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ONLINE SIZE RAMSEY NUMBER FOR C_4 AND P_6

Mateusz Litka

Adam Mickiewicz University, Poznań
e-mail: matlit@amu.edu.pl

Abstract

In this paper we consider a game played on the edge set of the infinite clique $K_{\mathbb{N}}$ by two players, *Builder* and *Painter*. In each round of the game, Builder *chooses* an edge and Painter *colors* it red or blue. Builder wins when Painter creates a red copy of G or a blue copy of H, for some fixed graphs G and H. Builder wants to win in as few rounds as possible, and Painter wants to delay Builder for as many rounds as possible.

The *online size Ramsey number* $\tilde{r}(G, H)$, is the minimum number of rounds within which Builder can win, assuming both players play optimally.

So far it has been proven by Dybizbański, Dzido and Zakrzewska that $11 \leq \tilde{r}(C_4, P_6) \leq 13$ [J. Dybizbański, T. Dzido and R. Zakrzewska, *Online Ramsey numbers for paths and short cycles*, Discrete Appl. Math. 282 (2020) 265–270]. In this paper, we refine this result and show the exact value, namely we will present the theorem that $\tilde{r}(C_4, P_6) = 11$, with the details of the proof.

Keywords: graph theory, Ramsey theory, combinatorial games, online size Ramsey number.

2020 Mathematics Subject Classification: 05C57, 91A43, 91A46.

1. Introduction

In this paper, we consider the following generalization of Ramsey numbers, which was introduced by Beck [1]. Consider a game played on the edge set of the infinite clique $K_{\mathbb{N}}$ by two players, *Builder* and *Painter*. In each round of the game, Builder *chooses* an edge and Painter *colors* it red or blue. Builder wins when Painter creates a red copy of G or a blue copy of H, for some fixed graphs G and H. Builder wants to win in as few rounds as possible, and Painter wants to delay Builder for as many rounds as possible.

By red-blue *edge coloring* of a graph G we mean a function defined on the set of edges $E(G)$ that assigns one of the colors — red or blue — to each of the edges. We say that a graph G is *colored* if each of its edges is either blue or red. A graph is *red* if all its edges are red. A graph is *blue* if all its edges are blue. Such graphs are called *monochromatic* (1-colored).

We say that Builder *forces* a red edge if, after Builder selects that edge, then Painter coloring it blue will create a blue copy of H. Similarly, we say Builder *forces* a blue edge if, after Builder selects that edge, then Painter coloring it red will create a red copy of G.

We say that Builder *forces* a red copy of G or a blue copy of H in a given number of rounds t if it has a strategy that ensures that after at most t rounds the edge selected by Builder forces both red and blue edge at the same time. Then Painter coloring it blue will create a blue H, and coloring it red will create a red G. This prevents Painter from making another non-losing move.

The *online size Ramsey number* of two graphs G and H, denoted by $\tilde{r}(G,H)$, is the minimum number of rounds in which Builder forces a red copy of G or a blue copy of H, assuming both Builder and Painter are playing optimally.

We will call this game $\tilde{R}(G,H)$-*game*. First note that $\tilde{r}(G,H) = \tilde{r}(H,G)$. One can also consider a symmetric version of the numbers defined above. If $H = G$, then instead of $\tilde{r}(G,G)$ we write $\tilde{r}(G)$.

2. Main Result

As mentioned above, online size Ramsey number was introduced by Beck [1]. The exact values of $\tilde{r}(G,H)$ are known for several graphs G and H. In [5], Kurek and Ruciński considered the case where G and H are cliques, but except for the trivial $\tilde{r}(K_2,K_k) = \binom{k}{2}$, they were able to determine only one more value, namely $\tilde{r}(K_3,K_3) = 8$. So far, only one more value is known, namely $\tilde{r}(K_3,K_4) = 17$, which was obtained by Prałat [9] with computer support. Grytczuk, Kierstead and Prałat [4] gave exact values for several short paths and showed that $\tilde{r}(P_6) = 10$. Prałat also showed that $\tilde{r}(P_7) = 12$, $\tilde{r}(P_8) = 15$ [8] and $\tilde{r}(P_9) = 17$ [10].

We now turn to the subject of this paper, i.e., to the online size Ramsey numbers of the form $\tilde{r}(C_4,P_k)$. To avoid any confusion, note that (following the notation like in the mentioned articles) C_n is a cycle on n vertices and P_k is a path on k vertices. Values $\tilde{r}(C_4,P_3) = 6$ and $\tilde{r}(C_4,P_4) = 8$ were obtained by Cyman, Dzido, Lapinskas and Lo [2]. Then in [3] Dybizbański, Dzido and Zakrzewska proved that $\tilde{r}(C_4,P_5) = 9$. Also in [3] were obtained bounds for longer paths.

Proposition 1 [3]. *For $k \geq 6$ we have*

$$3k-5 \geq \tilde{r}(C_4,P_k) \geq \begin{cases} 2k-1 & \text{if } k = 6,7; \\ 2k-2 & \text{if } k \geq 8. \end{cases}$$

Thus, from the results obtained so far, it follows that $13 \geq \tilde{r}(C_4, P_6) \geq 11$. In this paper, we prove the following theorem.

Theorem 2. $\tilde{r}(C_4, P_6) = 11$.

The proof of the above theorem is the main result of the Master's Thesis [6] (in Polish). It was translated into English and can be found here [7]. Then in the first five rounds Builder constructs the path P_6 by selecting five edges in sequence. Basically, one of the 19 significantly different color patterns appears (up to the symmetry). The rest of the result is obtained by analyzing the individual subcases of the last six rounds, which lead us to 50 subcases. We now present an improved proof of this theorem that requires far fewer cases to consider.

3. PROOF OF THEOREM 2

By Proposition 1 $11 \leq \tilde{r}(C_4, P_6) \leq 13$. Therefore, it suffices to prove that Builder can win the $\tilde{R}(C_4, P_6)$-game in 11 rounds. We will present a strategy for Builder that forces Painter to create a red copy of C_4 or a blue copy of P_6 in at most 11 rounds. The upper bound ($\tilde{r}(C_4, P_6) \leq 11$) is obtained by analyzing individual cases, presented in the table in Figure 1 (first five rounds) and in the graphs below (last five rounds).

First five rounds

In the first five rounds Builder constructs P_6, but this time we have some kind of strategy to get the initial path P_6. In the first two rounds Builder chooses two consecutive edges (columns marked with 1&2 in the table in Figure 1). So we will either start with two edges of the same color (BB and RR case) or with edges of different colors.

In the BB case, we are left with 3 edges to color in $2^3 = 8$ ways. We extend the path to the right until red (the opposite color) is used — then we start extending left. The RR case is analogous when we swap the colors.

If we have two different colors, then Builder in the third round chooses as if the fourth edge of the path (disjointed with the first two), which Painter colors blue or red. Then Builder selects the edge connecting the third edge with this one of the first two edges that has the opposite color to it. So we connect blue to red (BR case) or red to blue (RB case).

In the BR case, we are left with 2 edges to color in $2^2 = 4$ ways. If the fourth edge is blue (the color of the third edge), then in the fifth round we extend right, otherwise we extend left. The RB case is analogous when we swap the colors.

Basically, one of the 12 possible color patterns appears, as indicated in the table by the appropriate case number (numbers from 0 to 11). The first occurrence of a given case is highlighted in green. Note that we have discarded the 0 case, because in this case Builder would win just after the first 5 rounds. So we actually get 11 significantly different P_6 paths.

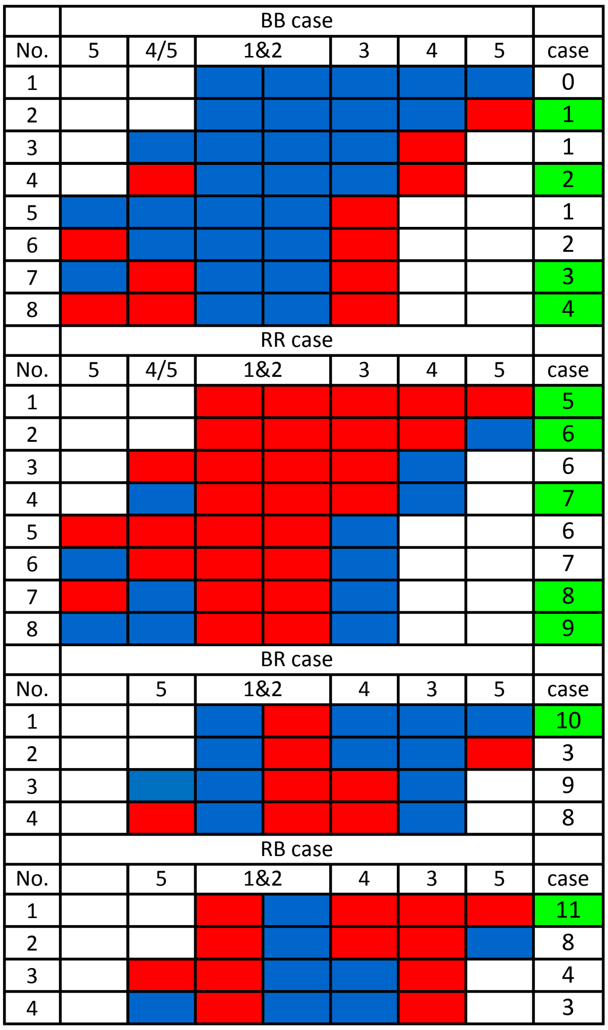


Figure 1. First five rounds.

Sixth round

In the sixth round we close the path P_6 to the cycle C_6. The sixth edge will be either blue or red, which we denote by adding the appropriate letter (B or R respectively) to the case number of the path we close with a given color.

Considering our paths in the order: 1, 10, 2, 3, 9, 5, 6, 11, 7, 8, 4, and then closing them to the cycle with blue and red edge respectively, it can be quickly and easily checked by hand that we get the following cycles:

($1B$ is discarded because we get blue P_6); $1R$; ($10B$ is discarded because we get blue P_6); $10R$; ($2B$ boils down to $10R$, which we will briefly denote by $2B \longrightarrow 10R$); $2R$; $3B$; $3R$; ($9B \longrightarrow 1R$); ($9R \longrightarrow 3R$); case 5 is an exception in which we do not close the path to the cycle; $6B$; $6R$; $11B$; ($11R \longrightarrow 6R$); ($7B \longrightarrow 2R$); ($7R \longrightarrow 11B$); ($8B \longrightarrow 3R$); $8R$; ($4B \longrightarrow 3R$); ($4R \longrightarrow 6B$).

Thus, considering the closing of the paths one by one, we actually get 10 significantly different C_6 cycles, namely $1R, 10R, 2R, 3B, 3R, 5, 6B, 6R, 11B, 8R$.

Last five rounds

The rest of the result is obtained by analyzing the individual subcases of the last five rounds, which are shown in the graphs below. The edges marked with the numbers 1 to 5 correspond to five edges of the path P_6 from the appropriate row in the table in Figure 1. At this stage, we have six vertices of the path P_6 marked with successive (uppercase) letters of the alphabet (A, B, C, D, E, F). The sixth edge (A, F), marked with number 6, corresponds to the edge, which form a cycle C_6. Remember that case 5 is an exception.

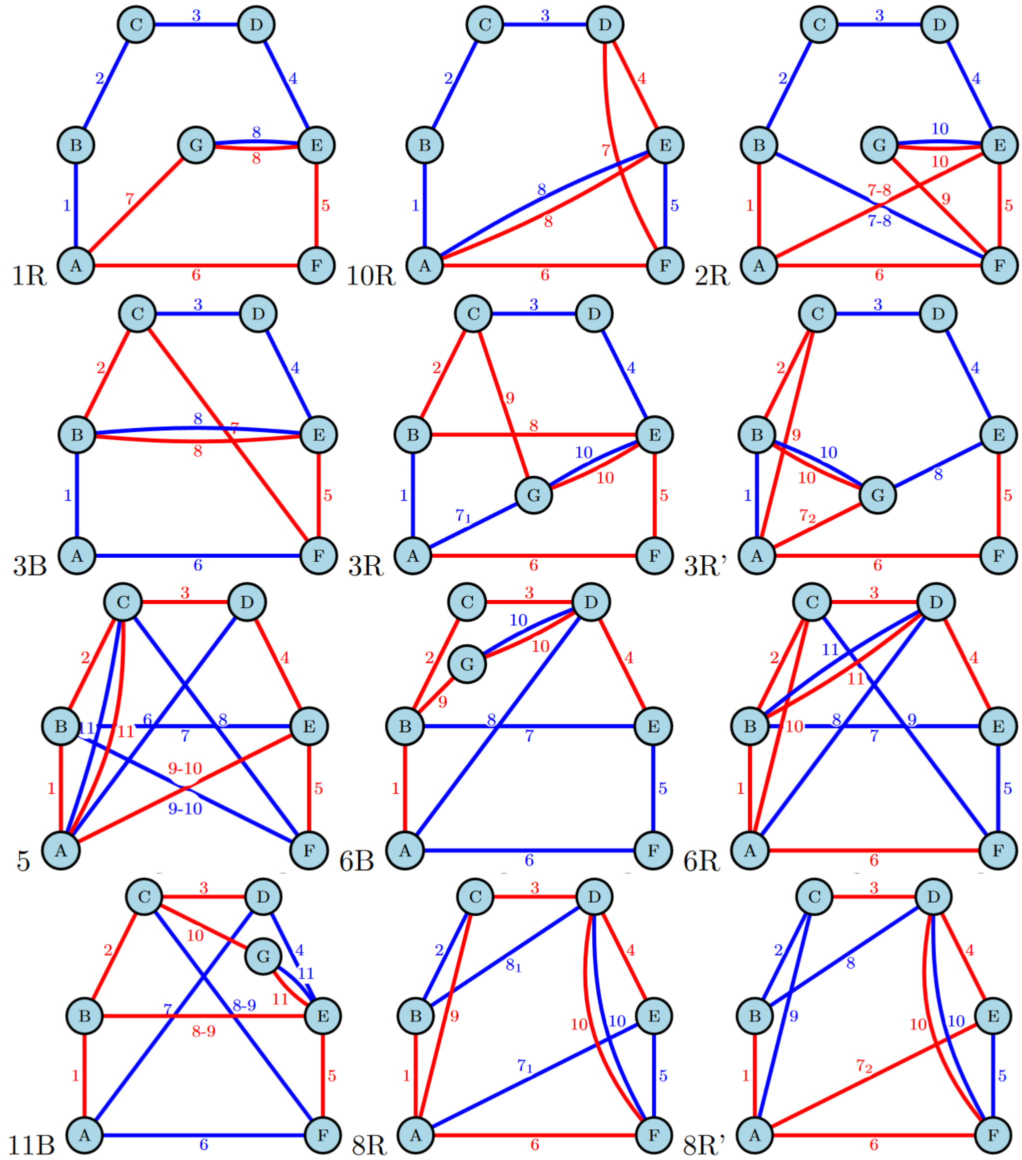
The edges marked with numbers from 7 to 11 correspond to the next edges selected by Builder. Note that the drawing software always puts a number exactly on the middle of an edge. Their colors have been forced by the fact that Painter wants to delay Builder for as many rounds as possible. So red color of the edge means that if Painter had colored this edge blue, he would have created blue P_6. Similarly, blue color of the edge means that if Painter had colored this edge red, he would have created red C_4. If a given subcase required it, we added a seventh vertex (marked by G). We have only 12 subcases to consider.

Firstly, because the Builder does not always have the option to choose the edge whose color is forced — we use a number with a subscript (N_1 and N_2), which means that Painter had a choice on the color of that edge in this round. This leads to branching into two subcases, which we denote by adding the symbol ' (apostrophe). Whereas (for the sake of order), we always assume that the N_1 edge is blue and the N_2 edge is red.

Secondly, let us note that sometimes taking into consideration two consecutively selected edges, Painter by coloring it both blue, will create blue P_6, and by coloring it both red, he will create red C_4. So we must have two different colors of these edges. In addition, there is symmetry (both before and after selecting these two edges). So, regardless of the color choice of the first edge, it is enough to consider just one subcase, e.g. red and blue respectively.

Therefore Builder forces a blue path P_6 or a red cycle C_4 over the next five rounds, as shown in the graphs above. The last edge is drawn in two colors. Note that after Builder selects this edge, Painter by coloring it blue, will create a blue

P_6, and by coloring it red, he will create a red C_4. This prevents Painter from making another non-losing move.



4. Comments and Open Questions

It would be too long and impractical to discuss each subcase in detail. Therefore, we will now make only some comments about our cases in the above proof. Cases $1R$, $10R$, $3B$, $6B$ and $6R$ are straightforward.

In the $2R$ case we are taking into consideration edges (A, E) and (B, F). Note that Painter by coloring it both blue, will create blue $P_6\colon (F, B, C, D, E, A)$, and by coloring it both red, he will create red $C_4\colon (A, E, F, B)$. So we must have two different colors of these edges. In addition, there is symmetry, so it is enough to consider just one subcase, e.g. red and blue, respectively.

In the $3R$ and $3R'$ cases we have a number with a subscript (7_1 and 7_2), which means that Painter had a choice about the color of the edge in this round, which led to branching into two subcases. By convention, we assume that 7_1 is blue and 7_2 is red.

In the 5 case we have an exception because we do not close the red path P_6 into the cycle. We are taking into consideration edges (A, E) and (B, F). Note that Painter by coloring it both blue, will create blue $P_6\colon (C, F, B, E, A, D)$, and by coloring it both red, he will create red $C_4\colon (A, E, F, B)$. So we must have two different colors of these edges. In addition, there is symmetry, so it is enough to consider just one subcase, e.g. red and blue, respectively.

In the $11B$ case we are taking into consideration edges (B, E) and (C, F). Note that Painter by coloring it both blue, will create blue $P_6\colon (C, F, A, D, E, B)$, and by coloring it both red, he will create red $C_4\colon (B, E, F, C)$. So we must have two different colors of these edges. In addition, there is symmetry, so it is enough to consider just one subcase, e.g. red and blue, respectively.

In the $8R$ and $8R'$ cases we have a number with a subscript (7_1 and 7_2), which means that Painter had a choice about the color of the edge in this round, which led to branching into two subcases. By convention, we assume that 7_1 is blue and 7_2 is red. In the first case, Painter then had a choice about the color of the edge (B, D). Note that it is enough to consider only the subcase when 8_1 is blue, because when 8_2 is red, then it boils down to the subcase $8R'$ (with the accuracy of a rotation).

Since the paper may inspire some future results, we will now pose some questions regarding not only the above result, but also possible future generalizations in the proofs for longer paths.

One can ask whether we have applied the optimal strategy in each of the cases considered. Note, however, that we do not really need optimal strategies. Since we know by Proposition 1 that $11 \leq \tilde{r}(C_4, P_6)$, we only need to point out strategies for Builder to win in 11 rounds. Another thing to consider is that we always start with P_6. However, as above, it suffices to show that Builder can win in 11 rounds.

One might rightly ask if we need to have an exception in case 5. On one hand, of course, case $5B$ boils down to $6R$, but on the other hand, case $5R$ is actually our case 5 with an extra red edge (A, F), so we would need 12 rounds here. One can wonder whether 12 is the minimum number of subcases that need to be considered. Note that no two graphs are isomorphic. Also, because Painter wants to delay Builder for as many rounds as possible, he will always play in such a way to delay Builder for 11 rounds, which are only required in 3 subcases.

One might think to minimize the maximum number of vertices that we used for the game. We have 6 or 7 vertices in our graphs. Less than 6 is not possible, because there must be somewhere forced a blue copy of P_6, which requires 6 vertices. It seems that in our case we will not omit the use of the seventh vertex. Maybe an interesting observation is that we always have exactly 4 blue edges (not counting the last edge, which is drawn in two colors). In the previous version of the proof, there were also 4 blue edges in all 50 subcases.

In conclusion, one might consider using our proof for longer paths. Possibly the optimal strategy would be to start with a path P_7. What is the smallest possible number of significantly different P_{k+1} paths that one can end up with, starting from P_k paths? It is an interesting combinatorial problem and might be relevant to the number of cases to consider when bounding $\tilde{r}(C_4, P_{k+1})$. However, it seems that simply extending our 11 significantly different P_6 paths left or right would be too easy. In addition, we would no longer be able to use moves with disjointed edge, which reuse (e.g. in the fifth round) might seems helpful.

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