

## CHARACTERIZATION OF $\alpha$ -EXCELLENT 2-TREES

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### Abstract

A graph is  $\alpha$ -excellent if every vertex of the graph is contained in some maximum independent set of the graph. In this paper, we present two characterizations of the  $\alpha$ -excellent 2-trees.

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## 1. INTRODUCTION

For notation and graph theory terminology we, in general, follow the recent books [9–11]. Specifically, let  $G = (V(G), E(G))$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . For a vertex  $v$  of  $G$ , its *neighborhood*, denoted by  $N_G(v)$ , is the set of all vertices adjacent to  $v$ . The *closed neighborhood* of  $v$ , denoted by  $N_G[v]$ , is the set  $N_G(v) \cup \{v\}$ . For a set  $S \subseteq V(G)$ , the *open neighborhood* of  $S$  is the set  $N_G(S) = \bigcup_{v \in S} N_G(v)$ , and the *closed neighborhood*  $N_G[S] = N_G(S) \cup S$ . For a positive integer  $k$ , we let  $[k] = \{1, \dots, k\}$ .

A subset  $D$  of the vertex set  $V(G)$  of a graph  $G$  is called a *dominating set* of  $G$  if every vertex belonging to  $V(G) \setminus D$  is adjacent to at least one vertex in  $D$ . A subset  $I$  of  $V(G)$  is *independent* if no two vertices belonging to  $I$  are adjacent in  $G$ . The cardinality of a largest (i.e., maximum) independent set of  $G$ , denoted by  $\alpha(G)$ , is called the *independence number* of  $G$ . Every largest independent set of a graph is called an  $\alpha$ -set of the graph. A dominating set  $D$  of a graph  $G$  is called an *independent dominating set* of  $G$  if  $D$  is also independent. The *independent domination number* of  $G$ , denoted by  $i(G)$ , is the cardinality of the smallest independent dominating set of  $G$  (or equivalently, the cardinality of a minimum maximal independent set of vertices in  $G$ ). The *common independence number* of a graph  $G$ , denoted by  $\alpha_c(G)$ , is the greatest integer  $r$  such that every vertex of  $G$  belongs to some independent subset  $X$  of  $V(G)$  with  $|X| \geq r$ . It follows immediately from the above definitions that the common independence number is bounded below by the independent domination number and above by the independence number. Formally, for any graph  $G$ ,

$$(1) \quad i(G) \leq \alpha_c(G) \leq \alpha(G).$$

The study of independent sets in graphs was begun by Berge [1,2] (see also [3]) and Ore [12]. We refer the reader to the book [11] and to the survey [8] of results on independent domination in graphs published in 2013 by Goddard and Henning. A graph  $G$  is said to be *well-covered* if  $i(G) = \alpha(G)$ . Equivalently,  $G$  is well-covered if every maximal independent set of  $G$  is a maximum independent set of  $G$ . The concept of well-covered graphs was introduced by Plummer [13] in 1970. Since then the well-covered graphs were very extensively investigated in many papers. We refer the reader to the excellent (but already older) survey on well-covered graphs by Plummer [14]. We are interested in characterization of  $\alpha$ -*excellent graphs*, that is, graphs  $G$  for which  $\alpha_c(G) = \alpha(G)$ . Thus, if  $G$  is an  $\alpha$ -excellent graph, then every vertex belongs to some  $\alpha$ -set of  $G$ . It follows from Inequalities (1) that every well-covered graph is an  $\alpha$ -excellent graph. The example of the cycle  $C_6$  shows that the set of well-covered graphs is properly contained in the set of  $\alpha$ -excellent graphs. The  $\alpha$ -excellent graphs have been studied in [4–7, 15, 16] and in [1, 2] as  $B$ -graphs. In this paper, we begin the study of  $\alpha$ -excellent  $k$ -trees.

## 2. PRELIMINARY RESULTS

A vertex  $v$  of a graph  $G$  is a *simplicial vertex* if every two vertices belonging to  $N_G(v)$  are adjacent in  $G$ . Equivalently, a simplicial vertex is a vertex that appears in exactly one clique of a graph, where a *clique* of a graph  $G$  is a maximal complete subgraph of  $G$ . A clique of a graph  $G$  containing at least one simplicial vertex of  $G$  is called a *simplex* of  $G$ . Note that if  $v$  is a simplicial vertex of  $G$ , then  $G[N_G[v]]$  is the unique simplex of  $G$  containing  $v$ . We begin with a simple proposition.

**Proposition 1.** *No  $\alpha$ -excellent graph contains a vertex belonging to at least two its simplexes.*

**Proof.** Assume that a vertex  $v$  of a graph  $G$  belongs to two simplexes of  $G$ , say to  $G[N_G[u]]$  and  $G[N_G[w]]$ . If  $I$  is a maximum independent set that contains  $v$ , then  $(I \setminus \{v\}) \cup \{u, w\}$  is an independent set of greater cardinality. Thus,  $\alpha(G) \geq |(I \setminus \{v\}) \cup \{u, w\}| > |I|$ , implying that  $v$  does not belong to any  $\alpha$ -set of  $G$ , and proving that  $G$  is not an  $\alpha$ -excellent graph. ■

For a positive integer  $k$ , a graph  $G$  is called a *k-tree* if it can be obtained from the complete graph  $K_k$  by a finite number of applications of the following operation: add a new vertex and join it to  $k$  mutually adjacent vertices of the existing graph. Certainly, every 1-tree is a tree and vice versa. In [17], Rose proved that a graph  $G$  is a  $k$ -tree if and only if the following conditions are fulfilled: (i)  $G$  is connected, (ii)  $G$  contains  $K_k$  as a subgraph and does not contain  $K_{k+2}$  as a subgraph, (iii) if  $v$  and  $u$  are nonadjacent vertices of  $G$ , then the subgraph induced by the smallest  $v-u$  separator is a complete graph on  $k$  vertices. Recall that a  $v-u$  separator in a connected graph  $G$  is a subset  $S$  of  $V(G)$  such that  $u$  and  $v$  are in distinct components of  $G[V(G) \setminus S]$ . Note that  $K_k$  and  $K_{k+1}$  are the only  $k$ -trees of order  $k$  and  $k+1$ , respectively.

It was proved in [4] that a bipartite graph (and, in particular, a tree) is an  $\alpha$ -excellent graph if and only if it has a perfect matching. On the other hand, it was observed in [18] that a  $k$ -tree  $G$  is a well-covered graph if and only if every vertex of  $G$  belongs to exactly one simplex of  $G$ . In this paper, we are interested in possible extensions of that characterization to a characterization of  $\alpha$ -excellent  $k$ -trees for every positive integer  $k$ . We begin with the following definition.

A set  $\mathcal{P}$  of complete subgraphs of a graph  $G$  is said to be a *perfect  $(k+1)$ -cover* of  $G$  if each subgraph belonging to  $\mathcal{P}$  is of order  $k+1$  and every vertex of  $G$  belongs to exactly one subgraph in  $\mathcal{P}$ . It is obvious that for  $k=1$  there exists a one-to-one correspondence between perfect 2-covers of a graph and perfect matchings of the graph. Here we are interested in the existence of perfect  $(k+1)$ -covers in  $k$ -trees. First of all, one can prove that every  $k$ -tree has at most one perfect  $(k+1)$ -cover. In the following proposition, we present the first relationship between  $k$ -trees having perfect  $(k+1)$ -covers and  $\alpha$ -excellent graphs.

**Proposition 2.** *If a  $k$ -tree  $G$  has a perfect  $(k+1)$ -cover, then  $G$  is an  $\alpha$ -excellent graph.*

**Proof.** Let  $G$  be a connected  $k$ -tree of order  $n \geq k+1$ . Then  $G$  is a  $(k+1)$ -partite graph, say  $A_1, A_2, \dots, A_{k+1}$  are partite sets of  $G$  and assume that  $|A_1| \geq |A_2| \geq \dots \geq |A_{k+1}| \geq 1$ . In addition, since  $A_1, A_2, \dots, A_{k+1}$  are independent sets of vertices and  $|A_1| + |A_2| + \dots + |A_{k+1}| = n$ , it follows that  $|A_1| \geq n/(k+1)$ , and, therefore,  $\alpha(G) \geq |A_1| \geq n/(k+1)$ . Let  $I$  be an  $\alpha$ -set of  $G$ . Assume now that  $\mathcal{P} = \{P_1, \dots, P_\ell\}$  is a perfect  $(k+1)$ -cover of  $G$ . Then,  $\ell = n/(k+1)$ ,  $|I \cap V(P_i)| \leq 1$  for each  $i \in [\ell]$ , and

$$\alpha(G) = |I| = \left| I \cap \bigcup_{i=1}^{\ell} V(P_i) \right| = \sum_{i=1}^{\ell} |I \cap V(P_i)| \leq \ell = \frac{n}{k+1}.$$

Consequently,  $|A_1| = |A_2| = \dots = |A_{k+1}| = n/(k+1) = \alpha(G)$ , and each of the sets  $A_1, A_2, \dots, A_{k+1}$  is an  $\alpha$ -set of  $G$ . This implies that every vertex of  $G$  belongs to an  $\alpha$ -set of  $G$  and, therefore,  $G$  is an  $\alpha$ -excellent graph. ■

### 3. $\alpha$ -EXCELLENT 2-TREES

Proposition 2 shows that a  $k$ -tree having a perfect  $(k+1)$ -cover is an  $\alpha$ -excellent  $k$ -tree. It is not clear to us whether the converse of this statement is true. That is, we do not know if every  $\alpha$ -excellent  $k$ -tree of order at least  $k+1$  has a perfect  $(k+1)$ -cover if  $k \geq 3$ . However, when  $k = 2$ , we provide in this paper a characterization of  $\alpha$ -excellent  $k$ -trees. For notational simplicity, in what follows if three vertices  $a, b$ , and  $c$  are mutually adjacent in a graph  $G$ , then the induced subgraph  $G[\{a, b, c\}]$  of  $G$  is isomorphic to  $K_3$  and is called a *triangle* in  $G$ , and we simply write  $abc$  rather than  $G[\{a, b, c\}]$ . To every triangle in a graph  $G$ , we assign label  $R$  or  $B$  (as red or blue, respectively), and by  $R(G)$  and  $B(G)$  we denote the set of all triangles in  $G$  that have label  $R$  and  $B$ , respectively. We also say that  $R(G)$  and  $B(G)$  are the sets of all “red” and “blue” triangles in  $G$ , respectively.

We are now in position to present a constructive characterization of  $\alpha$ -excellent 2-trees. For this purpose, let  $\mathcal{E}$  be the family of labeled 2-trees defined recursively as follows.

- (1) The family  $\mathcal{E}$  contains the 2-tree of order 3 in which the only triangle is red, that is, it has label  $R$ .
- (2) The family  $\mathcal{E}$  is closed under the operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$  defined below.

**Operation  $\mathcal{O}_1$ .** If a graph  $G'$  belongs to  $\mathcal{E}$  and  $v_1v_2$  is an edge of  $G'$ , then  $\mathcal{O}_1 = \mathcal{O}_1(v_1, v_2)$  forms a graph  $G$  by adding three new vertices  $u_1, u_2, u_3$  to

$G'$  in such a way that  $v_1v_2u_1$ ,  $v_2u_1u_2$  and  $u_1u_2u_3$  are three new triangles, while  $R(G) = R(G') \cup \{u_1u_2u_3\}$  and  $B(G) = B(G') \cup \{v_1v_2u_1, v_2u_1u_2\}$ . In this case we apply the operation  $\mathcal{O}_1$  to the edge  $v_1v_2$  of  $G'$ .

**Operation  $\mathcal{O}_2$ .** If a graph  $G'$  belongs to  $\mathcal{E}$ ,  $v_1v_2v_3$  is a red triangle in  $G'$  (that is,  $v_1v_2v_3 \in R(G')$ ), and  $v_4$  is a neighbor of  $v_3$  (it is possible that  $v_4 \in \{v_1, v_2\} \subseteq N_{G'}(v_3)$ ), then  $\mathcal{O}_2 = \mathcal{O}_2(v_1v_2, v_3v_4)$  forms a graph  $G$  by adding to  $G'$  three new vertices  $u_0$ ,  $u_1$  and  $u_2$  in such a way that  $u_0v_1v_2$ ,  $v_3v_4u_1$ , and  $v_3u_1u_2$  are new triangles, while  $R(G) = (R(G') \setminus \{v_1v_2v_3\}) \cup \{u_0v_1v_2, v_3u_1u_2\}$  and  $B(G) = B(G') \cup \{v_1v_2v_3, v_3v_4u_1\}$ . In this case we apply the operation  $\mathcal{O}_2$  to the edge  $v_1v_2$  of the triangle  $v_1v_2v_3$  and to the edge  $v_3v_4$ .

The operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are illustrated in Figure 1. Note that the operation  $\mathcal{O}_2$  changes “colors” of certain triangles, and the red triangle  $v_1v_2v_3$  in  $G'$  is recolored blue in  $G$ .

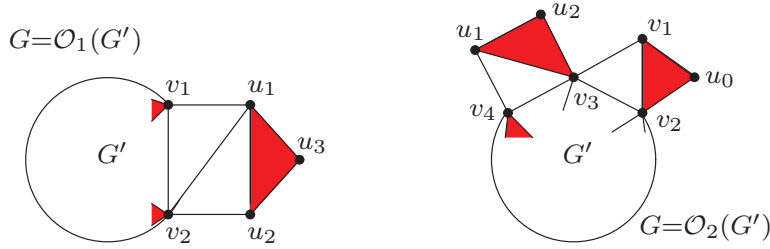


Figure 1. The operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

From the recursive definition of the graphs belonging to the family  $\mathcal{E}$  it follows readily that if a 2-tree  $G$  belongs to  $\mathcal{E}$ , then the set  $R(G)$  of red triangles in  $G$  is a perfect 3-cover of  $G$ . From this and from Proposition 2 it follows that  $G$  is an  $\alpha$ -excellent graph. Thus we have the following proposition that we will need when proving our main theorem.

**Proposition 3.** *Every 2-tree belonging to the family  $\mathcal{E}$  has a perfect 3-cover and it is an  $\alpha$ -excellent graph.*

The following theorem is the main result of this paper, and it presents two characterizations of the  $\alpha$ -excellent 2-trees: a constructive characterization, and a characterization in terms of perfect 3-covers.

**Theorem 4.** *If  $G$  is a 2-tree of order  $n \geq 3$ , then the following statements are equivalent.*

- (a)  $G$  has a perfect 3-cover.
- (b)  $G$  belongs to the family  $\mathcal{E}$ .

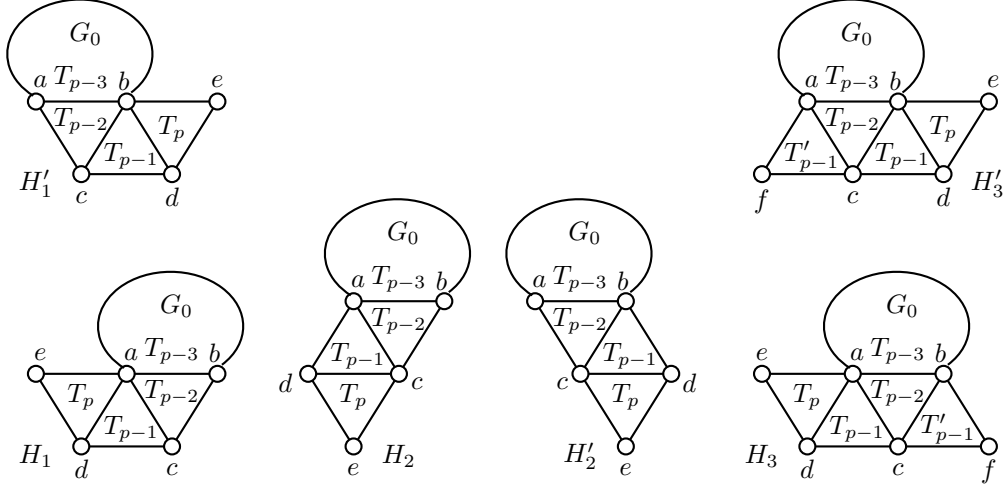
(c)  $G$  is an  $\alpha$ -excellent graph.

**Proof.** The implications (a)  $\Rightarrow$  (c), (b)  $\Rightarrow$  (a), and (b)  $\Rightarrow$  (c) are obvious from Propositions 2 and 3. Thus it suffices to prove the implication (c)  $\Rightarrow$  (b) (but we prove the implications (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (b) at the same time).

Thus assume that  $G$  is an  $\alpha$ -excellent 2-tree of order at least 3. By induction on the order of  $G$  we shall prove that  $G$  has a perfect 3-cover and that  $G$  belongs to the family  $\mathcal{E}$ . It is straightforward to observe that the implications (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (b) are true if  $G$  is a 2-tree of order  $n \leq 6$ . Now let  $G$  be an  $\alpha$ -excellent 2-tree of order greater than 6 and assume that the implications (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (b) are true for smaller  $\alpha$ -excellent 2-trees. Let  $(T_1, T_2, \dots, T_p)$  be a longest 3-path in  $G$ , that is, a longest sequence  $T_1, T_2, \dots, T_p$  of triangles in  $G$ , where  $|V(T_i) \cap V(T_j)| = 2$  if  $|i - j| = 1$ , and  $|V(T_i) \cap V(T_j)| \leq 1$  if  $|i - j| \geq 2$  ( $i, j \in [p]$ ). From the fact that  $(T_1, T_2, \dots, T_p)$  is a longest 3-path in  $G$  (which is an  $\alpha$ -excellent 2-tree of order at least 7) it follows that  $p \geq 4$ . Assume that  $a, b, c, d$ , and  $e$  are vertices of  $G$  for which  $V(T_{p-3}) \cap V(T_{p-2}) = \{a, b\}$ ,  $V(T_{p-2}) \setminus V(T_{p-3}) = \{c\}$ ,  $V(T_{p-1}) \setminus V(T_{p-2}) = \{d\}$ , and  $V(T_p) \setminus V(T_{p-1}) = \{e\}$ . From the choice of  $(T_1, \dots, T_p)$ , the vertex  $e$  is of degree 2 and  $T_p$  is a simplex in  $G$ . Let  $G'_0, G'_1, \dots, G'_\ell$  be the components of  $G - \{a, b\}$ , where  $G'_0$  is that component which contains at least one vertex of  $T_1$ . It is clear that  $\ell$  is positive integer.

We now let  $G_i$  denote the subgraph of  $G$  induced by  $V(G'_i) \cup \{a, b\}$  for  $i \in \{0\} \cup [\ell]$ . Among the graphs  $G_1, \dots, G_\ell$ , let  $H$  be the graph that contains the triangles of the 3-path  $\mathcal{P}_0 = (T_{p-2}, T_{p-1}^0, T_p^0)$ , where  $T_{p-1}^0 = T_{p-1}$  and  $T_p^0 = T_p$ . It is obvious that if  $T_{p-2}$ ,  $T_{p-1}$ , and  $T_p$  are the only triangles of  $H$ , then it is possible that  $H$  is one of the graphs  $H_1$ ,  $H'_1$ ,  $H_2$ , and  $H'_2$  shown in Figure 2. If  $H$  contains the triangle  $T_{p-2}$ ,  $T_{p-1}$ ,  $T_p$  and a simplex  $T_{p-1}'$  that shares an edge with the triangle  $T_{p-2}$  in  $H$ , then it follows readily from Proposition 1 that  $H$  is one of the graphs  $H_3$  or  $H'_3$  in Figure 2.

Thus assume that no simplex of  $H$  shares an edge with the triangle  $T_{p-2}$ . In this case,  $H$  is a subgraph induced by the triangles belonging to  $\mathcal{P}_0$  and to some additional 3-paths  $\mathcal{P}_i = (T_{p-2}, T_{p-1}^i, T_p^i)$ , where  $i \in [n]$  and  $n$  is a positive integer. It follows from Proposition 1 that if  $n = 1$ , then  $H$  is isomorphic to one of the graphs  $H_4$ ,  $H_5$  or  $H_6$  in Figure 3. Similarly, if  $n = 2$ , then, as can easily be verified,  $H$  is isomorphic to the graph  $H_7$  shown in Figure 3. Finally, we claim that the case  $n \geq 3$  is impossible. Suppose, to the contrary, that  $n \geq 3$ . Then let us first observe that if one of the edges  $ac$  and  $bc$  of the triangle  $T_{p-2}$  belongs to at least 3 of the triangles  $T_{p-1}^0 = T_{p-1}$ ,  $T_{p-1}^1, \dots, T_{p-1}^n$ , say to  $T_{p-1}^i, T_{p-1}^j, T_{p-1}^k$  (where  $0 \leq i < j < k \leq n$ ), then at least two of the simplexes  $T_p^i, T_p^j, T_p^k$  of  $H$  (and of  $G$ ) have a common vertex, which is impossible in an  $\alpha$ -excellent graph  $G$ . Thus assume that neither  $ac$  nor  $bc$  belongs to three of the triangles  $T_{p-1}^0 = T_{p-1}$ ,

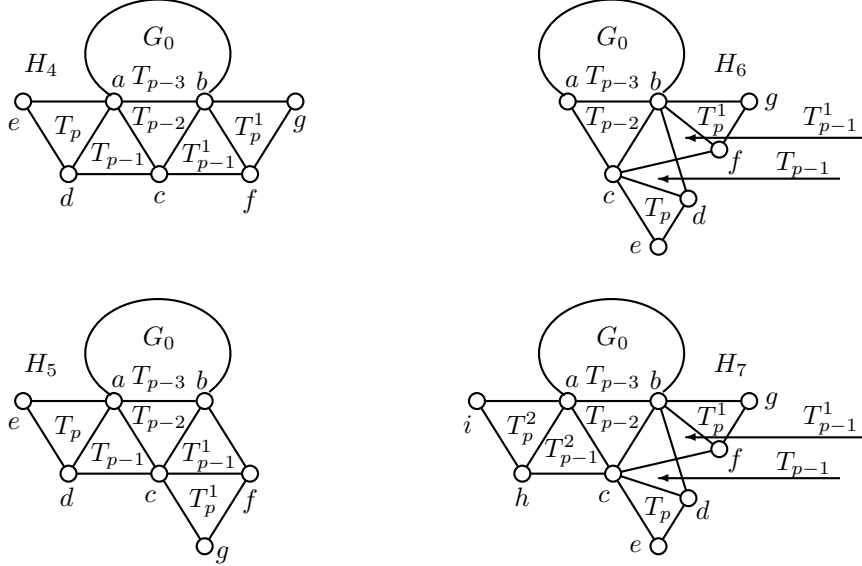
Figure 2. Graphs  $H_1$ ,  $H_1'$ ,  $H_2$ ,  $H_2'$ ,  $H_3$ , and  $H_3'$ .

$T_{p-1}^1, \dots, T_{p-1}^n$ . Then necessarily  $n = 3$  and each of the edges  $ac$  and  $bc$  belongs to exactly two of the triangles  $T_{p-1}^0, T_{p-1}^1, T_{p-1}^2, T_{p-1}^3$ , say  $ac$  belongs to  $T_{p-1}^0$  and  $T_{p-1}^1$ , while  $bc$  belongs to  $T_{p-1}^2$  and  $T_{p-1}^3$ . Therefore, as it is easy to verify, if neither the simplexes  $T_p^0$  and  $T_p^1$  have a common vertex nor the simplexes  $T_p^2$  and  $T_p^3$  have a common vertex, then one of the simplexes  $T_p^0$  and  $T_p^1$  has a common vertex with one of the simplexes  $T_p^2$  and  $T_p^3$ , which again by Proposition 1 is impossible as  $G$  is an  $\alpha$ -excellent graph. This proves that the case  $n \geq 3$  is impossible.

There are now several cases to consider depending on the structure of  $H$ . We begin showing that  $H$  cannot be any of the graphs  $H_1, H_1', H_4$ . Suppose, to the contrary, that  $H = H_1$  (where  $H_1$  is as illustrated in Figure 2). Then, since  $G$  is an  $\alpha$ -excellent graph, there exists an  $\alpha$ -set  $I$  in  $G$  that contains  $a$ . However in this case,  $(I \setminus \{a\}) \cup \{c, e\}$  is an independent set in  $G$ , and so  $\alpha(G) \geq |(I \setminus \{a\}) \cup \{c, e\}| > |I| = \alpha(G)$ , a contradiction which proves that  $H \neq H_1$ . Analogously,  $H \neq H_1'$ . Similarly, suppose that  $H = H_4$  (where  $H_4$  is shown in Figure 3). Let  $I_d$  be an  $\alpha$ -set that contains  $d$  in  $G$ . We note that  $|I_d \cap \{b, f, g\}| = 1$ . If  $b \in I_d$ , then let  $I_d' = (I_d \setminus \{b, d\}) \cup \{c, e, g\}$ . If  $f \in I_d$ , then let  $I_d' = (I_d \setminus \{d, f\}) \cup \{c, e, g\}$ . If  $g \in I_d$ , then let  $I_d' = (I_d \setminus \{d\}) \cup \{c, e\}$ . In all cases, the resulting set  $I_d'$  is an independent set in  $G$ , and so  $\alpha(G) \geq |I_d'| > |I_d| = \alpha(G)$ , a contradiction. Hence,  $H \neq H_4$ .

In each of the next five cases (corresponding to the possible graphs  $H$ , that is, to the graphs  $H_2$  (and  $H_2'$ ),  $H_3$ ,  $H_5$ ,  $H_6$ , and  $H_7$ ) we prove that  $G$  belongs to the family  $\mathcal{E}$  and has a perfect 3-cover.

*Case 1.*  $H = H_2$  or  $H = H_2'$ . Without loss of generality, assume that

Figure 3. Graphs  $H_4$ ,  $H_5$ ,  $H_6$ , and  $H_7$ .

$H = H_2$  (see Figure 2). In this case, let  $G'$  denote the subgraph  $G - \{c, d, e\}$  of  $G$ . In the following three claims we explain the main relations between properties of the graphs  $G$  and  $G'$ , that is, we prove that: (1)  $\alpha(G') = \alpha(G) - 1$ ; (2)  $G'$  is an  $\alpha$ -excellent graph; (3)  $G$  has a perfect 3-cover and belongs to the family  $\mathcal{E}$ .

**Claim 1.1.**  $\alpha(G') = \alpha(G) - 1$ .

If  $I'$  is an  $\alpha$ -set of  $G'$ , then  $I' \cup \{e\}$  is an independent set in  $G$  (as  $N_G[e] \cap I' = \{c, d, e\} \cap I' = \emptyset$ ) and therefore  $\alpha(G) \geq |I' \cup \{e\}| = \alpha(G') + 1$ . On the other hand, let  $I$  be an  $\alpha$ -set of  $G$ . In this case,  $|I \cap \{c, d, e\}| = 1$  and  $I \setminus \{c, d, e\}$  is independent in  $G'$ . Thus,  $\alpha(G') \geq |I \setminus \{c, d, e\}| = |I| - 1 = \alpha(G) - 1$ . Consequently,  $\alpha(G') = \alpha(G) - 1$ .

**Claim 1.2.**  $G'$  is an  $\alpha$ -excellent graph.

Let  $v$  be an arbitrary vertex of  $G'$ . We show that  $v$  belongs to an  $\alpha$ -set of  $G'$ . Since  $\alpha(G') = \alpha(G) - 1$  (by Claim 1.1), it suffices to show that  $v$  belongs to an independent set of cardinality  $\alpha(G) - 1 = \alpha(G')$  in  $G'$ . Let  $I_v$  be an  $\alpha$ -set of  $G$  that contains  $v$ , and let  $I'_v$  denote the set  $I_v \setminus \{c, d, e\}$ . Thus,  $v \in I'_v$  and the set  $I'_v$  is independent in  $G'$ . Furthermore,  $I'_v$  is an  $\alpha$ -set of  $G'$  noting that  $|I_v \cap \{c, d, e\}| = 1$  and  $|I'_v| = |I_v \setminus \{c, d, e\}| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$ . This proves that  $G'$  is an  $\alpha$ -excellent graph.

**Claim 1.3.**  $G$  has a perfect 3-cover and  $G$  belongs to the family  $\mathcal{E}$ .

By Claim 1.2,  $G'$  is an  $\alpha$ -excellent graph. Since the order of  $G'$  is less than



the order of  $G$ , applying the induction hypothesis we infer that  $G'$  has a perfect 3-cover and  $G'$  belongs to the family  $\mathcal{E}$ . Now, if  $\mathcal{P}'$  is a perfect 3-cover of  $G'$ , then  $\mathcal{P}' \cup \{cde\}$  is a perfect 3-cover of  $G$ . In addition, since  $G'$  belongs to the family  $\mathcal{E}$ , the graph  $G'$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and, since  $G$  can be obtained from  $G'$  by the operation  $\mathcal{O}_1 = \mathcal{O}_1(a, b)$  (that is, by  $\mathcal{O}_1$  applied to the edge  $ab$  of  $G'$ ), the graph  $G$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Thus,  $G$  belongs to the family  $\mathcal{E}$ .

*Case 2.*  $H = H_3$  or  $H = H_3'$ . Without loss of generality, assume that  $H = H_3$  (see Figure 2). This time, let  $G'$  denote the subgraph  $G - \{d, e, f\}$  of  $G$ . As in Case 1, we study desired relations between properties of the graphs  $G$  and  $G'$ .

**Claim 2.1.**  $\alpha(G') = \alpha(G) - 1$ .

Let  $I'$  be an  $\alpha$ -set of  $G'$ . In this case,  $|I' \cap \{a, b, c\}| = 1$  and either  $a \in I'$  or  $\{b, c\} \cap I' \neq \emptyset$ . Consequently, either  $I' \cup \{f\}$  or  $I' \cup \{e\}$  is an independent set in  $G$ , respectively, and therefore  $\alpha(G) \geq |I' \cup \{f\}| = |I' \cup \{e\}| = \alpha(G') + 1$ . This proves that  $\alpha(G) \geq \alpha(G') + 1$ . Now, let  $I$  be an  $\alpha$ -set of  $G$ . Then  $|I \cap \{a, b, c\}| \leq 1$  and we consider four cases. If  $I \cap \{a, b, c\} = \emptyset$ , then  $|I \cap \{d, e, f\}| = 2$  and  $(I \setminus \{d, e, f\}) \cup \{c\}$  is an independent set of cardinality  $\alpha(G) - 1$  in  $G'$ . If  $a \in I$ , then  $f \in I$ , and  $I \setminus \{f\}$  is an independent set of cardinality  $\alpha(G) - 1$  in  $G'$ . If  $b \in I$ , then  $|I \cap \{d, e\}| = 1$ ,  $f \notin I$ , and  $I \setminus \{d, e\}$  is an independent set of cardinality  $\alpha(G) - 1$  in  $G'$ . If  $c \in I$ , then  $e \in I$ ,  $f \notin I$ , and  $I \setminus \{e\}$  is an independent set of cardinality  $\alpha(G) - 1$  in  $G'$ . Consequently,  $\alpha(G') \geq \alpha(G) - 1$ . This proves that  $\alpha(G') = \alpha(G) - 1$ .

**Claim 2.2.**  $G'$  is an  $\alpha$ -excellent graph.

Let  $v$  be an arbitrary vertex of  $G'$ . As in the proof of Claim 1.2, it suffices to show that  $v$  belongs to an  $\alpha$ -set of  $G'$ , that is, to an independent set of cardinality  $\alpha(G') = \alpha(G) - 1$  in  $G'$ . Since  $G$  is  $\alpha$ -excellent, every vertex of  $G$  belongs to some  $\alpha$ -set in  $G$ . Let  $I_x$  be an  $\alpha$ -set of  $G$  that contains  $x$ , where  $x \in V(G)$ . We note that  $I_a \cap \{d, e, f\} = \{f\}$  and, consequently,  $I_a \setminus \{f\}$  is an  $\alpha$ -set of  $G'$  that contains the vertex  $a$ . Similarly, from the fact that  $|I_b \cap \{d, e\}| = 1$  and  $I_c \cap \{d, e, f\} = \{e\}$  it follows that  $I_b \setminus \{d, e\}$  and  $I_c \setminus \{e\}$  are  $\alpha$ -sets of  $G'$  and they contain  $b$  and  $c$ , respectively. If  $v \in V(G') \setminus \{a, b, c\}$  and  $I_v \cap \{a, b, c\} = \emptyset$ , then  $|I_v \cap \{d, e, f\}| = 2$  and  $(I_v \setminus \{d, e, f\}) \cup \{c\}$  is an  $\alpha$ -set of  $G'$  and it contains  $v$ . Finally, if  $v \in V(G') \setminus \{a, b, c\}$  and  $I_v \cap \{a, b, c\} \neq \emptyset$ , then  $|I_v \cap \{a, b, c\}| = 1$ ,  $|I_v \cap \{d, e, f\}| = 1$ , and, therefore,  $I_v \setminus \{d, e, f\}$  is an  $\alpha$ -set of  $G'$  and it contains  $v$ . Consequently,  $G'$  is an  $\alpha$ -excellent graph.

**Claim 2.3.**  $G$  has a perfect 3-cover and  $G$  belongs to the family  $\mathcal{E}$ .

Now, similarly as in the proof of Claim 1.3, since  $G'$  is an  $\alpha$ -excellent graph of order less than the order of  $G$ , the induction hypothesis implies that  $G'$  has a

perfect 3-cover and that  $G'$  belongs to the family  $\mathcal{E}$ . Certainly, if  $\mathcal{P}'$  is a perfect 3-cover of  $G'$ , then  $abc \in \mathcal{P}'$  and  $(\mathcal{P}' \setminus \{abc\}) \cup \{ade, bcf\}$  is a perfect 3-cover of  $G$ . In addition, since  $G'$  belongs to the family  $\mathcal{E}$ , the graph  $G'$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . From this and from the obvious fact that  $G$  can be obtained from  $G'$  by the operation  $\mathcal{O}_2 = \mathcal{O}_2(bc, ac)$  (that is, by  $\mathcal{O}_2$  applied to the edge  $bc$  of the simplex  $abc$  of  $G'$  and to the edge  $ac$  incident with the third vertex  $c$  of the triangle  $abc$ ), the graph  $G$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Thus,  $G$  belongs to the family  $\mathcal{E}$ .

*Case 3.*  $H = H_5$ . In this case, we let  $G'$  denote the subgraph  $G - \{e, f, g\}$  of  $G$  and, as before, we study desired relations between properties of the graphs  $G$  and  $G'$ .

**Claim 3.1.**  $\alpha(G') = \alpha(G) - 1$ .

Let  $I'$  be an  $\alpha$ -set of  $G'$ . In this case,  $|I' \cap \{a, c, d\}| = 1$ . If  $a \in I'$  or  $d \in I'$ , then the set  $I' \cup \{g\}$  is an independent set of  $G$ , while if  $c \in I'$ , then the set  $I' \cup \{e\}$  is an independent set of  $G$ , implying that  $\alpha(G) \geq \alpha(G') + 1$ . It remains to prove that  $\alpha(G') \geq \alpha(G) - 1$ . To prove this, let  $I$  be an  $\alpha$ -set of  $G$ . Thus,  $|I \cap \{a, d, e\}| = 1$  and  $|I \cap \{c, f, g\}| = 1$ . If  $a \in I$  or  $d \in I$ , then  $|I \cap \{f, g\}| = 1$  and we let  $I' = I \setminus \{f, g\}$ . If  $e \in I$  and  $c \in I$ , then we let  $I' = I \setminus \{e\}$ . If  $e \in I$  and  $b \in I$ , then  $g \in I$  and we let  $I' = (I \setminus \{e, g\}) \cup \{d\}$ . If  $e \in I$  and  $I \cap \{b, c\} = \emptyset$ , then  $|I \cap \{f, g\}| = 1$  and we let  $I' = (I \setminus \{e, f, g\}) \cup \{c\}$ . In all the above cases, the set  $I'$  is an independent set in  $G'$  and  $|I'| = |I| - 1$ , implying that  $\alpha(G') \geq |I'| = |I| - 1 = \alpha(G) - 1$ . As observed earlier,  $\alpha(G') \leq \alpha(G) - 1$ . Consequently,  $\alpha(G') = \alpha(G) - 1$ .

**Claim 3.2.**  $G'$  is an  $\alpha$ -excellent graph.

Since  $G$  is an  $\alpha$ -excellent graph, every vertex of  $G$  belongs to an  $\alpha$ -set of  $G$ . Let  $I_x$  denote an  $\alpha$ -set of  $G$  that contains the vertex  $x$  of  $G$ . Let  $v$  be an arbitrary vertex of  $G'$ . We show that  $v$  belongs to an  $\alpha$ -set of  $G'$ , that is, to an independent set of cardinality  $\alpha(G') = \alpha(G) - 1$  in  $G'$ . We note that  $|I_v \cap \{a, d, e\}| = 1$ ,  $|I_v \cap \{c, f, g\}| = 1$ , and  $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$ . If  $|I_v \cap \{a, b, c, d\}| = 2$ , then  $I_v \cap \{a, b, c, d\} = \{b, d\}$ ,  $g \in I_v$  and we let  $I'_v = I_v \setminus \{g\}$ . If  $I_v \cap \{a, b, c, d\} = \emptyset$ , then  $e \in I_v$  and  $|I_v \cap \{f, g\}| = 1$  and we let  $I'_v = (I_v \setminus \{e, f, g\}) \cup \{c\}$ . If  $I_v \cap \{a, b, c, d\} = \{a\}$  or  $I_v \cap \{a, b, c, d\} = \{d\}$ , then  $|I_v \cap \{f, g\}| = 1$  and we let  $I'_v = I_v \setminus \{f, g\}$ . If  $I_v \cap \{a, b, c, d\} = \{b\}$ , then  $e \in I_v$ ,  $g \in I_v$ , and we let  $I'_v = (I_v \setminus \{e, g\}) \cup \{d\}$ . If  $I_v \cap \{a, b, c, d\} = \{c\}$ , then  $e \in I_v$  and we let  $I'_v = I_v \setminus \{e\}$ . In all cases, the set  $I'_v$  is an independent set in  $G'$  that contains the vertex  $v$  and  $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$ , implying that  $v$  belongs to an  $\alpha$ -set of  $G'$ , as desired.

**Claim 3.3.**  $G$  has a perfect 3-cover and  $G$  belongs to the family  $\mathcal{E}$ .

Similarly as in the proofs of Claims 1.3 and 2.3, since  $G'$  is an  $\alpha$ -excellent graph of order less than the order of  $G$ , the induction hypothesis implies that  $G'$  has a perfect 3-cover and that  $G'$  belongs to the family  $\mathcal{E}$ . Now, if  $\mathcal{P}'$  is a perfect 3-cover of  $G'$ , then  $acd \in \mathcal{P}'$  and  $(\mathcal{P}' \setminus \{acd\}) \cup \{ade, cfg\}$  is a perfect 3-cover of  $G$ . In addition, since  $G'$  belongs to the family  $\mathcal{E}$ , the graph  $G'$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and, since  $G$  can be obtained from  $G'$  by the operation  $\mathcal{O}_2 = \mathcal{O}_2(ad, cb)$  (that is, by  $\mathcal{O}_2$  applied to the edge  $ad$  of the simplex  $acd$  of  $G'$  and to the edge  $cb$  incident with the third vertex  $c$  of the triangle  $acd$ ), the graph  $G$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Thus,  $G$  belongs to the family  $\mathcal{E}$ .

*Case 4.*  $H = H_6$ . In this case, we let  $G'$  denote the subgraph  $G - \{e, f, g\}$  of  $G$  and, as before, we study desired relations between properties of the graphs  $G$  and  $G'$ .

**Claim 4.1.**  $\alpha(G') = \alpha(G) - 1$ .

Let  $I'$  be an  $\alpha$ -set of  $G'$ . In this case,  $|I' \cap \{b, c, d\}| = 1$ . If  $c \in I'$  or  $d \in I'$ , then the set  $I' \cup \{g\}$  is an independent set of  $G$ , while if  $b \in I'$ , then the set  $I' \cup \{e\}$  is an independent set of  $G$ , implying that  $\alpha(G) \geq \alpha(G') + 1$ . It remains to prove that  $\alpha(G') \geq \alpha(G) - 1$ . By supposition,  $G$  is an  $\alpha$ -excellent graph, and so every vertex of  $G$  belongs to some  $\alpha$ -set of  $G$ . Let  $I$  be an  $\alpha$ -set of  $G$  that contains the vertex  $c$ . Necessarily,  $g \in I$  and  $\{b, e, f\} \cap I = \emptyset$ . Thus,  $I \setminus \{g\}$  is an independent set in  $G'$ , implying that  $\alpha(G') \geq |I| - 1 = \alpha(G) - 1$ . As observed earlier,  $\alpha(G') \leq \alpha(G) - 1$ . Consequently,  $\alpha(G') = \alpha(G) - 1$ .

**Claim 4.2.**  $G'$  is an  $\alpha$ -excellent graph.

Since  $G$  is an  $\alpha$ -excellent graph, every vertex of  $G$  belongs to an  $\alpha$ -set of  $G$ . Let  $I_x$  denote an  $\alpha$ -set of  $G$  that contains the vertex  $x$  of  $G$ . Let  $v$  be an arbitrary vertex of  $G'$ . We show that  $v$  belongs to an  $\alpha$ -set of  $G'$ , that is, to an independent set of cardinality  $\alpha(G') = \alpha(G) - 1$  in  $G'$ . This time we note that  $|I_v \cap \{c, d, e\}| = 1$ ,  $|I_v \cap \{b, f, g\}| = 1$ , and  $|I_v \cap \{a, b, c, d\}| \in \{0, 1, 2\}$ . If  $|I_v \cap \{a, b, c, d\}| = 2$  (that is, if  $I_v \cap \{a, b, c, d\} = \{a, d\}$  or  $I_v \cap \{a, b, c, d\} = \{d\}$ ), then  $|I_v \cap \{f, g\}| = 1$  and we let  $I'_v = I_v \setminus \{f, g\}$ . If  $I_v \cap \{a, b, c, d\} = \emptyset$  or  $I_v \cap \{a, b, c, d\} = \{a\}$ , then  $e \in I_v$ ,  $|I_v \cap \{f, g\}| = 1$  and we let  $I'_v = (I_v \setminus \{e, f, g\}) \cup \{d\}$ . If  $I_v \cap \{a, b, c, d\} = \{b\}$ , then  $e \in I_v$  and we let  $I'_v = I_v \setminus \{e\}$ . If  $I_v \cap \{a, b, c, d\} = \{c\}$ , then  $g \in I_v$  and we let  $I'_v = I_v \setminus \{g\}$ . In all cases, the set  $I'_v$  is an independent set in  $G'$  that contains the vertex  $v$  and  $|I'_v| = |I_v| - 1 = \alpha(G) - 1 = \alpha(G')$ , implying that  $v$  belongs to an  $\alpha$ -set of  $G'$ , as desired.

**Claim 4.3.**  $G$  has a perfect 3-cover and  $G$  belongs to the family  $\mathcal{E}$ .

Similarly as in the proofs of Claims 1.3, 2.3 and 3.3, since  $G'$  is an  $\alpha$ -excellent graph of order less than the order of  $G$ , the induction hypothesis implies that  $G'$

has a perfect 3-cover and that  $G'$  belongs to the family  $\mathcal{E}$ . Now, if  $\mathcal{P}'$  is a perfect 3-cover of  $G'$ , then  $bcd \in \mathcal{P}'$  and  $(\mathcal{P}' \setminus \{bcd\}) \cup \{cde, bfg\}$  is a perfect 3-cover of  $G$ . In addition, since  $G'$  belongs to the family  $\mathcal{E}$ , the graph  $G'$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and, since  $G$  can be obtained from  $G'$  by the operation  $\mathcal{O}_2 = \mathcal{O}_2(cd, bc)$  (that is, by  $\mathcal{O}_2$  applied to the edge  $cd$  of the simplex  $bcd$  of  $G'$  and to the edge  $bc$  incident with the third vertex  $b$  of the triangle  $bcd$ ), the graph  $G$  can be obtained recursively from a red triangle by operations  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Thus,  $G$  belongs to the family  $\mathcal{E}$ .

*Case 5.*  $H = H_7$ . In this case, we let  $G'$  denote the subgraph  $G - \{e, f, g\}$  of  $G$ . Using identical arguments as in the proof of Case 4, we prove that  $G$  has a perfect 3-cover and it belongs to the family  $\mathcal{E}$ . We omit the details.

Thus all possible cases have been considered. This completes the proof of Theorem 4.  $\blacksquare$

It is easy to observe that the corona graph  $H \circ K_1$  (the graph formed from  $H$  by adding for each vertex  $v$  in  $H$  a new vertex  $v'$  and the edge  $vv'$ ) is an  $\alpha$ -excellent graph for every graph  $H$ . This implies that every graph is an induced subgraph of an  $\alpha$ -excellent graph. As a consequence of Theorem 4, we have the following property of 2-trees.

**Corollary 5.** *Every 2-tree is an induced subgraph of an  $\alpha$ -excellent 2-tree.*

**Proof.** Let  $G$  be a 2-tree. The statement is obvious if  $G$  has order 2. Thus assume that  $G$  is a 2-tree of order at least 3. Let  $\mathcal{P}$  be a maximal family of vertex-disjoint triangles of  $G$ , and let  $Q$  be the set of all vertices in  $G$  that do not belong to any triangle in  $\mathcal{P}$ . If the set  $Q$  is empty, then, by Theorem 4,  $G$  is an  $\alpha$ -excellent 2-tree itself (and, therefore, it has the desired property). Thus assume that  $Q$  is nonempty. For each vertex  $v \in Q$ , we do the following. Let  $v'$  be an arbitrary neighbor of  $v$  in  $G - Q$ . We now add two new vertices  $x_{vv'}$  and  $y_{vv'}$ , and four edges  $x_{vv'}v$ ,  $x_{vv'}v'$ ,  $y_{vv'}v$  and  $y_{vv'}x_{vv'}$ . We note that the newly constructed triangle  $vx_{vv'}y_{vv'}$  covers the vertex  $v$  and the two new vertices  $x_{vv'}$  and  $y_{vv'}$ . Let  $G'$  be a resulting 2-tree obtained from  $G$  by performing this operation for all vertices  $v \in Q$ . Then the set  $\mathcal{P} \cup \bigcup_{v \in Q} \{vx_{vv'}y_{vv'}\}$  is a perfect 3-cover of  $G'$ . Thus, by Theorem 4, the graph  $G'$  is an  $\alpha$ -excellent 2-tree. By construction, the 2-tree  $G$  is an induced subgraph of the  $\alpha$ -excellent 2-tree  $G'$ . This completes the proof of Corollary 5.  $\blacksquare$

We close this paper with the following open question that we have yet to settle. Is it true that if  $k \geq 3$  is an integer and  $G$  is an  $\alpha$ -excellent  $k$ -tree of order at least  $k + 1$ , then  $G$  has a perfect  $(k + 1)$ -cover?

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