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ON NON-HAMILTONIAN POLYHEDRA WITHOUT CUBIC VERTICES AND THEIR VERTEX-DELETED SUBGRAPHS

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In loving memory of Johan Robaey.

Abstract

Thomassen proved in 1978 that if in an n-vertex planar graph G whose minimum degree is at least 4, all vertex-deleted subgraphs of G are hamiltonian, then G is hamiltonian. It was recently shown that in the preceding sentence, "all" can be replaced by "at least $n-5$". In this note we prove that, even if 3-connectedness is assumed, it cannot be replaced by $n-24$ (or any other integer greater than 24). The exact threshold remains unknown. We show that the same conclusion holds for triangulations and use computational means to prove that, under a natural restriction, this result is best possible.

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1. Introduction

Tutte proved in 1956 his influential result that if a planar graph is 4-connected, then it must contain a hamiltonian cycle [12]. Various extensions and generalisations of this theorem have been proven since then. Nelson [8] observed that one can infer from Tutte's work that in a planar 4-connected graph also every vertex-deleted subgraph must be hamiltonian. Viewing Nelson's observation from a different perspective, Thomassen showed in 1978 that if G is a planar graph on n vertices whose minimum degree is at least 4 and in which all n of its vertex-deleted subgraphs contain a hamiltonian cycle, then G itself contains a hamiltonian cycle [10]. This result, which we abbreviate by (†), extends Tutte's theorem. Subsequently, Thomassen proved that (†) is not true if 4 is replaced by 3, see [11] (whilst also providing the first cubic counterexample to a conjecture of Grünbaum [6]). In [10], he asked whether in (†) planarity can be dropped as a restriction; this remains an open problem. The last author extended (†) in several directions [13]. Investigating the interaction between the hamiltonian cycles occurring in a graph and the hamiltonicity of the graph's vertex-deleted subgraphs has a long history; we refer to the survey by Holton and Sheehan [7], and for two recent examples, see [3, 5].

We observe that the graphs in (†) are 3-connected, and this shall be our primary focus in this note, but we will also succinctly present results for the substantially larger family of 2-connected graphs; the latter case has also been studied in [14].

It was recently shown that in the statement of (†), the second "n" can be replaced by $n-5$, see [15]. It is natural to wonder how much more this threshold can be lowered. In Section 2 we show that, even assuming 3-connectedness, it *cannot* be lowered to $n-24$, and that this result also holds if we restrict ourselves to planar triangulations. In Section 3 we present our computational findings, among which is the observation that for planar triangulations, our aforementioned result is, in a certain sense, best possible. It is also shown that the smallest non-hamiltonian n-vertex polyhedral graph without cubic vertices and in which exactly one vertex-deleted subgraph admits a hamiltonian cycle, satisfies $23 \leq n \leq 25$.

Note that the exact threshold in the generalisation of (†) that we are discussing here remains unknown: if ζ is the largest integer such that every planar graph whose minimum degree is at least 4 and in which $n - \zeta$ vertex-deleted subgraphs contain a hamiltonian cycle, must itself contain a hamiltonian cycle, then we only know that $5 \leq \zeta \leq 23$. We do have an asymptotic solution to this problem, i.e., for every $c < 1$ there is a polyhedral n-vertex graph without vertices of degree 3 in which at least cn vertex-deleted subgraphs contain a hamiltonian cycle, but which itself contains no hamiltonian cycle [15].

By *polyhedral* we mean plane and 3-connected, and a *planar triangulation* is a planar graph which is rendered non-planar by the addition of any edge. All triangulations in this paper are planar, so we will simply write *triangulation*. Moreover, we will assume henceforth that our triangulations have at least four vertices, and are thus 3-connected. Given a graph G, the longest cycle length occurring in G is called the *circumference* of G. Consider a 2-connected n-vertex graph G with circumference $n-1$. We denote by $\mathrm{exc}(G) \subset V(G)$ the set of all vertices $v \in V(G)$ with the property that $G - v$ does not contain a hamiltonian cycle. A graph G containing no hamiltonian cycle and with $|\mathrm{exc}(G)| = k$ is called a *k-graph*. A vertex from $\mathrm{exc}(G)$ is called *exceptional*, and a vertex that is not exceptional, i.e., every vertex in $V(G) \setminus \mathrm{exc}(G)$, will be called *non-exceptional*. 0-graphs are also known as *hypohamiltonian* graphs. If a path has terminal vertex v we call the path a *v-path*, and a v-path whose second terminal vertex is $w \neq v$ is called a *vw-path*.

2. Theoretical Results

Let us now present an approach which yields the best available upper bound for the smallest number of exceptional vertices in a polyhedral (that is: plane and 3-connected) k-graph whose minimum degree is at least 4—this addresses a question raised in [13]. We require a variation of an operation introduced by Thomassen [11]. Consider a graph G which contains a cycle $C = v_1v_2v_3v_4v_1$. We write $\mathrm{Th}^+(G_C)$ for the graph that is obtained from G by considering a cycle $v_1'v_2'v_3'v_4'v_1'$ disjoint from G, together with the edges v_iv_i', with $1 \leq i \leq 4$. Results similar to the following lemma are proven in [3] and [13], so the case analysis constituting its proof is omitted.

Lemma 1. *Consider a polyhedral k-graph G which contains a facial cycle $C = v_1v_2v_3v_4v_1$ such that v_1 and v_3 are non-exceptional, v_2 and v_4 are exceptional, and v_1, v_2, v_3 are cubic. Then $\mathrm{Th}^+(G_C)$ is a polyhedral $(k+2)$-graph with exceptional vertices $\mathrm{exc}(G) \cup \{v_1', v_3'\}$.*

Consider a graph G and $v \in V(G)$. For a graph J disjoint from G and a subset W of $V(J)$ with $|W| = \deg(v)$, we *W-replace* v by J if we consider $G - v$ and J, and identify (with a bijection) $N(v)$ with W. If, after this identification, multiple edges occur between two vertices, these are replaced by a single edge.

For a set S, we say that $X \subset S$ and $Y \subset S$ *partition* S if $X \cup Y = S$ and $X \cap Y = \emptyset$. For a graph G and $W \subset V(G)$ we denote by $C(W)$ the set of all connected components of $G[W]$, and put $c(W) := |C(W)|$. A *single-vertex component* is a component containing exactly one vertex, and a *multiple-vertex component* contains at least two vertices.

Lemma 2. *Let G be a graph whose vertex set can be partitioned into two sets A and B such that $c(B) = |A| + 1 \geq 2$, at least two components of $G[B]$ are single-vertex components, every single-vertex component in $G[B]$ contains a non-exceptional vertex, and every multiple-vertex component in $G[B]$ contains only exceptional vertices. Consider any single-vertex component $(\{b\}, \emptyset)$ of $G[B]$. Let J be a graph disjoint from G containing a set W of vertices such that* (i) $|V(J)| \geq \deg_G(b) + 2$, (ii) $|W| = \deg_G(b)$, *and* (iii) *for any distinct vertices $v, w \in W$ we have that the graph $J - (W - v - w)$ contains a hamiltonian vw-path. Then the graph G' obtained by W-replacing b by J is a $(|V(J - W)| + |A| + |B_m|)$-graph, where B_m denotes the set of all vertices of B contained in multiple-vertex components of $G[B]$.*

Proof. Henceforth we see $G - b$, and thus A and $B - b$, as lying in G', and J as a subgraph of G'. In G' there exist $|A|$ vertices (namely the set A) whose removal yields $c(B) > |A|$ components, so G' is non-hamiltonian. The same argument shows that any vertex in $A \cup B_m \cup V(J)$ is exceptional, where for the vertices in J we use (i). Note that due to (i), G and G' cannot be isomorphic.

Due to (i) and (ii), $W' := V(J) \setminus W \neq \emptyset$. As every single-vertex component $(\{u\}, \emptyset)$ in $G[B - b]$ contains a non-exceptional vertex (we emphasise: the vertex is non-exceptional *in* G), there exists a hamiltonian cycle in $G - u$ which by definition must pass through b, so we obtain a hamiltonian vw-path $\mathfrak{p}$ in the graph $G' - W' - u$ for some distinct vertices $v, w \in W$. By (iii) we can extend $\mathfrak{p}$ to a hamiltonian cycle in $G' - u$, whence, u is non-exceptional in G'. ■

Theorem 3. *For any integer $t \geq 1$ there is a polyhedral $(23 + t)$-graph whose minimum degree is 4 and whose order is $23 + 2t$.*

Proof. Let H be the 13-vertex 6-graph from Figure 1 (right-hand side). Consider Herschel's graph shown in Figure 1 (left-hand side) and apply the operation Th^+ to one of its 4-cycles (all of which are facial) containing exactly one non-cubic vertex in order to obtain the graph H'. (We do not use Herschel's graph directly as it has no quartic non-exceptional vertices—an essential ingredient in our proof.) H' is a 15-vertex 7-graph. Each of the graphs H and H' have a 4-cycle C and C', respectively, satisfying the properties of Lemma 1.

We apply Th^+ to H and C, then apply Th^+ to $\mathrm{Th}^+(H_C)$ and the new 4-cycle formed when applying Th^+ (all vertices of this cycle are cubic), and so on, iterating this ad infinitum. The same procedure is applied to H' and C'. By

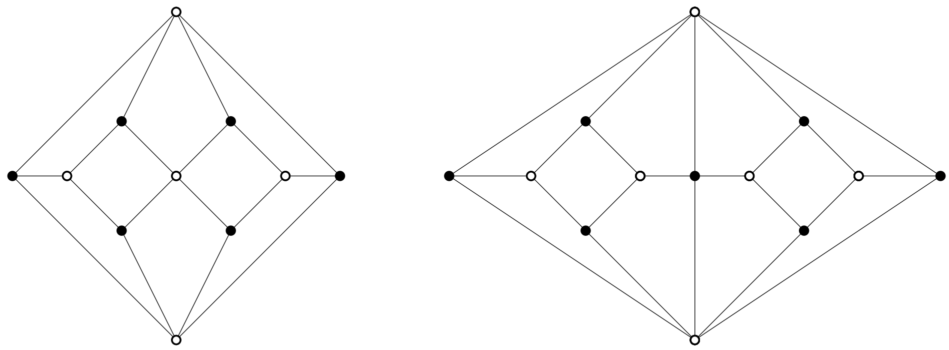

Figure 1. Herschel's graph (left-hand side), which is a polyhedral 5-graph on 11 vertices, and a polyhedral 6-graph on 13 vertices. In both graphs, exceptional (respectively, non-exceptional) vertices are coloured white (respectively, black).

Lemma 1 we get an infinite family of polyhedral n-vertex $\frac{n-1}{2}$-graphs, for every odd $n \geq 13$. Denote this family by $\mathcal{G}$. For every $G \in \mathcal{G}$, we apply Lemma 2 (six times), where A is the set of exceptional vertices initially contained in H and H' together with all exceptional vertices obtained by applying Th^+ (two new such exceptional vertices per application); B is the set of all vertices not in A, but note that we are not applying the lemma to all vertices in B, but only the cubic non-exceptional ones, of which there are exactly six; the chosen graph J is, for every replacement, the octahedron minus the edges of a triangle; and the identified vertices W are the three 2-valent vertices of J. It is elementary to verify that every condition of the lemma is met.

The effect of the (sixfold) application of Lemma 2 on the order of G and its number of exceptional vertices is as follows. We begin by describing the first application of our lemma. Regarding the order, we are replacing a vertex (one of the cubic non-exceptional vertices contained in the set B) by the three vertices residing in $J-W$, which implies that the graph we obtain has order $|V(G)|+2$. For the number of exceptional vertices in the graph we obtain, Lemma 2 states that this is $|V(J-W)|+|A|+|B_m|$. In our situation we have $|V(J-W)| = 6-3 = 3$, $|A| = \frac{|V(G)|-1}{2}$, and $|B_m| = 0$. So the graph we obtain after the first application of Lemma 2 is a $\frac{|V(G)|+5}{2}$-graph.

For each of the remaining applications we also add two vertices, so for a sixfold application the order of the graph increases by 12. Moreover, in each of the remaining applications of Lemma 2, a cubic non-exceptional vertex is replaced by three exceptional vertices inducing a triangle, so $|B_m|$ increases in each application by 3. Thus, when performing the sixth application of Lemma 2, as $|V(J-W)|+|A|+|B_m| = 3+\frac{|V(G)|-1}{2}+15$, we obtain a $\frac{|V(G)|+35}{2}$-graph. The (sixfold) application of Lemma 2 to H is illustrated in Figure 2.

As already mentioned, Lemma 1 yields an infinite family of polyhedral n-

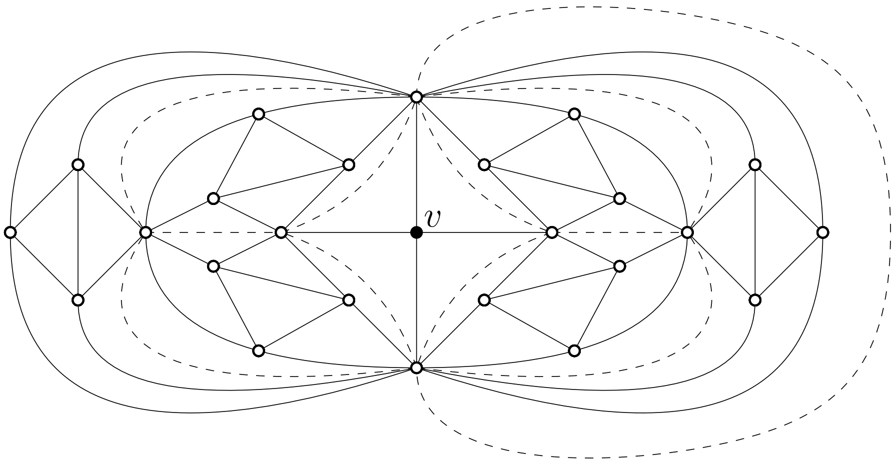

Figure 2. Considering only the solid edges in the above figure, a polyhedral 24-graph G of order 25 and minimum degree 4 is shown; its only non-exceptional vertex is v. Removing from G any edge yields a graph that is not a polyhedral 24-graph of minimum degree 4. G remains a polyhedral 24-graph of minimum degree 4 even after adding to it any subset of the set of dashed edges; if all such edges are added, a triangulation is obtained.

vertex $\frac{n-1}{2}$-graphs, for every odd $n \geq 13$. The sixfold application of Lemma 2 thus gives $(n+12)$-vertex $\frac{n+35}{2}$-graphs for every odd $n \geq 13$. Setting $t := \frac{n-11}{2}$ we obtain the theorem's statement. ■

We briefly comment on the observation used in Figure 2 to obtain a triangulation—for results on the interaction between n- and $(n-1)$-cycles in n-vertex triangulations we refer to [15]. For all graphs proving the statement of Theorem 3, we chose the graph J to be, for every replacement, the octahedron minus the edges of a triangle (as before, the identified vertices shall be the three 2-valent vertices of the octahedron minus the edges of a triangle). But we may also choose the octahedron itself as it satisfies all required properties. Thus, as long as we start with a quadrangulation (i.e., a plane graph satisfying the property that each of its faces is a 4-face), the resulting graph after performing all replacements will be a triangulation. By these observations, it is easy to deduce the following result.

Proposition 4. *For any integer $t \geq 1$ there is a triangulation whose minimum degree is* 4*, which is a $(23+t)$-graph, and which has order $23+2t$.*

Note that Proposition 4 is neither stronger nor weaker than Theorem 3. The graphs we are dealing with are on the one hand non-hamiltonian, so the more edges occur the stronger the result, but on the other hand we are also interested in hamiltonian vertex-deleted subgraphs, which are less likely to occur in graphs with fewer edges, so the fewer edges, the stronger the result.

We shall discuss in the next section that, in a certain sense, Proposition 4 is best possible.

In 1978 Thomassen raised the question whether there are hypohamiltonian graphs without vertices of degree 3 (equivalently: whose minimum degree is at least 4) [10]; this problem remains open. By (†), such a graph cannot be planar. In this context, we mention the following consequences of Theorem 3.

Corollary 5. *There is a polyhedral non-hamiltonian graph of order n and whose minimum degree is 4 which has the property that $n - 24 > 0$ of its vertex-deleted subgraphs contain a hamiltonian cycle.*

Corollary 6. *For every positive integer t there exists an n such that there is a non-hamiltonian polyhedral graph of order n and minimum degree 4 in which precisely t vertex-deleted subgraphs contain a hamiltonian cycle.*

Arguing as above, Corollaries 5 and 6 also hold if we replace "polyhedral graph" by "triangulation".

3. Computational Results

In order to ascertain how good the bounds given in the previous section are, and to study more generally the circumference of small polyhedral graphs and triangulations, we will use a thoroughly tested computational framework to generate planar graphs exhaustively and investigate their hamiltonian properties. We mainly focus on the 3-connected case but, in Subsection 3.2, also briefly discuss the more general 2-connected case.

In particular, for each of the following computational results listed in the observations below we used Brinkmann and McKay's generator *plantri* [1] to exhaustively generate all non-isomorphic graphs up to a given order for various subclasses of planar graphs whose minimum degree is at least 4 and then applied separate programs to filter the non-hamiltonian graphs and to determine their number of exceptional vertices. The latter is done by testing for every $v \in V(G)$, for a given non-hamiltonian graph G, whether the graph $G - v$ is hamiltonian or not. The filter programs we used to test the hamiltonicity of the generated graphs and their vertex-deleted subgraphs were already extensively tested before, for instance in [4].

3.1. The 3-connected case

Observation 7. *All polyhedral graphs with minimum degree at least 4 on at most 22 vertices are hamiltonian*[2]*.*

[2]For the counts of these graphs, see sequence A007025 (https://oeis.org/A007025) in the On-Line Encyclopedia of Integer Sequences [9].

From Theorem 3 and Observation 7 we can immediately infer the following.

Corollary 8. *The smallest n-vertex polyhedral $(n-1)$-graph satisfies $23 \leq n \leq 25$.*

It would be interesting to determine the exact value.

The computational results for triangulations with minimum degree at least 4 are summarised in Table 1; the graphs from Table 1 with circumference $n-1$ can also be obtained from the database of interesting graphs from the *House of Graphs* [2] by searching for the keywords "triangulation * circumference $n-1$ and minimum degree 4". These findings lead to the following observations.

Order	# non-ham.	# circ. $n-1$	types of k-graphs
6–22	0	0	
23	1	0	
24	4	0	
25	76	3	3 24-graphs
26	964	40	40 25-graphs
27	11 949	781	14 25-graphs & 767 26-graphs

Table 1. The number of n-vertex triangulations admitting no hamiltonian cycle and whose minimum degree is at least 4, the number of all such graphs with circumference $n-1$, and the types of k-graphs in the latter class. (For the counts of the triangulations with minimum degree at least 4, see sequence A000103 (https://oeis.org/A000103) in the On-Line Encyclopedia of Integer Sequences [9].)

Observation 9. *The smallest non-hamiltonian triangulation whose minimum degree is at least 4 has 23 vertices. There is exactly one such graph of that order. It has circumference 20 and is shown in Figure 3*[3].

In Figure 3 we have emphasised a subgraph S of the given triangulation G. Note that G is obtained from S by inserting an octahedron into each of its faces, i.e., for each face F of S we consider an octahedron (disjoint from S) and a triangle T therein, and identify the boundary of F with T in the natural way, whilst maintaining the planarity of the resulting graph. This is very similar to our approach in the proof of Theorem 3 and supports the idea that this technique is, in some sense, efficient in producing examples of small order.

Observation 10. *The smallest n-vertex triangulations whose minimum degree is at least 4 and whose circumference is $n-1$ have order $n = 25$. There are exactly three such graphs, each with exactly one non-exceptional vertex. One of these graphs is shown in Figure 2 while the other two are given in Figure 4.*

[3] This graph can also be accessed directly at the *House of Graphs* [2] at https://houseofgraphs.org/graphs/45707

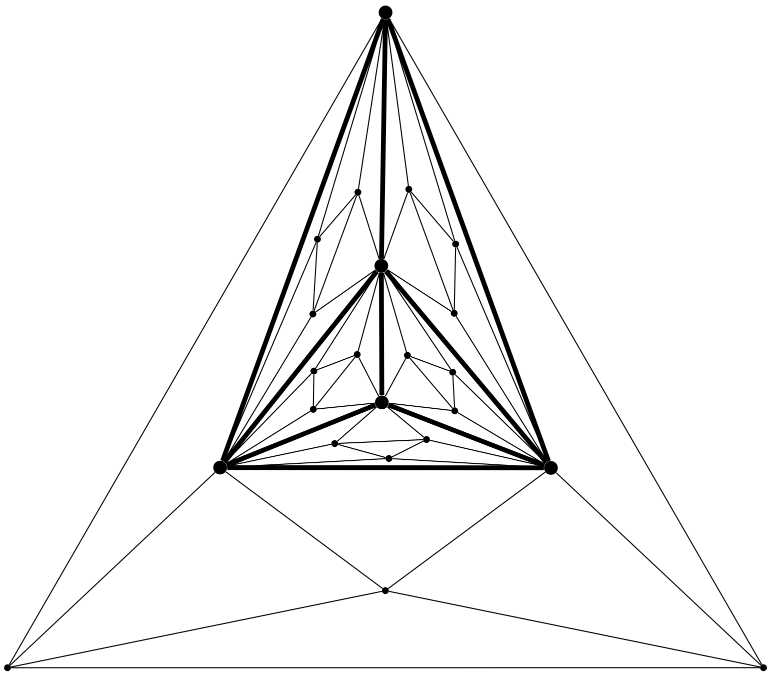

Figure 3. The smallest non-hamiltonian triangulation with minimum degree at least 4. It has order 23 and circumference 20.

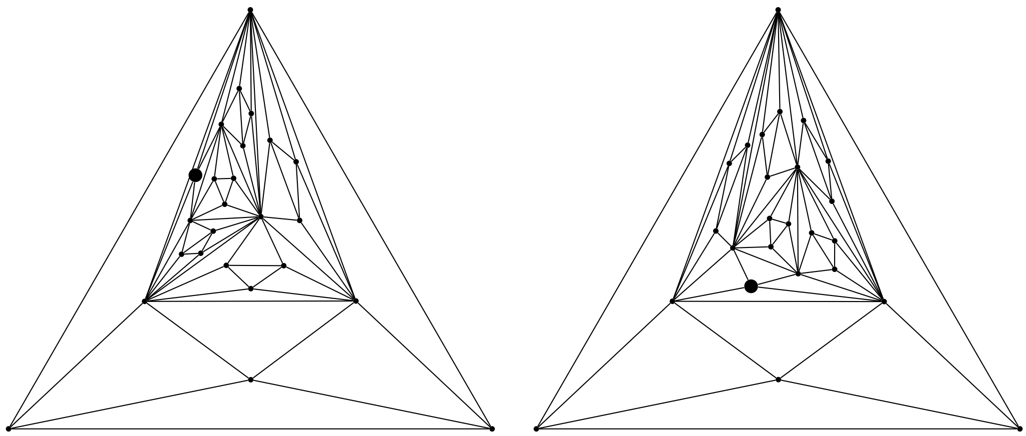

Figure 4. Two of the three smallest n-vertex triangulations with minimum degree at least 4 and circumference $n-1$. They have order $n = 25$ and exactly one non-exceptional vertex, which is marked in bold in the figure. The third such graph of that order is shown in Figure 2.

Corollary 11. *The smallest n-vertex triangulation with minimum degree at least 4 that is an $(n-1)$-graph has $n = 25$.*

This shows that for n-vertex $(n-1)$-graphs which have minimum degree at least 4 and which are triangulations, the result given in Proposition 4 is optimal in the sense that "23" cannot be replaced by "22".

By Euler's formula, a planar graph cannot have minimum degree greater than 5. We conclude this section by naturally complementing our findings on planar graphs with minimum degree 4 with results concerning the minimum degree 5 case.

Observation 12. *All triangulations with minimum degree* 5 *on up to at least* 41 *vertices are hamiltonian*[4]. *All polyhedral graphs with minimum degree* 5 *on up to at least* 35 *vertices are hamiltonian*[5].

In the context of the following proposition we note that a straightforward toughness-based construction using icosahedra yields the existence of a non-hamiltonian triangulation with minimum degree 5 and of order 59. Whether this is minimal is, as far as the authors are aware, unknown.

Proposition 13. *The smallest n-vertex triangulation with minimum degree* 5 *that is an $(n-1)$-graph satisfies* $42 \leq n \leq 71$.

Proof. The lower bound follows from Observation 12. For the upper bound, we follow the same strategy as in the proof of Theorem 3, so here we only give the main steps. Let G be the 15-vertex 7-graph from Figure 5. We replace (as defined in Section 2) each non-exceptional vertex occurring in $G - v$, depicted in Figure 5 as a black vertex, by an icosahedron. ■

3.2. The 2-connected case

It is easy to see that a graph satisfying the property that each of its vertex-deleted subgraphs admits a hamiltonian cycle, has to be 3-connected. However, if not all vertex-deleted subgraphs must be hamiltonian, then examples of connectivity 2 appear. Therefore, one might investigate generalisations of Thomassen's result (†) also in the more general 2-connected setting. In this section we complement, by computational means, results from [14].

In Table 2 we list the results for 2-connected graphs that are planar and whose minimum degree is at least 4. In [14, Figure 3] the last author presents a planar 18-vertex 16-graph whose connectivity is 2 and whose minimum degree is 4—it is the graph shown in Figure 7, bottom left. However, it was not known if this was the smallest example and/or only example of that order. Observation 15 settles this question based on the results from Table 2.

[4]For the counts of these graphs, see sequence A081621 (https://oeis.org/A081621) in the On-Line Encyclopedia of Integer Sequences [9].

[5]For the counts of these graphs, see [1].

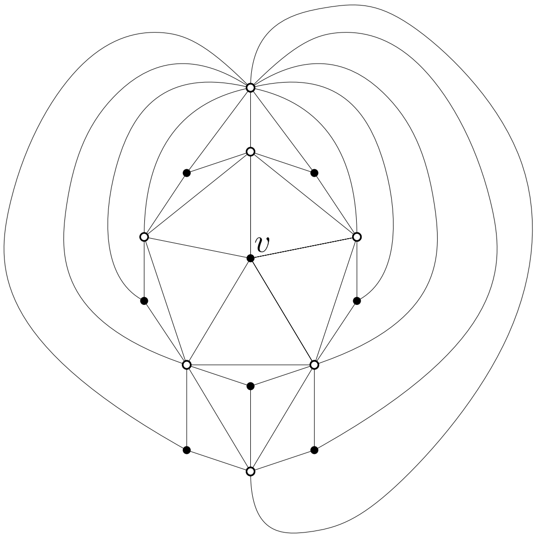

Figure 5. The polyhedral 7-graph on 15 vertices used in the proof of Proposition 13. Exceptional (non-exceptional) vertices are depicted in white (black).

Order	# non-ham. pl. 2-conn.	# circ. $n-1$	types of k-graphs
6-13	0	0	
14	2	0	
15	26	0	
16	238	0	
17	3 050	0	
18	38 432	12	12 16-graphs
19	469 271	392	336 17-graphs + 56 18-graphs
20	5 656 405	9 554	40 17-graphs + 7 066 18-graphs + 2 448 19-graphs

Table 2. The number of n-vertex non-hamiltonian 2-connected graphs that are planar and whose minimum degree is at least 4, all such graphs with circumference $n-1$, and the types of k-graphs in the latter class. The numbers given in this table refer to pairwise non-isomorphic abstract graphs (not embedded graphs).

Observation 14. *The smallest non-hamiltonian* 2*-connected graphs that are planar and whose minimum degree is at least* 4 *have* 14 *vertices. There are exactly two such graphs of that order. They both have circumference* 10 *and are shown in Figure* 6 *in the Appendix*[6].

[6]These graphs can also be accessed directly at the *House of Graphs* [2] at https://houseofgraphs.org/graphs/45709 and https://houseofgraphs.org/graphs/45711

Observation 15. *The smallest n-vertex* 2*-connected graphs that are planar, whose minimum degree is at least* 4*, and whose circumference is $n-1$ have order $n = 18$. There are exactly twelve such graphs, and each has exactly two non-exceptional vertices. These twelve graphs are shown in Figures* 7–8 *in the Appendix.*

The graphs from Table 2 with circumference $n-1$ and $n < 20$ can also be obtained from the database of interesting graphs from the *House of Graphs* [2] by searching for the keywords "2-connected planar * circumference $n-1$ and minimum degree 4".

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Appendix

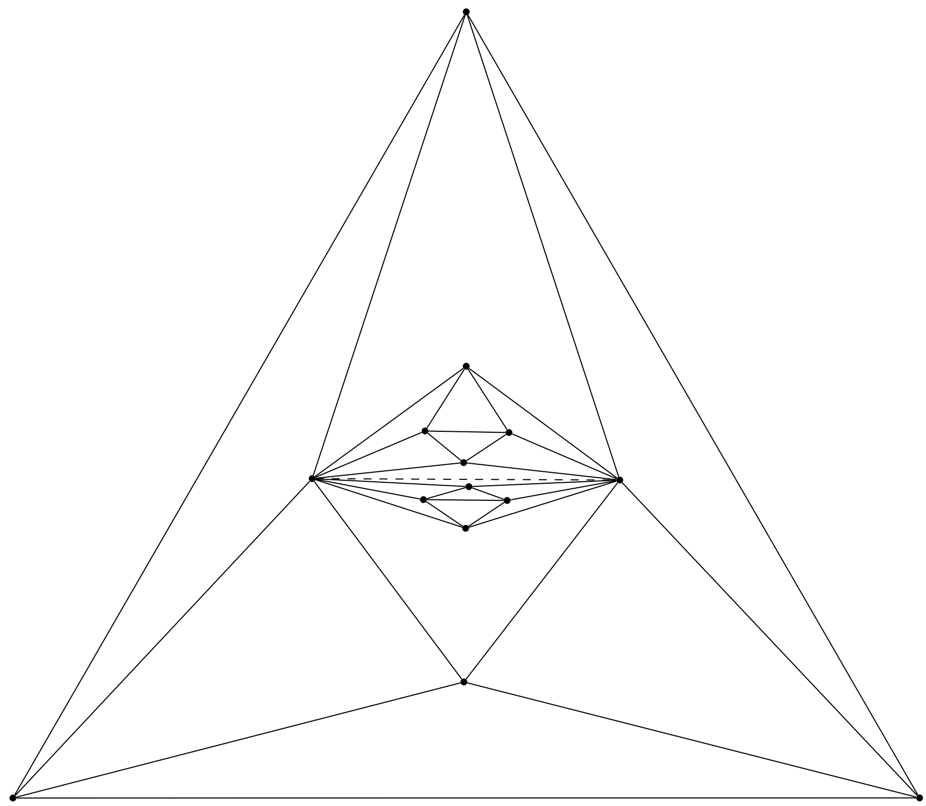

Figure 6. The smallest non-hamiltonian 2-connected graphs that are planar and whose minimum degree is at least 4. The second graph is obtained by removing the dashed edge. Both graphs have order 14 and circumference 10.

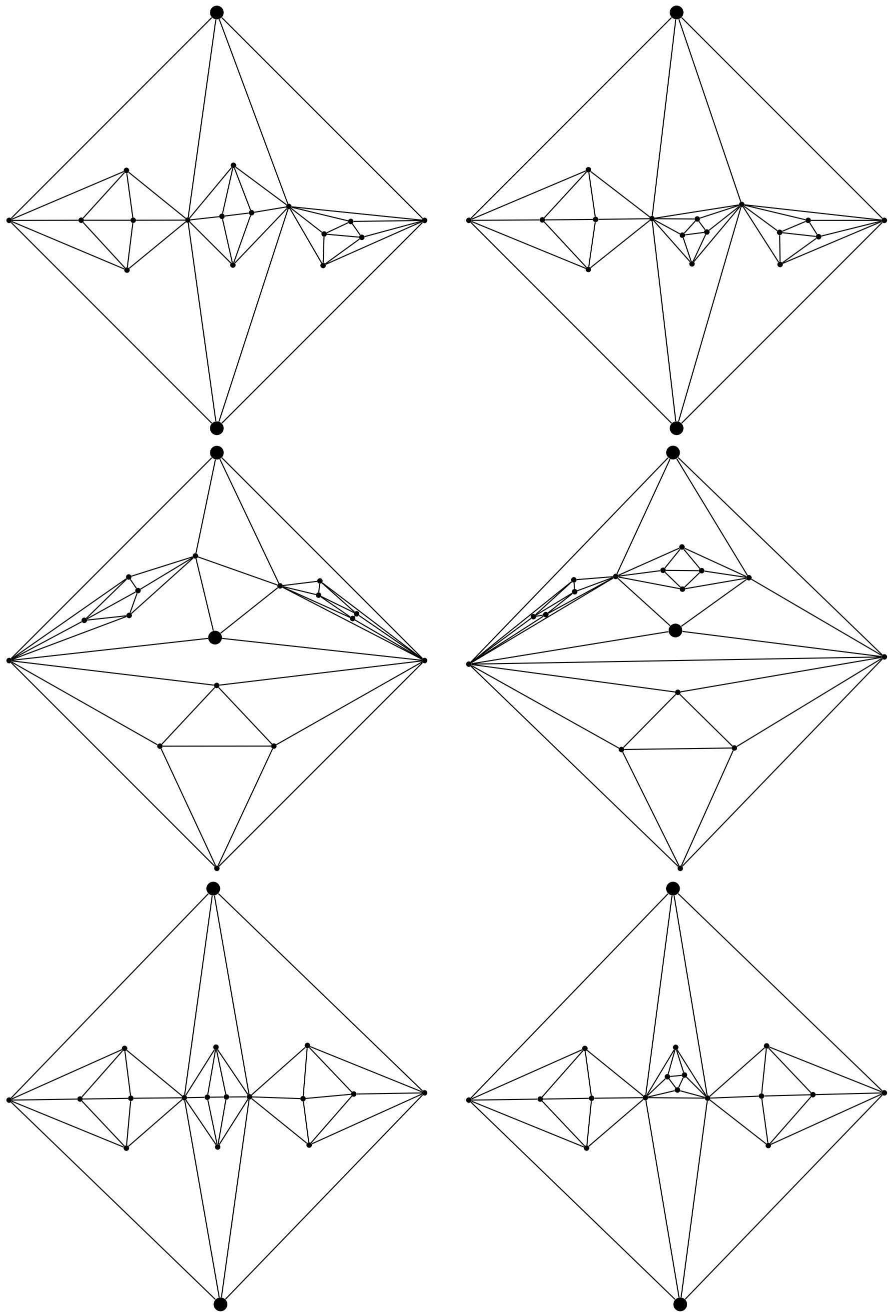

Figure 7. The smallest n-vertex graphs that are planar, whose minimum degree is at least 4, and whose circumference is $n-1$ (part 1 of 2). They have order $n = 18$ and exactly two non-exceptional vertices, which are marked in bold in the figure.

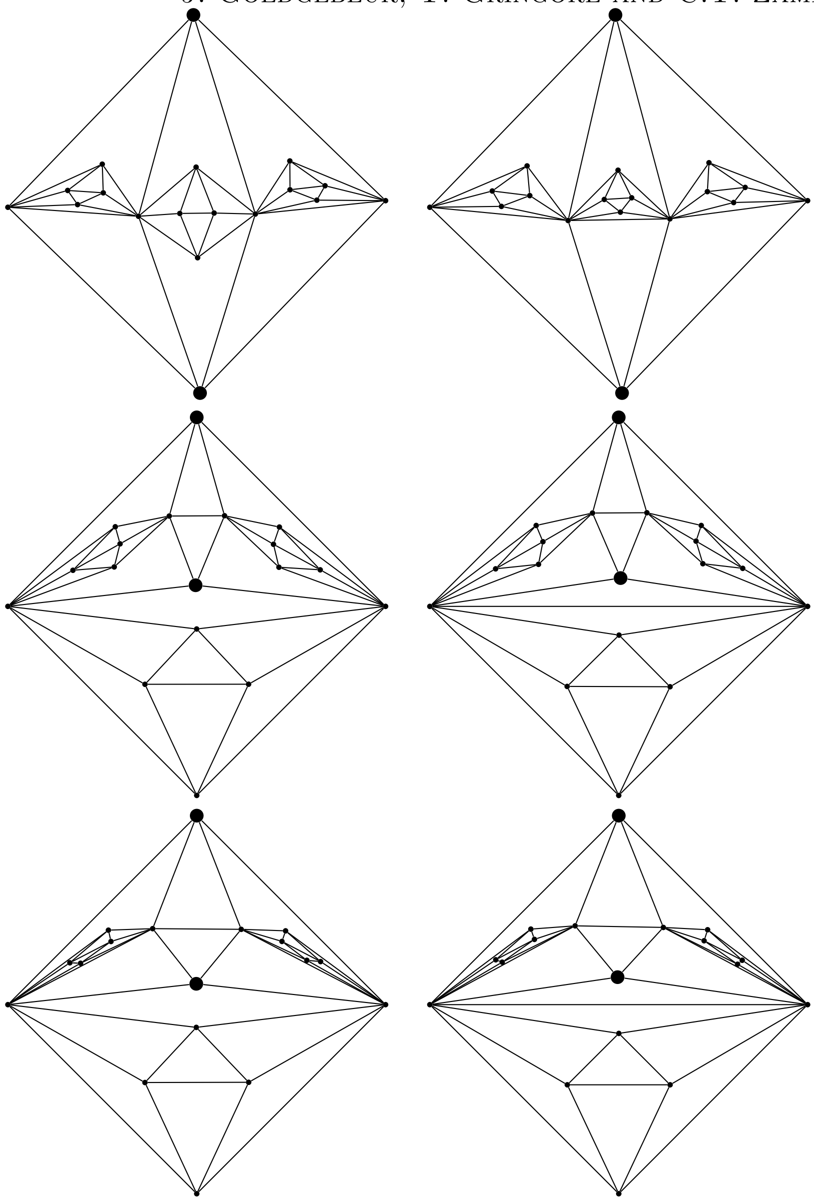

Figure 8. The smallest n-vertex graphs that are planar, whose minimum degree is at least 4, and whose circumference is $n-1$ (part 2 of 2). They have order $n = 18$ and exactly two non-exceptional vertices, which are marked in bold in the figure.

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