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CHARACTERIZATION OF α-EXCELLENT 2-TREES

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Abstract

A graph is α-excellent if every vertex of the graph is contained in some maximum independent set of the graph. In this paper, we present two characterizations of the α-excellent 2-trees.

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1. Introduction

For notation and graph theory terminology we, in general, follow the recent books [9–11]. Specifically, let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, its *neighborhood*, denoted by $N_G(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the *open neighborhood* of S is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the *closed neighborhood* $N_G[S] = N_G(S) \cup S$. For a positive integer k, we let $[k] = \{1, \ldots, k\}$.

A subset D of the vertex set $V(G)$ of a graph G is called a *dominating set* of G if every vertex belonging to $V(G) \setminus D$ is adjacent to at least one vertex in D. A subset I of $V(G)$ is *independent* if no two vertices belonging to I are adjacent in G. The cardinality of a largest (i.e., maximum) independent set of G, denoted by $\alpha(G)$, is called the *independence number* of G. Every largest independent set of a graph is called an α-set of the graph. A dominating set D of a graph G is called an *independent dominating set* of G if D is also independent. The *independent domination number* of G, denoted by $i(G)$, is the cardinality of the smallest independent dominating set of G (or equivalently, the cardinality of a minimum maximal independent set of vertices in G). The *common independence number* of a graph G, denoted by $\alpha_c(G)$, is the greatest integer r such that every vertex of G belongs to some independent subset X of $V(G)$ with $|X| \geq r$. It follows immediately from the above definitions that the common independence number is bounded below by the independent domination number and above by the independence number. Formally, for any graph G,

$$i(G) \leq \alpha_c(G) \leq \alpha(G). \tag{1}$$

The study of independent sets in graphs was begun by Berge [1,2] (see also [3]) and Ore [12]. We refer the reader to the book [11] and to the survey [8] of results on independent domination in graphs published in 2013 by Goddard and Henning. A graph G is said to be *well-covered* if $i(G) = \alpha(G)$. Equivalently, G is well-covered if every maximal independent set of G is a maximum independent set of G. The concept of well-covered graphs was introduced by Plummer [13] in 1970. Since then the well-covered graphs were very extensively investigating in many papers. We refer the reader to the excellent (but already older) survey on well-covered graphs by Plummer [14]. We are interested in characterization of *α-excellent graphs*, that is, graphs G for which $\alpha_c(G) = \alpha(G)$. Thus, if G is an α-excellent graph, then every vertex belongs to some α-set of G. It follows from Inequalities (1) that every well-covered graph is an α-excellent graph. The example of the cycle C_6 shows that the set of well-covered graphs is properly contained in the set of α-excellent graphs. The α-excellent graphs have been studied in [4–7,15,16] and in [1,2] as B-graphs. In this paper, we begin the study of α-excellent k-trees.

2. Preliminary results

A vertex v of a graph G is a *simplicial vertex* if every two vertices belonging to $N_G(v)$ are adjacent in G. Equivalently, a simplicial vertex is a vertex that appears in exactly one clique of a graph, where a *clique* of a graph G is a maximal complete subgraph of G. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G. Note that if v is a simplicial vertex of G, then $G[N_G[v]]$ is the unique simplex of G containing v. We begin with a simple proposition.

Proposition 1. *No α-excellent graph contains a vertex belonging to at least two its simplexes.*

Proof. Assume that a vertex v of a graph G belongs to two simplexes of G, say to $G[N_G[u]]$ and $G[N_G[w]]$. If I is a maximum independent set that contains v, then $(I \setminus \{v\}) \cup \{u, w\}$ is an independent set of greater cardinality. Thus, $\alpha(G) \geq |(I \setminus \{v\}) \cup \{u, w\}| > |I|$, implying that v does not belong to any α-set of G, and proving that G is not an α-excellent graph. ■

For a positive integer k, a graph G is called a *k-tree* if it can be obtained from the complete graph K_k by a finite number of applications of the following operation: add a new vertex and join it to k mutually adjacent vertices of the existing graph. Certainly, every 1-tree is a tree and vice versa. In [17], Rose proved that a graph G is a k-tree if and only if the following conditions are fulfilled: (i) G is connected, (ii) G contains K_k as a subgraph and does not contain K_{k+2} as a subgraph, (iii) if v and u are nonadjacent vertices of G, then the subgraph induced by the smallest $v-u$ separator is a complete graph on k vertices. Recall that a $v-u$ separator in a connected graph G is a subset S of $V(G)$ such that u and v are in distinct components of $G[V(G) \setminus S]$. Note that K_k and K_{k+1} are the only k-trees of order k and $k+1$, respectively.

It was proved in [4] that a bipartite graph (and, in particular, a tree) is an α-excellent graph if and only if it has a perfect matching. On the other hand, it was observed in [18] that a k-tree G is a well-covered graph if and only if every vertex of G belongs to exactly one simplex of G. In this paper, we are interested in possible extensions of that characterization to a characterization of α-excellent k-trees for every positive integer k. We begin with the following definition.

A set $\mathcal{P}$ of complete subgraphs of a graph G is said to be a *perfect* $(k+1)$*-cover* of G if each subgraph belonging to $\mathcal{P}$ is of order $k+1$ and every vertex of G belongs to exactly one subgraph in $\mathcal{P}$. It is obvious that for $k = 1$ there exists a one-to-one correspondence between perfect 2-covers of a graph and perfect matchings of the graph. Here we are interested in the existence of perfect $(k+1)$-covers in k-trees. First of all, one can prove that every k-tree has at most one perfect $(k+1)$-cover. In the following proposition, we present the first relationship between k-trees having perfect $(k+1)$-covers and α-excellent graphs.

Proposition 2. *If a k-tree G has a perfect $(k+1)$-cover, then G is an α-excellent graph.*

Proof. Let G be a connected k-tree of order $n \geq k+1$. Then G is a $(k+1)$-partite graph, say $A_1, A_2, \ldots, A_{k+1}$ are partite sets of G and assume that $|A_1| \geq |A_2| \geq \cdots \geq |A_{k+1}| \geq 1$. In addition, since $A_1, A_2, \ldots, A_{k+1}$ are independent sets of vertices and $|A_1| + |A_2| + \cdots + |A_{k+1}| = n$, it follows that $|A_1| \geq n/(k+1)$, and, therefore, $\alpha(G) \geq |A_1| \geq n/(k+1)$. Let I be an α-set of G. Assume now that $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ is a perfect $(k+1)$-cover of G. Then, $\ell = n/(k+1)$, $|I \cap V(P_i)| \leq 1$ for each $i \in [\ell]$, and

$$\alpha(G) = |I| = \left| I \cap \bigcup_{i=1}^{\ell} V(P_i) \right| = \sum_{i=1}^{\ell} |I \cap V(P_i)| \leq \ell = \frac{n}{k+1}.$$

Consequently, $|A_1| = |A_2| = \cdots = |A_{k+1}| = n/(k+1) = \alpha(G)$, and each of the sets $A_1, A_2, \ldots, A_{k+1}$ is an α-set of G. This implies that every vertex of G belongs to an α-set of G and, therefore, G is an α-excellent graph. ∎

3. α-Excellent 2-Trees

Proposition 2 shows that a k-tree having a perfect $(k+1)$-cover is an α-excellent k-tree. It is not clear to us whether the converse of this statement is true. That is, we do not know if every α-excellent k-tree of order at least $k+1$ has a perfect $(k+1)$-cover if $k \geq 3$. However, when $k = 2$, we provide in this paper a characterization of α-excellent k-trees. For notational simplicity, in what follows if three vertices a, b, and c are mutually adjacent in a graph G, then the induced subgraph $G[\{a,b,c\}]$ of G is isomorphic to K_3 and is called a *triangle* in G, and we simply write abc rather than $G[\{a,b,c\}]$. To every triangle in a graph G, we assign label R or B (as red or blue, respectively), and by $R(G)$ and $B(G)$ we denote the set of all triangles in G that have label R and B, respectively. We also say that $R(G)$ and $B(G)$ are the sets of all "red" and "blue" triangles in G, respectively.

We are now in position to present a constructive characterization of α-excellent 2-trees. For this purpose, let $\mathcal{E}$ be the family of labeled 2-trees defined recursively as follows.

(1) The family $\mathcal{E}$ contains the 2-tree of order 3 in which the only triangle is red, that is, it has label R.

(2) The family $\mathcal{E}$ is closed under the operations $\mathcal{O}_1$ and $\mathcal{O}_2$ defined below.

Operation $\mathcal{O}_1$. If a graph G' belongs to $\mathcal{E}$ and v_1v_2 is an edge of G', then $\mathcal{O}_1 = \mathcal{O}_1(v_1, v_2)$ forms a graph G by adding three new vertices u_1, u_2, u_3 to

G' in such a way that $v_1v_2u_1$, $v_2u_1u_2$ and $u_1u_2u_3$ are three new triangles, while $R(G) = R(G') \cup \{u_1u_2u_3\}$ and $B(G) = B(G') \cup \{v_1v_2u_1, v_2u_1u_2\}$. In this case we apply the operation $\mathcal{O}_1$ to the edge v_1v_2 of G'.

Operation $\mathcal{O}_2$. If a graph G' belongs to $\mathcal{E}$, $v_1v_2v_3$ is a red triangle in G' (that is, $v_1v_2v_3 \in R(G')$), and v_4 is a neighbor of v_3 (it is possible that $v_4 \in \{v_1, v_2\} \subseteq N_{G'}(v_3)$), then $\mathcal{O}_2 = \mathcal{O}_2(v_1v_2, v_3v_4)$ forms a graph G by adding to G' three new vertices u_0, u_1 and u_2 in such a way that $u_0v_1v_2$, $v_3v_4u_1$, and $v_3u_1u_2$ are new triangles, while $R(G) = (R(G') \setminus \{v_1v_2v_3\}) \cup \{u_0v_1v_2, v_3u_1u_2\}$ and $B(G) = B(G') \cup \{v_1v_2v_3, v_3v_4u_1\}$. In this case we apply the operation $\mathcal{O}_2$ to the edge v_1v_2 of the triangle $v_1v_2v_3$ and to the edge v_3v_4.

The operations $\mathcal{O}_1$ and $\mathcal{O}_2$ are illustrated in Figure 1. Note that the operation $\mathcal{O}_2$ changes "colors" of certain triangles, and the red triangle $v_1v_2v_3$ in G' is recolored blue in G.

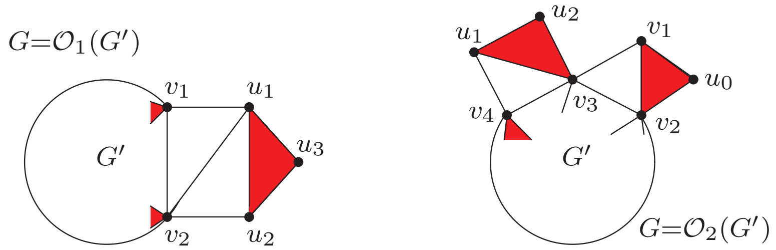

Figure 1. The operations $\mathcal{O}_1$ and $\mathcal{O}_2$.

From the recursive definition of the graphs belonging to the family $\mathcal{E}$ it follows readily that if a 2-tree G belongs to $\mathcal{E}$, then the set $R(G)$ of red triangles in G is a perfect 3-cover of G. From this and from Proposition 2 it follows that G is an α-excellent graph. Thus we have the following proposition that we will need when proving our main theorem.

Proposition 3. *Every* 2*-tree belonging to the family* $\mathcal{E}$ *has a perfect* 3*-cover and it is an* α*-excellent graph.*

The following theorem is the main result of this paper, and it presents two characterizations of the α-excellent 2-trees: a constructive characterization, and a characterization in terms of perfect 3-covers.

Theorem 4. *If G is a* 2*-tree of order* $n \geq 3$*, then the following statements are equivalent.*

(a) *G has a perfect* 3*-cover.*

(b) *G belongs to the family* $\mathcal{E}$*.*

(c) *G is an α-excellent graph.*

Proof. The implications (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (a), and (b) $\Rightarrow$ (c) are obvious from Propositions 2 and 3. Thus it suffices to prove the implication (c) $\Rightarrow$ (b) (but we prove the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) at the same time).

Thus assume that G is an α-excellent 2-tree of order at least 3. By induction on the order of G we shall prove that G has a perfect 3-cover and that G belongs to the family $\mathcal{E}$. It is straightforward to observe that the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are true if G is a 2-tree of order $n \le 6$. Now let G be an α-excellent 2-tree of order greater than 6 and assume that the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are true for smaller α-excellent 2-trees. Let $(T_1, T_2, \ldots, T_p)$ be a longest 3-path in G, that is, a longest sequence $T_1, T_2, \ldots, T_p$ of triangles in G, where $|V(T_i) \cap V(T_j)| = 2$ if $|i - j| = 1$, and $|V(T_i) \cap V(T_j)| \le 1$ if $|i - j| \ge 2$ ($i, j \in [p]$). From the fact that $(T_1, T_2, \ldots, T_p)$ is a longest 3-path in G (which is an α-excellent 2-tree of order at least 7) it follows that $p \ge 4$. Assume that a, b, c, d, and e are vertices of G for which $V(T_{p-3}) \cap V(T_{p-2}) = \{a, b\}$, $V(T_{p-2}) \setminus V(T_{p-3}) = \{c\}$, $V(T_{p-1}) \setminus V(T_{p-2}) = \{d\}$, and $V(T_p) \setminus V(T_{p-1}) = \{e\}$. From the choice of $(T_1, \ldots, T_p)$, the vertex e is of degree 2 and T_p is a simplex in G. Let $G'_0, G'_1, \ldots, G'_\ell$ be the components of $G - \{a, b\}$, where G'_0 is that component which contains at least one vertex of T_1. It is clear that ℓ is positive integer.

We now let G_i denote the subgraph of G induced by $V(G'_i) \cup \{a, b\}$ for $i \in \{0\} \cup [\ell]$. Among the graphs $G_1, \ldots, G_\ell$, let H be the graph that contains the triangles of the 3-path $\mathcal{P}_0 = (T_{p-2}, T^0_{p-1}, T^0_p)$, where $T^0_{p-1} = T_{p-1}$ and $T^0_p = T_p$. It is obvious that if T_{p-2}, T_{p-1}, and T_p are the only triangles of H, then it is possible that H is one of the graphs H_1, H'_1, H_2, and H'_2 shown in Figure 2. If H contains the triangle T_{p-2}, T_{p-1}, T_p and a simplex T'_{p-1} that shares an edge with the triangle T_{p-2} in H, then it follows readily from Proposition 1 that H is one of the graphs H_3 or H'_3 in Figure 2.

Thus assume that no simplex of H shares an edge with the triangle T_{p-2}. In this case, H is a subgraph induced by the triangles belonging to $\mathcal{P}_0$ and to some additional 3-paths $\mathcal{P}_i = (T_{p-2}, T^i_{p-1}, T^i_p)$, where $i \in [n]$ and n is a positive integer. It follows from Proposition 1 that if $n = 1$, then H is isomorphic to one of the graphs H_4, H_5 or H_6 in Figure 3. Similarly, if $n = 2$, then, as can easily be verified, H is isomorphic to the graph H_7 shown in Figure 3. Finally, we claim that the case $n \ge 3$ is impossible. Suppose, to the contrary, that $n \ge 3$. Then let us first observe that if one of the edges ac and bc of the triangle T_{p-2} belongs to at least 3 of the triangles $T^0_{p-1} = T_{p-1}, T^1_{p-1}, \ldots, T^n_{p-1}$, say to $T^i_{p-1}, T^j_{p-1}, T^k_{p-1}$ (where $0 \le i < j < k \le n$), then at least two of the simplexes T^i_p, T^j_p, T^k_p of H (and of G) have a common vertex, which is impossible in an α-excellent graph G. Thus assume that neither ac nor bc belongs to three of the triangles $T^0_{p-1} = T_{p-1}$,

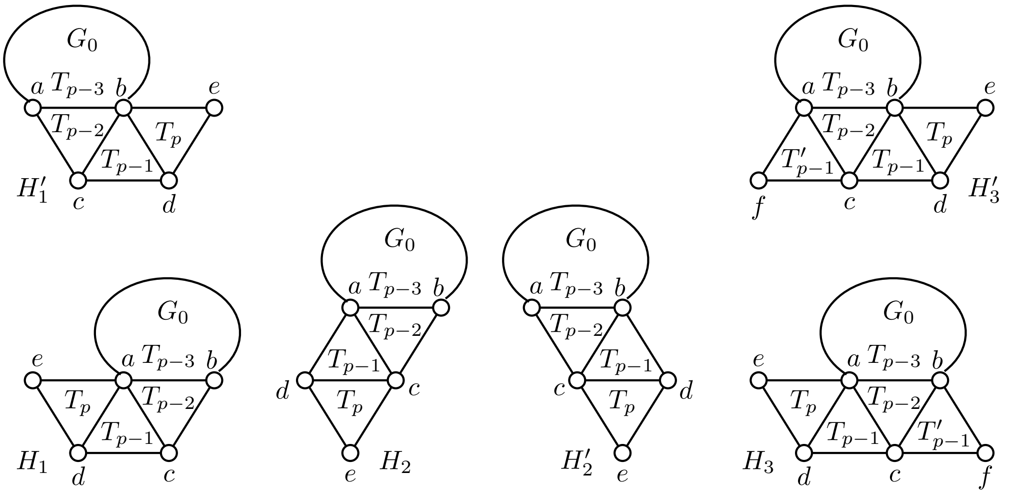

Figure 2. Graphs H_1, H_1', H_2, H_2', H_3, and H_3'.

$T^1_{p-1}, \ldots, T^n_{p-1}$. Then necessarily $n = 3$ and each of the edges ac and bc belongs to exactly two of the triangles $T^0_{p-1}, T^1_{p-1}, T^2_{p-1}, T^3_{p-1}$, say ac belongs to T^0_{p-1} and T^1_{p-1}, while bc belongs to T^2_{p-1} and T^3_{p-1}. Therefore, as it is easy to verify, if neither the simplexes T^0_p and T^1_p have a common vertex nor the simplexes T^2_p and T^3_p have a common vertex, then one of the simplexes T^0_p and T^1_p has a common vertex with one of the simplexes T^2_p and T^3_p, which again by Proposition 1 is impossible as G is an α-excellent graph. This proves that the case $n \geq 3$ is impossible.

There are now several cases to consider depending on the structure of H. We begin showing that H cannot be any of the graphs H_1, H_1', H_4. Suppose, to the contrary, that $H = H_1$ (where H_1 is as illustrated in Figure 2). Then, since G is an α-excellent graph, there exists an α-set I in G that contains a. However in this case, $(I \setminus \{a\}) \cup \{c, e\}$ is an independent set in G, and so $\alpha(G) \geq |(I \setminus \{a\}) \cup \{c, e\}| > |I| = \alpha(G)$, a contradiction which proves that $H \neq H_1$. Analogously, $H \neq H_1'$. Similarly, suppose that $H = H_4$ (where H_4 is shown in Figure 3). Let I_d be an α-set that contains d in G. We note that $|I_d \cap \{b, f, g\}| = 1$. If $b \in I_d$, then let $I_d' = (I_d \setminus \{b, d\}) \cup \{c, e, g\}$. If $f \in I_d$, then let $I_d' = (I_d \setminus \{d, f\}) \cup \{c, e, g\}$. If $g \in I_d$, then let $I_d' = (I_d \setminus \{d\}) \cup \{c, e\}$. In all cases, the resulting set I_d' is an independent set in G, and so $\alpha(G) \geq |I_d'| > |I_d| = \alpha(G)$, a contradiction. Hence, $H \neq H_4$.

In each of the next five cases (corresponding to the possible graphs H, that is, to the graphs H_2 (and H_2'), H_3, H_5, H_6, and H_7) we prove that G belongs to the family $\mathcal{E}$ and has a perfect 3-cover.

Case 1. $H = H_2$ *or* $H = H_2'$. Without loss of generality, assume that

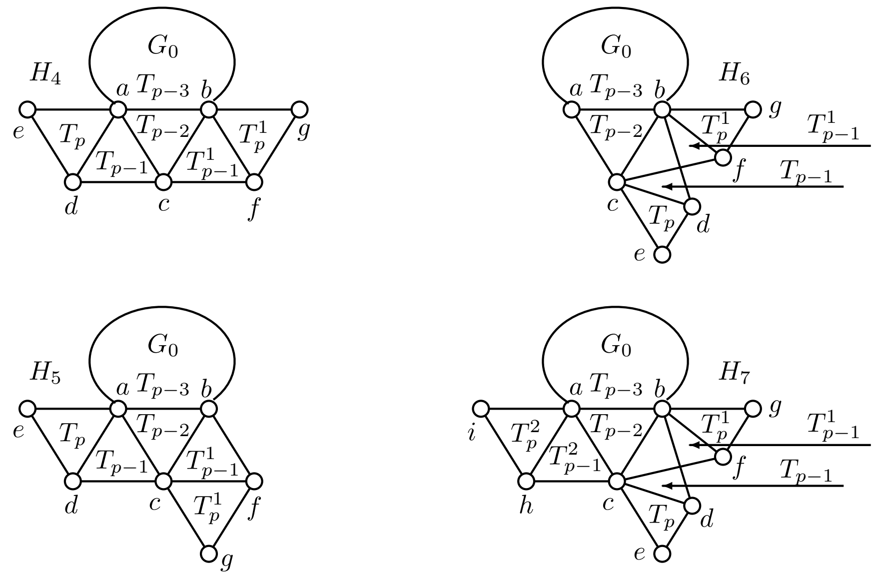

Figure 3. Graphs H_4, H_5, H_6, and H_7.

$H = H_2$ (see Figure 2). In this case, let G' denote the subgraph $G - \{c, d, e\}$ of G. In the following three claims we explain the main relations between properties of the graphs G and G', that is, we prove that: (1) $\alpha(G') = \alpha(G) - 1$; (2) G' is an α-excellent graph; (3) G has a perfect 3-cover and belongs to the family $\mathcal{E}$.

Claim 1.1. $\alpha(G') = \alpha(G) - 1$.

If I' is an α-set of G', then $I' \cup \{e\}$ is an independent set in G (as $N_G[e] \cap I' = \{c, d, e\} \cap I' = \emptyset$) and therefore $\alpha(G) \geq |I' \cup \{e\}| = \alpha(G') + 1$. On the other hand, let I be an α-set of G. In this case, $|I \cap \{c, d, e\}| = 1$ and $I \setminus \{c, d, e\}$ is independent in G'. Thus, $\alpha(G') \geq |I \setminus \{c, d, e\}| = |I| - 1 = \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 1.2. *G' is an α-excellent graph.*

Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G'. Since $\alpha(G') = \alpha(G) - 1$ (by Claim 1.1), it suffices to show that v belongs to an independent set of cardinality $\alpha(G) - 1 = \alpha(G')$ in G'. Let I_v be an α-set of G that contains v, and let I'_v denote the set $I_v \setminus \{c, d, e\}$. Thus, $v \in I'_v$ and the set I'_v is independent in G'. Furthermore, I'_v is an α-set of G' noting that $|I_v \cap \{c, d, e\}| = 1$ and $|I'_v| = |I_v \setminus \{c, d, e\}| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$. This proves that G' is an α-excellent graph.

Claim 1.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

By Claim 1.2, G' is an α-excellent graph. Since the order of G' is less than the order of G, applying the induction hypothesis we infer that G' has a perfect 3-cover and G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $\mathcal{P}' \cup \{cde\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_1 = \mathcal{O}_1(a, b)$ (that is, by $\mathcal{O}_1$ applied to the edge ab of G'), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 2. *$H = H_3$ or $H = H_3'$.* Without loss of generality, assume that $H = H_3$ (see Figure 2). This time, let G' denote the subgraph $G - \{d, e, f\}$ of G. As in Case 1, we study desired relations between properties of the graphs G and G'.

Claim 2.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{a, b, c\}| = 1$ and either $a \in I'$ or $\{b, c\} \cap I' \neq \emptyset$. Consequently, either $I' \cup \{f\}$ or $I' \cup \{e\}$ is an independent set in G, respectively, and therefore $\alpha(G) \geq |I' \cup \{f\}| = |I' \cup \{e\}| = \alpha(G') + 1$. This proves that $\alpha(G) \geq \alpha(G') + 1$. Now, let I be an α-set of G. Then $|I \cap \{a, b, c\}| \leq 1$ and we consider four cases. If $I \cap \{a, b, c\} = \emptyset$, then $|I \cap \{d, e, f\}| = 2$ and $(I \setminus \{d, e, f\}) \cup \{c\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $a \in I$, then $f \in I$, and $I \setminus \{f\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $b \in I$, then $|I \cap \{d, e\}| = 1$, $f \notin I$, and $I \setminus \{d, e\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. If $c \in I$, then $e \in I$, $f \notin I$, and $I \setminus \{e\}$ is an independent set of cardinality $\alpha(G) - 1$ in G'. Consequently, $\alpha(G') \geq \alpha(G) - 1$. This proves that $\alpha(G') = \alpha(G) - 1$.

Claim 2.2. *G' is an α-excellent graph.*

Let v be an arbitrary vertex of G'. As in the proof of Claim 1.2, it suffices to show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. Since G is α-excellent, every vertex of G belongs to some α-set in G. Let I_x be an α-set of G that contains x, where $x \in V(G)$. We note that $I_a \cap \{d, e, f\} = \{f\}$ and, consequently, $I_a \setminus \{f\}$ is an α-set of G' that contains the vertex a. Similarly, from the fact that $|I_b \cap \{d, e\}| = 1$ and $I_c \cap \{d, e, f\} = \{e\}$ it follows that $I_b \setminus \{d, e\}$ and $I_c \setminus \{e\}$ are α-sets of G' and they contain b and c, respectively. If $v \in V(G') \setminus \{a, b, c\}$ and $I_v \cap \{a, b, c\} = \emptyset$, then $|I_v \cap \{d, e, f\}| = 2$ and $(I_v \setminus \{d, e, f\}) \cup \{c\}$ is an α-set of G' and it contains v. Finally, if $v \in V(G') \setminus \{a, b, c\}$ and $I_v \cap \{a, b, c\} \neq \emptyset$, then $|I_v \cap \{a, b, c\}| = 1$, $|I_v \cap \{d, e, f\}| = 1$, and, therefore, $I_v \setminus \{d, e, f\}$ is an α-set of G' and it contains v. Consequently, G' is an α-excellent graph.

Claim 2.3. *G has a perfect* 3*-cover and G belongs to the family $\mathcal{E}$.*

Now, similarly as in the proof of Claim 1.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G' has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Certainly, if $\mathcal{P}'$ is a perfect 3-cover of G', then $abc \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{abc\}) \cup \{ade, bcf\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. From this and from the obvious fact that G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(bc, ac)$ (that is, by $\mathcal{O}_2$ applied to the edge bc of the simplex abc of G' and to the edge ac incident with the third vertex c of the triangle abc), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 3. $H = H_5$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G and, as before, we study desired relations between properties of the graphs G and G'.

Claim 3.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{a, c, d\}| = 1$. If $a \in I'$ or $d \in I'$, then the set $I' \cup \{g\}$ is an independent set of G, while if $c \in I'$, then the set $I' \cup \{e\}$ is an independent set of G, implying that $\alpha(G) \geq \alpha(G') + 1$. It remains to prove that $\alpha(G') \geq \alpha(G) - 1$. To prove this, let I be an α-set of G. Thus, $|I \cap \{a, d, e\}| = 1$ and $|I \cap \{c, f, g\}| = 1$. If $a \in I$ or $d \in I$, then $|I \cap \{f, g\}| = 1$ and we let $I' = I \setminus \{f, g\}$. If $e \in I$ and $c \in I$, then we let $I' = I \setminus \{e\}$. If $e \in I$ and $b \in I$, then $g \in I$ and we let $I' = (I \setminus \{e, g\}) \cup \{d\}$. If $e \in I$ and $I \cap \{b, c\} = \emptyset$, then $|I \cap \{f, g\}| = 1$ and we let $I' = (I \setminus \{e, f, g\}) \cup \{c\}$. In all the above cases, the set I' is an independent set in G' and $|I'| = |I| - 1$, implying that $\alpha(G') \geq |I'| = |I| - 1 = \alpha(G) - 1$. As observed earlier, $\alpha(G') \leq \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 3.2. *G' is an α-excellent graph.*

Since G is an α-excellent graph, every vertex of G belongs to an α-set of G. Let I_x denote an α-set of G that contains the vertex x of G. Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. We note that $|I_v \cap \{a, d, e\}| = 1$, $|I_v \cap \{c, f, g\}| = 1$, and $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$. If $|I_v \cap \{a, b, c, d\}| = 2$, then $I_v \cap \{a, b, c, d\} = \{b, d\}$, $g \in I_v$ and we let $I'_v = I_v \setminus \{g\}$. If $I_v \cap \{a, b, c, d\} = \emptyset$, then $e \in I_v$ and $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = (I_v \setminus \{e, f, g\}) \cup \{c\}$. If $I_v \cap \{a, b, c, d\} = \{a\}$ or $I_v \cap \{a, b, c, d\} = \{d\}$, then $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = I_v \setminus \{f, g\}$. If $I_v \cap \{a, b, c, d\} = \{b\}$, then $e \in I_v$, $g \in I_v$, and we let $I'_v = (I_v \setminus \{e, g\}) \cup \{d\}$. If $I_v \cap \{a, b, c, d\} = \{c\}$, then $e \in I_v$ and we let $I'_v = I_v \setminus \{e\}$. In all cases, the set I'_v is an independent set in G' that contains the vertex v and $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$, implying that v belongs to an α-set of G', as desired.

Claim 3.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

Similarly as in the proofs of Claims 1.3 and 2.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G' has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $acd \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{acd\}) \cup \{ade, cfg\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(ad, cb)$ (that is, by $\mathcal{O}_2$ applied to the edge ad of the simplex acd of G' and to the edge cb incident with the third vertex c of the triangle acd), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 4. $H = H_6$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G and, as before, we study desired relations between properties of the graphs G and G'.

Claim 4.1. $\alpha(G') = \alpha(G) - 1$.

Let I' be an α-set of G'. In this case, $|I' \cap \{b, c, d\}| = 1$. If $c \in I'$ or $d \in I'$, then the set $I' \cup \{g\}$ is an independent set of G, while if $b \in I'$, then the set $I' \cup \{e\}$ is an independent set of G, implying that $\alpha(G) \geq \alpha(G') + 1$. It remains to prove that $\alpha(G') \geq \alpha(G) - 1$. By supposition, G is an α-excellent graph, and so every vertex of G belongs to some α-set of G. Let I be an α-set of G that contains the vertex c. Necessarily, $g \in I$ and $\{b, e, f\} \cap I = \emptyset$. Thus, $I \setminus \{g\}$ is an independent set in G', implying that $\alpha(G') \geq |I| - 1 = \alpha(G) - 1$. As observed earlier, $\alpha(G') \leq \alpha(G) - 1$. Consequently, $\alpha(G') = \alpha(G) - 1$.

Claim 4.2. *G' is an α-excellent graph.*

Since G is an α-excellent graph, every vertex of G belongs to an α-set of G. Let I_x denote an α-set of G that contains the vertex x of G. Let v be an arbitrary vertex of G'. We show that v belongs to an α-set of G', that is, to an independent set of cardinality $\alpha(G') = \alpha(G) - 1$ in G'. This time we note that $|I_v \cap \{c, d, e\}| = 1$, $|I_v \cap \{b, f, g\}| = 1$, and $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$. If $|I_v \cap \{a, b, c, d\}| = 2$ (that is, if $I_v \cap \{a, b, c, d\} = \{a, d\}$) or $I_v \cap \{a, b, c, d\} = \{d\}$, then $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = I_v \setminus \{f, g\}$. If $I_v \cap \{a, b, c, d\} = \emptyset$ or $I_v \cap \{a, b, c, d\} = \{a\}$, then $e \in I_v$, $|I_v \cap \{f, g\}| = 1$ and we let $I'_v = (I_v \setminus \{e, f, g\}) \cup \{d\}$. If $I_v \cap \{a, b, c, d\} = \{b\}$, then $e \in I_v$ and we let $I'_v = I_v \setminus \{e\}$. If $I_v \cap \{a, b, c, d\} = \{c\}$, then $g \in I_v$ and we let $I'_v = I_v \setminus \{g\}$. In all cases, the set I'_v is an independent set in G' that contains the vertex v and $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$, implying that v belongs to an α-set of G', as desired.

Claim 4.3. *G has a perfect 3-cover and G belongs to the family $\mathcal{E}$.*

Similarly as in the proofs of Claims 1.3, 2.3 and 3.3, since G' is an α-excellent graph of order less than the order of G, the induction hypothesis implies that G'

has a perfect 3-cover and that G' belongs to the family $\mathcal{E}$. Now, if $\mathcal{P}'$ is a perfect 3-cover of G', then $bcd \in \mathcal{P}'$ and $(\mathcal{P}' \setminus \{bcd\}) \cup \{cde, bfg\}$ is a perfect 3-cover of G. In addition, since G' belongs to the family $\mathcal{E}$, the graph G' can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$, and, since G can be obtained from G' by the operation $\mathcal{O}_2 = \mathcal{O}_2(cd, bc)$ (that is, by $\mathcal{O}_2$ applied to the edge cd of the simplex bcd of G' and to the edge bc incident with the third vertex b of the triangle bcd), the graph G can be obtained recursively from a red triangle by operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Thus, G belongs to the family $\mathcal{E}$.

Case 5. $H = H_7$. In this case, we let G' denote the subgraph $G - \{e, f, g\}$ of G. Using identical arguments as in the proof of Case 4, we prove that G has a perfect 3-cover and it belongs to the family $\mathcal{E}$. We omit the details.

Thus all possible cases have been considered. This completes the proof of Theorem 4. ■

It is easy to observe that the corona graph $H \circ K_1$ (the graph formed from H by adding for each vertex v in H a new vertex v' and the edge vv') is an α-excellent graph for every graph H. This implies that every graph is an induced subgraph of an α-excellent graph. As a consequence of Theorem 4, we have the following property of 2-trees.

Corollary 5. *Every* 2*-tree is an induced subgraph of an* α*-excellent* 2*-tree.*

Proof. Let G be a 2-tree. The statement is obvious if G has order 2. Thus assume that G is a 2-tree of order at least 3. Let $\mathcal{P}$ be a maximal family of vertex-disjoint triangles of G, and let Q be the set of all vertices in G that do not belong to any triangle in $\mathcal{P}$. If the set Q is empty, then, by Theorem 4, G is an α-excellent 2-tree itself (and, therefore, it has the desired property). Thus assume that Q is nonempty. For each vertex $v \in Q$, we do the following. Let v' be an arbitrary neighbor of v in $G-Q$. We now add two new vertices $x_{vv'}$ and $y_{vv'}$, and four edges $x_{vv'}v$, $x_{vv'}v'$, $y_{vv'}v$ and $y_{vv'}x_{vv'}$. We note that the newly constructed triangle $vx_{vv'}y_{vv'}$ covers the vertex v and the two new vertices $x_{vv'}$ and $y_{vv'}$. Let G' be a resulting 2-tree obtained from G by performing this operation for all vertices $v \in Q$. Then the set $\mathcal{P} \cup \bigcup_{v \in Q}\{vx_{vv'}y_{vv'}\}$ is a perfect 3-cover of G'. Thus, by Theorem 4, the graph G' is an α-excellent 2-tree. By construction, the 2-tree G is an induced subgraph of the α-excellent 2-tree G'. This completes the proof of Corollary 5. ■

We close this paper with the following open question that we have yet to settle. Is it true that if $k \geq 3$ is an integer and G is an α-excellent k-tree of order at least $k+1$, then G has a perfect $(k+1)$-cover?

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