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THE MAXIMUM NUMBER OF EDGES IN A $\{K_{r+1}, M_{k+1}\}$ -FREE GRAPH

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Abstract

Let G be a graph and \mathcal{F} be a family of graphs. We say G is \mathcal{F} -free if it does not contain F as subgraph for any $F \in \mathcal{F}$. The Turán number $ex(n, \mathcal{F})$ is defined as the maximum number of edges in an \mathcal{F} -free graph on n vertices. Let K_{r+1} denote the complete graph on r+1 vertices and let M_{k+1} denote the graph on 2k + 2 vertices with k + 1 pairwise disjoint edges. By using the alternating path technique and the Zykov symmetrization, we determine that for n > 3k,

$$ex(n, \{M_{k+1}, K_{r+1}\}) = t_{r-1}(k) + k(n-k),$$

where $t_{r-1}(k)$ is the number of edges in an (r-1)-partite k-vertex Turán graph. Let $\nu(G)$, $\tau(G)$ denote the matching number and the vertex cover number of G, respectively. For $n \geq 2k$, we prove that if $\nu(G) \leq k$ and $\tau(G) \geq k + r$, then

$$e(G) \le \max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\}.$$

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1. INTRODUCTION

Let G(V, E) be a simple undirected graph with vertex set V(G) and edge set E(G). We use e(G) to denote the size of E(G). For a graph H, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, then H is called a subgraph of G. Let \mathcal{F} be a family of graphs. If for every $F \in \mathcal{F}$, G does not contain F as subgraph, then we say G is \mathcal{F} -free. The Turán number of \mathcal{F} , denoted by $ex(n, \mathcal{F})$, is defined as the maximum number of edges in an \mathcal{F} -free graph on n vertices. For $\mathcal{F} = \{F\}$, we simply write ex(n, F). The study of the Turán numbers plays a central role in the extremal graph theory. The Turán number of many graphs have been determined, see [4, 10, 11, 15, 16, 18, 19, 22, 23, 24, etc.]

Let $T_r(n)$ denote the Turán graph on n vertices, i.e., the complete r-partite graph of order n with each partite of sizes $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Note that for n < r, $T_r(n)$ represents the complete graph on n vertices. We use $t_r(n)$ to denote the number of edges of $T_r(n)$. In 1941, Turán [19] showed that the Turán graph $T_r(n)$ is the only K_{r+1} -free graph attaining the maximum number of edges.

Theorem 1.1 [19]. $ex(n, K_{r+1}) = t_r(n)$.

For any $M \subset E(G)$, if the edges of M are pairwise disjoint, then M is called a matching of G. The matching number $\nu(G)$ is the size of a maximum matching in G. We often use M_{k+1} to denote the graph on 2k + 2 vertices with k + 1pairwise disjoint edges. In 1959, Erdős-Gallai [10] determined the Turán number of M_{k+1} .

Theorem 1.2 [10]. For $n \ge 2k + 1$,

$$\operatorname{ex}(n, M_{k+1}) = \max\left\{ \binom{2k+1}{2}, \binom{k}{2} + k(n-k) \right\}.$$

It should be mentioned that by using the shifting technique, Akiyama and Frankl [2] give a short proof of Theorem 1.2 and their proof also works for an rainbow extension of Theorem 1.2. That is, if $G_1, G_2, \ldots, G_{k+1}$ are k+1 graphs on the same vertex set of size n and $e(G_i) > \max\left\{\binom{2k+1}{2}, \binom{k}{2} + k(n-k)\right\}$ for $i = 1, 2, \ldots, k+1$, then there is a rainbow matching of size k+1.

Let G_1 and G_2 be two disjoint subgraphs of G. We use $G_1 \cup G_2$ to denote the union of G_1 and G_2 with the vertex set being $V(G_1) \cup V(G_2)$ and the edge set being $E(G_1) \cup E(G_2)$. We use $G_1 \vee G_2$ to denote the join graph of G_1 and G_2 with the vertex set being $V(G_1) \cup V(G_2)$ and the edge set being $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote by E_n the empty graph on n vertices. It is easy to see that one of $K_{2k+1} \cup E_{n-2k-1}$ and $K_k \vee E_{n-k}$ achieves the maximum number of edges among all M_{k+1} -free graphs.

In this paper, we determine the Turán number of $\mathcal{F} = \{K_{r+1}, M_{k+1}\}$ for n > 3k.

Theorem 1.3. For $n \ge 3k + 1$,

$$ex(n, \{M_{k+1}, K_{r+1}\}) = t_{r-1}(k) + k(n-k).$$

Note that for $r \ge k+1$ and $n \ge 3k+1$, by Theorem 1.2 and Theorem 1.3 we infer

$$\exp(n, \{M_{k+1}, K_{r+1}\}) = \binom{k}{2} + k(n-k) = \exp(n, M_{k+1}).$$

Obviously, $T_{r-1}(k) \vee E_{n-k}$ is an $\{M_{k+1}, K_{r+1}\}$ -free graph that achieves the maximum number of edges.

For any $K \subseteq V(G)$, K is called a vertex cover set of G if each edge of G has at least one endpoint in K. A vertex cover set with the minimum size is called a minimum vertex cover set. The vertex covering number $\tau(G)$ is defined as the size of a minimum vertex cover set of G. In [12], Fănică found the relation between the matching number and the vertex covering number. In this paper, we determine the maximum number of edges in a graph G with $\nu(G) \leq k$ and $\tau(G) \geq k + r$.

Theorem 1.4. Let G be an n-vertex graph with $\nu(G) \leq k$ and $\tau(G) \geq k+r$. For $n \geq 2k$ and $r \leq k$,

$$e(G) \le \max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\}.$$

For sets A_1, A_2 , let $A_1 \triangle A_2$ denote the symmetric difference set of A_1 and A_2 , i.e., $(A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

Let us recall two techniques that are needed in our proofs. For self-containedness, we give a formal definition of the alternating path used in [1, 6, 7].

Definition 1.5 [1, 6, 7]. Let G be a graph with $\nu(G) = s < n/2$. Let M be a maximum matching of G and let Y be the set of vertices that are not covered by M. A directed path $P = v_0 v_1 v_2 \cdots v_m$ in G is called an M-alternating path if it satisfies conditions (i), (ii) and (iii).

- (i) $v_0 \in Y;$
- (ii) $v_i v_{i+1} \in M$ for any odd *i* with $1 \le i \le m 1$;
- (iii) $v_i v_{i+1} \notin M$ for any even *i* with $0 \le i \le m 1$.

When it is clear from the context, we simply call P an alternating path. Clearly, $v_m \notin Y$. Otherwise, $E(P) \bigtriangleup M$ will be a matching of size |M| + 1, a contradiction.

An *M*-augmenting path is an *M*-alternating path whose origin v_0 and terminus v_m are in *Y*. Clearly, if *M* is a maximum matching of *G*, then there is no

M-augmenting path in *G*. Otherwise, let *P* be an *M*-augmenting path. Then $E(P) \bigtriangleup M$ is a matching of size |M| + 1, contradicting the maximality of |M|.

If a matching of G covers all the vertices, then it is called a perfect matching of G. The Tutte-Berge formula is a central result concerning the maximum matchings in graphs. Let odd(G) denote the number of connected components of odd order in G. In 1947, Tutte [20] obtained a sufficient and necessary condition for G to guarantee a perfect matching. That is, $odd(G - A) \leq |A|$ for all $A \subseteq$ V(G). In other words, if $odd(G - A) \leq |A|$ holds for all $A \subseteq V(G)$, then there is a perfect matching of G.

In 1958, Berge extended Tutte's result to graphs without perfect matchings and determined a formula for the matching number of G.

Theorem 1.6 [6]. Let M be a maximum matching of G. Let G - A denote the subgraph obtained from G by deleting vertices in A from G. Then

$$|M| = \frac{1}{2} \min_{A \subseteq V(G)} \{ |A| - \text{odd}(G - A) + |V(G)| \}.$$

This result is known as the Tutte-Berge formula. For related researches please see [5, 8, 9, 13, 14, 17, 21].

Another technique we need in the proofs is the Zykov symmetrization. In 1949, Zykov [25] invented this method to show that $T_r(n)$ is the only K_{r+1} -free graph of order n which maximizes the number of copies of K_s with $2 \leq s \leq r$, which is a generalized version of Theorem 1.1.

In our proofs, we also need the following lemma.

Lemma 1.7. For $x \ge 0$, $t_r(x)$ is a convex function.

Proof. Note that

$$t_r(x+1) - t_r(x) = x + 1 - \left\lceil \frac{x+1}{r} \right\rceil, \ t_r(x) - t_r(x-1) = x - \left\lceil \frac{x}{r} \right\rceil.$$

It follows that

$$t_r(x+1) - 2t_r(x) + t_r(x-1) = 1 - \left(\left\lceil \frac{x+1}{r} \right\rceil - \left\lceil \frac{x}{r} \right\rceil \right) \ge 0.$$

Thus, $t_r(x)$ is a convex function for $x \ge 0$.

Finally, let us recall some notations. For any $X \subseteq V(G)$, we use G[X] to denote the subgraph with vertex set X and edge set $\{uv \in E(G) : u, v \in X\}$. When the content is clear, we often use e(X) to denote e(G[X]). Let $G - X = G[V(G) \setminus X]$. Let

$$N_G(X) = \{ v \in V(G) \setminus X : \text{ there exists a } u \in X \text{ such that } uv \in E(G) \}.$$

For $X = \{x\}$, we simply write $N_G(x)$. We use $\deg_G(x)$ to denote the cardinality of $N_G(x)$. We often omit subscripts when there is no confusion.

2. Proof of Theorem 1.3

In this section, we study the Turán number of $\{M_{k+1}, K_{r+1}\}$ by using the alternating path technique and the Zykov symmetrization.

Proof of Theorem 1.3. Let G be an $\{M_{k+1}, K_{r+1}\}$ -free graph with maximum number of edges. Let

$$M = \{x_1y_1, x_2y_2, \dots, x_sy_s\}$$

be a maximum matching of G. Since G is M_{k+1} -free, we infer $s \leq k$. Let

$$X = \{x_1, y_1, x_2, y_2, \dots, x_s, y_s\}, \ Y = V(G) \setminus X.$$

Obviously, Y is an independent set.

Now let us partition V(G) into four classes by the alternating path method. For every *M*-alternating path $P = v_0 v_1 v_2 \cdots v_{2m}$, label vertices $v_1, v_3, \ldots, v_{2m-1}$ with the symbol *l*, label vertices v_0, v_2, \ldots, v_{2m} with the symbol *b*. Then the vertices in *G* are partitioned into four types: vertices labeled *l*, vertices labeled *b*, vertices labeled *l* and *b* and unlabeled vertices. If a vertex is labeled *l* and *b*, we also say that it is labeled *lb*. We use *L*, *B*, *LB* and Φ to denote the set of these four types of vertices, respectively. Obviously (L, B, Φ, LB) is a partition of V(G) and $Y \subset B$.

Claim 1. B is an independent set.

Proof. Suppose for contradiction that $xy \in E(G[B])$. Note that by $x, y \in B$, x, y are both labeled b. Since Y is an independent set, $\{x, y\} \cap (B \setminus Y) \neq \emptyset$. Without loss of generality, we assume that $x \in B \setminus Y$. Let $P_1 = v_0v_1 \cdots v_{2m}x'x$ be an alternating path with terminal vertex x. Clearly $x'x \in M$. If y is not in $V(P_1)$, then $P = P_1y$ is an alternating path. It implies that y is also labeled l, which contradicts $y \in B$. Thus y is in $V(P_1)$. But then let P_2 be the sub-path of P_1 with terminal vertex y. It follows that P_2x is an alternating path. By the label of y, we infer that x is labeled l as well, contradicting $x \in B$.

The following claim is a well known result (see, e.g., [7]). Here we give a proof for self-containedness.

Claim 2. There are three kinds of connected components of G - L.

- (I) An isolated vertex in B.
- (II) A connected component consisting of even number of vertices in Φ .
- (III) A connected component consisting of a vertex in B and even number of vertices in LB.

Proof. Let $C \subset V(G)$ be a connected component of G - L.

(I) By Claim 1, B is an independent set. If $C \subset B$, then C contains exactly one vertex.

(II) Suppose that C contains some unlabeled vertex. Let x be such a vertex. Note that all the vertices in Y are labeled b. It follows that $x \in X$. To show (II), it suffices to show that all the neighbors of x are also unlabeled. Let $xy \in M$. Clearly y is also unlabeled. For any $z \in N_C(x) \setminus \{y\}$, if z is labeled then z is labeled either b or lb. In either case there exists an alternating path $P = v_0v_1 \cdots v_{2m}z'z$ with terminal vertex z and $z'z \in M$. Since x is unlabeled, $x \notin V(P)$. Then Pxis also an alternating path. It follows that x is labeled l, a contradiction. Thus all the neighbors of x are also unlabeled. Obviously, if one endpoint of some edge in M is unlabeled, so is the other endpoint and they have to fall into the same component of G - L. Therefore, C is an connected component with even number of unlabeled vertices.

(III) If there is a vertex in C labeled lb, noting that the vertices with label lb appear in pairs (two endpoints of a matching edge), we see that the number of vertices in C with label lb is even.

Let $xx' \in M$ with $x, x' \in C \cap LB$. First we show that there is a vertex in C labeled b. Let $P = v_0v_1 \cdots v_{2m}x'x$ be an alternating path. Clearly $v_0 \in Y$ is labeled b. Let v_i be the last vertex on P that is not labeled lb. Since C is a connected component of G - L, $v_{i+1}, \ldots, v_{2m} \in V(C)$. Since the vertices labeled lb appear in pairs in $P, v_i \in B$. Moreover, if $i \geq 1$ then $v_{i-1}v_i \in M$ and $v_{i-1} \in L$. Thus $v_i \in V(C) \cap B$.

We are left to show that $|V(C) \cap B| = 1$. First we show that one can choose a vertex $y \in V(C) \cap B$ and an alternating path P_0 with terminal vertex y such that $V(P_0) \cap V(C) = y$. If $V(C) \cap Y \neq \emptyset$ then choose $y \in V(C) \cap Y$ and simply set $P_0 = y$. If $V(C) \cap Y = \emptyset$ then choose $z \in V(C) \cap B$. Since $z \in B \setminus Y$, there is an alternating path $P = u_0u_1 \cdots u_{2p-1}u_{2p}z'z$. Choose y be the first vertex on P that is in V(C) and let P_0 be the sub-path of P with terminal vertex y. Clearly $V(P_0) \cap V(C) = \{y\}$. Since C is a connected component in G - L, the predecessor of y on P_0 has to be in L. It follows that $y \in B$. Thus we find a vertex $y \in V(C) \cap B$ and an alternating path P_0 with terminal vertex y satisfying $V(P_0) \cap V(C) = \{y\}$.

Let S be the set of all vertices x in V(C) so that there exists an alternating path $P = P_0 P'x$ and $V(P') \subset V(C)$. Clearly $y \in S$. We claim that every vertex in $S \setminus \{y\}$ is labeled *lb*. Indeed, otherwise there exists $x \in S \cap B$. Let $x'x \in M$ and let P_0P_1x be an alternating path. Since $x' \in L$, x' is not a vertex on path P_0P_1x . It follows that x has to be labeled *l* on P_0P_1x , contradicting $x \in B$. Thus y is the unique vertex in S that is labeled b.

Since C is connected, there exists $x \in N_C(y)$. Note that B is an independent set. It follows that $x \in LB$. Thus $S \cap LB \neq \emptyset$. Next we show that for each $x, x' \in S \cap LB$ with $xx' \in M$, there exist alternating paths P_0P_1xx' and $P_0P'_1x'x$

such that $V(P_1) \subset V(C)$ and $V(P'_1) \subset V(C)$. Since $x, x' \in S$, by symmetry there is an alternating path P_0P_1xx' with $V(P_1) \subset V(C)$. We are left to find an alternating path ending with x'x. Since x, x' are labeled lb, there exists an alternating path $P = v_0v_1 \cdots v_{2p}x'x$. Let z' be the last vertex on P that is not in V(C) and let z be the successor of z' on P. Then clearly $z'z \in M$ and $z \in B \cap V(C)$. Let $P_2x'x$ be the sub-path of P with start vertex z. If $V(P_2) \cap V(P_1) = \emptyset$ then $P_0P_1xx'P_2^{-1}$ is an alternating path with terminal vertex z, where P_2^{-1} is the reverse path of P_2 . It follows that $z \in S$, contradicting $S \cap B = \{y\}$. Hence $V(P_2) \cap V(P_1) \neq \emptyset$.

If z = y then $P_0P_2x'x$ is an alternating path ending with x'x. Thus we assume $z \neq y$. Note that $V(P_2) \cap V(P_1) \neq \emptyset$ implies that P_1 and P_2 intersect in a matching edge in C whose endpoints are both labeled lb. Let ww' be the first edge appeared in both P_1 and P_2 . Now by symmetry there are two cases: (a) $P_1 = P_{11}ww'P_{12}, P_2 = P_{21}ww'P_{22}$ and (b) $P_1 = P_{11}ww'P_{12}, P_2 = P_{21}w'wP_{22}$. For case (a), $P_{11}ww'P_{22}x'x$ is an alternating path ending with x'x. For case (b) $P_{11}ww'P_{21}^{-1}$ is an alternating path connecting y and z. It implies that $z \in S$, contradicting $S \cap B = \{y\}$. Thus, for each $x, x' \in S \cap LB$ with $xx' \in M$, there exist alternating paths P_0P_1xx' and $P_0P_1'x'x$ such that $V(P_1) \subset V(C)$ and $V(P_1') \subset V(C)$.

Finally, we show that S = V(C). Suppose to the contrary that there exist $z \in S$ and $w \in V(C) \setminus S$ such that $zw \in E(G)$. Let $zz' \in M$. Clearly $z' \in S$. It follows that $zw \notin M$. Since there exists an alternating path $P_0P_1z'z$ with $V(P_1) \subset V(C)$, $P_0P_1z'zw$ is an alternating path. It implies that $w \in S$, contradicting our assumption that $w \notin S$. Thus S = V(C). Together with $S \cap B = \{y\}$, we conclude that (III) holds.

Definition 2.1. For any $u, v \in V(G)$, define $G[u \leftarrow v]$ as a new graph obtained from G by removing edges adjacent to u and adding all edges in $\{ux : x \in N_G(v)\}$.

Claim 3. If G is $\{M_{k+1}, K_{r+1}\}$ -free, then for any $u, v \in L$, $G[u \leftarrow v]$ is also $\{M_{k+1}, K_{r+1}\}$ -free.

Proof. Let $G' = G[u \leftarrow v]$. Suppose to the contrary that $G[u \leftarrow v]$ is not $\{M_{k+1}, K_{r+1}\}$ -free. Then $G[u \leftarrow v]$ has either a copy of K_{r+1} or a matching of size k+1.

If there exists some $R \subset V(G')$ such that $G'[R] \cong K_{r+1}$, then $u \in R$. Since $uv \notin E(G'), v \notin R$. Since the neighbors of u in G' are also neighbors of v in G, it follows that $G[R \setminus \{u\} \cup \{v\}] \cong K_{r+1}$, a contradiction. Therefore G' is K_{r+1} -free.

Since $u, v \in L$, we see that E(G) and E(G') differ only in edges with at least one endpoint in L. It implies that G' - L = G - L. According to Claim 2, there are only three types of connected components of G' - L: isolated vertices in B, connected components consisting of even number of vertices in Φ , and odd connected components consisting of a vertex in B and even number of vertices in LB. Consequently,

$$\nu(G') \le \frac{|\Phi|}{2} + \frac{|LB|}{2} + |L| = \nu(G) \le k,$$

that is, G' is M_{k+1} -free, a contradiction. Thus the claim holds.

Claim 4. G[L] is a complete multi-partite graph.

Proof. Let G be a $\{K_{r+1}, M_{k+1}\}$ -free graph with maximal number of edges. We define a binary relation \sim on L. For any $u, v \in L$, $u \sim v$ if and only if $uv \notin E(G)$. We claim that \sim is an equivalence relation. For any $u \in L$, $uu \notin E(G)$, so \sim is reflexive. If $vu \notin E(G)$, then $uv \notin E(G)$, so \sim is symmetric.

Now we prove \sim is transitive. Otherwise, suppose there are $u, v, w \in L$ such that $vu, uw \notin E(G)$, then $vw \in E(G)$.

Case 1. $d_G(u) < d_G(v)$ or $d_G(u) < d_G(w)$. By symmetry, assume that $d_G(u) < d_G(v)$. Let $G' = G[u \leftarrow v]$. By Claim 3, we know that G' is also $\{M_{k+1}, K_{r+1}\}$ -free and e(G') > e(G), contradicting the maximality of e(G).

Case 2. $d_G(u) \ge d_G(v)$ and $d_G(u) \ge d_G(w)$. Let $G'' = G[v \leftarrow u][w \leftarrow u]$. By Claim 3 again, G'' is also $\{M_{k+1}, K_{r+1}\}$ -free and e(G'') > e(G), a contradiction.

Thus, the relation \sim on L is an equivalence relation and G[L] is a complete multi-partite graph.

Let $|L| = x, 0 \le x \le s$. Since G is K_{r+1} -free, by Theorem 1.1 we infer $e(L) \le t_r(x)$. By Claim 2, there are three types of connected components in G-L. Let t be the number of connected components of type (II) and let y_1, y_2, \ldots, y_t be the number of vertices in these connected components, respectively. Clearly, $|\Phi| = y = y_1 + y_2 + \cdots + y_t$.

For $y_1, y_2 > 0$, the following inequality can be checked directly:

(2.1)
$$\binom{y_1}{2} + \binom{y_2}{2} \le \binom{y_1 + y_2}{2}.$$

Applying (2.1) repeatedly, we obtain that

(2.2)
$$e(\Phi) \le \begin{pmatrix} y_1 \\ 2 \end{pmatrix} + \begin{pmatrix} y_2 \\ 2 \end{pmatrix} + \dots + \begin{pmatrix} y_t \\ 2 \end{pmatrix} \le \begin{pmatrix} y \\ 2 \end{pmatrix}.$$

Let p be the number of connected components of type (III) and let $2z_1 + 1, 2z_2 + 1, \ldots, 2z_p + 1$ be the number of vertices in these connected components, respectively. Let B_0 be the set of vertices labeled b in the union of connected components of type (III). Since each connected components of type (III) has

exactly one vertex labeled b, we infer $|B_0| = p$ and $|LB| = 2(z_1 + z_2 + \dots + z_p) = 2s - 2x - y$.

For $z_1, z_2 \ge 1$, it is easy to verify that

(2.3)
$$\binom{2z_1+1}{2} + \binom{2z_2+1}{2} \le \binom{2z_1+2z_2+1}{2}$$

Let $LB \cup B_0$ be the union of connected components of type (III). By applying (2.3) repeatedly, we get

(2.4)
$$e(LB \cup B_0) \leq \binom{2z_1+1}{2} + \binom{2z_2+1}{2} + \dots + \binom{2z_p+1}{2} \\ \leq \binom{2(z_1+\dots+z_p)+1}{2} = \binom{2s-2x-y+1}{2}.$$

By Claim 4, G[L] is a complete multi-partite graph. Since G is K_{r+1} -free, G[L] is t-partite with $t \leq r$. We distinguish two cases according to t.

Case 1. $t \leq r-1$. Since G[L] is a complete t-partite graph and $t \leq r-1$, we see that $e(L) \leq t_{r-1}(x)$. By (2.2) and (2.4), we obtain that

$$e(G) = e(L) + e(LB \cup B_0) + e(\Phi) + e(L, V(G) \setminus L)$$

$$\leq t_{r-1}(x) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + |L|(n - |L|)$$

$$\leq t_{r-1}(x) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + x(n - x).$$

By (2.1) we infer that

(2.5)
$$e(G) \le t_{r-1}(x) + \binom{2s-2x+1}{2} + x(n-x) =: g(x).$$

By Lemma 1.7 we know that $t_{r-1}(x)$ is a convex function. Since both $t_{r-1}(x)$ and $\binom{2s-2x+1}{2} + x(n-x)$ are convex functions, we infer that g(x) is also a convex function for $0 \le x \le s \le k$. Hence,

$$e(G) \le \max\{g(0), g(k)\} = \max\left\{\binom{2s+1}{2}, t_{r-1}(k) + k(n-k)\right\}$$
$$\le \max\left\{\binom{2k+1}{2}, t_{r-1}(k) + k(n-k)\right\}.$$

For $n \ge 3k + 1$, we conclude that

$$\binom{2k+1}{2} = k(2k+1) \le k(n-k) < t_{r-1}(k) + k(n-k),$$

and Theorem 1.3 holds.

Case 2. t = r. Let L_1, L_2, \ldots, L_r be r partite sets of G[L]. It is obvious that $|L| = x \ge r$. Since G[L] is a complete r-partite graph, for any $u \in V(G) \setminus L$ there exists some i such that $L_i \cap N_G(u) = \emptyset$. Otherwise if there exists $v_i \in L_i$ such that $uv_i \in E(G)$ for each $i = 1, 2, \ldots, r$, then $\{u, v_1, v_2, \ldots, v_r\}$ spans a copy of K_{r+1} , a contradiction. By symmetry, assume that $|L_1| \le |L_2| \le \cdots \le |L_r|$. Let $|L_1| = z$. Clearly $1 \le z \le \lfloor x/r \rfloor$. Then

$$e(L, V(G) \setminus L) \le (n - |L|)(|L| - z) = (n - x)(x - z).$$

Note that |B| = n - 2s + x and $|B_0| \le |LB|/2 \le s - |L| = s - x$. By (2.2) and (2.4), we infer that

$$e(G) = e(L) + e(LB \cup B_0) + e(\Phi) + e(L, V(G) \setminus L) \le |L_1|(|L| - |L_1|) + t_{r-1}(|L| - |L_1|) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + (n - x)(x - z) = z(x - z) + t_{r-1}(x - z) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + (n - x)(x - z) = (n - x + z)(x - z) + t_{r-1}(x - z) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2}.$$

By (2.1), we infer that

(2.6)
$$e(G) \le (n-x+z)(x-z) + t_{r-1}(x-z) + \binom{2s-2x+1}{2} =: g(x,z).$$

Since for $r \leq x \leq s \leq k$, $n \geq 2k$ and $1 \leq z \leq \lfloor x/r \rfloor$, g(x, z) is a decreasing function with respect to z. Using $z \geq 1$, we get

$$g(x,z) \le g(x,1) = (n-x+1)(x-1) + t_{r-1}(x-1) + \binom{2s-2x+1}{2}$$
$$\le (n-x+1)(x-1) + t_{r-1}(x-1) + \binom{2k-2x+1}{2} =: h(x)$$

It is easy to verify that h(x) is also a convex function for $1 \le r \le x \le s \le k$. Therefore,

$$e(G) \le \max\{h(1), h(k)\} = \max\left\{\binom{2k-1}{2}, (n-k+1)(k-1) + t_{r-1}(k-1)\right\}.$$

Note that for $n \ge 3k$,

$$t_{r-1}(k) + k(n-k) - h(k) = t_{r-1}(k) + k(n-k) - t_{r-1}(k-1) - (n-k+1)(k-1)$$
$$= n - 2k + 1 + k - \left\lceil \frac{k}{r-1} \right\rceil \ge n - 2k - \frac{(r-2)k}{r-1} > 0.$$

It follows that $h(k) \leq t_{r-1}(k) + k(n-k)$.

For $n \geq 3k - 2$, we have

$$t_{r-1}(k) + k(n-k) - \binom{2k-1}{2} > kn - k^2 - 2k^2 + 3k - 1 \ge 0.$$

It follows that $h(1) \leq t_{r-1}(k) + k(n-k)$.

Consequently $e(G) \leq t_{r-1}(k) + k(n-k)$ for $n \geq 3k+1$ and the theorem holds.

3. Proof of Theorem 1.4

By a similar approach as in the proof of Theorem 1.3, we determine the maximum number of edges in a graph with $\nu(G) \leq k$ and $\tau(G) \geq k + r$.

Proof of Theorem 1.4. Let M be a maximal matching in G and let L, LB, Φ, B be the partition of V(G) obtained by the alternating path method.

By Claim 1, we infer that $L \cup LB \cup \Phi$ is a vertex cover of G. Since $\tau(G) \ge k+r$, it follows that $|L| + |LB| + |\Phi| \ge k + r$. By Claim 2, for any $xy \in M$, there are three possibilities: one of x, y is labeled l and the other is labeled b, or x, y are both labeled lb, or x, y are both unlabeled. Since $\nu(G) \le k$ implies $|L| + |LB|/2 + |\Phi|/2 \le k$, it follows that $|LB|/2 + |\Phi|/2 \ge r$ and $|L| \le k - |LB|/2 - |\Phi|/2 \le k - r$.

Let |L| = x and $|\Phi| = y$. Then $0 \le x \le k - r$, $0 \le y \le 2k - 2x$. By (2.2) and (2.4), we obtain that

(3.1)

$$e(G) = e(L) + e(\Phi) + e(LB \cup B_0) + e(L, V(G) \setminus L)$$

$$\leq \binom{|L|}{2} + \binom{y}{2} + \binom{2k - 2x - y + 1}{2} + |L|(n - |L|)$$

$$\leq \binom{x}{2} + \binom{y}{2} + \binom{2k - 2x - y + 1}{2} + x(n - x).$$

By (2.1), we have

(3.2)
$$e(G) \le \binom{2k-x+1}{2} + x(n-x) =: f(x).$$

It is easy to verify that f(x) is a convex function for $0 \le x \le k - r$. Therefore,

$$e(G) \le \max\{f(0), f(k-r)\}\$$

= $\max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\},\$

and Theorem 1.4 holds.

Remark. Very recently, Alon and Frankl [3] determined $ex(n, \{M_{k+1}, K_{r+1}\})$ for all values of n by a very nice argument.

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