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ADJACENT VERTEX STRONGLY DISTINGUISHING TOTAL COLORING OF GRAPHS WITH LOWER AVERAGE DEGREE

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Abstract

An *adjacent vertex strongly distinguishing total-coloring* of a graph G is a proper *total-coloring* such that no two adjacent vertices meet the same color set, where the color set of a vertex consists of all colors assigned on the vertex and its incident edges and neighbors. The minimum number of the colors required is called *adjacent vertex strongly distinguishing total chromatic number*, denoted by $\chi_{ast}(G)$. Let mad(G) and $\Delta(G)$ denote the maximum average degree and the maximum degree of graph G, respectively. In this paper, we prove the following results. (1) If G is a graph with mad$(G)< \frac{7}{3}$ and $\Delta(G) \geq 5$, then $\chi_{ast}(G) \leq \max\{\Delta(G)+2, 8\}$. (2) If G is a graph with mad$(G)< \frac{9}{4}$ and $4 \leq \Delta(G) \leq 5$, then $\chi_{ast}(G) \leq \Delta(G)+2$. (3) If G is a graph with mad$(G)< \frac{9}{4}$ and $\Delta(G) = 3$, then $\chi_{ast}(G) \leq 6$.

Keywords: adjacent vertex strongly distinguishing total-coloring, adjacent vertex strongly distinguishing total chromatic number, maximum average degree, discharging method.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. We use $d_G(v)$ to denote the degree of vertex v in G. Call $v \in V(G)$ a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex* if its degree $d_G(v)$ is equal to k, or at least k, or at most k, respectively. A 1-vertex is also said to be a *leaf.* Denote by $\Delta(G) = \max\{d(x)|x \in V(G)\}$ the maximum degree of G and $\delta(G) = \min\{d(x)|x \in V(G)\}$ the minimum degree of G. We define the *girth* $g(G)$ of a graph G to be the length of a shortest cycle in G. The *maximum average degree* $\mathrm{mad}(G)$ of G is defined by $\mathrm{mad}(G) = \max_{H\subseteq G}\left\{\frac{2|E(H)|}{|V(H)|}\right\}$. The terminologies and notations used but undefined in this paper can be found in [1].

An *adjacent vertex distinguishing edge-coloring* of a graph is a proper edge coloring such that no two adjacent vertices have the same color set, where the color set of each vertex consists of colors assigned on its incident edges. Let f be a *proper total-coloring* of G. Then f is called an *adjacent vertex distinguishing total-coloring* of G if $C_f(u) \neq C_f(v)$ for any $uv \in E(G)$, where $C_f(u) = \{f(u)\} \cup \{f(uv)|uv \in E(G)\}$. The minimum number of colors for the coloring required is called an *adjacent vertex distinguishing total chromatic number*, and denoted by $\chi_{at}(G)$ for short. The adjacent vertex distinguishing total chromatic number of paths, cycles, trees, complete graphs and complete bipartite graphs was characterized completely by Zhang *et al.* [15]. And they proposed the following conjecture.

Conjecture 1 [15]**.** *If G is a graph with at least two vertices, then $\chi_{at}(G) \leq \Delta(G) + 3$.*

Since then, researchers have conducted a lot of research on Conjecture 1 [2–9, 12–14]. Most notably, Wang and Wang [10] showed that if G is a graph with $\mathrm{mad}(G) < 3$, then $\chi_{at}(G) \leq \max\{\Delta(G)+2, 6\}$; and if G is a graph with $\mathrm{mad}(G) < \frac{8}{3}$ and $\Delta(G) \leq 3$, then $\chi_{at}(G) \leq 5$. After that, they also confirmed the adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree [11]. In this paper, motivated by the above two articles of Wang [10, 11], we mainly consider the adjacent vertex strongly distinguishing total coloring of graphs with lower average degree. Now, we first introduce the concept of adjacent vertex strongly distinguishing total-coloring of graphs, which was proposed by Zhang *et al.* [16] in 2008.

Definition [16]**.** Let $G = (V(G), E(G))$ be a simple connected graph with $|V(G)| \geq 3$, and k be a positive integer. If f is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \dots, k\}$ such that

(1) for any $uv \in E(G)$, $f(u) \neq f(v), f(u) \neq f(uv), f(v) \neq f(uv)$;

(2) for any adjacent edges $uv, uw \in E(G)$ $(v \neq w)$, $f(uv) \neq f(uw)$;

(3) for any edge $uv \in E(G)$, $C_f\langle u\rangle \neq C_f\langle v\rangle$

where $C_f\langle u\rangle = \{f(u)\} \cup \{f(uv), f(v) | uv \in E(G)\}$. Then f is called *adjacent vertex strongly distinguishing total-coloring* of G, denoted by k-AVSDTC of G for short, and

$$\chi_{ast}(G) = \min\{k \mid G \text{ has a } k\text{-AVSDTC}\}$$

is called the *adjacent vertex strongly distinguishing total chromatic number* of G.

It is worth noting that adjacent vertex strongly distinguishing total-coloring of a graph is more complex than its adjacent vertex distinguishing total-coloring of graphs since the former one whose color set consists of the colors not only assigned on the vertex and its incident edges, but also on its neighbors. In addition, if G is a disconnected graph, then G is not allowed to have isolated edges because the color sets of the endpoints of each isolated edge are always identical under f, and thus, for any graph G we always consider that G contains no isolated edges in what follows. Moreover, let f be a proper total-coloring of G. Suppose that any two adjacent vertices $u, v \in V(G)$ satisfy $C_f\langle u\rangle = C_f\langle v\rangle$, then we say that u and v are *indistinguishable*.

Zhang *et al.* in [16] first investigated the adjacent vertex strongly distinguishing total-coloring of graphs by determining the adjacent vertex distinguishing total chromatic number for cycles, paths, complete graphs and complete bipartite graphs. Based on these results, they also proposed the following two conjectures.

Conjecture 2 [16]**.** *Let G be a graph on $n\,(n \geq 3)$ vertices. Then $\chi_{ast}(G) \leq n + \lceil \log_2 n \rceil + 1$, and the equality holds if $n = 2^k - 2$.*

Conjecture 3 [16]**.** *Let G be a planar graph with maximum degree $\Delta(G)$. Then $\chi_{ast}(G) \leq \Delta(G) + 3$.*

From [7] we present a proposition as follows.

Proposition 4 [7]**.** *Let G be a planar graph. Then*

$$\mathrm{mad}(G) < \frac{2g(G)}{g(G) - 2}\,.$$

In this paper, we prove the following results.

Theorem 5. *Let G be a graph with maximum degree $\Delta(G)$.*

(1) *If* $\mathrm{mad}(G) < \frac{7}{3}$ *and* $\Delta(G) \geq 5$*, then* $\chi_{ast}(G) \leq \max\{\Delta(G) + 2, 8\}$.

(2) *If* $\mathrm{mad}(G) < \frac{9}{4}$ *and* $4 \leq \Delta(G) \leq 5$*, then* $\chi_{ast}(G) \leq \Delta(G) + 2$.

(3) *If* $\mathrm{mad}(G) < \frac{9}{4}$ *and* $\Delta(G) = 3$*, then* $\chi_{ast}(G) \leq 6$.

Combining with Proposition 4 one can deduce the following corollary.

Corollary 6. *Let G be a planar graph.*

(1) *If $g(G) \geq 14$ and $\Delta(G) \geq 5$, then $\chi_{ast}(G) \leq \max\{\Delta(G)+2, 8\}$.*

(2) *If $g(G) \geq 18$ and $4 \leq \Delta(G) \leq 5$, then $\chi_{ast}(G) \leq \Delta(G)+2$.*

(3) *If $g(G) \geq 18$ and $\Delta(G) = 3$, then $\chi_{ast}(G) \leq 6$.*

2. Some Lemmas and Main Results

In this section, we cite some lemmas which will be used in the following proofs.

Lemma 7 [16]**.** *For any connected graph G with $|V(G)| \geq 3$, $\chi_{ast}(G) \geq \Delta + 1$. Moreover, if G has adjacent maximum degree vertices, then $\chi_{ast}(G) \geq \Delta + 2$.*

Lemma 8. *For a graph G, suppose f is a proper total coloring of G. Let x be a leaf of G with $d(x) = 1$. If y is the neighbor of x with $d(y) \geq 3$, then $C_f\langle x\rangle \neq C_f\langle y\rangle$.*

Proof. For a vertex $y \in V(G)$, we have $|C_f\langle y\rangle| \geq d(y)+1$. However, $|C_f\langle x\rangle| = 3$ for the leaf x. If $d(y) \geq 3$, then $|C_f\langle y\rangle| \geq 4$. Therefore, $C_f\langle x\rangle \neq C_f\langle y\rangle$. ■

Lemma 9 [7]**.** *Let G be a graph.*

(1) *If v is a leaf of G, then* $\mathrm{mad}(G - v) \leq \mathrm{mad}(G)$.

(2) *If e is an edge of G, then* $\mathrm{mad}(G - e) \leq \mathrm{mad}(G)$.

We will use the following two theorems to prove that Theorem 5 holds.

Theorem 10. *If G is a graph with* $\mathrm{mad}(G) < \frac{9}{4}$, *$\Delta(G) \geq 3$ and $K(G) = \max\{\Delta(G)+2, 6\}$, then $\chi_{ast}(G) \leq K(G)$.*

Proof. Our proof proceeds by *reductio ad absurdum.* Let G be a counterexample such that $|T(G)| = |V(G)| + |E(G)|$ is as small as possible. Then, any subgraph G' of G with $\mathrm{mad}(G') < \frac{9}{4}$ has $\chi_{ast}(G') \leq K(G') \leq K(G)$ by the minimality of $T(G)$, where $K(G) \geq 6$.

We will analyze the structure of G with several claims, then derive a contradiction using the discharging method. In the proofs that follow, we usually construct proper total colorings to deduce contradictions by *reductio ad absurdum*, and then prove the absence of certain substructures.

Claim 11. *No vertex of degree at most 3 is adjacent to a leaf.*

Proof. Assume to the contrary that G contains a vertex v with $d_G(v) \leq 3$ adjacent to a leaf. Since G contains no isolated edges, $2 \leq d_G(v) \leq 3$. Then we consider the following two cases.

Case 1. $d_G(v) = 2$. Suppose that u_1 and u_2 are neighbors of v with $d_G(u_1) = 1$. Let $G^{'} = G - u_1$. Then $G^{'}$ is a subgraph with $\mathrm{mad}(G^{'}) \leq \mathrm{mad}(G) < \frac{9}{4}$ by Lemma 9(1). By the minimality of $T(G)$, there is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$. Suppose $f^{'}(v) = 1$, $f^{'}(vu_2) = 2$ and $f^{'}(u_2) = 3$, then we have $C_{f'}\langle v\rangle = \{1, 2, 3\}$. Here we will give an AVSDTC f of G from $f^{'}$. Not stated otherwise, $f(z) = f^{'}(z)$ for any $z \in T(G) \cap T(G^{'})$, so we would not mention it again in what follows.

From Definition 1 we know that f should first be a proper total coloring of G. Thus, there are 2 forbidden colors for vu_1 since $f(vu_1) \neq f(v)$ and $f(vu_1) \neq f(vu_2)$; and 2 forbidden colors for u_1 since $f(u_1) \neq f(v)$ and $f(u_1) \neq f(vu_1)$. Therefore, there are at least $(6 - 2) \times (6 - 2) = 16$ available color combinations for vu_1 and u_1.

Now, we consider the number of the forbidden color combinations such that the color set of the vertex v and that of the vertex u_1 are the same. One can see that only if $f(vu_1) = 3$ and $f(u_1) = 2$, then $C_f\langle v\rangle = C_f\langle u_1\rangle$. Hence, there are at most one forbidden color combination for vu_1 and u_1 to yield $C_f\langle v\rangle = C_f\langle u_1\rangle$.

Then we consider the number of the forbidden color combinations such that the color set of the vertex v and that of the vertex u_2 are the same. It is obvious that $C_f\langle v\rangle \neq C_f\langle u_2\rangle$ when $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| \geq 3$ and $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| \leq 0$.

(1) If $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| = 2$, without loss of generality, we may assume that $C_{f'}\langle u_2\rangle = \{1, 2, 3, x, y\}$, then there exist at most 2 forbidden color combinations (x, y) and (y, x) on $\{f(vu_1), f(u_1)\}$ such that $C_f\langle u_2\rangle = C_f\langle v\rangle$.

(2) If $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| = 1$, without loss of generality, we may assume that $C_{f'}\langle u_2\rangle = \{1, 2, 3, x\}$. Then $C_f\langle v\rangle = C_f\langle u_2\rangle$ holds only if vu_1 and u_1 have one of the corresponding forbidden color combinations $(x, 2)$, $(x, 3)$ and $(3, x)$. Thus, there are at most 3 forbidden color combinations such that both the color sets of v and u_2 are indistinguishable.

According to (1) and (2), we have at most $\max\{2, 3\} = 3$ forbidden color combinations such that $C_f\langle v\rangle = C_f\langle u_2\rangle$. Hence, there are at least $16 - 1 - 3 = 12$ available color combinations for vu_1 and u_1. Therefore, one can extend $f^{'}$ to be a $K(G)$-AVSDTC f of G, it contradicts the choice of G.

Case 2. $d_G(v) = 3$. We assume that u_1, u_2 and u_3 are neighbors of v with $d_G(u_1) = 1$. Let $G^{'} = G - u_1$. By the minimality of $T(G)$, there is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$. Suppose $f^{'}(v) = 1$, $f^{'}(vu_2) = 2$ and $f^{'}(vu_3) = 3$, then we get $C_{f'}\langle v\rangle = \{1, 2, 3, f^{'}(u_2), f^{'}(u_3)\}$. Next, we will extend $f^{'}$ to be an AVSDTC f of G.

Clearly, there are 3 forbidden colors for vu_1 since $f(vu_1) \neq f(v)$ and $f(vu_1) \neq f(vu_i)$ where $i = 2, 3$; and 2 forbidden colors for u_1. Therefore, there are at least $(6-3) \times (6-2) = 12$ available color combinations for vu_1 and u_1. Meanwhile, we have $C_f\langle u_1\rangle \neq C_f\langle v\rangle$ by Lemma 8.

We consider the number of the forbidden color combinations such that the color set of the vertex v and that of the vertex u_2 are the same. It is easy to see that $C_f\langle v\rangle \neq C_f\langle u_2\rangle$ when $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| \geq 3$ and $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| \leq 0$.

(a) If $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| = 2$, we may assume that $C_{f'}\langle u_2\rangle = \{1, 2, 3, f^{'}(u_2), f^{'}(u_3), x, y\}$, then there exist at most 2 forbidden color combinations (x, y) and (y, x) on $\{f(vu_1),\ f(u_1)\}$ such that $C_f\langle u_2\rangle = C_f\langle v\rangle$.

(b) If $|C_{f'}\langle u_2\rangle| - |C_{f'}\langle v\rangle| = 1$, we may assume that $C_{f'}\langle u_2\rangle = \{1, 2, 3, f^{'}(u_2), f^{'}(u_3), x\}$. Then $C_f\langle v\rangle = C_f\langle u_2\rangle$ holds only if vu_1 and u_1 have one of the corresponding forbidden color combinations $(x, 2)$, $(x, 3)$, $(x, f^{'}(u_2))$, $(x, f^{'}(u_3))$, $(f^{'}(u_2), x)$, $(f^{'}(u_3), x)$. Thus, there are at most 6 forbidden color combinations such that both the color sets of v and u_2 are indistinguishable.

According to (a) and (b), we have at most $\max\{2, 6\} = 6$ forbidden color combinations such that $C_f\langle v\rangle = C_f\langle u_2\rangle$. Similarly, we assume that $C_{f'}\langle u_3\rangle = \{1, 2, 3, f^{'}(u_2), f^{'}(u_3),\ y\}$, then $C_f\langle v\rangle = C_f\langle u_3\rangle$ holds only if vu_1 and u_1 have one of the corresponding forbidden color combinations $(y, 2)$, $(y, 3)$, $(y, f^{'}(u_2))$, $(y, f^{'}(u_3))$, $(f^{'}(u_2), y)$, $(f^{'}(u_3), y)$. Noting that $1 \notin \{x, y, f^{'}(u_2), f^{'}(u_3)\}$, there are at most $C_5^2 = 10$ different binary combinations on $\{2, 3, 4, 5, 6\}$, which implies that 2 of the 12 forbidden color combinations are repeated at least. In other words, there are at most 10 distinct combinations in these 12 forbidden color combinations. Hence, we have at least $12 - 10 = 2$ available colors for vu_1 and u_1 such that $C_f\langle v\rangle \neq C_f\langle u_2\rangle$ and $C_f\langle v\rangle \neq C_f\langle u_3\rangle$, contrary to the choice of G. $\square$

Claim 12. *There does not exist a 2-vertex v adjacent to two 2-vertices.*

Proof. Assume to the contrary that G contains a 2-vertex v with neighbors u_1, u_2 such that $d_G(u_1) = d_G(u_2) = 2$. Let w_i be the neighbor of u_i different from v in G for $i = 1, 2$, and let $G^{'} = G - v$. (See Figure 1.) Then by the minimality of $T(G)$, there is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$. In the following, we will color the edges u_1v, vu_2 and the vertex v to extend $f^{'}$ to be an AVSDTC f of G. According to whether or not the vertices u_1 and u_2 have been given the same color, we consider two cases in the following.

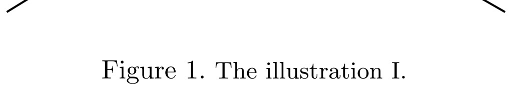

Figure 1. The illustration I.

Case 1. $f'(u_1) = f'(u_2)$. Without loss of generality, we suppose $f'(u_1) = f'(u_2) = 1$, $f'(w_1u_1) = x$ and $f'(u_2w_2) = y$. By Definition 1 we know that f should be a proper total coloring of G firstly. Thus, there are 2 forbidden colors for u_1v since $f(u_1v) \neq f(w_1u_1)$ and $f(u_1v) \neq f(u_1)$, where $f(w_1u_1) = f'(w_1u_1)$ and $f(u_1) = f'(u_1)$, 3 forbidden colors for vu_2 since $f(vu_2) \neq f(u_1v)$, $f(vu_2) \neq f(u_2)$ (note that $f(u_2) = f'(u_2)$) and $f(vu_2) \neq f(u_2w_2)$ (note that $f(u_2w_2) = f'(u_2w_2)$), and 3 forbidden colors for v since $f(v) \neq f(u_1)$ (note that $f(u_1) = f'(u_1)$), $f(v) \neq f(u_1v)$ and $f(v) \neq f(vu_2)$. Therefore, there are at least $(6-2) \times (6-3) \times (6-3) = 36$ available color combinations for u_1v, vu_2 and v.

First, we consider the number of the forbidden color combinations such that the color sets of the vertex w_1 and that of the vertex u_1 are the same. Clearly, it is trivial that $C_f\langle w_1\rangle \neq C_f\langle u_1\rangle$ when $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| \geq 3$ and $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| \leq 0$.

(i) If $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| = 2$, then there exist at most 2 forbidden color combinations $\{f(u_1v), f(v)\} \subset C_{f'}\langle w_1\rangle \setminus C_{f'}\langle u_1\rangle$ such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. Noticing that vu_2 has 3 available colors, thus there are at most $2 \times 3 = 6$ forbidden color combinations to yield that both the color sets of w_1 and u_1 are indistinguishable.

(ii) If $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| = 1$, then there exist at most 3 forbidden color combinations for u_1v and v such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. To maximize the number of forbidden color combinations as exhaustive as possible, we may assume that $f'(w_1) = z$, $C_{f'}\langle u_1\rangle = \{1, x, z\}$ and $C_{f'}\langle w_1\rangle = \{1, x, z, t\}$, where x, z and t are different from each other. Then $C_f\langle w_1\rangle = C_f\langle u_1\rangle$ holds only if u_1v and v have one of the corresponding forbidden color combinations (z, t), (t, z) and (t, x). Note that vu_2 has 3 available colors. Hence, there are at most $3 \times 3 = 9$ forbidden color combinations such that both the color sets of w_1 and u_1 are indistinguishable.

From (i) and (ii) we know that there are at most $\max\{6, 9\} = 9$ forbidden color combinations such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. It is important to remind that the calculation result is affected by the order of priority. Similarly, there exist at most 3 forbidden color combinations for vu_2 and v such that $C_f\langle w_2\rangle = C_f\langle u_2\rangle$. Note that u_1v has 4 available colors. Hence, there are at most $3 \times 4 = 12$ forbidden color combinations for w_2 and u_2 to yield $C_f\langle w_2\rangle = C_f\langle u_2\rangle$.

Next, we consider the number of the forbidden color combinations such that the color sets of the vertex u_1 and that of the vertex v are the same. Suppose that u_1v and v have been colored under f, then $3 \leq |C_f\langle u_1\rangle| \leq 5$.

(a) If $|C_f\langle u_1\rangle| = 3$, without loss of generality, we may assume that $C_f\langle u_1\rangle = \{1, x, z\}$, then $f'(w_1) = f(w_1) = f(u_1v) = z$ and $f(v) = x$. Since $f(u_2) = 1$ we get $f(vu_2) \neq 1$. Note that $f(vu_2) \neq x$ and $f(vu_2) \neq z$, so we have $f(vu_2) \notin C_f\langle u_1\rangle$. Hence, $C_f\langle u_1\rangle \neq C_f\langle v\rangle$.

(b) If $|C_f\langle u_1\rangle| = 4$, there exist at most 4 forbidden color combinations for

u_1v, v and vu_2 such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. In order to obtain the number of forbidden color combinations as exhaustive as possible, we may suppose that $C_f\langle u_1\rangle = \{1, x, z, t\}$. Then $C_f\langle u_1\rangle = C_f\langle v\rangle$ holds only if u_1v, v and vu_2 have one of the corresponding forbidden color combinations (z, x, t), (z, t, x), (t, x, z) and (t, z, x). Thus, there are at most 4 forbidden color combinations such that both the color sets of u_1 and v are indistinguishable.

(c) If $|C_f\langle u_1\rangle| = 5$, then $C_f\langle u_1\rangle \neq C_f\langle v\rangle$ since $f(u_1) = f(u_2) = 1$ implies that $|C_f\langle v\rangle| = 4$.

According to (a), (b) and (c), we have at most 4 forbidden color combinations such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. By a similar argument, there are at most 4 forbidden color combinations to yield $C_f\langle u_2\rangle = C_f\langle v\rangle$. Hence, we have at least $36 - 9 - 12 - 4 \times 2 = 7$ available color combinations for u_1v, vu_2 and v. Therefore, one can extend f' to be a $K(G)$-AVSDTC f of G, contrary to the choice of G.

Case 2. $f'(u_1) \neq f'(u_2)$. Without loss of generality, we suppose $f'(u_1) = 1$, $f'(u_2) = 2$, $f'(w_1u_1) = x$ and $f'(u_2w_2) = y$. In order to drive the coloring employed colors as more as possible, we set $x, y \neq 1, 2$. By Definition 1 we know that f should first be a proper total coloring of G. Thus, there are 2 forbidden colors for v since $f(v) \neq f(u_i)$ (note that $f(u_i) = f'(u_i)$) for $i = 1, 2$, and 3 forbidden colors for u_1v and 4 forbidden colors for vu_2. Consequently, there are at least $(6 - 2) \times (6 - 3) \times (6 - 4) = 24$ available color combinations for v, u_1v and vu_2.

Firstly, we consider the number of the forbidden color combinations such that the color sets of the vertex w_1 and that of the vertex u_1 are the same. Clearly, it is trivial that $C_f\langle w_1\rangle \neq C_f\langle u_1\rangle$ when $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| \geq 3$ and $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| \leq 0$.

(1) If $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| = 2$, then there exist at most 2 forbidden color combinations $\{f(u_1v), f(v)\} \subset C_{f'}\langle w_1\rangle \setminus C_{f'}\langle u_1\rangle$ such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. Noticing that vu_2 has 2 available colors, thus there are at most $2 \times 2 = 4$ forbidden color combinations to yield that both the color sets of w_1 and u_1 are indistinguishable.

(2) If $|C_{f'}\langle w_1\rangle| - |C_{f'}\langle u_1\rangle| = 1$, then there exist at most 3 forbidden color combinations such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. Without loss of generality, we may assume that $f'(w_1) = z$, $C_{f'}\langle u_1\rangle = \{1, x, z\}$ and $C_{f'}\langle w_1\rangle = \{1, x, z, t\}$. Then $C_f\langle w_1\rangle = C_f\langle u_1\rangle$ holds only if u_1v and v have one of the corresponding forbidden color combinations (z, t), (t, z) and (t, x). Note that vu_2 has 2 available colors. Hence, there are at most $3 \times 2 = 6$ forbidden color combinations such that both the color sets of w_1 and u_1 are indistinguishable.

From (1) and (2) we know that there are at most 6 forbidden color combinations such that $C_f\langle w_1\rangle = C_f\langle u_1\rangle$. Similar to as above, there are at most $3 \times 3 = 9$ forbidden color combinations for w_2 and u_2 to yield $C_f\langle w_2\rangle = C_f\langle u_2\rangle$.

Next, we consider the number of the forbidden color combinations such that the color sets of the vertex u_1 and that of the vertex v are the same. Suppose that u_1v and v have been colored under f, then $3 \leq |C_f\langle u_1\rangle| \leq 5$.

(i') If $|C_f\langle u_1\rangle| = 3$, there exist at most one forbidden color combination $(2, x, 1)$ for u_1v, v and vu_2. Since $f^{'}(u_2) = 2 \in C_f\langle v\rangle$, $2 \in C_f\langle v\rangle$. In order to ensure $C_f\langle u_1\rangle = C_f\langle v\rangle$, we have $f(u_1v) = 2$, further we get $f(v) = f^{'}(w_1u_1) = x$ and $f(vu_2) = f^{'}(u_1) = 1$. Thus, there exist at most one forbidden color combination such that both the color sets of u_1 and v are indistinguishable.

(ii') If $|C_f\langle u_1\rangle| = 4$, there exist at most 3 forbidden color combinations such that $C_f\langle u_1\rangle = C_f\langle v\rangle$ since $f^{'}(u_2) = 2$ implies $2 \in C_f\langle u_1\rangle$. Without loss of generality, let $C_f\langle u_1\rangle = \{1, 2, x, z\}$. Then $C_f\langle u_1\rangle = C_f\langle v\rangle$ holds only if u_1v, v and vu_2 have one of the corresponding forbidden color combinations $(z, x, 1)$, $(2, z, x)$ and $(2, x, z)$. Thus if $|C_f\langle u_1\rangle| = 4$, then there are at most 3 forbidden color combinations such that both the color sets of u_1 and v are indistinguishable.

(iii') If $|C_f\langle u_1\rangle| = 5$, without loss of generality, we may assume $C_f\langle u_1\rangle = \{1, 2, x, z, t\}$. Then $C_f\langle u_1\rangle = C_f\langle v\rangle$ holds only if u_1v, v and vu_2 have one of the corresponding forbidden color combinations (z, t, x) and (t, z, x). Thus, there are at most 2 forbidden color combinations such that both the color sets of u_1 and v are indistinguishable.

According to (a), (b) and (c), we have at most 3 forbidden color combinations such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. Similarly, there are at most 3 forbidden color combinations to yield $C_f\langle u_2\rangle = C_f\langle v\rangle$. Hence, we have at least $24 - 6 - 9 - 3 \times 2 = 3$ available color combinations for v, u_1v and u_2v. Therefore, one can extend $f^{'}$ to be a $K(G)$-AVSDTC f of G, and so, it is a contradiction to the choice of G. □

Claim 13. *There does not exist a k-vertex v adjacent to $(k-2)$ 1-vertices, where $k \geq 4$.*

Proof. Assume to the contrary that G contains a k-vertex v with neighbors $u_1, u_2, \ldots, u_k$ such that $d_G(u_i) = 1, i = 1, 2, \ldots, k-2$. Let $G^{'} = G - u_1$. By the minimality of $T(G)$, there is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$. Suppose that $f^{'}(v) = 1$, and $f^{'}(vu_i) = i$ for $i = 2, 3, \ldots, k$, then we get $|C_{f'}\langle v\rangle| \geq k$, see Figure 2 for instance.

If $|C_{f'}\langle v\rangle| \geq k+1$, we color $f(vu_1)$ with $C_{f'}\langle v\rangle \setminus \{1, 2, \ldots, k\}$, and color $f(u_1)$ with 2. The purpose is to ensure $C_{f'}\langle v\rangle = C_f\langle v\rangle$. Hence, we have $C_f\langle v\rangle \neq C_f\langle u_{k-1}\rangle$ and $C_f\langle v\rangle \neq C_f\langle u_k\rangle$.

If $|C_{f'}\langle v\rangle| = k$, then $C_{f'}\langle v\rangle = \{1, 2, \ldots, k\}$. One can see that there are $K(G) - d(v)$ forbidden colors for vu_1 since $f(vu_1) \neq f(v)$ and $f(vu_1) \neq f(vu_i)$ for $i = 2, 3, \ldots, k$; and 2 forbidden colors for u_1. Consequently, there are $(K(G) - d(v)) \times (K(G) - 2) \geq 2\Delta$ available color combinations for vu_1 and u_1.

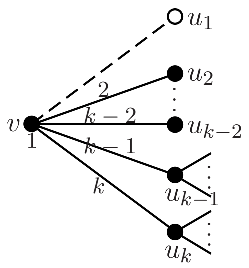

Figure 2. The illustration II.

We firstly consider the number of the forbidden color combinations such that the color set of the vertex v and that of the vertex u_{k-1} are the same. It is easy to see that $C_f\langle v\rangle \neq C_f\langle u_{k-1}\rangle$ when $|C_{f'}\langle u_{k-1}\rangle| - |C_{f'}\langle v\rangle| \geq 3$ and $|C_{f'}\langle u_{k-1}\rangle| - |C_{f'}\langle v\rangle| \leq 0$.

(1) If $|C_{f'}\langle u_{k-1}\rangle| - |C_{f'}\langle v\rangle| = 2$, we may assume that $C_{f'}\langle u_{k-1}\rangle = \{1, 2, \ldots, k, x, y\}$, then there exist at most 2 forbidden color combinations (x, y) and (y, x) on $\{f(vu_1), f(u_1)\}$ such that $C_f\langle v\rangle = C_f\langle u_{k-1}\rangle$.

(2) If $|C_{f'}\langle u_{k-1}\rangle| - |C_{f'}\langle v\rangle| = 1$, without loss of generality, we assume that $C_{f'}\langle u_{k-1}\rangle = \{1, 2, \ldots, k, x\}$. Then $C_f\langle v\rangle = C_f\langle u_{k-1}\rangle$ holds only if vu_1 and u_1 have one of the corresponding forbidden color combinations (x, i) where $i = 2, 3, \ldots, k$. Thus, there are at most $k-1$ forbidden color combinations such that both the color sets of v and u_{k-1} are indistinguishable.

According to (1) and (2), we have at most $\max\{2, k-1\} = k-1$ forbidden color combinations such that $C_f\langle v\rangle = C_f\langle u_{k-1}\rangle$. Similarly as above, there are at most $k-1$ forbidden color combinations such that $C_f\langle v\rangle = C_f\langle u_k\rangle$. Hence, we have $2\Delta - 2(k-1) \geq 2$ available color combinations for vu_1 and u_1. Therefore, f' can be extended a $K(G)$-AVSDTC f of G, and so, it contradicts the choice of G.

□

Claim 14. *Let H be the graph obtained by removing all leaves of G. Then we have*

(i) $\delta(H) \geq 2$.

(ii) *If $v \in V(G)$ with $2 \leq d_G(v) \leq 3$, then $v \in V(H)$ and $d_H(v) = d_G(v)$.*

(iii) *If $v \in V(H)$ with $d_H(v) = 2$, then $d_G(v) = 2$.*

(iv) *If $v \in V(G)$ with $d_G(v) \geq 4$, then $d_H(v) \geq 3$.*

Proof. (i) Assume to the contrary that $\delta(H) \leq 1$.

When $\delta(H) = 0$, H is a complete graph K_1 and G is a star $K_{1,n-1}$, where $n = |V(G)| \geq 3$. It is easy to check that $\chi_{ast}(K_{1,2}) = 4$ and $\chi_{ast}(K_{1,n-1}) = \Delta + 1$ for $n \geq 4$, which contradicts the choice of G.

Assume now that $\delta(H) = 1$. Suppose that $v \in V(H)$ with $d_H(v) = 1$, then $d_G(v) \geq 2$ and v adjacent to at least $d_G(v) - 1$ leaves in G. This contradicts Claims 1 and 3.

The statements (ii), (iii) and (iv) follow immediately from Claims 1 and 3. $\square$

We are going to make use of the discharging method to proof the theorem. First, we define an initial charge function $w(v) = d_H(v)$ for every $v \in V(H)$. Next, we design a discharging rule and redistribute weights accordingly. Once the discharging is finished, a new charge function $w^{'}$ is produced. However, the sum of all charges is kept fixed while the discharging is in progress.

The discharging rule is defined as follows.

(R) Every vertex v of degree at least 3 give $\frac{1}{4}$ to neighboring 2-vertices.

Let $v \in V(H)$. Then $d_H(v) \geq 2$ by Claim 14(1). If $d_H(v) = 2$, then v is adjacent to one vertex of degree at least 3 by Claims 11, 12 and 13. Thus, $w^{'}(v) \geq d_H(v) + \frac{1}{4} = \frac{9}{4}$. If $d_H(v) \geq 3$, then v is adjacent to at most $d_H(v)$ 2-vertices and hence $w^{'}(v) \geq d_H(v) - \frac{1}{4}d_H(v) = \frac{3}{4}d_H(v) \geq \frac{9}{4}$ by (R).

However, this leads to the following contradiction

$$\frac{9}{4} = \frac{\frac{9}{4}|V(H)|}{|V(H)|} \leq \frac{\sum_{v\in V(H)} w^{'}(v)}{|V(H)|} = \frac{\sum_{v\in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \mathrm{mad}(H) < \frac{9}{4}\,.$$

Therefore, the conclusion holds. $\blacksquare$

Theorem 15. *If G is a graph with* $\mathrm{mad}(G) < \frac{7}{3}$*, $\Delta(G) \geq 5$ and $K(G) = \max\{\Delta(G) + 2, 8\}$, then $\chi_{ast}(G) \leq K(G)$.*

Proof. We use *reductio ad absurdum* to prove the theorem. Let G be a counterexample such that $|T(G)| = |V(G)| + |E(G)|$ is as small as possible. It is not hard to see that G also satisfies Claims 11, 12, 13, and 14.

Claim 16. *There does not exist a 3-vertex v adjacent to three 2-vertices.*

Proof. Assume to the contrary that G contains a 3-vertex v with neighbors u_1, u_2 and u_3 such that $d_G(u_i) = 2$, and w_i is the neighbor of u_i different from v in G, where $i = 1, 2, 3$. Let $G^{'} = G - v$. (See Figure 3.) Then by the minimality of $T(G)$, there is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$, where $K(G) \geq 8$. In the following, we keep the coloring $f^{'}$ unchange on $G^{'}$, and then extend $f^{'}$ to be an AVSDTC f of G.

Now, we color the vertex v and its incident edges vu_1, vu_2, vu_3 successively. From Definition 1 we have that f should first be a proper total coloring of G. So, there are 3 forbidden colors for v since $f(v) \neq f(u_i)$ (note that $f(u_i) = f^{'}(u_i)$) for $i = 1, 2, 3$, and 3 forbidden colors for vu_1, and 4 forbidden colors for vu_2 and 5 forbidden colors for vu_3. Thus, there are at least $8 - 3 = 5$, $8 - 3 = 5$, $8 - 4 = 4$

and $8-5=3$ available colors for v, vu_1, vu_2 and vu_3, respectively. Consequently, there are at least $5\times5\times4\times3=300$ available color combinations for v, vu_1, vu_2 and vu_3.

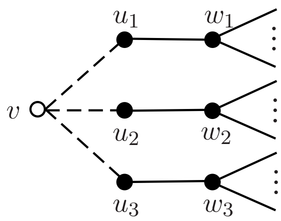

Figure 3. The illustration III.

First, we consider the number of the forbidden color combinations such that the color sets of vertex u_i $(i=1,2,3)$ and that of vertex w_i are the same. It is not hard to see that if $|C_{f'}\langle w_i\rangle|-|C_{f'}\langle u_i\rangle|\geq 3$ and $|C_{f'}\langle w_i\rangle|-|C_{f'}\langle u_i\rangle|\leq 0$, $C_f\langle w_i\rangle\neq C_f\langle u_i\rangle$ for $i=1,2,3$.

(1) For the color sets of u_1 and w_1, there are two cases to be considered.

(i) If $|C_{f'}\langle w_1\rangle|-|C_{f'}\langle u_1\rangle|=2$, then there exist at most 2 forbidden color combinations $\{f(vu_1),f(v)\}\subset C_{f'}\langle w_1\rangle\setminus C_{f'}\langle u_1\rangle$ such that $C_f\langle w_1\rangle=C_f\langle u_1\rangle$. Note that vu_2 has 4 available colors and vu_3 has 3 available colors. Thus, there are at most $2\times4\times3=24$ forbidden color combinations such that both the color sets of w_1 and u_1 are indistinguishable.

(ii) If $|C_{f'}\langle w_1\rangle|-|C_{f'}\langle u_1\rangle|=1$, then there exist at most 3 forbidden color combinations for v and vu_1 such that $C_f\langle w_1\rangle=C_f\langle u_1\rangle$ since, without loss of generality, let $f'(u_1)=1$, $f'(u_1w_1)=2$, $f'(w_1)=3$, $C_{f'}\langle u_1\rangle=\{1,2,3\}$ and $C_{f'}\langle w_1\rangle=\{1,2,3,z\}$. Then $C_f\langle w_1\rangle=C_f\langle u_1\rangle$ holds only if v and vu_1 have one of the corresponding forbidden color combinations $(2,z)$, $(3,z)$ and $(z,3)$. Noticing that vu_2 has 4 available colors and vu_3 has 3 available colors. Hence, there are at most $3\times4\times3=36$ forbidden color combinations such that both the color sets of w_1 and u_1 are indistinguishable.

(2) For the color sets of u_2 and w_2, there are two cases to be considered similarly.

(a) If $|C_{f'}\langle w_2\rangle|-|C_{f'}\langle u_2\rangle|=2$, there are at most $2\times5\times3=30$ forbidden color combinations such that both the color sets of w_2 and u_2 are indistinguishable.

(b) If $|C_{f'}\langle w_2\rangle|-|C_{f'}\langle u_2\rangle|=1$, there are at most $3\times5\times3=45$ forbidden color combinations such that both the color sets of w_2 and u_2 are indistinguishable.

(3) For the color sets of u_3 and w_3, there are also two cases to be considerd.

(i') If $|C_{f'}\langle w_3\rangle|-|C_{f'}\langle u_3\rangle|=2$, there are at most $2\times5\times4=40$ forbidden color combinations such that both the color sets of w_3 and u_3 are indistinguishable.

(ii') If $|C_{f'}\langle w_3\rangle| - |C_{f'}\langle u_3\rangle| = 1$, there are at most $3 \times 5 \times 4 = 60$ forbidden color combinations such that both the color sets of w_3 and u_3 are indistinguishable similarly.

From (1), (2) and (3) we know that there are at most $36 + 45 + 60 = 141$ forbidden color combinations such that $C_f\langle u_i\rangle = C_f\langle w_i\rangle$ for $i = 1, 2, 3$.

Next, we consider the number of the forbidden color combinations such that the color sets of vertex v and that of the vertex u_i are identical. Suppose that vu_i and v have been colored by the proper total coloring f, then $3 \leq |C_f\langle u_i\rangle| \leq 5$. Taking u_1 as an example, we consider the following. Without loss of generality, we assume that $f^{'}(u_1) = 1$, $f^{'}(u_1w_1) = 2$, $f^{'}(w_1) = 3$.

(a) If $|C_f\langle u_1\rangle| = 3$, then $C_f\langle u_1\rangle \neq C_f\langle v\rangle$ since $d_G(v) = 3$ implies that $|C_f\langle v\rangle| \geq 4$.

(b) If $|C_f\langle u_1\rangle| = 4$, there exist at most 3 forbidden color combinations for v and vu_1 such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. Without loss of generality, let $C_f\langle u_1\rangle = \{1, 2, 3, z\}$. Then $C_f\langle u_1\rangle = C_f\langle v\rangle$ holds only if v and vu_1 have one of the corresponding forbidden color combinations $(2, z)$, $(3, z)$ and $(z, 3)$. Noticing that vu_2 has 4 available colors and vu_3 has 3 available colors. Thus, there are at most $3 \times 4 \times 3 = 36$ forbidden color combinations such that both the color sets of u_1 and v are indistinguishable.

(c) If $|C_f\langle u_1\rangle| = 5$, there exist at most 2 forbidden color combinations for v and vu_1 such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. We may suppose $C_f\langle u_1\rangle = \{1, 2, 3, z, t\}$. Then $C_f\langle u_1\rangle = C_f\langle v\rangle$ holds only if v and vu_1 have one of the corresponding forbidden color combinations (z, t) and (t, z). Noticing that vu_2 has 4 available colors and vu_3 has 3 available colors. Thus, there are at most $2 \times 4 \times 3 = 24$ forbidden color combinations such that both the color sets of u_1 and v are indistinguishable.

According to (a), (b) and (c), we have at most 36 forbidden color combinations such that $C_f\langle u_1\rangle = C_f\langle v\rangle$. Similarly, we can get at most $3 \times 5 \times 3 = 45$ forbidden color combinations such that $C_f\langle u_2\rangle = C_f\langle v\rangle$ and $3 \times 5 \times 4 = 60$ forbidden color combinations such that $C_f\langle u_3\rangle = C_f\langle v\rangle$. Hence, there are at least $300 - 141 - 36 - 45 - 60 = 8$ available color combinations for v, vu_1, vu_2 and vu_3, contrary to the choice of G. □

Claim 17. *There does not exist a k-vertex v adjacent to $(k-3)$ 1-vertices and three 2-vertices, where $k \geq 4$.*

Proof. Assume to the contrary that G contains a k-vertex v with neighbors $u_1, u_2, \ldots, u_k$ such that $d_G(u_i) = 1, i = 1, 2, \ldots, k-3$ and $d_G(u_j) = 2, j = k-2, k-1, k$. Let $G^{'} = G - u_1$. There is a $K(G)$-AVSDTC $f^{'}$ of $G^{'}$ with the color set $C = \{1, 2, \ldots, K(G)\}$ by the minimality of $T(G)$, see Figure 4(1) for instance.

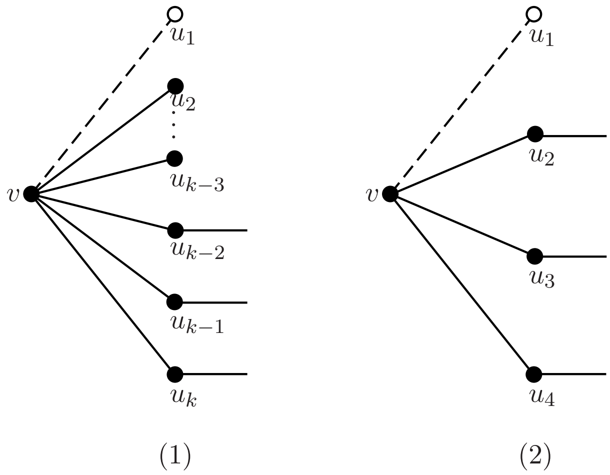

Figure 4. The illustration IV.

From Lemma 8 we know that $C_f\langle v\rangle \neq C_f\langle u_i\rangle$ where $i = 1, 2, \ldots, k-3$. Since $d_G(u_j) = 2$ we have $3 \leq |C_f\langle u_j\rangle| \leq 5$ for $j = k-2, k-1, k$. Note that $k \geq 4$. So $|C_f\langle v\rangle| \geq 5$. It is easy to see that $C_f\langle v\rangle \neq C_f\langle u_j\rangle$ when $|C_f\langle u_j\rangle| = 3, 4$ for $j = k-2, k-1, k$. Thus, if $k \geq 5$, then we have $C_f\langle v\rangle \neq C_f\langle u_j\rangle$ where $j = k-2, k-1, k$; if $k = 4$, see Figure 4(2) for example, then $|C_{f'}\langle v\rangle| \geq 4$ since the color set of $C_{f'}\langle v\rangle$ contains at least $\{f^{'}(v),\ f^{'}(vu_2), f^{'}(vu_3), f^{'}(vu_4)\}$. Now we color $f(vu_1)$ and $f(u_1)$ with two distinct colors from $C\backslash\{f^{'}(v), f^{'}(vu_2), f^{'}(vu_3), f^{'}(vu_4)\}$. Here $|C_f\langle v\rangle| \geq 6$, which leads to $C_f\langle v\rangle \neq C_f\langle u_j\rangle$ for $j = 2, 3, 4$. Hence, it contradicts the choice of G. □

Let H be the graph obtained by removing all leaves of G. Then $\mathrm{mad}(H) \leq \mathrm{mad}(G) < \frac{7}{3}$ by Lemma 9(1). Again, we define an initial charge $w(v) = d_H(v)$ for every vertex $v \in V(H)$ and design the following discharging rule.

$(R^{'})$ Every vertex v of degree at least 3 give $\frac{1}{3}$ to neighboring 2-vertices.

Let $v \in V(H)$. Then $d_H(v) \geq 2$ by Claim 14(1). If $d_H(v) = 2$, then $w^{'}(v) \geq d_H(v) + \frac{1}{3} = \frac{7}{3}$ by Claims 11, 12 and 13. If $d_H(v) = 3$, we know v is adjacent to at most two 2-vertices by Claims 16 and 17, thus $w^{'}(v) \geq d_H(v) - 2 \times \frac{1}{3} = \frac{7}{3}$. If $d_H(v) \geq 4$, then $w^{'}(v) \geq d_H(v) - \frac{1}{3}d_H(v) = \frac{2}{3}d_H(v) \geq \frac{8}{3}$.

This leads to the following obvious contradiction

$$\frac{7}{3} = \frac{\frac{7}{3}|V(H)|}{|V(H)|} \leq \frac{\sum_{v\in V(H)} w^{'}(v)}{|V(H)|} = \frac{\sum_{v\in V(H)} w(v)}{|V(H)|} = \frac{2|E(H)|}{|V(H)|} \leq \mathrm{mad}(H) < \frac{7}{3}.$$

Hence, we complete the proof. ■

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