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THE MAXIMUM NUMBER OF EDGES IN A $\{K_{r+1}, M_{k+1}\}$-FREE GRAPH

Lingting Fu, Jian Wang

and

Weihua Yang

Department of Mathematics
Taiyuan University of Technology
Taiyuan 030024, P.R. China

e-mail: ltfu925@163.com
wangjian01@tyut.edu.cn
yangweihua@tyut.edu.cn

Abstract

Let G be a graph and $\mathcal{F}$ be a family of graphs. We say G is $\mathcal{F}$-free if it does not contain F as subgraph for any $F \in \mathcal{F}$. The Turán number $\mathrm{ex}(n, \mathcal{F})$ is defined as the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. Let K_{r+1} denote the complete graph on $r+1$ vertices and let M_{k+1} denote the graph on $2k+2$ vertices with $k+1$ pairwise disjoint edges. By using the alternating path technique and the Zykov symmetrization, we determine that for $n > 3k$,

$$\mathrm{ex}(n, \{M_{k+1}, K_{r+1}\}) = t_{r-1}(k) + k(n-k),$$

where $t_{r-1}(k)$ is the number of edges in an $(r-1)$-partite k-vertex Turán graph. Let $\nu(G)$, $\tau(G)$ denote the matching number and the vertex cover number of G, respectively. For $n \geq 2k$, we prove that if $\nu(G) \leq k$ and $\tau(G) \geq k + r$, then

$$e(G) \leq \max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\}.$$

Keywords: Turán number, alternating path, Zykov symmetrization.

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1. Introduction

Let $G(V, E)$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. We use $e(G)$ to denote the size of $E(G)$. For a graph H, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, then H is called a subgraph of G. Let $\mathcal{F}$ be a family of graphs. If for every $F \in \mathcal{F}$, G does not contain F as subgraph, then we say G is $\mathcal{F}$-free. The Turán number of $\mathcal{F}$, denoted by $\mathrm{ex}(n, \mathcal{F})$, is defined as the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. For $\mathcal{F} = \{F\}$, we simply write $\mathrm{ex}(n, F)$. The study of the Turán numbers plays a central role in the extremal graph theory. The Turán number of many graphs have been determined, see [4, 10, 11, 15, 16, 18, 19, 22, 23, 24, etc.]

Let $T_r(n)$ denote the Turán graph on n vertices, i.e., the complete r-partite graph of order n with each partite of sizes $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$. Note that for $n < r$, $T_r(n)$ represents the complete graph on n vertices. We use $t_r(n)$ to denote the number of edges of $T_r(n)$. In 1941, Turán [19] showed that the Turán graph $T_r(n)$ is the only K_{r+1}-free graph attaining the maximum number of edges.

Theorem 1.1 [19]**.** $\mathrm{ex}(n, K_{r+1}) = t_r(n)$.

For any $M \subset E(G)$, if the edges of M are pairwise disjoint, then M is called a matching of G. The matching number $\nu(G)$ is the size of a maximum matching in G. We often use M_{k+1} to denote the graph on $2k + 2$ vertices with $k + 1$ pairwise disjoint edges. In 1959, Erdős-Gallai [10] determined the Turán number of M_{k+1}.

Theorem 1.2 [10]**.** *For $n \geq 2k + 1$,*

$$\mathrm{ex}(n, M_{k+1}) = \max\left\{ \binom{2k+1}{2}, \binom{k}{2} + k(n-k) \right\}.$$

It should be mentioned that by using the shifting technique, Akiyama and Frankl [2] give a short proof of Theorem 1.2 and their proof also works for an rainbow extension of Theorem 1.2. That is, if $G_1, G_2, \ldots, G_{k+1}$ are $k + 1$ graphs on the same vertex set of size n and $e(G_i) > \max\left\{ \binom{2k+1}{2}, \binom{k}{2} + k(n-k) \right\}$ for $i = 1, 2, \ldots, k + 1$, then there is a rainbow matching of size $k + 1$.

Let G_1 and G_2 be two disjoint subgraphs of G. We use $G_1 \cup G_2$ to denote the union of G_1 and G_2 with the vertex set being $V(G_1) \cup V(G_2)$ and the edge set being $E(G_1) \cup E(G_2)$. We use $G_1 \vee G_2$ to denote the join graph of G_1 and G_2 with the vertex set being $V(G_1) \cup V(G_2)$ and the edge set being $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Denote by E_n the empty graph on n vertices. It is easy to see that one of $K_{2k+1} \cup E_{n-2k-1}$ and $K_k \vee E_{n-k}$ achieves the maximum number of edges among all M_{k+1}-free graphs.

In this paper, we determine the Turán number of $\mathcal{F} = \{K_{r+1}, M_{k+1}\}$ for $n > 3k$.

Theorem 1.3. *For $n \geq 3k+1$,*

$$\mathrm{ex}(n, \{M_{k+1}, K_{r+1}\}) = t_{r-1}(k) + k(n-k).$$

Note that for $r \geq k+1$ and $n \geq 3k+1$, by Theorem 1.2 and Theorem 1.3 we infer

$$\mathrm{ex}(n, \{M_{k+1}, K_{r+1}\}) = \binom{k}{2} + k(n-k) = \mathrm{ex}(n, M_{k+1}).$$

Obviously, $T_{r-1}(k) \vee E_{n-k}$ is an $\{M_{k+1}, K_{r+1}\}$-free graph that achieves the maximum number of edges.

For any $K \subseteq V(G)$, K is called a vertex cover set of G if each edge of G has at least one endpoint in K. A vertex cover set with the minimum size is called a minimum vertex cover set. The vertex covering number $\tau(G)$ is defined as the size of a minimum vertex cover set of G. In [12], Fănică found the relation between the matching number and the vertex covering number. In this paper, we determine the maximum number of edges in a graph G with $\nu(G) \leq k$ and $\tau(G) \geq k+r$.

Theorem 1.4. *Let G be an n-vertex graph with $\nu(G) \leq k$ and $\tau(G) \geq k+r$. For $n \geq 2k$ and $r \leq k$,*

$$e(G) \leq \max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\}.$$

For sets A_1, A_2, let $A_1 \triangle A_2$ denote the symmetric difference set of A_1 and A_2, i.e., $(A_1 \setminus A_2) \cup (A_2 \setminus A_1)$.

Let us recall two techniques that are needed in our proofs. For self-containedness, we give a formal definition of the alternating path used in [1, 6, 7].

Definition 1.5 [1, 6, 7]**.** Let G be a graph with $\nu(G) = s < n/2$. Let M be a maximum matching of G and let Y be the set of vertices that are not covered by M. A directed path $P = v_0v_1v_2\cdots v_m$ in G is called an M-alternating path if it satisfies conditions (i), (ii) and (iii).

(i) $v_0 \in Y$;

(ii) $v_iv_{i+1} \in M$ for any odd i with $1 \leq i \leq m-1$;

(iii) $v_iv_{i+1} \notin M$ for any even i with $0 \leq i \leq m-1$.

When it is clear from the context, we simply call P an alternating path. Clearly, $v_m \notin Y$. Otherwise, $E(P) \triangle M$ will be a matching of size $|M|+1$, a contradiction.

An M-augmenting path is an M-alternating path whose origin v_0 and terminus v_m are in Y. Clearly, if M is a maximum matching of G, then there is no

M-augmenting path in G. Otherwise, let P be an M-augmenting path. Then $E(P) \triangle M$ is a matching of size $|M|+1$, contradicting the maximality of $|M|$.

If a matching of G covers all the vertices, then it is called a perfect matching of G. The Tutte-Berge formula is a central result concerning the maximum matchings in graphs. Let $\operatorname{odd}(G)$ denote the number of connected components of odd order in G. In 1947, Tutte [20] obtained a sufficient and necessary condition for G to guarantee a perfect matching. That is, $\operatorname{odd}(G-A) \leq |A|$ for all $A \subseteq V(G)$. In other words, if $\operatorname{odd}(G-A) \leq |A|$ holds for all $A \subseteq V(G)$, then there is a perfect matching of G.

In 1958, Berge extended Tutte's result to graphs without perfect matchings and determined a formula for the matching number of G.

Theorem 1.6 [6]**.** *Let M be a maximum matching of G. Let $G-A$ denote the subgraph obtained from G by deleting vertices in A from G. Then*

$$|M| = \frac{1}{2} \min_{A \subseteq V(G)} \{|A| - \operatorname{odd}(G-A) + |V(G)|\}.$$

This result is known as the Tutte-Berge formula. For related researches please see [5, 8, 9, 13, 14, 17, 21].

Another technique we need in the proofs is the Zykov symmetrization. In 1949, Zykov [25] invented this method to show that $T_r(n)$ is the only K_{r+1}-free graph of order n which maximizes the number of copies of K_s with $2 \leq s \leq r$, which is a generalized version of Theorem 1.1.

In our proofs, we also need the following lemma.

Lemma 1.7. *For $x \geq 0$, $t_r(x)$ is a convex function.*

Proof. Note that

$$t_r(x+1) - t_r(x) = x + 1 - \left\lceil \frac{x+1}{r} \right\rceil, \ t_r(x) - t_r(x-1) = x - \left\lceil \frac{x}{r} \right\rceil.$$

It follows that

$$t_r(x+1) - 2t_r(x) + t_r(x-1) = 1 - \left(\left\lceil \frac{x+1}{r} \right\rceil - \left\lceil \frac{x}{r} \right\rceil \right) \geq 0.$$

Thus, $t_r(x)$ is a convex function for $x \geq 0$. ∎

Finally, let us recall some notations. For any $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph with vertex set X and edge set $\{uv \in E(G) \colon u, v \in X\}$. When the content is clear, we often use $e(X)$ to denote $e(G[X])$. Let $G - X = G[V(G) \setminus X]$. Let

$$N_G(X) = \{v \in V(G) \setminus X : \text{ there exists a } u \in X \text{ such that } uv \in E(G)\}.$$

For $X = \{x\}$, we simply write $N_G(x)$. We use $\deg_G(x)$ to denote the cardinality of $N_G(x)$. We often omit subscripts when there is no confusion.

2. Proof of Theorem 1.3

In this section, we study the Turán number of $\{M_{k+1}, K_{r+1}\}$ by using the alternating path technique and the Zykov symmetrization.

***Proof of Theorem* 1.3.** Let G be an $\{M_{k+1}, K_{r+1}\}$-free graph with maximum number of edges. Let

$$M = \{x_1y_1, x_2y_2, \dots, x_sy_s\}$$

be a maximum matching of G. Since G is M_{k+1}-free, we infer $s \le k$. Let

$$X = \{x_1, y_1, x_2, y_2, \dots, x_s, y_s\},\ Y = V(G) \setminus X.$$

Obviously, Y is an independent set.

Now let us partition $V(G)$ into four classes by the alternating path method. For every M-alternating path $P = v_0v_1v_2\cdots v_{2m}$, label vertices $v_1, v_3, \dots, v_{2m-1}$ with the symbol l, label vertices $v_0, v_2, \dots, v_{2m}$ with the symbol b. Then the vertices in G are partitioned into four types: vertices labeled l, vertices labeled b, vertices labeled l and b and unlabeled vertices. If a vertex is labeled l and b, we also say that it is labeled lb. We use L, B, LB and Φ to denote the set of these four types of vertices, respectively. Obviously (L, B, Φ, LB) is a partition of $V(G)$ and $Y \subset B$.

Claim 1. *B is an independent set.*

Proof. Suppose for contradiction that $xy \in E(G[B])$. Note that by $x, y \in B$, x, y are both labeled b. Since Y is an independent set, $\{x, y\} \cap (B \setminus Y) \neq \emptyset$. Without loss of generality, we assume that $x \in B \setminus Y$. Let $P_1 = v_0v_1\cdots v_{2m}x'x$ be an alternating path with terminal vertex x. Clearly $x'x \in M$. If y is not in $V(P_1)$, then $P = P_1y$ is an alternating path. It implies that y is also labeled l, which contradicts $y \in B$. Thus y is in $V(P_1)$. But then let P_2 be the sub-path of P_1 with terminal vertex y. It follows that P_2x is an alternating path. By the label of y, we infer that x is labeled l as well, contradicting $x \in B$. □

The following claim is a well known result (see, e.g., [7]). Here we give a proof for self-containedness.

Claim 2. *There are three kinds of connected components of $G - L$.*

(I) An isolated vertex in B.

(II) A connected component consisting of even number of vertices in Φ.

(III) A connected component consisting of a vertex in B and even number of vertices in LB.

Proof. Let $C \subset V(G)$ be a connected component of $G - L$.

(I) By Claim 1, B is an independent set. If $C \subset B$, then C contains exactly one vertex.

(II) Suppose that C contains some unlabeled vertex. Let x be such a vertex. Note that all the vertices in Y are labeled b. It follows that $x \in X$. To show (II), it suffices to show that all the neighbors of x are also unlabeled. Let $xy \in M$. Clearly y is also unlabeled. For any $z \in N_C(x)\backslash\{y\}$, if z is labeled then z is labeled either b or lb. In either case there exists an alternating path $P = v_0v_1 \cdots v_{2m}z'z$ with terminal vertex z and $z'z \in M$. Since x is unlabeled, $x \notin V(P)$. Then Px is also an alternating path. It follows that x is labeled l, a contradiction. Thus all the neighbors of x are also unlabeled. Obviously, if one endpoint of some edge in M is unlabeled, so is the other endpoint and they have to fall into the same component of $G-L$. Therefore, C is an connected component with even number of unlabeled vertices.

(III) If there is a vertex in C labeled lb, noting that the vertices with label lb appear in pairs (two endpoints of a matching edge), we see that the number of vertices in C with label lb is even.

Let $xx' \in M$ with $x, x' \in C \cap LB$. First we show that there is a vertex in C labeled b. Let $P = v_0v_1 \cdots v_{2m}x'x$ be an alternating path. Clearly $v_0 \in Y$ is labeled b. Let v_i be the last vertex on P that is not labeled lb. Since C is a connected component of $G-L$, $v_{i+1}, \dots, v_{2m} \in V(C)$. Since the vertices labeled lb appear in pairs in P, $v_i \in B$. Moreover, if $i \geq 1$ then $v_{i-1}v_i \in M$ and $v_{i-1} \in L$. Thus $v_i \in V(C) \cap B$.

We are left to show that $|V(C) \cap B| = 1$. First we show that one can choose a vertex $y \in V(C) \cap B$ and an alternating path P_0 with terminal vertex y such that $V(P_0) \cap V(C) = y$. If $V(C) \cap Y \neq \emptyset$ then choose $y \in V(C) \cap Y$ and simply set $P_0 = y$. If $V(C) \cap Y = \emptyset$ then choose $z \in V(C) \cap B$. Since $z \in B \setminus Y$, there is an alternating path $P = u_0u_1 \cdots u_{2p-1}u_{2p}z'z$. Choose y be the first vertex on P that is in $V(C)$ and let P_0 be the sub-path of P with terminal vertex y. Clearly $V(P_0) \cap V(C) = \{y\}$. Since C is a connected component in $G-L$, the predecessor of y on P_0 has to be in L. It follows that $y \in B$. Thus we find a vertex $y \in V(C) \cap B$ and an alternating path P_0 with terminal vertex y satisfying $V(P_0) \cap V(C) = \{y\}$.

Let S be the set of all vertices x in $V(C)$ so that there exists an alternating path $P = P_0P'x$ and $V(P') \subset V(C)$. Clearly $y \in S$. We claim that every vertex in $S \setminus \{y\}$ is labeled lb. Indeed, otherwise there exists $x \in S \cap B$. Let $x'x \in M$ and let P_0P_1x be an alternating path. Since $x' \in L$, x' is not a vertex on path P_0P_1x. It follows that x has to be labeled l on P_0P_1x, contradicting $x \in B$. Thus y is the unique vertex in S that is labeled b.

Since C is connected, there exists $x \in N_C(y)$. Note that B is an independent set. It follows that $x \in LB$. Thus $S \cap LB \neq \emptyset$. Next we show that for each $x, x' \in S \cap LB$ with $xx' \in M$, there exist alternating paths P_0P_1xx' and $P_0P_1'x'x$

such that $V(P_1) \subset V(C)$ and $V(P_1') \subset V(C)$. Since $x, x' \in S$, by symmetry there is an alternating path P_0P_1xx' with $V(P_1) \subset V(C)$. We are left to find an alternating path ending with $x'x$. Since x, x' are labeled lb, there exists an alternating path $P = v_0v_1 \cdots v_{2p}x'x$. Let z' be the last vertex on P that is not in $V(C)$ and let z be the successor of z' on P. Then clearly $z'z \in M$ and $z \in B \cap V(C)$. Let $P_2x'x$ be the sub-path of P with start vertex z. If $V(P_2) \cap V(P_1) = \emptyset$ then $P_0P_1xx'P_2^{-1}$ is an alternating path with terminal vertex z, where P_2^{-1} is the reverse path of P_2. It follows that $z \in S$, contradicting $S \cap B = \{y\}$. Hence $V(P_2) \cap V(P_1) \neq \emptyset$.

If $z = y$ then $P_0P_2x'x$ is an alternating path ending with $x'x$. Thus we assume $z \neq y$. Note that $V(P_2) \cap V(P_1) \neq \emptyset$ implies that P_1 and P_2 intersect in a matching edge in C whose endpoints are both labeled lb. Let ww' be the first edge appeared in both P_1 and P_2. Now by symmetry there are two cases: (a) $P_1 = P_{11}ww'P_{12}$, $P_2 = P_{21}ww'P_{22}$ and (b) $P_1 = P_{11}ww'P_{12}$, $P_2 = P_{21}w'wP_{22}$. For case (a), $P_{11}ww'P_{22}x'x$ is an alternating path ending with $x'x$. For case (b) $P_{11}ww'P_{21}^{-1}$ is an alternating path connecting y and z. It implies that $z \in S$, contradicting $S \cap B = \{y\}$. Thus, for each $x, x' \in S \cap LB$ with $xx' \in M$, there exist alternating paths P_0P_1xx' and $P_0P_1'x'x$ such that $V(P_1) \subset V(C)$ and $V(P_1') \subset V(C)$.

Finally, we show that $S = V(C)$. Suppose to the contrary that there exist $z \in S$ and $w \in V(C) \setminus S$ such that $zw \in E(G)$. Let $zz' \in M$. Clearly $z' \in S$. It follows that $zw \notin M$. Since there exists an alternating path $P_0P_1z'z$ with $V(P_1) \subset V(C)$, $P_0P_1z'zw$ is an alternating path. It implies that $w \in S$, contradicting our assumption that $w \notin S$. Thus $S = V(C)$. Together with $S \cap B = \{y\}$, we conclude that (III) holds. □

Definition 2.1. For any $u, v \in V(G)$, define $G[u \leftarrow v]$ as a new graph obtained from G by removing edges adjacent to u and adding all edges in $\{ux : x \in N_G(v)\}$.

Claim 3. *If G is $\{M_{k+1}, K_{r+1}\}$-free, then for any $u, v \in L$, $G[u \leftarrow v]$ is also $\{M_{k+1}, K_{r+1}\}$-free.*

Proof. Let $G' = G[u \leftarrow v]$. Suppose to the contrary that $G[u \leftarrow v]$ is not $\{M_{k+1}, K_{r+1}\}$-free. Then $G[u \leftarrow v]$ has either a copy of K_{r+1} or a matching of size $k+1$.

If there exists some $R \subset V(G')$ such that $G'[R] \cong K_{r+1}$, then $u \in R$. Since $uv \notin E(G')$, $v \notin R$. Since the neighbors of u in G' are also neighbors of v in G, it follows that $G[R \setminus \{u\} \cup \{v\}] \cong K_{r+1}$, a contradiction. Therefore G' is K_{r+1}-free.

Since $u, v \in L$, we see that $E(G)$ and $E(G')$ differ only in edges with at least one endpoint in L. It implies that $G' - L = G - L$. According to Claim 2, there are only three types of connected components of $G' - L$: isolated vertices in B, connected components consisting of even number of vertices in Φ, and odd connected components consisting of a vertex in B and even number of vertices in LB. Consequently,

$$\nu(G') \leq \frac{|\Phi|}{2} + \frac{|LB|}{2} + |L| = \nu(G) \leq k,$$

that is, G' is M_{k+1}-free, a contradiction. Thus the claim holds. □

Claim 4. *$G[L]$ is a complete multi-partite graph.*

Proof. Let G be a $\{K_{r+1}, M_{k+1}\}$-free graph with maximal number of edges. We define a binary relation $\sim$ on L. For any $u, v \in L$, $u \sim v$ if and only if $uv \notin E(G)$. We claim that $\sim$ is an equivalence relation. For any $u \in L$, $uu \notin E(G)$, so $\sim$ is reflexive. If $vu \notin E(G)$, then $uv \notin E(G)$, so $\sim$ is symmetric.

Now we prove $\sim$ is transitive. Otherwise, suppose there are $u, v, w \in L$ such that vu, $uw \notin E(G)$, then $vw \in E(G)$.

Case 1. $d_G(u) < d_G(v)$ or $d_G(u) < d_G(w)$. By symmetry, assume that $d_G(u) < d_G(v)$. Let $G' = G[u \leftarrow v]$. By Claim 3, we know that G' is also $\{M_{k+1}, K_{r+1}\}$-free and $e(G') > e(G)$, contradicting the maximality of $e(G)$.

Case 2. $d_G(u) \geq d_G(v)$ and $d_G(u) \geq d_G(w)$. Let $G'' = G[v \leftarrow u][w \leftarrow u]$. By Claim 3 again, G'' is also $\{M_{k+1}, K_{r+1}\}$-free and $e(G'') > e(G)$, a contradiction.

Thus, the relation $\sim$ on L is an equivalence relation and $G[L]$ is a complete multi-partite graph. □

Let $|L| = x$, $0 \leq x \leq s$. Since G is K_{r+1}-free, by Theorem 1.1 we infer $e(L) \leq t_r(x)$. By Claim 2, there are three types of connected components in $G-L$. Let t be the number of connected components of type (II) and let $y_1, y_2, \ldots, y_t$ be the number of vertices in these connected components, respectively. Clearly, $|\Phi| = y = y_1 + y_2 + \cdots + y_t$.

For y_1, $y_2 > 0$, the following inequality can be checked directly:

$$\binom{y_1}{2} + \binom{y_2}{2} \leq \binom{y_1 + y_2}{2}. \tag{2.1}$$

Applying (2.1) repeatedly, we obtain that

$$e(\Phi) \leq \binom{y_1}{2} + \binom{y_2}{2} + \cdots + \binom{y_t}{2} \leq \binom{y}{2}. \tag{2.2}$$

Let p be the number of connected components of type (III) and let $2z_1 + 1, 2z_2 + 1, \ldots, 2z_p + 1$ be the number of vertices in these connected components, respectively. Let B_0 be the set of vertices labeled b in the union of connected components of type (III). Since each connected components of type (III) has exactly one vertex labeled b, we infer $|B_0| = p$ and $|LB| = 2(z_1 + z_2 + \cdots + z_p) = 2s - 2x - y$.

For z_1, $z_2 \geq 1$, it is easy to verify that

$$\binom{2z_1+1}{2} + \binom{2z_2+1}{2} \leq \binom{2z_1+2z_2+1}{2}. \tag{2.3}$$

Let $LB \cup B_0$ be the union of connected components of type (III). By applying (2.3) repeatedly, we get

$$\begin{aligned} e(LB \cup B_0) &\leq \binom{2z_1+1}{2} + \binom{2z_2+1}{2} + \cdots + \binom{2z_p+1}{2} \\ &\leq \binom{2(z_1+\cdots+z_p)+1}{2} = \binom{2s-2x-y+1}{2}. \end{aligned} \tag{2.4}$$

By Claim 4, $G[L]$ is a complete multi-partite graph. Since G is K_{r+1}-free, $G[L]$ is t-paritite with $t \leq r$. We distinguish two cases according to t.

Case 1. $t \leq r-1$. Since $G[L]$ is a complete t-partite graph and $t \leq r-1$, we see that $e(L) \leq t_{r-1}(x)$. By (2.2) and (2.4), we obtain that

$$\begin{aligned} e(G) &= e(L) + e(LB \cup B_0) + e(\Phi) + e(L, V(G) \setminus L) \\ &\leq t_{r-1}(x) + \binom{2s-2x-y+1}{2} + \binom{y}{2} + |L|(n-|L|) \\ &\leq t_{r-1}(x) + \binom{2s-2x-y+1}{2} + \binom{y}{2} + x(n-x). \end{aligned}$$

By (2.1) we infer that

$$e(G) \leq t_{r-1}(x) + \binom{2s-2x+1}{2} + x(n-x) =: g(x). \tag{2.5}$$

By Lemma 1.7 we know that $t_{r-1}(x)$ is a convex function. Since both $t_{r-1}(x)$ and $\binom{2s-2x+1}{2} + x(n-x)$ are convex functions, we infer that $g(x)$ is also a convex function for $0 \leq x \leq s \leq k$. Hence,

$$\begin{aligned} e(G) \leq \max\{g(0), g(k)\} &= \max\left\{\binom{2s+1}{2}, t_{r-1}(k) + k(n-k)\right\} \\ &\leq \max\left\{\binom{2k+1}{2}, t_{r-1}(k) + k(n-k)\right\}. \end{aligned}$$

For $n \geq 3k+1$, we conclude that

$$\binom{2k+1}{2} = k(2k+1) \leq k(n-k) < t_{r-1}(k) + k(n-k),$$

and Theorem 1.3 holds.

Case 2. $t = r$. Let $L_1, L_2, \ldots, L_r$ be r partite sets of $G[L]$. It is obvious that $|L| = x \geq r$. Since $G[L]$ is a complete r-partite graph, for any $u \in V(G) \setminus L$ there exists some i such that $L_i \cap N_G(u) = \emptyset$. Otherwise if there exists $v_i \in L_i$ such that $uv_i \in E(G)$ for each $i = 1, 2, \ldots, r$, then $\{u, v_1, v_2, \ldots, v_r\}$ spans a copy of K_{r+1}, a contradiction. By symmetry, assume that $|L_1| \leq |L_2| \leq \cdots \leq |L_r|$. Let $|L_1| = z$. Clearly $1 \leq z \leq \lfloor x/r \rfloor$. Then

$$e(L, V(G) \setminus L) \leq (n - |L|)(|L| - z) = (n - x)(x - z).$$

Note that $|B| = n - 2s + x$ and $|B_0| \leq |LB|/2 \leq s - |L| = s - x$. By (2.2) and (2.4), we infer that

$$\begin{aligned} e(G) &= e(L) + e(LB \cup B_0) + e(\Phi) + e(L, V(G) \setminus L) \leq |L_1|(|L| - |L_1|) \\ &\quad + t_{r-1}(|L| - |L_1|) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + (n - x)(x - z) \\ &= z(x - z) + t_{r-1}(x - z) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2} + (n - x)(x - z) \\ &= (n - x + z)(x - z) + t_{r-1}(x - z) + \binom{2s - 2x - y + 1}{2} + \binom{y}{2}. \end{aligned}$$

By (2.1), we infer that

$$e(G) \leq (n - x + z)(x - z) + t_{r-1}(x - z) + \binom{2s - 2x + 1}{2} =: g(x, z). \tag{2.6}$$

Since for $r \leq x \leq s \leq k$, $n \geq 2k$ and $1 \leq z \leq \lfloor x/r \rfloor$, $g(x, z)$ is a decreasing function with respect to z. Using $z \geq 1$, we get

$$\begin{aligned} g(x, z) \leq g(x, 1) &= (n - x + 1)(x - 1) + t_{r-1}(x - 1) + \binom{2s - 2x + 1}{2} \\ &\leq (n - x + 1)(x - 1) + t_{r-1}(x - 1) + \binom{2k - 2x + 1}{2} =: h(x). \end{aligned}$$

It is easy to verify that $h(x)$ is also a convex function for $1 \leq r \leq x \leq s \leq k$. Therefore,

$$e(G) \leq \max\{h(1), h(k)\} = \max\left\{\binom{2k - 1}{2}, (n - k + 1)(k - 1) + t_{r-1}(k - 1)\right\}.$$

Note that for $n \geq 3k$,

$$\begin{aligned} t_{r-1}(k) + k(n - k) - h(k) &= t_{r-1}(k) + k(n-k) - t_{r-1}(k-1) - (n-k+1)(k-1) \\ &= n - 2k + 1 + k - \left\lceil \frac{k}{r - 1} \right\rceil \geq n - 2k - \frac{(r - 2)k}{r - 1} > 0. \end{aligned}$$

It follows that $h(k) \le t_{r-1}(k) + k(n-k)$.

For $n \ge 3k-2$, we have

$$t_{r-1}(k) + k(n-k) - \binom{2k-1}{2} > kn - k^2 - 2k^2 + 3k - 1 \ge 0.$$

It follows that $h(1) \le t_{r-1}(k) + k(n-k)$.

Consequently $e(G) \le t_{r-1}(k) + k(n-k)$ for $n \ge 3k+1$ and the theorem holds. ■

3. Proof of Theorem 1.4

By a similar approach as in the proof of Theorem 1.3, we determine the maximum number of edges in a graph with $\nu(G) \le k$ and $\tau(G) \ge k+r$.

***Proof of Theorem* 1.4.** Let M be a maximal matching in G and let L, LB, Φ, B be the partition of $V(G)$ obtained by the alternating path method.

By Claim 1, we infer that $L \cup LB \cup \Phi$ is a vertex cover of G. Since $\tau(G) \ge k+r$, it follows that $|L| + |LB| + |\Phi| \ge k + r$. By Claim 2, for any $xy \in M$, there are three possibilities: one of x, y is labeled l and the other is labeled b, or x, y are both labeled lb, or x, y are both unlabeled. Since $\nu(G) \le k$ implies $|L| + |LB|/2 + |\Phi|/2 \le k$, it follows that $|LB|/2 + |\Phi|/2 \ge r$ and $|L| \le k - |LB|/2 - |\Phi|/2 \le k - r$.

Let $|L| = x$ and $|\Phi| = y$. Then $0 \le x \le k - r$, $0 \le y \le 2k - 2x$. By (2.2) and (2.4), we obtain that

$$\begin{aligned} e(G) &= e(L) + e(\Phi) + e(LB \cup B_0) + e(L, V(G) \setminus L) \\ &\le \binom{|L|}{2} + \binom{y}{2} + \binom{2k-2x-y+1}{2} + |L|(n-|L|) \\ &\le \binom{x}{2} + \binom{y}{2} + \binom{2k-2x-y+1}{2} + x(n-x). \end{aligned} \tag{3.1}$$

By (2.1), we have

$$e(G) \le \binom{2k-x+1}{2} + x(n-x) =: f(x). \tag{3.2}$$

It is easy to verify that $f(x)$ is a convex function for $0 \le x \le k-r$. Therefore,

$$\begin{aligned} e(G) &\le \max\{f(0), f(k-r)\} \\ &= \max\left\{\binom{2k+1}{2}, \binom{k+r+1}{2} + (k-r)(n-k-r-1)\right\}, \end{aligned}$$

and Theorem 1.4 holds. ■

Remark. Very recently, Alon and Frankl [3] determined $\mathrm{ex}(n, \{M_{k+1}, K_{r+1}\})$ for all values of n by a very nice argument.

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