# THE MAXIMUM NUMBER OF EDGES IN A $\left\{K_{r+1}, M_{k+1}\right\}$-FREE GRAPH 

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## AND

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#### Abstract

Let $G$ be a graph and $\mathcal{F}$ be a family of graphs. We say $G$ is $\mathcal{F}$-free if it does not contain $F$ as subgraph for any $F \in \mathcal{F}$. The Turán number ex $(n, \mathcal{F})$ is defined as the maximum number of edges in an $\mathcal{F}$-free graph on $n$ vertices. Let $K_{r+1}$ denote the complete graph on $r+1$ vertices and let $M_{k+1}$ denote the graph on $2 k+2$ vertices with $k+1$ pairwise disjoint edges. By using the alternating path technique and the Zykov symmetrization, we determine that for $n>3 k$, $$
\operatorname{ex}\left(n,\left\{M_{k+1}, K_{r+1}\right\}\right)=t_{r-1}(k)+k(n-k)
$$ where $t_{r-1}(k)$ is the number of edges in an $(r-1)$-partite $k$-vertex Turán graph. Let $\nu(G), \tau(G)$ denote the matching number and the vertex cover number of $G$, respectively. For $n \geq 2 k$, we prove that if $\nu(G) \leq k$ and $\tau(G) \geq k+r$, then $$
e(G) \leq \max \left\{\binom{2 k+1}{2},\binom{k+r+1}{2}+(k-r)(n-k-r-1)\right\} .
$$


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## 1. Introduction

Let $G(V, E)$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. We use $e(G)$ to denote the size of $E(G)$. For a graph $H$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, then $H$ is called a subgraph of $G$. Let $\mathcal{F}$ be a family of graphs. If for every $F \in \mathcal{F}, G$ does not contain $F$ as subgraph, then we say $G$ is $\mathcal{F}$ free. The Turán number of $\mathcal{F}$, denoted by $\operatorname{ex}(n, \mathcal{F})$, is defined as the maximum number of edges in an $\mathcal{F}$-free graph on $n$ vertices. For $\mathcal{F}=\{F\}$, we simply write ex $(n, F)$. The study of the Turán numbers plays a central role in the extremal graph theory. The Turán number of many graphs have been determined, see $[4,10,11,15,16,18,19,22,23,24$, etc.]

Let $T_{r}(n)$ denote the Turán graph on $n$ vertices, i.e., the complete $r$-partite graph of order $n$ with each partite of sizes $\lceil n / r\rceil$ or $\lfloor n / r\rfloor$. Note that for $n<r$, $T_{r}(n)$ represents the complete graph on $n$ vertices. We use $t_{r}(n)$ to denote the number of edges of $T_{r}(n)$. In 1941, Turán [19] showed that the Turán graph $T_{r}(n)$ is the only $K_{r+1}$-free graph attaining the maximum number of edges.
Theorem 1.1 [19]. $\operatorname{ex}\left(n, K_{r+1}\right)=t_{r}(n)$.
For any $M \subset E(G)$, if the edges of $M$ are pairwise disjoint, then $M$ is called a matching of $G$. The matching number $\nu(G)$ is the size of a maximum matching in $G$. We often use $M_{k+1}$ to denote the graph on $2 k+2$ vertices with $k+1$ pairwise disjoint edges. In 1959, Erdős-Gallai [10] determined the Turán number of $M_{k+1}$.

Theorem 1.2 [10]. For $n \geq 2 k+1$,

$$
\operatorname{ex}\left(n, M_{k+1}\right)=\max \left\{\binom{2 k+1}{2},\binom{k}{2}+k(n-k)\right\} .
$$

It should be mentioned that by using the shifting technique, Akiyama and Frankl [2] give a short proof of Theorem 1.2 and their proof also works for an rainbow extension of Theorem 1.2. That is, if $G_{1}, G_{2}, \ldots, G_{k+1}$ are $k+1$ graphs on the same vertex set of size $n$ and $e\left(G_{i}\right)>\max \left\{\binom{2 k+1}{2},\binom{k}{2}+k(n-k)\right\}$ for $i=1,2, \ldots, k+1$, then there is a rainbow matching of size $k+1$.

Let $G_{1}$ and $G_{2}$ be two disjoint subgraphs of $G$. We use $G_{1} \cup G_{2}$ to denote the union of $G_{1}$ and $G_{2}$ with the vertex set being $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and the edge set being $E\left(G_{1}\right) \cup E\left(G_{2}\right)$. We use $G_{1} \vee G_{2}$ to denote the join graph of $G_{1}$ and $G_{2}$ with the vertex set being $V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and the edge set being $E\left(G_{1}\right) \cup E\left(G_{2}\right) \cup\{x y$ : $\left.x \in V\left(G_{1}\right), y \in V\left(G_{2}\right)\right\}$. Denote by $E_{n}$ the empty graph on $n$ vertices. It is easy to see that one of $K_{2 k+1} \cup E_{n-2 k-1}$ and $K_{k} \vee E_{n-k}$ achieves the maximum number of edges among all $M_{k+1}$-free graphs.

In this paper, we determine the Turán number of $\mathcal{F}=\left\{K_{r+1}, M_{k+1}\right\}$ for $n>3 k$.

Theorem 1.3. For $n \geq 3 k+1$,

$$
\operatorname{ex}\left(n,\left\{M_{k+1}, K_{r+1}\right\}\right)=t_{r-1}(k)+k(n-k) .
$$

Note that for $r \geq k+1$ and $n \geq 3 k+1$, by Theorem 1.2 and Theorem 1.3 we infer

$$
\operatorname{ex}\left(n,\left\{M_{k+1}, K_{r+1}\right\}\right)=\binom{k}{2}+k(n-k)=\operatorname{ex}\left(n, M_{k+1}\right) .
$$

Obviously, $T_{r-1}(k) \vee E_{n-k}$ is an $\left\{M_{k+1}, K_{r+1}\right\}$-free graph that achieves the maximum number of edges.

For any $K \subseteq V(G), K$ is called a vertex cover set of $G$ if each edge of $G$ has at least one endpoint in $K$. A vertex cover set with the minimum size is called a minimum vertex cover set. The vertex covering number $\tau(G)$ is defined as the size of a minimum vertex cover set of $G$. In [12], Fănică found the relation between the matching number and the vertex covering number. In this paper, we determine the maximum number of edges in a graph $G$ with $\nu(G) \leq k$ and $\tau(G) \geq k+r$.

Theorem 1.4. Let $G$ be an n-vertex graph with $\nu(G) \leq k$ and $\tau(G) \geq k+r$. For $n \geq 2 k$ and $r \leq k$,

$$
e(G) \leq \max \left\{\binom{2 k+1}{2},\binom{k+r+1}{2}+(k-r)(n-k-r-1)\right\} .
$$

For sets $A_{1}, A_{2}$, let $A_{1} \triangle A_{2}$ denote the symmetric difference set of $A_{1}$ and $A_{2}$, i.e., $\left(A_{1} \backslash A_{2}\right) \cup\left(A_{2} \backslash A_{1}\right)$.

Let us recall two techniques that are needed in our proofs. For self-containedness, we give a formal definition of the alternating path used in $[1,6,7]$.

Definition $1.5[1,6,7]$. Let $G$ be a graph with $\nu(G)=s<n / 2$. Let $M$ be a maximum matching of $G$ and let $Y$ be the set of vertices that are not covered by $M$. A directed path $P=v_{0} v_{1} v_{2} \cdots v_{m}$ in $G$ is called an $M$-alternating path if it satisfies conditions (i), (ii) and (iii).
(i) $v_{0} \in Y$;
(ii) $v_{i} v_{i+1} \in M$ for any odd $i$ with $1 \leq i \leq m-1$;
(iii) $v_{i} v_{i+1} \notin M$ for any even $i$ with $0 \leq i \leq m-1$.

When it is clear from the context, we simply call $P$ an alternating path. Clearly, $v_{m} \notin Y$. Otherwise, $E(P) \triangle M$ will be a matching of size $|M|+1$, a contradiction.

An $M$-augmenting path is an $M$-alternating path whose origin $v_{0}$ and terminus $v_{m}$ are in $Y$. Clearly, if $M$ is a maximum matching of $G$, then there is no
$M$-augmenting path in $G$. Otherwise, let $P$ be an $M$-augmenting path. Then $E(P) \triangle M$ is a matching of size $|M|+1$, contradicting the maximality of $|M|$.

If a matching of $G$ covers all the vertices, then it is called a perfect matching of $G$. The Tutte-Berge formula is a central result concerning the maximum matchings in graphs. Let odd $(G)$ denote the number of connected components of odd order in $G$. In 1947, Tutte [20] obtained a sufficient and necessary condition for $G$ to guarantee a perfect matching. That is, odd $(G-A) \leq|A|$ for all $A \subseteq$ $V(G)$. In other words, if odd $(G-A) \leq|A|$ holds for all $A \subseteq V(G)$, then there is a perfect matching of $G$.

In 1958, Berge extended Tutte's result to graphs without perfect matchings and determined a formula for the matching number of $G$.
Theorem 1.6 [6]. Let $M$ be a maximum matching of $G$. Let $G-A$ denote the subgraph obtained from $G$ by deleting vertices in $A$ from $G$. Then

$$
|M|=\frac{1}{2} \min _{A \subseteq V(G)}\{|A|-\operatorname{odd}(G-A)+|V(G)|\} .
$$

This result is known as the Tutte-Berge formula. For related researches please see $[5,8,9,13,14,17,21]$.

Another technique we need in the proofs is the Zykov symmetrization. In 1949, Zykov [25] invented this method to show that $T_{r}(n)$ is the only $K_{r+1}$-free graph of order $n$ which maximizes the number of copies of $K_{s}$ with $2 \leq s \leq r$, which is a generalized version of Theorem 1.1.

In our proofs, we also need the following lemma.
Lemma 1.7. For $x \geq 0, t_{r}(x)$ is a convex function.
Proof. Note that

$$
t_{r}(x+1)-t_{r}(x)=x+1-\left\lceil\frac{x+1}{r}\right\rceil, t_{r}(x)-t_{r}(x-1)=x-\left\lceil\frac{x}{r}\right\rceil .
$$

It follows that

$$
t_{r}(x+1)-2 t_{r}(x)+t_{r}(x-1)=1-\left(\left\lceil\frac{x+1}{r}\right\rceil-\left\lceil\frac{x}{r}\right\rceil\right) \geq 0
$$

Thus, $t_{r}(x)$ is a convex function for $x \geq 0$.
Finally, let us recall some notations. For any $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph with vertex set $X$ and edge set $\{u v \in E(G): u, v \in X\}$. When the content is clear, we often use $e(X)$ to denote $e(G[X])$. Let $G-X=$ $G[V(G) \backslash X]$. Let

$$
N_{G}(X)=\{v \in V(G) \backslash X: \text { there exists a } u \in X \text { such that } u v \in E(G)\} .
$$

For $X=\{x\}$, we simply write $N_{G}(x)$. We use $\operatorname{deg}_{G}(x)$ to denote the cardinality of $N_{G}(x)$. We often omit subscripts when there is no confusion.

## 2. Proof of Theorem 1.3

In this section, we study the Turán number of $\left\{M_{k+1}, K_{r+1}\right\}$ by using the alternating path technique and the Zykov symmetrization.
Proof of Theorem 1.3. Let $G$ be an $\left\{M_{k+1}, K_{r+1}\right\}$-free graph with maximum number of edges. Let

$$
M=\left\{x_{1} y_{1}, x_{2} y_{2}, \ldots, x_{s} y_{s}\right\}
$$

be a maximum matching of $G$. Since $G$ is $M_{k+1}$ free, we infer $s \leq k$. Let

$$
X=\left\{x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{s}, y_{s}\right\}, Y=V(G) \backslash X .
$$

Obviously, $Y$ is an independent set.
Now let us partition $V(G)$ into four classes by the alternating path method. For every $M$-alternating path $P=v_{0} v_{1} v_{2} \cdots v_{2 m}$, label vertices $v_{1}, v_{3}, \ldots, v_{2 m-1}$ with the symbol $l$, label vertices $v_{0}, v_{2}, \ldots, v_{2 m}$ with the symbol $b$. Then the vertices in $G$ are partitioned into four types: vertices labeled $l$, vertices labeled $b$, vertices labeled $l$ and $b$ and unlabeled vertices. If a vertex is labeled $l$ and $b$, we also say that it is labeled $l b$. We use $L, B, L B$ and $\Phi$ to denote the set of these four types of vertices, respectively. Obviously $(L, B, \Phi, L B)$ is a partition of $V(G)$ and $Y \subset B$.

Claim 1. $B$ is an independent set.
Proof. Suppose for contradiction that $x y \in E(G[B])$. Note that by $x, y \in B$, $x, y$ are both labeled $b$. Since $Y$ is an independent set, $\{x, y\} \cap(B \backslash Y) \neq \emptyset$. Without loss of generality, we assume that $x \in B \backslash Y$. Let $P_{1}=v_{0} v_{1} \cdots v_{2 m} x^{\prime} x$ be an alternating path with terminal vertex $x$. Clearly $x^{\prime} x \in M$. If $y$ is not in $V\left(P_{1}\right)$, then $P=P_{1} y$ is an alternating path. It implies that $y$ is also labeled $l$, which contradicts $y \in B$. Thus $y$ is in $V\left(P_{1}\right)$. But then let $P_{2}$ be the sub-path of $P_{1}$ with terminal vertex $y$. It follows that $P_{2} x$ is an alternating path. By the label of $y$, we infer that $x$ is labeled $l$ as well, contradicting $x \in B$.

The following claim is a well known result (see, e.g., [7]). Here we give a proof for self-containedness.

Claim 2. There are three kinds of connected components of $G-L$.
(I) An isolated vertex in $B$.
(II) A connected component consisting of even number of vertices in $\Phi$.
(III) A connected component consisting of a vertex in $B$ and even number of vertices in $L B$.

Proof. Let $C \subset V(G)$ be a connected component of $G-L$.
(I) By Claim 1, $B$ is an independent set. If $C \subset B$, then $C$ contains exactly one vertex.
(II) Suppose that $C$ contains some unlabeled vertex. Let $x$ be such a vertex. Note that all the vertices in $Y$ are labeled $b$. It follows that $x \in X$. To show (II), it suffices to show that all the neighbors of $x$ are also unlabeled. Let $x y \in M$. Clearly $y$ is also unlabeled. For any $z \in N_{C}(x) \backslash\{y\}$, if $z$ is labeled then $z$ is labeled either $b$ or $l b$. In either case there exists an alternating path $P=v_{0} v_{1} \cdots v_{2 m} z^{\prime} z$ with terminal vertex $z$ and $z^{\prime} z \in M$. Since $x$ is unlabeled, $x \notin V(P)$. Then $P x$ is also an alternating path. It follows that $x$ is labeled $l$, a contradiction. Thus all the neighbors of $x$ are also unlabeled. Obviously, if one endpoint of some edge in $M$ is unlabeled, so is the other endpoint and they have to fall into the same component of $G-L$. Therefore, $C$ is an connected component with even number of unlabeled vertices.
(III) If there is a vertex in $C$ labeled $l b$, noting that the vertices with label $l b$ appear in pairs (two endpoints of a matching edge), we see that the number of vertices in $C$ with label $l b$ is even.

Let $x x^{\prime} \in M$ with $x, x^{\prime} \in C \cap L B$. First we show that there is a vertex in $C$ labeled $b$. Let $P=v_{0} v_{1} \cdots v_{2 m} x^{\prime} x$ be an alternating path. Clearly $v_{0} \in Y$ is labeled $b$. Let $v_{i}$ be the last vertex on $P$ that is not labeled $l b$. Since $C$ is a connected component of $G-L, v_{i+1}, \ldots, v_{2 m} \in V(C)$. Since the vertices labeled $l b$ appear in pairs in $P, v_{i} \in B$. Moreover, if $i \geq 1$ then $v_{i-1} v_{i} \in M$ and $v_{i-1} \in L$. Thus $v_{i} \in V(C) \cap B$.

We are left to show that $|V(C) \cap B|=1$. First we show that one can choose a vertex $y \in V(C) \cap B$ and an alternating path $P_{0}$ with terminal vertex $y$ such that $V\left(P_{0}\right) \cap V(C)=y$. If $V(C) \cap Y \neq \emptyset$ then choose $y \in V(C) \cap Y$ and simply set $P_{0}=y$. If $V(C) \cap Y=\emptyset$ then choose $z \in V(C) \cap B$. Since $z \in B \backslash Y$, there is an alternating path $P=u_{0} u_{1} \cdots u_{2 p-1} u_{2 p} z^{\prime} z$. Choose $y$ be the first vertex on $P$ that is in $V(C)$ and let $P_{0}$ be the sub-path of $P$ with terminal vertex $y$. Clearly $V\left(P_{0}\right) \cap V(C)=\{y\}$. Since $C$ is a connected component in $G-L$, the predecessor of $y$ on $P_{0}$ has to be in $L$. It follows that $y \in B$. Thus we find a vertex $y \in V(C) \cap B$ and an alternating path $P_{0}$ with terminal vertex $y$ satisfying $V\left(P_{0}\right) \cap V(C)=\{y\}$.

Let $S$ be the set of all vertices $x$ in $V(C)$ so that there exists an alternating path $P=P_{0} P^{\prime} x$ and $V\left(P^{\prime}\right) \subset V(C)$. Clearly $y \in S$. We claim that every vertex in $S \backslash\{y\}$ is labeled $l b$. Indeed, otherwise there exists $x \in S \cap B$. Let $x^{\prime} x \in M$ and let $P_{0} P_{1} x$ be an alternating path. Since $x^{\prime} \in L, x^{\prime}$ is not a vertex on path $P_{0} P_{1} x$. It follows that $x$ has to be labeled $l$ on $P_{0} P_{1} x$, contradicting $x \in B$. Thus $y$ is the unique vertex in $S$ that is labeled $b$.

Since $C$ is connected, there exists $x \in N_{C}(y)$. Note that $B$ is an independent set. It follows that $x \in L B$. Thus $S \cap L B \neq \emptyset$. Next we show that for each $x, x^{\prime} \in S \cap L B$ with $x x^{\prime} \in M$, there exist alternating paths $P_{0} P_{1} x x^{\prime}$ and $P_{0} P_{1}^{\prime} x^{\prime} x$
such that $V\left(P_{1}\right) \subset V(C)$ and $V\left(P_{1}^{\prime}\right) \subset V(C)$. Since $x, x^{\prime} \in S$, by symmetry there is an alternating path $P_{0} P_{1} x x^{\prime}$ with $V\left(P_{1}\right) \subset V(C)$. We are left to find an alternating path ending with $x^{\prime} x$. Since $x, x^{\prime}$ are labeled $l b$, there exists an alternating path $P=v_{0} v_{1} \cdots v_{2 p} x^{\prime} x$. Let $z^{\prime}$ be the last vertex on $P$ that is not in $V(C)$ and let $z$ be the successor of $z^{\prime}$ on $P$. Then clearly $z^{\prime} z \in M$ and $z \in B \cap V(C)$. Let $P_{2} x^{\prime} x$ be the sub-path of $P$ with start vertex $z$. If $V\left(P_{2}\right) \cap V\left(P_{1}\right)=\emptyset$ then $P_{0} P_{1} x x^{\prime} P_{2}^{-1}$ is an alternating path with terminal vertex $z$, where $P_{2}^{-1}$ is the reverse path of $P_{2}$. It follows that $z \in S$, contradicting $S \cap B=\{y\}$. Hence $V\left(P_{2}\right) \cap V\left(P_{1}\right) \neq \emptyset$.

If $z=y$ then $P_{0} P_{2} x^{\prime} x$ is an alternating path ending with $x^{\prime} x$. Thus we assume $z \neq y$. Note that $V\left(P_{2}\right) \cap V\left(P_{1}\right) \neq \emptyset$ implies that $P_{1}$ and $P_{2}$ intersect in a matching edge in $C$ whose endpoints are both labeled $l b$. Let $w w^{\prime}$ be the first edge appeared in both $P_{1}$ and $P_{2}$. Now by symmetry there are two cases: (a) $P_{1}=P_{11} w w^{\prime} P_{12}, P_{2}=P_{21} w w^{\prime} P_{22}$ and (b) $P_{1}=P_{11} w w^{\prime} P_{12}, P_{2}=P_{21} w^{\prime} w P_{22}$. For case (a), $P_{11} w w^{\prime} P_{22} x^{\prime} x$ is an alternating path ending with $x^{\prime} x$. For case (b) $P_{11} w w^{\prime} P_{21}^{-1}$ is an alternating path connecting $y$ and $z$. It implies that $z \in S$, contradicting $S \cap B=\{y\}$. Thus, for each $x, x^{\prime} \in S \cap L B$ with $x x^{\prime} \in M$, there exist alternating paths $P_{0} P_{1} x x^{\prime}$ and $P_{0} P_{1}^{\prime} x^{\prime} x$ such that $V\left(P_{1}\right) \subset V(C)$ and $V\left(P_{1}^{\prime}\right) \subset V(C)$.

Finally, we show that $S=V(C)$. Suppose to the contrary that there exist $z \in S$ and $w \in V(C) \backslash S$ such that $z w \in E(G)$. Let $z z^{\prime} \in M$. Clearly $z^{\prime} \in$ $S$. It follows that $z w \notin M$. Since there exists an alternating path $P_{0} P_{1} z^{\prime} z$ with $V\left(P_{1}\right) \subset V(C), P_{0} P_{1} z^{\prime} z w$ is an alternating path. It implies that $w \in S$, contradicting our assumption that $w \notin S$. Thus $S=V(C)$. Together with $S \cap B=\{y\}$, we conclude that (III) holds.

Definition 2.1. For any $u, v \in V(G)$, define $G[u \leftarrow v]$ as a new graph obtained from $G$ by removing edges adjacent to $u$ and adding all edges in $\left\{u x: x \in N_{G}(v)\right\}$.

Claim 3. If $G$ is $\left\{M_{k+1}, K_{r+1}\right\}$-free, then for any $u, v \in L, G[u \leftarrow v]$ is also $\left\{M_{k+1}, K_{r+1}\right\}$-free.

Proof. Let $G^{\prime}=G[u \leftarrow v]$. Suppose to the contrary that $G[u \leftarrow v]$ is not $\left\{M_{k+1}, K_{r+1}\right\}$-free. Then $G[u \leftarrow v]$ has either a copy of $K_{r+1}$ or a matching of size $k+1$.

If there exists some $R \subset V\left(G^{\prime}\right)$ such that $G^{\prime}[R] \cong K_{r+1}$, then $u \in R$. Since $u v \notin E\left(G^{\prime}\right), v \notin R$. Since the neighbors of $u$ in $G^{\prime}$ are also neighbors of $v$ in $G$, it follows that $G[R \backslash\{u\} \cup\{v\}] \cong K_{r+1}$, a contradiction. Therefore $G^{\prime}$ is $K_{r+1}$-free.

Since $u, v \in L$, we see that $E(G)$ and $E\left(G^{\prime}\right)$ differ only in edges with at least one endpoint in $L$. It implies that $G^{\prime}-L=G-L$. According to Claim 2, there are only three types of connected components of $G^{\prime}-L$ : isolated vertices in $B$, connected components consisting of even number of vertices in $\Phi$, and odd
connected components consisting of a vertex in $B$ and even number of vertices in $L B$. Consequently,

$$
\nu\left(G^{\prime}\right) \leq \frac{|\Phi|}{2}+\frac{|L B|}{2}+|L|=\nu(G) \leq k
$$

that is, $G^{\prime}$ is $M_{k+1}$-free, a contradiction. Thus the claim holds.
Claim 4. $G[L]$ is a complete multi-partite graph.
Proof. Let $G$ be a $\left\{K_{r+1}, M_{k+1}\right\}$-free graph with maximal number of edges. We define a binary relation $\sim$ on $L$. For any $u, v \in L, u \sim v$ if and only if $u v \notin E(G)$. We claim that $\sim$ is an equivalence relation. For any $u \in L, u u \notin E(G)$, so $\sim$ is reflexive. If $v u \notin E(G)$, then $u v \notin E(G)$, so $\sim$ is symmetric.

Now we prove $\sim$ is transitive. Otherwise, suppose there are $u, v, w \in L$ such that $v u, u w \notin E(G)$, then $v w \in E(G)$.

Case 1. $d_{G}(u)<d_{G}(v)$ or $d_{G}(u)<d_{G}(w)$. By symmetry, assume that $d_{G}(u)<d_{G}(v)$. Let $G^{\prime}=G[u \leftarrow v]$. By Claim 3, we know that $G^{\prime}$ is also $\left\{M_{k+1}, K_{r+1}\right\}$-free and $e\left(G^{\prime}\right)>e(G)$, contradicting the maximality of $e(G)$.

Case 2. $d_{G}(u) \geq d_{G}(v)$ and $d_{G}(u) \geq d_{G}(w)$. Let $G^{\prime \prime}=G[v \leftarrow u][w \leftarrow u]$. By Claim 3 again, $G^{\prime \prime}$ is also $\left\{M_{k+1}, K_{r+1}\right\}$-free and $e\left(G^{\prime \prime}\right)>e(G)$, a contradiction.

Thus, the relation $\sim$ on $L$ is an equivalence relation and $G[L]$ is a complete multi-partite graph.

Let $|L|=x, 0 \leq x \leq s$. Since $G$ is $K_{r+1}$-free, by Theorem 1.1 we infer $e(L) \leq t_{r}(x)$. By Claim 2, there are three types of connected components in $G-L$. Let $t$ be the number of connected components of type (II) and let $y_{1}, y_{2}, \ldots, y_{t}$ be the number of vertices in these connected components, respectively. Clearly, $|\Phi|=y=y_{1}+y_{2}+\cdots+y_{t}$.

For $y_{1}, y_{2}>0$, the following inequality can be checked directly:

$$
\begin{equation*}
\binom{y_{1}}{2}+\binom{y_{2}}{2} \leq\binom{ y_{1}+y_{2}}{2} . \tag{2.1}
\end{equation*}
$$

Applying (2.1) repeatedly, we obtain that

$$
\begin{equation*}
e(\Phi) \leq\binom{ y_{1}}{2}+\binom{y_{2}}{2}+\cdots+\binom{y_{t}}{2} \leq\binom{ y}{2} . \tag{2.2}
\end{equation*}
$$

Let $p$ be the number of connected components of type (III) and let $2 z_{1}+$ $1,2 z_{2}+1, \ldots, 2 z_{p}+1$ be the number of vertices in these connected components, respectively. Let $B_{0}$ be the set of vertices labeled $b$ in the union of connected components of type (III). Since each connected components of type (III) has
exactly one vertex labeled $b$, we infer $\left|B_{0}\right|=p$ and $|L B|=2\left(z_{1}+z_{2}+\cdots+z_{p}\right)=$ $2 s-2 x-y$.

For $z_{1}, z_{2} \geq 1$, it is easy to verify that

$$
\begin{equation*}
\binom{2 z_{1}+1}{2}+\binom{2 z_{2}+1}{2} \leq\binom{ 2 z_{1}+2 z_{2}+1}{2} \tag{2.3}
\end{equation*}
$$

Let $L B \cup B_{0}$ be the union of connected components of type (III). By applying (2.3) repeatedly, we get

$$
\begin{align*}
e\left(L B \cup B_{0}\right) & \leq\binom{ 2 z_{1}+1}{2}+\binom{2 z_{2}+1}{2}+\cdots+\binom{2 z_{p}+1}{2}  \tag{2.4}\\
& \leq\binom{ 2\left(z_{1}+\cdots+z_{p}\right)+1}{2}=\binom{2 s-2 x-y+1}{2}
\end{align*}
$$

By Claim 4, $G[L]$ is a complete multi-partite graph. Since $G$ is $K_{r+1}$-free, $G[L]$ is $t$-paritite with $t \leq r$. We distinguish two cases according to $t$.

Case 1. $t \leq r-1$. Since $G[L]$ is a complete $t$-partite graph and $t \leq r-1$, we see that $e(L) \leq t_{r-1}(x)$. By (2.2) and (2.4), we obtain that

$$
\begin{aligned}
e(G) & =e(L)+e\left(L B \cup B_{0}\right)+e(\Phi)+e(L, V(G) \backslash L) \\
& \leq t_{r-1}(x)+\binom{2 s-2 x-y+1}{2}+\binom{y}{2}+|L|(n-|L|) \\
& \leq t_{r-1}(x)+\binom{2 s-2 x-y+1}{2}+\binom{y}{2}+x(n-x)
\end{aligned}
$$

By (2.1) we infer that

$$
\begin{equation*}
e(G) \leq t_{r-1}(x)+\binom{2 s-2 x+1}{2}+x(n-x)=: g(x) \tag{2.5}
\end{equation*}
$$

By Lemma 1.7 we know that $t_{r-1}(x)$ is a convex function. Since both $t_{r-1}(x)$ and $\binom{2 s-2 x+1}{2}+x(n-x)$ are convex functions, we infer that $g(x)$ is also a convex function for $0 \leq x \leq s \leq k$. Hence,

$$
\begin{aligned}
e(G) \leq \max \{g(0), g(k)\} & =\max \left\{\binom{2 s+1}{2}, t_{r-1}(k)+k(n-k)\right\} \\
& \leq \max \left\{\binom{2 k+1}{2}, t_{r-1}(k)+k(n-k)\right\}
\end{aligned}
$$

For $n \geq 3 k+1$, we conclude that

$$
\binom{2 k+1}{2}=k(2 k+1) \leq k(n-k)<t_{r-1}(k)+k(n-k)
$$

and Theorem 1.3 holds.
Case 2. $t=r$. Let $L_{1}, L_{2}, \ldots, L_{r}$ be $r$ partite sets of $G[L]$. It is obvious that $|L|=x \geq r$. Since $G[L]$ is a complete $r$-partite graph, for any $u \in V(G) \backslash L$ there exists some $i$ such that $L_{i} \cap N_{G}(u)=\emptyset$. Otherwise if there exists $v_{i} \in L_{i}$ such that $u v_{i} \in E(G)$ for each $i=1,2, \ldots, r$, then $\left\{u, v_{1}, v_{2}, \ldots, v_{r}\right\}$ spans a copy of $K_{r+1}$, a contradiction. By symmetry, assume that $\left|L_{1}\right| \leq\left|L_{2}\right| \leq \cdots \leq\left|L_{r}\right|$. Let $\left|L_{1}\right|=z$. Clearly $1 \leq z \leq\lfloor x / r\rfloor$. Then

$$
e(L, V(G) \backslash L) \leq(n-|L|)(|L|-z)=(n-x)(x-z) .
$$

Note that $|B|=n-2 s+x$ and $\left|B_{0}\right| \leq|L B| / 2 \leq s-|L|=s-x$. By (2.2) and (2.4), we infer that

$$
\begin{aligned}
e(G) & =e(L)+e\left(L B \cup B_{0}\right)+e(\Phi)+e(L, V(G) \backslash L) \leq\left|L_{1}\right|\left(|L|-\left|L_{1}\right|\right) \\
& +t_{r-1}\left(|L|-\left|L_{1}\right|\right)+\binom{2 s-2 x-y+1}{2}+\binom{y}{2}+(n-x)(x-z) \\
& =z(x-z)+t_{r-1}(x-z)+\binom{2 s-2 x-y+1}{2}+\binom{y}{2}+(n-x)(x-z) \\
& =(n-x+z)(x-z)+t_{r-1}(x-z)+\binom{2 s-2 x-y+1}{2}+\binom{y}{2} .
\end{aligned}
$$

By (2.1), we infer that

$$
\begin{equation*}
e(G) \leq(n-x+z)(x-z)+t_{r-1}(x-z)+\binom{2 s-2 x+1}{2}=: g(x, z) . \tag{2.6}
\end{equation*}
$$

Since for $r \leq x \leq s \leq k, n \geq 2 k$ and $1 \leq z \leq\lfloor x / r\rfloor, g(x, z)$ is a decreasing function with respect to $z$. Using $z \geq 1$, we get

$$
\begin{aligned}
g(x, z) \leq g(x, 1) & =(n-x+1)(x-1)+t_{r-1}(x-1)+\binom{2 s-2 x+1}{2} \\
& \leq(n-x+1)(x-1)+t_{r-1}(x-1)+\binom{2 k-2 x+1}{2}=: h(x) .
\end{aligned}
$$

It is easy to verify that $h(x)$ is also a convex function for $1 \leq r \leq x \leq s \leq k$. Therefore,

$$
e(G) \leq \max \{h(1), h(k)\}=\max \left\{\binom{2 k-1}{2},(n-k+1)(k-1)+t_{r-1}(k-1)\right\} .
$$

Note that for $n \geq 3 k$,

$$
\begin{aligned}
t_{r-1}(k)+k(n-k)-h(k) & =t_{r-1}(k)+k(n-k)-t_{r-1}(k-1)-(n-k+1)(k-1) \\
& =n-2 k+1+k-\left\lceil\frac{k}{r-1}\right\rceil \geq n-2 k-\frac{(r-2) k}{r-1}>0 .
\end{aligned}
$$

It follows that $h(k) \leq t_{r-1}(k)+k(n-k)$.
For $n \geq 3 k-2$, we have

$$
t_{r-1}(k)+k(n-k)-\binom{2 k-1}{2}>k n-k^{2}-2 k^{2}+3 k-1 \geq 0 .
$$

It follows that $h(1) \leq t_{r-1}(k)+k(n-k)$.
Consequently $e(G) \leq t_{r-1}(k)+k(n-k)$ for $n \geq 3 k+1$ and the theorem holds.

## 3. Proof of Theorem 1.4

By a similar approach as in the proof of Theorem 1.3, we determine the maximum number of edges in a graph with $\nu(G) \leq k$ and $\tau(G) \geq k+r$.
Proof of Theorem 1.4. Let $M$ be a maximal matching in $G$ and let $L, L B, \Phi, B$ be the partition of $V(G)$ obtained by the alternating path method.

By Claim 1, we infer that $L \cup L B \cup \Phi$ is a vertex cover of $G$. Since $\tau(G) \geq k+r$, it follows that $|L|+|L B|+|\Phi| \geq k+r$. By Claim 2, for any $x y \in M$, there are three possibilities: one of $x, y$ is labeled $l$ and the other is labeled $b$, or $x, y$ are both labeled $l b$, or $x, y$ are both unlabeled. Since $\nu(G) \leq k$ implies $|L|+|L B| / 2+$ $|\Phi| / 2 \leq k$, it follows that $|L B| / 2+|\Phi| / 2 \geq r$ and $|L| \leq k-|L B| / 2-|\Phi| / 2 \leq k-r$.

Let $|L|=x$ and $|\Phi|=y$. Then $0 \leq x \leq k-r, 0 \leq y \leq 2 k-2 x$. By (2.2) and (2.4), we obtain that

$$
\begin{align*}
e(G) & =e(L)+e(\Phi)+e\left(L B \cup B_{0}\right)+e(L, V(G) \backslash L) \\
& \leq\binom{|L|}{2}+\binom{y}{2}+\binom{2 k-2 x-y+1}{2}+|L|(n-|L|)  \tag{3.1}\\
& \leq\binom{ x}{2}+\binom{y}{2}+\binom{2 k-2 x-y+1}{2}+x(n-x) .
\end{align*}
$$

By (2.1), we have

$$
\begin{equation*}
e(G) \leq\binom{ 2 k-x+1}{2}+x(n-x)=: f(x) \tag{3.2}
\end{equation*}
$$

It is easy to verify that $f(x)$ is a convex function for $0 \leq x \leq k-r$. Therefore,

$$
\begin{aligned}
e(G) & \leq \max \{f(0), f(k-r)\} \\
& =\max \left\{\binom{2 k+1}{2},\binom{k+r+1}{2}+(k-r)(n-k-r-1)\right\},
\end{aligned}
$$

and Theorem 1.4 holds.

Remark. Very recently, Alon and Frankl [3] determined $\operatorname{ex}\left(n,\left\{M_{k+1}, K_{r+1}\right\}\right)$ for all values of $n$ by a very nice argument.

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