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CORRIGENDUM TO "MINIMUM EDGE CUTS IN DIAMETER 2 GRAPHS"

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Abstract

In [*Minimum edge cuts in diameter* 2 *graphs*, Discuss. Math. Graph Theory 39 (2019) 605–608] we proved several results on the structure of diameter 2 graphs with a nontrivial minimum edge cut. An error led to several results being incorrect. Here we state and prove the corresponding correct results.

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Let G be a graph. For S, $T \subseteq V(G)$, let $[S,T]$ be the set of edges with one end in S and the other in T. An *edge cut* of a graph G is a set $X = [S,T]$, of edges so that $G - X$ has more components than G. The *edge connectivity* $\lambda(G)$ of a connected graph is the smallest size of an edge cut. Often we can express an edge cut as $\left[S,\overline{S}\right]$, where $\overline{S} = V(G) \setminus S$. A *trivial edge cut* is an edge cut whose deletion isolates a single vertex. Denote the *minimum degree* of G by $\delta(G)$.

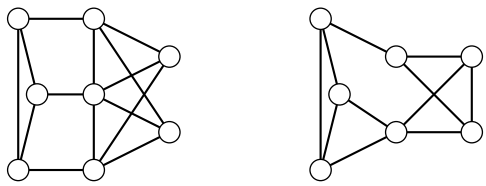
In [1], we presented a short proof of the result of Plesnik [3] that if a graph G has diameter 2, then $\lambda(G) = \delta(G)$. We also presented several results on the structure of diameter 2 graphs with a nontrivial minimum edge cut. In Theorem 3 of [1], we incorrectly assumed that $n(H_1) = d$, when actually we only have $n(H_1) \leq d$. This led to Theorem 3, Corollary 4, and Corollary 5 being incorrect. Here, we state and prove the corresponding correct results.

Corrected Results

Corollary 1 [2]**.** *If G has diameter 2, then one of the subgraphs on one side of a minimum edge cut is either K_1 or $K_{\delta(G)}$.*

Definition. Let $\mathbb{G}$ be the set of graphs that contains the Cartesian product $K_{\frac{n}{2}} \square K_2$, $n \geq 4$, and those graphs with the following structure. The vertices can be partitioned into three sets, S_1, S_2, and S_3. Set S_1 induces K_d, S_2 has $n_2 \leq d$ vertices, and S_3 has n_3 vertices, $n_2 + n_3 > d$. There are d edges joining a vertex of S_1 and a vertex of S_2 so that each vertex in $S_1 \cup S_2$ is incident with at least one edge. All possible edges between S_2 and S_3 are present. There are enough extra edges with both ends in S_2 or S_3 so that $\delta(G) \geq d$.

In the examples of graphs in $\mathbb{G}$ below, the sets (S_1, S_2, S_3) induce graphs $\left(K_3, P_3, \overline{K}_2\right)$ at left and $\left(K_3, \overline{K}_2, K_2\right)$ at right.



Theorem 2. *A graph has diameter 2 and contains a non-trivial minimum edge cut if and only if it is in set $\mathbb{G}$.*

Proof. ($\Leftarrow$) It is readily checked that a graph $G \in \mathbb{G}$ has $\delta(G) = d = \lambda(G)$, and contains a nontrivial minimum edge cut. Each graph G has diameter 2 since each pair of vertices in S_1 and S_3 has a unique common neighbor.

($\Rightarrow$) Let G have diameter 2 and contain a non-trivial minimum edge cut $X = \left[S_1, \overline{S}_1\right]$, and let $d = \delta(G)$. Then (say) $G[S_1] = K_d$, and the order of $\overline{S}$ is at least d. If it is exactly d, then $G = K_{\frac{n}{2}} \square K_2$.

If not, then $\overline{S}$ contains vertices not adjacent to any vertex of K_d. Let S_3 be the set of these vertices and $S_2 = \overline{S}_1 \setminus S_3$. Then each vertex of S_2 is incident with at least one edge of X, and each vertex of S_1 is incident with exactly one edge of X. Then each vertex of S_3 is adjacent to each vertex of S_2 since otherwise some pair of vertices in S_1 and S_3 will have distance more than 2. Since $\delta(G) = d$, there are enough extra edges with both ends in S_2 or S_3 so that each vertex has degree at least d. ■

Corollary 3. *If $G \in \mathbb{G}$, it has between d and* $\max\{n-1, 3d-1\}$ *trivial minimum edge cuts.*

Proof. The number of trivial minimum edge cuts is the number of vertices of minimum degree. All the vertices of K_d have minimum degree, so this is at least

d. Now $K_{\frac{n}{2}} \square K_2$ has $n = 2d$ such vertices. If G is regular, then it has at most $d + d + (d-1)$ vertices since each vertex in S_2 has degree at least $1 + |S_3|$. If $|S_3| \geq d$, then each vertex in S_2 has degree more than d, so there are at most $n - 1$ minimum degree vertices. ■

Theorem 4. *All graphs in set $\mathbb{G}$ have a single non-trivial minimum edge cut except for C_4 and those constructed as follows. Let a vertex v be adjacent to s, $\frac{d}{2} \leq s \leq d$, vertices each in two copies of K_d, $d \geq 2$, and add a matching between $d - s$ vertices in each K_d not adjacent to v.*

For $d = 2$, the three possible graphs in $\mathbb{G}$ with more than one non-trivial minimum edge cut are C_4, C_5, and $K_1 + 2K_2$.

Proof. Let $G \in \mathbb{G}$, so $\delta(G) \geq 2$. Let $\delta(G) = 2$ and $|S_2| = 2$. Note that C_4 and C_5 have two and five nontrivial edge cuts, respectively. Now $C_5 + e$ has a single non-trivial minimum edge cut. Let u and v be the vertices in S_2. If there are at least two vertices in S_3, then G has a spanning subgraph with $n - 4$ $u - v$ paths of length 2 and one $u - v$ path of length 3. Hence the result holds for $\delta(G) = 2$.

Let $\delta(G) = 2$, $|S_2| = 1$, and $v \in S_2$. If there is another nontrivial edge cut, it must separate $S_1 \cup v$ from K_2 (by Corollary 1). Thus $G = K_1 + 2K_2$.

Let $d = \delta(G) > 2$. Then no nontrivial minimum edge cut separates vertices in K_d. Assume there is another nontrivial edge cut X. One component of $G - X$ contains all vertices of S_1 and at least one of S_2. Thus the other component of $G - X$ must be $H = K_d$ by Corollary 1. Now there are $s \leq d$ vertices of H in S_3 and $d - s$ vertices of H in S_2. If there are r other vertices in S_2, then X contains at least $rs + d - s \geq d$ edges. Equality requires $r = 1$, so let v be the one vertex in $S_2 - H$. Also, each vertex in $S_2 \setminus v$ is adjacent to exactly one vertex of S_1. Then v is adjacent to exactly s vertices in S_1, so $s \geq \frac{d}{2}$. Then G can be constructed as described and has exactly two non-trivial minimum edge cuts. ■

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