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DOMINATION IN GRAPHS AND THE REMOVAL OF A MATCHING

Geoffrey Boyer and Wayne Goddard

School of Mathematical and Statistical Sciences
Clemson University

Abstract

We consider how the domination number of an undirected graph changes on the removal of a maximal matching. It is straightforward that there are graphs where no matching removal increases the domination number, and where some matching removal doubles the domination number. We show that in a nontrivial tree there is always a matching removal that increases the domination number; and if a graph has domination number at least 2 there is always a maximal matching removal that does not double the domination number. We show that these results are sharp and discuss related questions.

Keywords: domination, matching, edge removal, trees.

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1. Introduction

If G is a graph, then a *dominating set* D of G is a subset of vertices such that every vertex of G is either in D or adjacent to a vertex in D. The *domination number* of G, denoted $\gamma(G)$, is the cardinality of a minimum dominating set of G, and a set achieving this is a γ*-set.* Note that in general we consider simple undirected connected graphs.

One fundamental area of study as regards the domination number of a graph is how it changes under graph operations. In particular, both the removal of vertices, and the removal or addition of edges, have been the focus of many papers. For example, vertex-critical graphs are ones where the deletion of any vertex decreases the domination number (see for example [2]) while critical graphs are ones where the addition of any edge decreases the domination number (see for example [5]). One specific parameter in this regard is the *bondage number* of a graph, which is defined to be the minimum number of edges whose removal increases the domination number. For example, it is known that the bondage number of a tree is at most 2 [3].

We study an extension of edge removal by considering what happens when one removes a matching from a graph, where a *matching* is a set of edges such that each vertex is incident to at most one edge. This is of course a special case of more general questions such as: what is the smallest k such that there exists a subset of edges, inducing a subgraph with maximum degree k, whose removal increases γ by at least ℓ? But even for trees, the question of how much or how little the least-impact or the most-impact matching changes the domination number seems interesting. For example, we show that: (a) for a tree there always exists a matching whose removal increases the domination number (indeed any maximum matching suffices); (b) there are trees of arbitrarily large domination number where no matching removal increases it by more than 1; (c) in any graph a matching removal can at most double the domination number; and (d) while there is no graph with domination number bigger than 1 where for every maximal matching its removal doubles the domination number, there are trees of arbitrarily large domination number r where the removal of any maximal matching increases the domination number to at least $2r - 1$.

2. The Fundamentals

The removal of a matching can both change and not change the domination number. There are graphs, such as the 5-cycle, where removing any matching does not change the domination number. Or, for example, if the 2-domination number of a graph (meaning the smallest size of a set D such that every vertex outside D has at least two neighbor in D) equals its domination number, then the removal of a matching cannot change the domination number. But there are graphs, such as the complete graph K_n for n even, where the removal of any maximal matching does increase the domination number.

There is a simple limit though on how much the removal of a matching can increase the domination number. Since each vertex in a dominating set loses at most one neighbor, one immediately has the following.

Lemma 1. *Removing a matching from a graph can at most double the domination number of the graph.*

For example, Figure 1 shows a tree T with $\gamma(T) = 3$, where there are maximal matchings $M_3 = \{c, d\}$, $M_4 = \{c, e\}$, $M_5 = \{a, b, d\}$, and $M_6 = \{a, b, e\}$ such that $\gamma(T - M_i) = i$ for $3 \leq i \leq 6$.

Now, in the case that one is removing a matching that has the most impact (that is, increases the domination number as much as possible), it is immediate

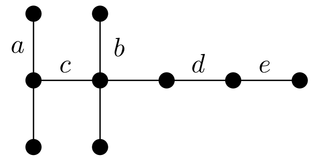


Figure 1. A tree with multiple maximal matchings.

that one may assume that the matching is maximal. On the other hand, in the case where one is removing a matching that has the least impact, it is necessary to add some condition on the matching, else the empty matching will be the answer. In this paper:

> *The focus is on the case that the matching is maximal.*

Some of the results apply to all matchings, and so we state them as such. We also occasionally consider the case that the matching is required to be maximum, but this is a different problem. For example, Figure 2 shows a tree that has a unique maximum matching up to symmetry, and removing it increases the domination number from 3 to 5. On the other hand, removing the depicted maximal matching, consisting of two leaf-edges and the central edge, increases the domination number to 6. (We use *end-vertex* to mean a vertex of degree 1, and *leaf-edge* to mean an edge incident to such a vertex.)

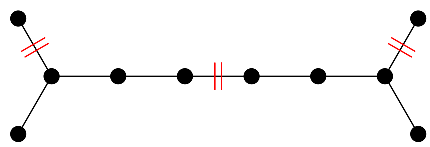


Figure 2. A tree where the matching with most impact is not maximum.

There is a partial connection with the bondage number. If the bondage number of a graph is 1, then it is immediate that the removal of any matching containing that one edge increases the domination number. As noted above, a tree has bondage number at most 2. But the two edges might need to be adjacent. Indeed, if one has to remove independent edges, then there are trees T that require the removal of $\gamma(T)$ edges to increase the domination number. One example is the star where every edge has been subdivided once, as shown in Figure 3.

It is well-known that the domination number of a graph without isolates is at most half its order. One can ask a similar question for a matching-deleted graph.

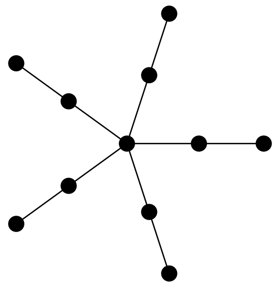


Figure 3. A subdivided star.

Theorem 2. *If G is a connected graph of order $n \geq 3$ and M is a matching of G, then $\gamma(G-M) \leq 3n/4$.*

Proof. Since the removal of an edge cannot decrease the domination number, we may assume G is a tree. The proof is by induction on the order n. If G is a star, then $\gamma(G-M)=2$. So we may assume G is not a star. If G has diameter 3, then $\gamma(G)=2$ and so $\gamma(G-M) \leq 4$. The bound follows if $n \geq 6$. But it can be easily checked that the bound is true if n is 4 or 5. So we may assume that G has diameter at least 4. Consider a longest path P in G with penultimate vertex v.

Case 1. Vertex v has at least two end-vertex neighbors. Then let L be the set consisting of v and its end-vertex neighbors. In $G-M$ it is still possible to dominate L using at most two vertices. Let $G' = G - L$, and let M' be the matching M restricted to $V(G')$. Since G has diameter at least 4, the graph G' has at least three vertices. So by the induction hypothesis, $G'-M'$ can be dominated with at most 3/4 its order. It follows that $\gamma(G-M) \leq 2 + 3(n-|L|)/4 < 3n/4$.

Case 2. For all choices of P, vertex v has exactly one end-vertex neighbor. That is, it has degree 2 in G. Let w be the other neighbor of v, and x the other neighbor of w on P. If P has length 5 or more, then let J be the set of vertices separated from x by the bridge xw. Then let $G' = G - J$ and let M' be the matching M restricted to $V(G')$. By the induction hypothesis, $G'-M'$ can be dominated with at most 3/4 its order. If P has length 4, then let $J = V(G)$ and vacuously the graph $G' = G - J$ can be dominated with at most 3/4 its order.

Say J contains ℓ neighbors of w of degree 1 and t neighbors of w of degree 2. (Since we are in Case 2 it holds that $t \geq 1$.) Note that $|J| = 2t+\ell+1$. If $\ell = 0$, in $G-M$ one can dominate J with at most $t+1$ vertices; since $(t+1) < 3(2t+1)/4$ it follows that then $\gamma(G-M) < 3|J|/4 + 3|G'|/4 = 3n/4$. If $\ell > 0$, in $G-M$ one can dominate J with at most $t+2$ vertices; since $(t+2) \leq 3(2t+2)/4$ it follows that then $\gamma(G-M) \leq 3|J|/4 + 3|G'|/4 = 3n/4$. ■

The bound in Theorem 2 is best possible. Given a graph G, we define the *k-corona* of G as the graph obtained by, for each vertex, adding k new end-vertices adjacent to that vertex. If $k = 1$, we call it simply the corona. The value in Theorem 2 is achieved for the corona of a corona of a graph. Figure 4 shows an example where removing all leaf-edges increases the domination number to 15.

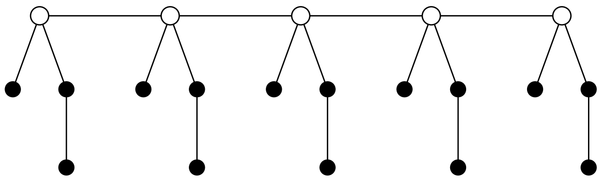


Figure 4. A corona of a corona.

3. Matchings Whose Removal (Almost) Double the Domination Number

In this section we provide some insight into the maximum change that removing a matching can cause. Consider for example a tree T where every vertex in the (necessarily unique) γ-set has (at least) two end-vertex neighbors. Fink *et al.* [3] observed that for such a tree the bondage number is 1. We note that if M is a matching consisting of a leaf-edge incident with each vertex in the dominating set, it is immediate that $\gamma(T - M) = 2\gamma(T)$.

There is a bound on the domination number of graphs that have a matching whose removal doubles the domination number. We will need the following terminology: given a vertex set D and a vertex v of D, an *external private neighbor* of v with respect to D is defined to be a vertex outside D whose only neighbor in D is v.

Theorem 3. *If a connected graph G with order $n \geq 3$ has a matching M such that $\gamma(G - M) = 2\gamma(G)$, then $\gamma(G) \leq n/3$.*

Proof. Consider a γ-set D of G; we can choose D to not contain an end-vertex. For each vertex $v \in D$, if v is incident with an edge of M, then let v' be the end of that edge. Let the set X consist of D and all the v'. This set dominates $G - M$. By the hypothesis, this means $|X| = 2|D|$, so that all the vertices v' exist; that is, every vertex of D is incident with M. Further, the set X must be minimal dominating. So it follows that each v' has no neighbor in $D \setminus v$; that is, in G the vertex v' is an external private neighbor of v with respect to D.

Let D' be the set of vertices of D that have only one external private neighbor in G. If D' is nonempty, consider any vertex $v \in D'$. By the choice of D, the vertex v is not an end-vertex and thus has a neighbor other than v', say w. Since the set X is minimal dominating in $G - M$, and vertex v is in X only to dominate itself, it must be that w is not in D. Further, we claim that w has only one neighbor in D'. Since, if w also has neighbor $u \in D'$, one can take X and replace u, v by w and get a smaller dominating set, which is a contradiction.

For each vertex $v \in D'$, define the triple $R_v = \{v, v', w\}$. For each vertex $v \in D \setminus D'$, define the triple R_v as v and two of its external private neighbors in G. By the above discussion these triples are disjoint. Thus the order of G is at least $3|D|$, whence the desired bound. ∎

There are graphs and indeed trees that achieve the bound in Theorem 3. The simplest example is the 2-corona of any graph; an example is shown in Figure 5. But another example is given in Figure 1.

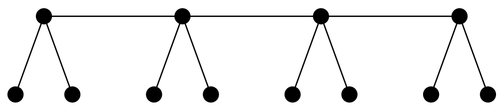


Figure 5. A 2-corona.

It is also natural to look for graphs where the removal of any **maximum** matching doubles the domination number. Or even stronger, where the removal of any **maximal** matching doubles the domination number.

The graphs with domination number 1 are a special case. If a graph has domination number 1 and even order, then the removal of any maximal matching M increases the domination number to 2. For suppose there is a still a dominating vertex v after the removal of M. Then v is not covered by M, and since the order is even, there is another uncovered vertex, necessarily a neighbor of v, which contradicts the maximality of M. If a graph has domination 1 and odd order, then both possibilities can occur. For example, the removal of any maximal/maximum matching from a star increases the domination number; but the removal of any maximal/maximum matching from a complete graph keeps it at 1.

So consider graphs with domination number at least 2. Here there are graphs where the removal of any maximum matching doubles the domination number. For example, define the octopus O_r by taking the star $K_{1,r}$ and subdividing every edge except one exactly three times. The result has $\gamma(O_r) = r$, achieved uniquely by taking the support vertices (where a support vertex is defined as one with an end-vertex neighbor). Further, the graph has a unique perfect matching, and its removal increases the domination number to $2r$. Figure 6 shows O_5.

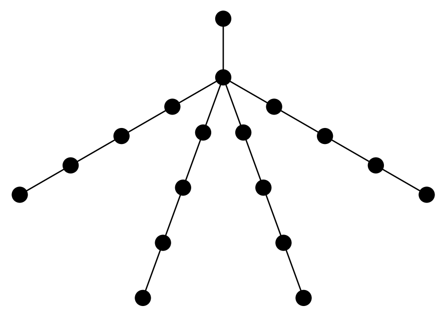


Figure 6. The octopus O_5.

But the situation is (slightly) different if one considers all maximal matchings.

Theorem 4. *If G is a connected graph with $\gamma(G) > 1$, then there exists a maximal matching M such that $\gamma(G - M) < 2\gamma(G)$.*

Proof. Consider a γ-set D of G. If G is not a tree, choose a spanning tree T such that D dominates T. It then follows that there exist two vertices u and v of D that are distance at most 3 apart in T, say joined by path P. Construct a maximal matching M by starting with the first edge of P and, if the length of P is 3, also the last edge of P. Extend to a maximal matching arbitrarily.

Create a dominating set D' of $T - M$ as follows. Start with $D' = D$. For every vertex of D other than u or v, they lose at most one neighbor when M is removed; if they do lose a neighbor, add that neighbor to D'. At this point $|D'| \leq 2|D| - 2$. The only possible vertices not dominated by D' are the neighbors of u and v in M; call these u' and v', respectively. If u and v are adjacent in T, then we constructed M to include the edge uv, and D' to include both u and v, and so there is no undominated vertex. If u and v are at distance 2 in T, then u' is dominated by v, and so only v' is not dominated by D', and we can add it to D'. If u and v are at distance 3, then u' and v' are the undominated vertices. But they are adjacent, and so can be dominated by adding one of them to D'. In all cases it follows for the final D' that $|D'| < 2|D|$. ■

The bound in Theorem 4 is best possible. That is, for all $r \geq 2$ there is a connected graph with $\gamma(G) = r$ and for every maximal matching M of G it holds that $\gamma(G - M) \geq 2r - 1$. For example, consider the octopus O_r defined above and a maximal matching M of O_r. By the maximality of M, it must contain an edge incident with every support vertex. It follows that a dominating set of $O_r - M$ must contain at least two vertices from each subdivided edge, and one more vertex to dominate the end-vertex neighbor of the center. That is, $\gamma(O_r - M) \geq 2r - 1$.

4. Trees With Unique γ-Sets

We observed earlier that if a tree has a γ-set where every vertex is adjacent to at least two end-vertices, then the tree has a matching whose removal doubles the domination number. Such a tree is a special case of a tree with a unique γ-set. However, it is not the case that having a unique γ-set forces there to be a "doubling matching". The smallest example is the path P_9, where $\gamma = 3$ but the removal of a matching can only increase the domination number to 5.

It is immediate that if a graph has a unique γ-set, then the bondage number is 1 (first observed in [6]) and hence there is a matching whose removal increases the domination number. But one can say more in trees.

Theorem 5. *If a tree T has a unique γ-set, then there is a matching M in T such that $\gamma(T - M) > \frac{3}{2}\gamma(T)$.*

This result is best possible as shown by the paths P_{3m} for m odd. Then $\gamma(P_{3m}) = m$, but removing a matching can only increase the domination number to $(3m + 1)/2$.

For the proof of Theorem 5, it is easier to work in a slightly more general setting. Gunther *et al.* [4] observed that in any graph if D is the unique γ-set (and there are no isolated vertices), then every vertex in D has (at least) two external private neighbors. Thus Theorem 5 follows from the following theorem. (Note that the statement of the theorem does not require D to be dominating.)

Theorem 6. *If a tree T has a set D such that every vertex in D has at least two external private neighbors, then there is a matching M in T such that $\gamma(T-M) > \frac{3}{2}|D|$.*

Proof. The proof is by induction on the order of T. Trivially we may assume D is not empty. Indeed, we have the following.

Claim 7. *We may assume that every support vertex is in D.*

Proof. The condition on D precludes an end-vertex x from being in D. So if such x is not dominated by D, just delete it from T and apply the induction hypothesis (and note that deleting an end-vertex cannot increase the domination number). □

Consider the base case. If each vertex in D has two neighbors that are end-vertices, then the matching M consisting of one leaf-edge incident with each vertex of D is such that $\gamma(T - M) \geq 2|D|$. It follows that we may assume the diameter of T is at least 5. If the diameter is exactly 5 and there is a vertex in D without two end-vertex neighbors, then let M consist of the central edge and one leaf-edge incident with each vertex of D, and again it follows that $\gamma(T - M) \geq 2|D|$. So we may assume the diameter is at least 6.

For the inductive step, most of the time we proceed as follows. We identify a subset S of the vertices such that $T' = T - S$ is a tree that satisfies the hypothesis with $D \setminus S$. The inductive hypothesis yields a matching M' of T', which we extend to a matching M of T by adding a set N of edges all of whose ends are in S. Further, S and M are such that the vertices of T' that have a neighbor in S can be dominated in $T - M$ by one vertex of $V(T')$. We say S and M with all the above properties are *standard*.

For any subset S and matching M, let ψ_S^M denote the minimum possible number of vertices of S in any dominating set of $T - M$.

Claim 8. *Let S and M be standard.*

(a) *It holds that $\gamma(T - M) \geq \psi_S^M + \gamma(T' - M') - 1$.*

(b) *If $\psi_S^M \geq \frac{3}{2}|D \cap S| + 1$, then the inductive step is valid.*

(c) *Further, if no set that achieves ψ_S^M has a neighbor in $V(T')$, then for the inductive step it is sufficient that $\psi_S^M \geq \frac{3}{2}|D \cap S|$.*

Proof. (a) Consider a γ-set of $T - M$ and let X be the restriction of it to $V(T')$. Then $|X| \leq \gamma(T - M) - \psi_S^M$. Further, X dominates all of $T' - M$ except possibly those vertices of T' with neighbors in S, and these can be dominated by one vertex of $V(T')$ by the definition of standard. Thus $\gamma(T' - M') \leq |X| + 1$. It follows that $\gamma(T - M) - \psi_S^M \geq \gamma(T' - M') - 1$, which re-arranged gives one the desired inequality.

(b) We have that $\gamma(T - M) \geq \psi_S^M + \gamma(T' - M') - 1 > \psi_S^M + \frac{3}{2}(|D \setminus S|) - 1 = \frac{3}{2}|D| + \psi_S^M - \frac{3}{2}|D \cap S| - 1$. The claim follows.

(c) The calculation is similar, except that $\gamma(T - M) \geq \psi_S^M + \gamma(T' - M')$. □

It is common in trees to induct by focussing on the vertices at the end of a longest path. We generalize this slightly. Given a pair of adjacent vertices v and w, we define a (v, w)*-peripheral path* as a path starting with edge vw of longest possible length, and call its length the (v, w)-peripheral length. For a vertex v that is not an end-vertex, we define the *peripherality of* v, denoted $per(v)$, by considering the multiset of (v, w)-peripheral lengths for all neighbors w, and taking the **second-largest** length (which might equal the largest). We define $per(v) = 0$ if v is an end-vertex. For example, if $v_0v_1 \cdots v_d$ is a longest path in T, then $per(v_i) = per(v_{n-i}) = i$ for $0 \leq i \leq d/2$. Further, we designate the neighbor w of v with the largest (v, w)-peripheral length its *free* neighbor. Note that, if the diameter of T is more than $2\,per(v)$, then the free neighbor of v is uniquely determined; otherwise designate one arbitrarily, if necessary.

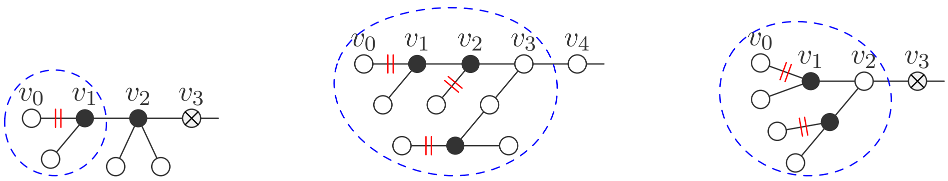
We continue the proof of the theorem. Since the diameter is at least 6 there is some vertex v_3 with $per(v_3) = 3$. Note that in all the figures in the proof of this theorem, solid vertices are vertices that are definitely in D and hollow vertices are vertices that are definitely not in D.

Claim 9. *If $per(v_3) = 3$ with peripheral path $v_3v_2v_1v_0$, then we may assume both v_1 and v_2 have degree 2.*

Proof. By Claim 7, $v_1 \in D$.

(i) We first prove that v_1 has degree 2. So suppose that v_1 has at least two end-vertex neighbors. There are three cases.

Case A. *Assume v_2 is in D and v_2 has two end-vertex neighbors.* Then induct with S consisting of all vertices separated from v_2 by the bridge v_1v_2, and $N = \{v_0v_1\}$. (See Figure 7a for an example.) It is immediate that $\psi_S^M = 2$ while $|D \cap S| = 1$. Since one may assume that v_2 is in the dominating set of $T - M$ (as it has an end-vertex neighbor), the bound holds by Claim 8b.



(a) (b) (c)

Figure 7. Three possible choices of S.

Case B. *Assume v_2 is in D but has only one end-vertex neighbor.* Then v_3 must be the other external private neighbor of v_2. Let v_4 be the free neighbor of v_3. Then, by the peripherality of v_3, all other neighbors of v_3, if any, are within distance 2 of an end-vertex. Since v_3 has no other neighbor in D and is not in D itself, it follows that each of these neighbors must be distance exactly 2 from an end-vertex, have a neighbor in D and all other neighbors of that neighbor must be end-vertices. (See Figure 7b for an example.) Induct with S consisting of all vertices separated from v_4 by the bridge v_3v_4, and N consisting of one leaf-edge incident with each vertex in $D \cap S$. Then every vertex of $D \cap S$ except v_2 has two end-vertex neighbors in T and hence at least one end-vertex neighbor in $T - M$; thus $\psi_S^M = 2|D \cap S| - 1$. However, any set achieving this does not dominate v_4. So the bound holds by Claim 8b, since $|D \cap S| \geq 2$.

Case C. *Assume $v_2 \notin D$.* By the peripherality of v_3, it must be that any neighbor of v_2, other than v_3, is in D and has at least two end-vertex neighbors. (See Figure 7c for an example.) Induct with S consisting of all vertices separated from v_3 by the bridge v_2v_3, and N consisting of one leaf-edge incident with each vertex of $D \cap S$. We have $\psi_S^M = 2|D \cap S|$; but a set achieving this cannot dominate v_3. So the bound holds by Claim 8b.

Hence we have shown that one may assume that v_1 has degree 2.

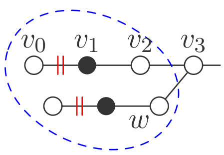
(ii) It follows that v_2 is the external private neighbor of v_1. Thus v_2 cannot have another neighbor with peripherality at most 1, and so it has degree 2. □

It follows that a vertex of peripherality 3 cannot be in D.

Claim 10. *We may assume that every vertex v_3 of peripherality 3 has degree 2.*

Proof. Let $v_3v_2v_1v_0$ be a peripheral path, and let v_4 denote the free neighbor of v_3. Suppose v_3 has a third neighbor w.

(i) Assume that w has peripherality 2. Then by Claim 9, w has degree 2 and its other neighbor has degree 2 too. Then induct with S consisting of the vertices separated from v_3 by the bridge v_2v_3 together with the vertices separated from v_3 by the bridge wv_3, and N consisting of a leaf-edge incident with each vertex of $D \cap S$. (See Figure 8a.) It follows that $\psi_S^M = 4$ while $|D \cap S| = 2$, and so the bound holds by Claim 8a.



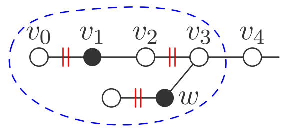


Figure 8. Two possible choices of S.

(ii) Assume that w has peripherality 1 and two end-vertex neighbors. Then induct as in the previous case.

(iii) Assume v_3 has no neighbor of these two types. It follows that w has peripherality 1 and only one end-vertex neighbor, and thus that v_3 is an external private neighbor of w. In particular, v_3 has degree 3. (See Figure 8b.) Then induct with S consisting of the vertices separated from v_4 by the bridge v_3v_4, and N consisting of v_0v_1, v_2v_3, and the leaf-edge incident with w. It follows that $\psi_S^M = 4$ while $|D \cap S| = 2$, and so the bound holds by Claim 8b. □

If the tree has diameter 6 or 7, then it follows from the above claim that T is a path, that is, P_7 or P_8. It is easy to observe that the largest D satisfying the hypothesis of the theorem has two vertices, and that removing a maximum matching yields a forest with domination number at least 4. So we may assume that T has diameter at least 8. In particular, there exists some vertex v_4 with $per(v_4) = 4$.

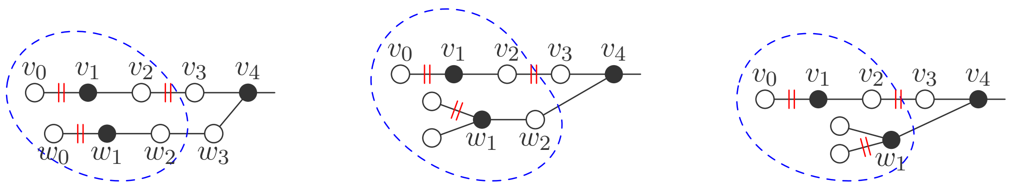
Claim 11. *We may assume that every vertex v_4 of peripherality 4 is in D and has degree 2.*

Proof. Let $v_4v_3v_2v_1v_0$ be a peripheral path.

(i) Suppose that $v_4 \notin D$. Then induct with $S = \{v_0, v_1, v_2, v_3\}$ and $N = \{v_0v_1, v_2v_3\}$. Then $\psi_S^M = 2$ and no set achieving this contains v_3, while $|D \cap S| = 1$. Thus the bound follows from Claim 8c. So we may assume v_4 is in D.

(ii) Suppose v_4 has degree more than 2. Let v_5 be its free neighbor. Let m be the length of the longest path that starts with v_4 and does not use v_3 or v_5. Since $per(v_4) = 4$, it follows that $m \le 4$. There are four cases.

Case A. $m = 4$. Say we have path $v_4w_3w_2w_1w_0$. Then $per(w_3) = 3$, and so by the above claim has degree 2. We use a non-standard inductive step. Let $S = \{v_0, v_1, v_2, w_0, w_1, w_2\}$. (See Figure 9a.) Then induct on $T' = T - S$ to produce matching M'. Then, since M' can contain only one edge incident with v_4, we may assume without loss of generality that the edge v_3v_4 is not in M'. So we can define $N = \{v_0v_1, v_2v_3, w_0w_1\}$. It follows that $\psi_S^M = 4$, but only one vertex of T' can be dominated by S in $T - M$. Thus $\gamma(T - M) \ge \gamma(T' - M') + 3$, while $|D \cap S| = 2$, and so the bound follows.



(a) (b) (c)

Figure 9. Three possible choices of S.

Case B. $m = 3$. Say we have path $v_4w_2w_1w_0$. Then w_1 is a support vertex and so it is in D; and thus w_1 has (at least) two end-vertex neighbors, while w_2 is not an external private neighbor of v_4. Indeed, $per(y) \le 1$ for all neighbors y of w_2 other than v_4. Again we use a non-standard inductive step. Let S consist of v_0, v_1, v_2 together with all vertices separated from v_4 by the bridge w_2v_4. (See Figure 9b.) Then induct on $T' = T - S$ to produce matching M'. Now form M^* from M' by deleting edge v_3v_4 if present. Then form M from M^* by adding v_2v_3 and one leaf-edge incident with each vertex of $D \cap S$. Then $\psi_S^M = 2|D \cap S|$. But since in $T - M$ vertex v_3 is an end-vertex adjacent to v_4, we may assume v_4 is in the dominating set of $T - M$; thus a set achieving ψ_S^M cannot help with dominating $V(T')$. It follows that $\gamma(T - M) \ge \psi_S^M + \gamma(T' - M^*) \ge \psi_S^M + \gamma(T' - M') - 1$, and the bound follows since $|D \cap S| \ge 2$.

Case C. $m = 2$. Say we have path $v_4w_1w_0$. Then w_1 is in D and has (at least) two end-vertex neighbors. Again we use a non-standard inductive step. Let S consist of v_0, v_1, v_2 together with all vertices separated from v_4 by the bridge w_1v_4. (See Figure 9c.) Then induct on $T' = T - S$ to produce matching M'. Now, form M^* from M' by deleting edge v_3v_4 if present. Then form M from M^* by adding edges v_0v_1, v_2v_3, and w_0w_1. Then $\psi_S^M = 4$ while $|D \cap S| = 2$; and by a similar argument to the previous case, it follows that $\gamma(T - M) \geq \gamma(T' - M') + 3$, so that the bound follows similarly.

Case D. $m = 1$. That is, all other neighbors of v_4 are end-vertices. Then induct with S consisting of all vertices separated from v_5 by the bridge v_4v_5, and N consisting of a leaf-edge incident with each of v_1 and v_4. Again $\psi_S^M = 4$ while $|D \cap S| = 2$. Thus the claim holds. □

If the tree has diameter 8 or 9, then it follows from the above claim that T is a path, that is, P_9 or P_{10}. It is easy to observe that the largest D satisfying the hypothesis of the theorem has three vertices, and that removing a maximum matching yields a forest with domination number at least 5. So we may assume that T has diameter at least 10. In particular, there exists some vertex v_5 with $per(v_5) = 5$.

Say v_5 has peripheral path $v_5v_4v_3v_2v_1v_0$ and free neighbor v_6. Let y be a neighbor of v_5 other than v_6, if it exists. Then $per(y) \leq 4$. Since v_5 cannot have a second neighbor in D, it follows that y is not in D. Hence by the above claim $per(y) \neq 4$. Also y does not have an end-vertex neighbor. It follows that, for each neighbor z of y apart from v_5, $per(z) \leq 2$. If $per(z) = 2$, then $per(y) = 3$, and so z is not in D. (See Figure 10.) If there is a z with $per(z) = 1$ that does not have at least two end-vertex neighbors, then it is the only z with $per(z) = 1$. Induct with S consisting of all vertices separated from v_6 by the bridge v_5v_6, and N consisting of v_0v_1, v_2v_3, v_4v_5, and one leaf-edge incident with each vertex in $(D \cap S) \setminus \{v_1, v_4\}$. Then $\psi_S^M = 2|D \cap S| - 1$ and any set achieving this does not contain v_5. So the bound follows from Claim 8c, since $|D \cap S| \geq 3$.

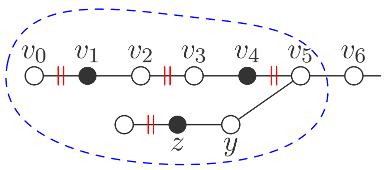


Figure 10. One possible choice of S.

If y does not exist, then induct similarly with $S = \{v_0, \ldots, v_5\}$. This concludes the proof. ■

5. Trees where Matching Removal Has Little Impact

We considered earlier the case where the removal of a matching doubles or nearly doubles the domination number. We consider here the other extreme but only for trees. We show first that there is always a matching whose removal increases the domination number.

We will need the following fact, originally obtained by Bollobás and Cockayne.

Lemma 12 [1]**.** *The domination number of a graph without isolated vertices is at most its matching number.*

This yields the following.

Theorem 13. *Removing any maximum matching M from a nontrivial tree T increases the domination number.*

Proof. It is immediate that $T - M$ has $|M| + 1$ components. Thus $\gamma(T - M) \geq |M| + 1$, since we need at least one vertex in each component. By the above lemma, $|M| \geq \gamma(T)$. ■

As noted earlier, it is not true that every maximal matching removal increases the domination number.

We next consider the trees where removing any maximal matching only increases the domination number by at most 1. We have already seen that the star is an example. We define two families of trees.

• Let $\mathcal{S}$ denote the set of all trees that have radius at most two and at most one vertex of degree more than 2. We call the vertex of degree more than 2 the "hub".

• Let $\mathcal{T}$ denote the set of all trees obtained from a tree of diameter 3 by subdividing each edge once. (Note that all trees in $\mathcal{T}$ have odd order.) We call the starting tree the *base tree.*

Figure 11 gives a picture of a tree in $\mathcal{S}$ and a tree in $\mathcal{T}$.

Lemma 14. *If T is a tree in $\mathcal{S} \cup \mathcal{T}$, then the removal of a matching M from T either leaves the domination number unchanged or increases it by* 1.

Proof. Assume $T \in \mathcal{S}$. Then $\gamma(T)$ equals the number of vertices of degree 2, plus one if the hub has a neighbor that is an end-vertex. Further, $\gamma(T - M)$ is at most the number of vertices of degree 2 plus one for the hub, plus one if M contains an edge joining the hub to an end-vertex neighbor.

Assume $T \in \mathcal{T}$ with order n. Then $\gamma(T) = (n-1)/2$ as, for example, the end-vertices and the central vertex form a dominating set while we need a different

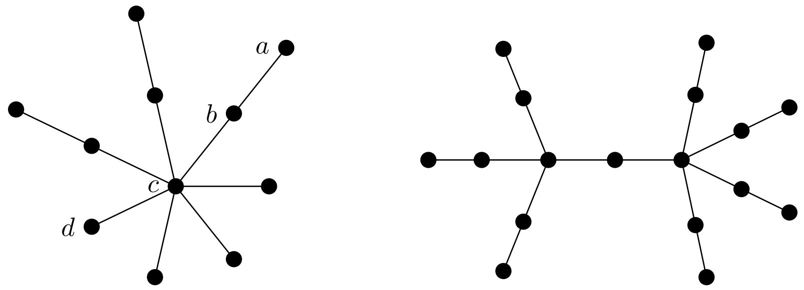


Figure 11. A tree in $\mathcal{S}$ and in $\mathcal{T}$.

vertex to dominate each of them. On the other hand, $\gamma(T - M) \leq (n + 1)/2$, as the original vertices of the base tree dominate T even after the removal of a matching. ■

It is well-known and trivial that adding an edge to a graph can reduce the domination number by at most 1.

Theorem 15. *A tree T has the property that $\gamma(T - M) \leq \gamma(T) + 1$ for all matchings M if and only if T is in $\mathcal{S} \cup \mathcal{T}$.*

Proof. Let us say a matching is "bad" if its removal increases the domination number by more than 1, and a tree is "good" if it has no bad matching. By Lemma 14, it remains to show that if a tree T is good, then it is in one of the two families. The proof is by induction on the order of T.

For the base case consider a tree of diameter at most 3. If such a tree has at most one vertex of degree more than 2, then it is in $\mathcal{S}$. If such a tree has two vertices of degree more than 2, then a maximum matching is bad. Hence we may assume T has diameter at least 4.

Consider a longest path P of good T with penultimate edge uv where all other neighbors of v are end-vertices. Say $T - uv$ consists of trees T_u and T_v. By Theorem 13, there exists a matching of T_u whose removal increases the domination number of T_u; let M_u be any such matching (not necessarily maximum). Extend M_u to matching M of T by adding a leaf-edge incident with v. Then

$$\gamma(T - M) \geq \gamma(T_u - M_u) + 1 > \gamma(T_u) + 1 \geq \gamma(T).$$

In particular, $\gamma(T - M) = \gamma(T) + 1$ requires both that $\gamma(T) = \gamma(T_u) + 1$ and that $\gamma(T_u - M_u) = \gamma(T_u) + 1$. Since M_u was any matching such that $\gamma(T_u - M_u) > \gamma(T_u)$, this implies that T_u is good, and thus $T_u \in \mathcal{S} \cup \mathcal{T}$.

Let D_ℓ denote the set of support vertices of T, let M_ℓ be a matching consisting of a leaf-edge incident with each support vertex, and let U_ℓ be the tree obtained from T by removing each end-vertex incident to an edge of M_ℓ. We say that the tree T is *leafy* if D_ℓ forms a dominating set. If so, $\gamma(T - M_\ell) = |M_\ell| + \gamma(U_\ell) = \gamma(T) + \gamma(U_\ell)$. So U_ℓ must have $\gamma(U_\ell) = 1$ and thus be a star if T is leafy.

Assume first that $T_u \in \mathcal{S}$. If T_u is a star, then T is leafy and $\gamma(T) = 2$. It is easily checked that the only way T can be good is that it is P_5, which is in $\mathcal{S}$. So assume T_u is not a star. In T_u, let a be a vertex at distance 2 from the hub, c the hub, and b their common neighbor. Let d be an end-vertex neighbor of the hub, if it exists.

Up to symmetry there are four possibilities for u. (a) Assume $u = a$. Then $vabc$ is in U_ℓ, and so T is not leafy; thus the hub has no end-vertex neighbor. So if v has degree 2 in T the tree T is in $\mathcal{T}$. Otherwise it can be checked that any maximal matching containing the edge bc and a leaf-edge incident with v is bad. (b) Assume $u = b$. Then T is leafy. The only way U_ℓ can be a star is that both v and c have degree 2 in T, and c's other neighbor is an end-vertex; so T is in $\mathcal{S}$. (c) Assume $u = c$. Then T is leafy. If v has degree 2 in T, then T is in $\mathcal{S}$; otherwise U_ℓ is not a star. (d) Assume $u = d$. Then T is leafy, but U_ℓ is not a star, which contradicts the above.

Assume second that $T_u \in \mathcal{T}$. By the choice of P, u cannot be the central vertex of T_u. If u is an end-vertex of T_u, then it is easily checked that $\gamma(T) = \gamma(T_u)$, which contradicts the above. If u is a subdivision vertex in T_u, then one can check that there is maximum matching whose removal increases the domination number of T by 2. If u is one of the large-degree vertices, then we are in $\mathcal{T}$ if v has degree 2 in T, and there is a bad matching otherwise. ■

6. Further Thoughts

There are several directions to consider further. For example, can sharper bounds be obtained for other families of graphs? Even for trees, one could restrict the maximum degree and then ask about better bounds for the impact of matching removal. There is also the question of an algorithm for finding the matching in a tree whose removal increases the domination number the most, and the complexity of that task in general graphs.

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