-tree of order at least $k + 1$, then G has a perfect $(k + 1)$-cover?

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Received 31 December 2022
Revised 28 August 2023
Accepted 29 August 2023
Available online 3 October 2023