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SEMITOTAL DOMINATION IN CLAW-FREE GRAPHS

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Abstract

In an isolate-free graph G, a subset S of vertices is a *semitotal dominating set* of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set in G. We prove that if G is a connected claw-free graph of order n with minimum degree $\delta(G) \geqslant 2$ and is not one of fourteen exceptional graphs (ten of which are cycles), then $\gamma_{t2}(G) \leqslant \frac{3}{7}n$, and we also characterize the graphs achieving equality, which are an infinite family of graphs. In particular, if we restrict $\delta(G) \geqslant 3$ and $G \neq K_4$, then we can improve the result to $\gamma_{t2}(G) \leqslant \frac{2}{5}n$, solving the conjecture for the case of claw-free graphs, proposed by Goddard, Henning and McPillan in [*Semitotal domination in graphs*, Util. Math. 94 (2014) 67–81].

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1. Introduction

In this paper, we consider only finite simple undirected graphs. A subset D of vertices in a graph G is a *dominating set* (respectively, *total dominating set*) of G if every vertex of $V(G) \setminus D$ (respectively, $V(G)$) is adjacent to a vertex in

D. The minimum cardinality of a dominating set (respectively, total dominating set) is called the *dominating number* (respectively, *total dominating number*), represented as $\gamma(G)$ (respectively, $\gamma_t(G)$) of G. Since 1997, domination and total domination have been extensively studied, and those who are interested can see [1, 2, 4–6].

The semitotal domination was introduced by Goddard, Henning and McPillan [3] in 2014. A subset D of vertices in an isolate-free graph G is a *semitotal dominating set*, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set in G. We refer to a minimum semi-TD-set of G as a $\gamma_{t2}(G)$-*set*. Since every total dominating set is a semi-TD-set and every semi-TD-set is a dominating set, $\gamma(G) \leqslant \gamma_{t2}(G) \leqslant \gamma_t(G)$. However, the semitotal domination number is very different from the domination and total domination number. For example, the total domination number cannot be compared with the matching number, while the semitotal domination number is comparable with the matching number and cannot be greater than the matching number plus one (see [7]). That makes the study of semitotal domination number interesting.

Goddard *et al.* [3] proved that if G is a connected graph of order $n \geqslant 4$, then $\gamma_{t2}(G) \leqslant \frac{n}{2}$ and characterized the graphs of minimum degree 2 achieving this bound. Also, they proposed a conjecture as follows.

Conjecture 1. *For all graphs $G \neq K_4$ of order n with minimum degree $\delta(G) \geqslant 3$, $\gamma_{t2}(G) \leqslant \frac{2}{5}n$.*

As usual, $K_{1,3}$ is called a *claw*. A graph is *claw-free* if it does not contain the $K_{1,3}$ as an induced graph. Let u be the vertex of degree 3 of a claw. Then we can say that the claw is *centered* at u. Zhu *et al.* [10] proved that if G is a connected claw-free cubic graph of order n other than two graphs, then $\gamma_{t2}(G) \leqslant \frac{n}{3}$. Henning [8] established the tight upper bounds on the upper semitotal domination number of a regular graph using edge weighting functions. In [9], Henning and Pandey showed the semitotal domination problem is NP-complete for chordal graphs and bipartite graphs.

This paper is organized as follows. In Section 2, we establish a upper bound on the semitotal domination number of claw-free graphs G with minimum degree $\delta(G) \geqslant 3$ (i.e., Theorem 3). In Section 3, we prove that if $G \neq C_n$ is a connected claw-free graph of order n with $\delta(G) \geqslant 2$ and is not one of four exceptional graphs, then $\gamma_{t2}(G) \leqslant \frac{3}{7}n$, and we characterize the infinite families of graphs that achieve equality in this bound (i.e., Theorem 8). Combining this result with the semitotal domination number of cycles, we can obtain a general conclusion (i.e., Theorem 9). In Section 4, we conclude the paper with one problem as possible future work.

2. Claw-Free Graphs G with $\delta(G) \geqslant 3$

In this section, we discuss the semitotal domination number in connected claw-free graphs with minimum degree at least three. Before that, we introduce some definitions.

Let S_1 and S_2 be two subsets of $V(G)$. We denote by $E[S_1, S_2]$ the subset of edges of G with one end in S_1 and the other end in S_2. Let S be a subset of $V(G)$ and v be a vertex in S. The *S-external private neighborhood* of v, denoted by $epn(v, S)$, is the set of all vertices in $V(G) \setminus S$ that are adjacent to v but to no other vertex of S. In other words, if $u \in epn(v, S)$, then $u \in V(G) \setminus S$ and $N_G(u) \cap S = \{v\}$. The *S-internal private 2-neighborhood* of v, denoted by $ipn_2(v, S)$, is the set of all vertices in $S \setminus \{v\}$ that are within distance 2 of v in G but at distance greater than 2 from every other vertex of S. In other words, if $u \in ipn_2(v, S)$, then $u \in S \setminus \{v\}$, $d(v, u) \leqslant 2$, and $d(u, w) > 2$ for any vertex $w \in S \setminus \{u, v\}$.

A semi-TD-set in a graph G is a *minimal semi-TD-set* if it contains no semi-TD-set of G as a proper subset. The following result in [7] provides a characterization of minimal semi-TD-sets.

Lemma 2 [7]. *Let S be a semi-TD-set in a graph G. Then, S is a minimal semi-TD-set of G if and only if every vertex $v \in S$ satisfies at least one of the following three properties.*

(a) *The vertex v is isolated in $G[S]$.*

(b) $ipn_2(v, S) \neq \emptyset$.

(c) $epn(v, S) \neq \emptyset$.

Theorem 3. *If $G \neq K_4$ is a connected claw-free graph of order n with $\delta(G) \geqslant 3$, then $\gamma_{t2}(G) \leqslant \frac{2}{5}n$.*

Proof. Let $G \neq K_4$ be a connected claw-free graph of order n with $\delta(G) \geqslant 3$. We choose a $\gamma_{t2}(G)$-set S such that $\lambda(S)$ is minimum, where $\lambda(S)$ is the number of edges in $G[S]$. For convenience, set $\overline{S} = V \setminus S$ and $N_{\overline{S}}(w) = N(w) \cap \overline{S}$ and $N_S(w) = N(w) \cap S$, where w is a vertex in $V(G)$.

We first define an edge weight function on S in G. The edge weight function is the function ψ_S: $E(G) \to [0, 1]$ that assigns a weight of 0 for each edge in $E[S, S] \cup E[\overline{S}, \overline{S}]$, and assigns a weight of $\psi_S(e) = \frac{1}{|N_S(u)|}$ for each edge $e \in E[S, \overline{S}]$ that joins a vertex u in $\overline{S}$ to a vertex in S. The vertex weight function is the function ϕ_S: $V(G) \to [0, \infty]$ that assigns to each vertex $w \in V(G)$ the sum of the weights of the edges incident with w. Finally, we denote the vertex weight sum of S by $\xi(S) = \sum_{v \in S} \phi_S(v)$. Thus $\xi(S) = \sum_{v \in S} \phi_S(v) = \sum_{e \in E(G)} \psi_S(e) = \sum_{u \in \overline{S}} \phi_S(u) = |\overline{S}| = n - |S|$. The strategy is to show that each vertex of S has a weight of at least $\frac{3}{2}$ on average. Then $\frac{3}{2}|S| \leqslant \xi(S) = n - |S|$. It follows that $|S| \leqslant \frac{2}{5}n$.

Let $A = \{v \in S \mid ipn_2(v,S) \neq \emptyset\}$ and $A_1 = \{v \in A \mid v \in ipn_2(v',S)$ for some vertex $v' \in S\}$ and $A_2 = A \backslash A_1$. For each vertex $v \in A_2$, let $S_v = ipn_2(v,S) \cup \{v\}$. If $v' \in ipn_2(v,S)$, then $v' \notin A$. Otherwise, $v' \in A$ and $v \in ipn_2(v',S)$ since v is the only vertex of S within distance 2 from v' in G, contradicting the fact that $v \in A_2$. Further, we note that if v and v' are distinct vertices in A_2, then $ipn_2(v,S) \cap ipn_2(v',S) = \emptyset$. Hence, $S_v \cap S'_v = \emptyset$ for each pair of different vertices $v, v' \in A_2$. Let $B = \bigcup_{v \in A_2} S_v$ and $C = S \setminus (A_1 \cup B)$.

Claim 1. *If $v \in A_1$, then $\phi_S(v) \geqslant \frac{3}{2}$ on average.*

Proof. Let $v \in A_1$ and $v \in ipn_2(v',S)$. Then v' is the only vertex of S within distance 2 from v in G. Since $ipn_2(v,S) \neq \emptyset$, $v' \in ipn_2(v,S)$ and $v' \in A_1$. This implies that the vertices in A_1 are paired off. Thus all neighbors of v and v' in $\overline{S}$ are adjacent to no vertex of $S \setminus \{v, v'\}$. It follows that $\phi_S(v) + \phi_S(v') = |N_{\overline{S}}(v) \cup N_{\overline{S}}(v')|$. If $|N_{\overline{S}}(v) \cup N_{\overline{S}}(v')| \leqslant 2$, then $N_{\overline{S}}(v) = N_{\overline{S}}(v')$, $|N_{\overline{S}}(v)| = 2$ and $vv' \in E(G)$ since $\delta(G) \geqslant 3$. Let $\{u_1, u_2\} = N_{\overline{S}}(v) \cup N_{\overline{S}}(v')$. Since $G \neq K_4$ and $\delta(G) \geqslant 3$, without loss of generality, u_2 has a neighbor u_3 in $\overline{S}$. Note, u_3 is adjacent to a vertex of $S \backslash \{v, v'\}$. When $u_1u_2 \notin E(G)$, $S' = (S \backslash \{v, v'\}) \cup \{u_1, u_2\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S') < \lambda(S)$, a contradiction. When $u_1u_2 \in E(G)$, $S' = (S \setminus \{v, v'\}) \cup \{u_2\}$ is a semi-TD-set of G with $|S'| < |S|$, a contradiction. Thus $|N_{\overline{S}}(v) \cup N_{\overline{S}}(v')| \geqslant 3$. Further, $\phi_S(v) + \phi_S(v') \geqslant 3$, as desired. $\square$

Claim 2. *If $v \in A_2$, then $\phi_S(v') \geqslant \frac{3}{2}$ on average for each vertex $v' \in S_v$.*

Proof. Let $v \in A_2$, $S_v = \{v_1, \ldots, v_{|S_v|}\}$ and $v_1 = v$. Note that $|S_v| \geqslant 2$. Since $\{v_2, \ldots, v_{|S_v|}\} \subseteq ipn_2(v)$, at most one vertex of $\{v_2, \ldots, v_{|S_v|}\}$ is adjacent to v; all neighbors of v_i in $\overline{S}$ are adjacent to no vertex of $S \backslash \{v, v_i\}$, where $i \in \{2, \ldots, |S_v|\}$. It follows that $\sum_{i \in \{2,\ldots,|S_v|\}} \phi_S(v_i) = \left|\bigcup_{i \in \{2,\ldots,|S_v|\}} N_{\overline{S}}(v_i)\right|$. If $|S_v| \geqslant 3$, then the vertex weight sum of S_v is at least $\sum_{i \in \{2,\ldots,|S_v|\}} \phi_S(v_i) \geqslant 2 + 3(|S_v| - 2) = 3|S_v| - 4$. Since $\frac{3|S_v|-4}{|S_v|} > \frac{3}{2}$, each vertex of S_v has a weight of at least $\frac{3}{2}$ on average.

Next consider $|S_v| = 2$, and then $S_v = \{v, v_2\}$. If $|N_{\overline{S}}(v_2)| \leqslant 2$, then $|N_{\overline{S}}(v_2)| = 2$ and $vv_2 \in E(G)$ since $\delta(G) \geqslant 3$. Let $N(v_2) = \{v, u_1, u_2\}$. When $vu_1 \in E(G)$ and $vu_2 \in E(G)$, $S' = S \setminus \{v_2\}$ is a dominating set of G. Since $v \in A_2$, $v \notin ipn_2(v_2)$ and there exists a vertex v' distinct from v_2 in S such that $d(v, v') \leqslant 2$. Hence S' is a semi-TD-set of G with $|S'| < |S|$, a contradiction. When $vu_1 \notin E(G)$ or $vu_2 \notin E(G)$, without loss of generality, consider $vu_1 \notin E(G)$. Then $vu_2 \in E(G)$ or $u_1u_2 \in E(G)$ since G is claw-free. Now, $S' = (S \setminus \{v_2\}) \cup \{u_1\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S') < \lambda(S)$, a contradiction. Thus $|N_{\overline{S}}(v_2)| \geqslant 3$. Then $\phi_S(v) + \phi_S(v_2) \geqslant 3$, as desired. $\square$

By Claim 2 and $B = \bigcup_{v\in A_2} S_v$, each vertex of B has a weight of at least $\frac{3}{2}$ on average. Hence, next we consider the vertex in C. Before this, we give the following claim.

Claim 3. *If v is a vertex of S with $epn(v, S) \neq \emptyset$, then v is isolated in $G[S]$.*

Proof. By contradiction, we suppose that there exists a vertex $v_1 \in S$ with $epn(v_1, S) \neq \emptyset$ such that v_1 has a neighbor v_2 in S. Let $u_1 \in epn(v_1, S)$. Then $E[\{u_1\}, S \setminus \{v_1\}] = \emptyset$. Since G is claw-free and $u_1v_2 \notin E(G)$, each vertex of $N(v_1) \setminus \{u_1, v_2\}$ is adjacent to u_1 or v_2. Now, $S' = (S \setminus \{v_1\}) \cup \{u_1\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S') < \lambda(S)$, a contradiction. □

Claim 4. *If $v \in C$, then $\phi_S(v) \geqslant \frac{3}{2}$.*

Proof. Let $v \in C$. Since $v \notin A$, $ipn_2(v, S) = \emptyset$. If v is not isolated in $G[S]$, then $epn(v, S) \neq \emptyset$ by Lemma 2, and it contradicts Claim 3. Thus v is isolated in $G[S]$.

If there exists a vertex $u_1 \in N_{\overline{S}}(v)$ such that $|N_S(u_1)| \geqslant 3$, let $\{v_1, v_2\} \subseteq N_S(u_1)$. Since G is claw-free and v is isolated in $G[S]$, $v_1v_2 \in E(G)$. By Claim 3, $epn(v_2, S) = \emptyset$. According to Lemma 2, $ipn_2(v_2, S) \neq \emptyset$. Since $d(v, v_1) = 2$, $v_1 \notin ipn_2(v_2, S)$. Let $v_3 \in ipn_2(v_2, S)$. Then $d(v_2, v_3) = 2$, otherwise $d(v_1, v_3) = 2$ and it is a contradiction. Let u_2 be a vertex in $\overline{S}$ connecting v_2 and v_3. Note, all neighbors of v_3 in $\overline{S}$ are adjacent to no vertex of $S \setminus \{v_2, v_3\}$. Then $u_2v_1 \notin E(G)$. Since G is claw-free, each vertex of $N(v_2) \setminus \{v_1, u_2\}$ is adjacent to v_1 or u_2. When $N(v_3) \subseteq N[u_2]$, $S' = (S \setminus \{v_2, v_3\}) \cup \{u_2\}$ is a semi-TD-set of G with $|S'| < |S|$, a contradiction. When $N(v_3) \nsubseteq N[u_2]$, there exists a vertex $u_3 \in N(v_3)$ such that $u_3u_2 \notin E(G)$. Since G is claw-free, each vertex of $N(v_3) \setminus \{u_2, u_3\}$ is adjacent to u_2 or u_3. Then $S' = (S \setminus \{v_2, v_3\}) \cup \{u_2, u_3\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S') < \lambda(S)$, a contradiction. Thus for each vertex u_1 of $N_{\overline{S}}(v)$, $|N_S(u_1)| \leqslant 2$ and $\psi_S(u_1v) \geqslant \frac{1}{2}$. Since v is isolated in $G[S]$ and $\delta(G) \geqslant 3$, $\phi_S(v) \geqslant \frac{3}{2}$. □

According to Claims 1, 2 and 4, each vertex of S has a weight of at least $\frac{3}{2}$ on average. This completes the proof of Theorem 3. ■

3. CLAW-FREE GRAPHS G WITH $\delta(G) \geqslant 2$

In this section, we consider connected claw-free graphs with minimum degree at least two. In order to state clearly, we first introduce some basic conclusions.

Lemma 4. *If there exists a vertex v in G such that $N(v)$ is a clique, then we can find a $\gamma_{t2}(G)$-set containing vertices of $N(v)$.*

Proof. Let D be a $\gamma_{t2}(G)$-set. In order to dominate v, $N[v] \cap D \neq \emptyset$. If $N(v) \cap D \neq \emptyset$, there is noting to prove. Thus consider $N(v) \cap D = \emptyset$. Then $v \in D$ and $(D \setminus \{v\}) \cup \{v_1\}$ is a $\gamma_{t2}(G)$-set of G, where $v_1 \in N(v)$, as required. ■

The semitotal domination number of a cycle is established by Goddard *et al.* [3].

Proposition 5 [3]**.** *For* $n \geqslant 3$, $\gamma_{t2}(C_n) = \left\lceil \frac{2}{5}n \right\rceil$.

We define six graphs G_1-G_6 as the following graphs.

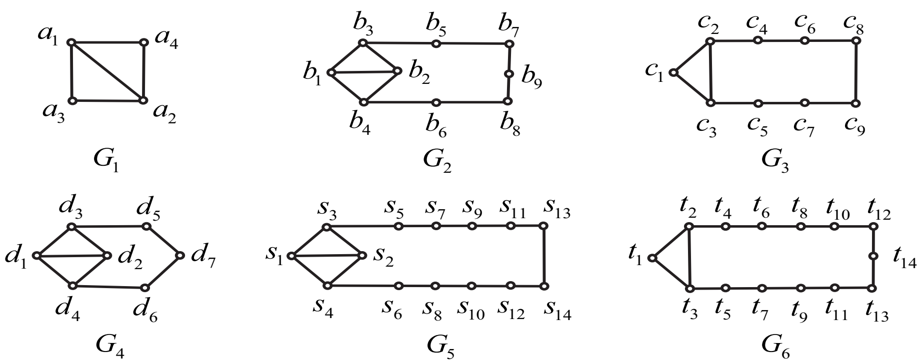

Figure 1. Six graphs: G_1–G_6.

Observation 6. *Let G be a graph in $\{G_2, G_3\}$. Then the following properties hold.*

(a) $\gamma_{t2}(G) = 4$ *and there exists a $\gamma_{t2}(G)$-set containing v for any vertex $v \in V(G)$.*

(b) *Let G' be a graph obtained from G by adding a vertex u and some edges between $\{u\}$ and $V(G)$. Then $\gamma_{t2}(G') \leqslant 4$ and there exists a $\gamma_{t2}(G')$-set containing u.*

Observation 7. *Let G be a graph in $\{G_4, G_5, G_6\}$. Then the following properties hold.*

(a) $\gamma_{t2}(G_4) = 3 = \frac{3}{7}n$ *and* $\gamma_{t2}(G_5) = \gamma_{t2}(G_6) = 6 = \frac{3}{7}n$.

(b) *Let G' be a graph obtained from G by adding a vertex u and some edges between $\{u\}$ and $V(G)$ such that G' is claw-free. Then there exists a $\gamma_{t2}(G')$-set D' containing vertex u. Further, when $G = G_4$, $|D'| \leqslant 3$; when $G \in \{G_5, G_6\}$, $|D'| \leqslant 6$.*

The graph H is illustrated in Figure 2(a) and the vertex w is the *root vertex* of H. The *rooted product graph* $K_t \circ_w H$ is defined from K_t and H by taking t copies of H and identifying the i^{th} vertices of K_t with the vertex w in the i^{th} copy of H for each $i \in \{1, \ldots, t\}$, where t is a positive integer. For example, $K_3 \circ_w H$ is illustrated in Figure 2(b). The resulting graph G is a connected claw-free graph of order $n = 7t$ with minimum degree 2 and $\gamma_{t2}(G) = 3t = \frac{3}{7}n$. Let $\mathcal{G} = \{K_t \circ_w H\}$.

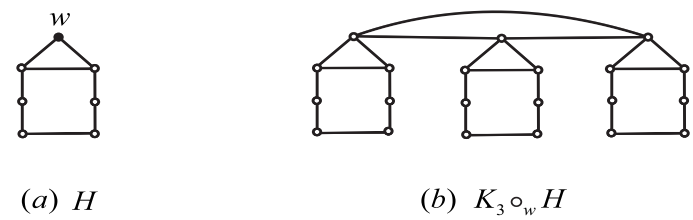

Figure 2. (a) Graph H, where the black vertex w is the root vertex of H. (b) Graph $K_3 \circ_w H$.

Theorem 8. *If $G \notin \{C_n, K_4, G_1, G_2, G_3\}$ is a connected claw-free graph of order n with $\delta(G) \geqslant 2$, then $\gamma_{t2}(G) \leqslant \frac{3}{7}n$, with equality if and only if $G \in \{G_4, G_5, G_6\} \cup \mathcal{G}$.*

Proof. For convenience, we call such a graph is normal if it is a connected claw-free graph with minimum degree at least 2 and is not isomorphic to a cycle, K_4, G_1, G_2, or G_3. Combined with Observation 7(a), to prove Theorem 8, it suffices to prove that if G is a normal graph of order n, then $\gamma_{t2}(G) < \frac{3}{7}n$ or $G \in \{G_4, G_5, G_6\} \cup \mathcal{G}$. We proceed by induction on the order $n \geqslant 3$ of a normal graph.

If $n = 3$ or $n = 4$, then $G \in \{C_3, C_4, K_4, G_1\}$, a contradiction. Thus $n \geqslant 5$. If $n = 5$, it is easy to verify that $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. If $\delta(G) \geqslant 3$, then $\gamma_{t2}(G) \leqslant \frac{2}{5}n < \frac{3}{7}n$ by Theorem 3. Hence, we may assume that $n \geqslant 6$ and $\delta(G) = 2$. Denote by S_2 the set of vertices of degree 2 in G. Then $G[S_2]$ is a disjoint unit of paths since G is not a cycle. Let $P = x_1 \cdots x_k$ be a longest path of $G[S_2]$ for some integer k. In what follows, we divide into five cases according to the size of k.

Case 1. $k \geqslant 5$. Let $N(x_1) = \{x_2, u_1\}$. Then $d(u_1) \geqslant 3$ since P is a longest path of $G[S_2]$. If $u_1x_5 \in E(G)$, then G has a claw centered at u_1 as $d(u_1) \geqslant 3$ and $\{x_1, x_5\} \subseteq S_2$, a contradiction. Thus $u_1x_5 \notin E(G)$. Let $N(x_5) = \{x_4, u_2\}$. (Possibly, $u_2 = x_6$.)

We construct G' from G by removing all vertices of $\{x_1, \ldots, x_5\}$ and adding the edge u_1u_2 when $u_1u_2 \notin E(G)$. If $\delta(G') = 1$, then $d_{G'}(u_2) = 1$ since $d(u_1) \geqslant 3$. Thus $N(u_2) = \{x_5, u_1\}$. Now, G has a claw centered at u_1, a contradiction. Hence G' is a connected claw-free graph of order $n' = n - 5$ with $\delta(G') \geqslant 2$.

If $u_1u_2 \in E(G)$, then $N_{G'}[u_1]$ is a clique of G' and $N_{G'}[u_1] = N_{G'}[u_2]$ since G is claw-free. Thus $G' \notin \{G_1, G_2, G_3\}$ and G' is not a cycle unless $G' = C_3$. When $G' \in \{C_3, K_4\}$, $\{u_1, x_2, x_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Hence consider G' is a normal graph.

If $u_1u_2 \notin E(G)$, then $d_{G'}(u_1) \geqslant 3$. Thus G' is not a cycle. Since G is claw-free, $N_{G'}[u_1] \setminus \{u_2\}$ and $N_{G'}[u_2] \setminus \{u_1\}$ are cliques of G'. When $G' = K_4$, we have $G = G_2$, a contradiction. When $G' = G_1$, we have $G = G_3$, a contradiction.

When $G' = G_2$, without loss of generality, u_1 and u_2 play the roles of b_3 and b_5 in G_2, respectively. Then $G = G_5$. When $G' = G_3$, without loss of generality, u_1 and u_2 play the roles of c_2 and c_4 in G_3, respectively. Then $G = G_6$. Hence consider G' is a normal graph.

Let D' be a $\gamma_{t2}(G')$-set. By the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-5)$. When $|\{u_1, u_2\} \cap D'| = 0$ and $|\{u_1, u_2\} \cap D'| = 2$, $D' \cup \{x_2, x_4\}$ is a semi-TD-set of G. When $|\{u_1, u_2\} \cap D'| = 1$, without loss of generality, consider $u_1 \in D'$ and $u_2 \notin D'$. Then, either $D' \cup \{x_3, x_5\}$ is a semi-TD-set of G in the case of $N[u_2] \cap D' = \emptyset$ or $D' \cup \{x_2, x_5\}$ is a semi-TD-set of G in the case of $N[u_2] \cap D' \neq \emptyset$. Anyway, $\gamma_{t2}(G) \leqslant |D'| + 2 \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Before discussing the following cases (i.e. $k \leqslant 4$), we prove two claims on the condition $k \leqslant 4$.

Claim 5. *If there exists a vertex $u \in V(G) \setminus S_2$ such that $|N(u) \cap S_2| \geqslant 2$, then $\gamma_{t2}(G) < \frac{3}{7}n$ or $G \in \mathcal{G}$.*

Proof. Let u be a vertex in $V(G) \setminus S_2$ with $|N(u) \cap S_2| \geqslant 2$. Firstly, suppose that $N(u) \cap S_2$ is not a clique of G. Then there exists two vertices u_1 and u_2 in $N(u) \cap S_2$ such that $u_1u_2 \notin E(G)$. Since $d(u) \geqslant 3$, u has a neighbor u_3 different from u_1 and u_2. As G is claw-free, $u_1u_3 \in E(G)$ or $u_2u_3 \in E(G)$. By symmetry, consider $u_1u_3 \in E(G)$. If $u_2u_3 \in E(G)$, then $n = 4$ since G is claw-free, a contradiction. Thus $u_2u_3 \notin E(G)$ and u_2 has a neighbor u_4 other than u. Then $N(u) \subseteq \{u_1, u_2, u_3, u_4\}$, otherwise G has a claw centered at u, a contradiction. Clearly, $n \geqslant 5$. If $n = 5$ or $n = 6$, then $\{u_2, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 2 < \frac{3}{7}n$. Thus we may assume that $n \geqslant 7$.

Consider $N(u_3) \nsubseteq \{u, u_1, u_4\}$ and $N(u_4) \nsubseteq \{u, u_2, u_3\}$. Let G' be the graph obtained from G by removing vertices u, u_1, u_2 and adding the edge u_3u_4 when $u_3u_4 \notin E(G)$. By construction, G' is a connected claw-free graph of order $n' = n - 3 \geqslant 4$ with $\delta(G') \geqslant 2$. If $G' = C_{n'}$, then $d_{G'}(u_3) = d_{G'}(u_4) = 2$. Since $n' \geqslant 4$, let $N_{G'}(u_4) = \{u_3, u_5\}$ and $N_{G'}(u_5) = \{u_4, u_6\}$. Then $u_3u_4 \notin E(G)$, otherwise G has a claw centered at u_4 and it is a contradiction. When $uu_4 \notin E(G)$, $d(u_4) = 2$ and $\{u_2, u_4, u_5, u_6\} \subseteq S_2$. Since $k \leqslant 4$, $u_5u_6 \in E(G)$. Now, $G = H \in \mathcal{G}$. When $uu_4 \in E(G)$, $n' \leqslant 6$ since $k \leqslant 4$. Then either $\{u, u_5\}$ is a semi-TD-set of G in the case of $n' = 4$ or $\{u, u_5, u_6\}$ is a semi-TD-set of G in the case of $4 < n' \leqslant 6$. Anyway, $\gamma_{t2}(G) < \frac{3}{7}n$.

Hence, we may assume that G' is not a cycle. When $G' \in \{K_4, G_1\}$, $\{u_2, u_3\}$ or $\{u, u_4\}$ is a semi-TD-set of G. So $\gamma_{t2}(G) \leqslant 2 < \frac{3}{7}n$, where $n = 7$. When $G' \in \{G_2, G_3\}$, $n = 12$. By Observation 6(a), we can find a $\gamma_{t2}(G')$-set D' containing u_3 and $|D'| = 4$. Then $D' \cup \{u_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$. When G' is a normal graph, by the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-3)$, where D' is a $\gamma_{t2}(G')$-set. Then either $D' \cup \{u_2\}$ is a semi-TD-set of G in the case of $u_3 \in D'$ or $D' \cup \{u\}$ is a semi-TD-set of G in the case of $u_3 \notin D'$. Anyway,

$\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3)+1 < \frac{3}{7}n$.

Consider $N(u_3) \subseteq \{u, u_1, u_4\}$. Let $X = N(u_4) \setminus \{u, u_2, u_3\}$. Since $n \geqslant 7$, $X \neq \emptyset$. Assume that $X \cap S_2 \neq \emptyset$. Let $u_5 \in X \cap S_2$ and $N(u_5) = \{u_4, u_6\}$. Then $N(u_4) \subseteq \{u, u_2, u_5, u_6\}$, otherwise G has a claw centered at u_4, a contradiction. If $n = 7$, then $u_4u_6 \in E(G)$. Further, $\{u, u_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 2 < \frac{3}{7}n$. Thus consider $n \geqslant 8$. Let $Y = N(u_6) \setminus \{u_4, u_5\}$. Since G is claw-free, Y is a clique of G. Let G' be the graph obtained from G by removing all vertices of $\{u, u_1, \ldots, u_4\}$ and adding edges between $\{u_5\}$ and Y such that $\{u_5\} \cup Y$ is a clique in G'. Then G' is a connected claw-free graph of order $n' = n-5$ with $\delta(G') \geqslant 2$. Note, for any vertex $y \in Y$, $d_{G'}(y) \geqslant 3$. Thus G' is not a cycle. Since $N_{G'}[u_5]$ is a clique and $N_{G'}[u_5] = N_{G'}[u_6]$, $G' \notin \{G_1, G_2, G_3\}$. If $G' = K_4$, then $\{u, u_4, u_6\}$ is a $\gamma_{t2}(G)$-set and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$, where $n = 9$. Thus we may assume that G' is a normal graph. Since $N_{G'}(u_5) = \{u_6\} \cup Y$ is a clique of G', by Lemma 4, we can find a $\gamma_{t2}(G')$-set D' containing vertices of $\{u_6\} \cup Y$. By the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-5)$. Then $D' \cup \{u, u_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5)+2 < \frac{3}{7}n$.

Assume that $X \cap S_2 = \emptyset$. Let $u_5 \in X$ and $\{u_6, u_7\} \subseteq N(u_5) \setminus \{u_4\}$. Now, $n \geqslant 8$. If $n = 8$, then $\{u, u_4, u_5\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Thus consider $n \geqslant 9$. Let $G' = G - \{u, u_1, \ldots, u_4\}$. Then G' is connected claw-free graph of order $n' = n-5 \geqslant 4$ with $\delta(G') \geqslant 2$. If G' is a cycle, then $N(u_5) = \{u_4, u_6, u_7\}$ and $u_6u_7 \notin E(G)$. Since G is claw-free, $N(u_4) \cap V(G') = \{u_5, u_6\}$ or $N(u_4) \cap V(G') = \{u_5, u_7\}$. Without loss of generality, consider $N(u_4) \cap V(G') = \{u_5, u_6\}$. Since $k \leqslant 4$, $n' \leqslant 6$. Then the structure of G is clear. Clearly, $\gamma_{t2}(G) < \frac{3}{7}n$. Thus we may assume that G' is not a cycle. When $G' \in \{K_4, G_1\}$, $n = 9$ and $\{u, u_4, u_5\}$ or $\{u, u_4, u_6\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. When $G' \in \{G_2, G_3\}$, by Observation 6(b), we can find a minimum semi-TD-set D'' of $G'' = G - \{u, u_1, u_2, u_3\}$ containing vertex u_4 with $|D''| \leqslant 4$. Then $D'' \cup \{u\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$, where $n = 14$. When G' is a normal graph, by the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-5)$, where D' is a $\gamma_{t2}(G')$-set. Then $D' \cup \{u, u_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5)+2 < \frac{3}{7}n$.

Consider $N(u_4) \subseteq \{u, u_2, u_3\}$. Since $n \geqslant 7$, $u_3u_4 \notin E(G)$, otherwise G has a claw centered at u_3 and it is a contradiction. Since $\delta(G) \geqslant 2$, $uu_4 \in E(G)$. Similar to the case of $N(u_3) \subseteq \{u, u_1, u_4\}$, we have $\gamma_{t2}(G) < \frac{3}{7}n$, where u_1, u_2, u_3, u_4 paly the roles of u_2, u_1, u_4, u_3, respectively.

Next, suppose that $N(u) \cap S_2$ is a clique of G. Then $|N(u) \cap S_2| = 2$. Let $N(u) \cap S_2 = \{u_1, u_2\}$ and $X = N(u) \setminus \{u_1, u_2\}$. As G is claw-free, X is a clique of G. Then $G' = G - \{u, u_1, u_2\}$ is a connected claw-free graph of order $n' = n-3$ with $\delta(G') \geqslant 2$. When $G' = C_{n'}$, $3 \leqslant n' \leqslant 6$ since $k \leqslant 4$. It is easy to obtain that $\gamma_{t2}(G) < \frac{3}{7}n$. When $G' \in \{K_4, G_1\}$, $n = 7$ and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. When $G' \in \{G_2, G_3\}$, we can find a minimum semi-TD-set D'' of $G'' = G - \{u_1, u_2\}$ containing u with $|D''| \leqslant 4$ by Observation 6(b). The set D'' is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 4 < \frac{3}{7}n$, where $n = 12$. When G' is a normal graph, by the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-3)$, where D' is a $\gamma_{t2}(G')$-set. Then $D' \cup \{u\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$. □

By Claim 5, we may assume that for any vertex $u \in V(G) \backslash S_2$, $|N(u) \cap S_2| \leqslant 1$, called *Assumption* 1.

Claim 6. *If there exists a vertex $v \in S_2$ such that v has two adjacent neighbors, then $\gamma_{t2}(G) < \frac{3}{7}n$.*

Proof. Let $N(v) = \{v_1, v_2\}$. According to the condition of Claim 6, $v_1v_2 \in E(G)$. Since $n \geqslant 6$, $d(v_1) \geqslant 3$ or $d(v_2) \geqslant 3$. Without loss of generality, consider $d(v_1) \geqslant 3$. By Assumption 1, $N(v_1) \cap S_2 = \{v\}$. Thus $d(v_2) \geqslant 3$. Further, $N(v_2) \cap S_2 = \{v\}$ by Assumption 1. Let $X_1 = N(v_1) \setminus \{v, v_2\}$ and $X_2 = N(v_2) \setminus \{v, v_1\}$. Then $X_1 \cap S_2 = \emptyset$ and $X_2 \cap S_2 = \emptyset$. Since G is claw-free, X_1 and X_2 are cliques of G. We construct G' from G by removing all vertices of $\{v, v_1, v_2\}$ and adding edges between X_1 and X_2 such that $X_1 \cup X_2$ is a clique of G'. Then G' is a connected claw-free graph of order $n' = n - 3$.

Suppose that $\delta(G') \geqslant 2$. When $G' = C_{n'}$, let $G' = u_1u_2u_3 \cdots u_{n'}u_1$ and $v_1u_1 \in E(G)$. Now, $u_1 \in X_1$. If $u_1u_2 \notin E(G)$, then $u_2 \in X_2$ and $u_1 \notin X_2$ and $u_1v_2 \notin E(G)$. Further, $d(u_1) = 2$ and it contradicts $X_1 \cap S_2 = \emptyset$. Thus $u_1u_2 \in E(G)$. Similarly, $u_1u_{n'} \in E(G)$. If $n' = 3$, then $n = 6$ and $\{v_1, u_1\}$ is a $\gamma_{t2}(G)$-set and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. Hence consider $n' \geqslant 4$. Since $u_2u_{n'} \notin E(G)$ and G is claw-free, $N_{G'}(v_1) = \{u_1, u_2\}$ or $N_{G'}(v_1) = \{u_1, u_{n'}\}$. By symmetry, consider $N_{G'}(v_1) = \{u_1, u_2\}$. Thus $\{u_1, u_2\} \subseteq X_1$. As $X_1 \cup X_2$ is a clique of G', $|X_1 \cup X_2| \leqslant 2$, and then $X_1 \cup X_2 = \{u_1, u_2\}$. Since $k \leqslant 4$, $4 \leqslant n' \leqslant 6$. Now, either $\{v_1, u_4\}$ is a semi-TD-set of G in the case of $n' = 4$ or $\{v_1, u_4, u_{n'}\}$ is a semi-TD-set of G in the case of $4 < n' \leqslant 6$. Anyway, $\gamma_{t2}(G) < \frac{3}{7}n$.

When $G' \in \{K_4, G_1\}$, $n = 7$ and $\gamma_{t2}(G) \leqslant 2 < \frac{3}{7}n$. When $G' \in \{G_2, G_3\}$, $X_1 \cup X_2$ is a clique of G. Otherwise, there exist two vertices $u_1 \in X_1 \setminus X_2$ and $u_2 \in X_2 \setminus X_1$ such that $u_1u_2 \notin E(G)$. Further, $d_{G'}(u_1) \geqslant 3$ and $d_{G'}(u_2) \geqslant 3$. Since $X_1 \cup X_2$ is a clique of G', $X_1 \cup X_2 \subseteq N_{G'}[u_1] \cap N_{G'}[u_2]$. According to the structure of G_2 and G_3, $k \geqslant 5$, a contradiction. By Observation 6(b), we can find a minimum semi-TD-set D'' of the graph $G - \{v, v_2\}$ containing v_1 with $|D''| \leqslant 4$. Clearly, D'' is a semi-TD-set of G. Then $\gamma_{t2}(G) \leqslant 4 < \frac{3}{7}n$, where $n = 12$. Thus consider G' is a normal graph. Let D' be a $\gamma_{t2}(G')$-set. By the inductive hypothesis, $|D'| \leqslant \frac{3}{7}(n-3)$. When $X_1 \cap D' \neq \emptyset$, $D' \cup \{v_2\}$ is a semi-TD-set of G. When $X_1 \cap D' = \emptyset$, $D' \cup \{v_1\}$ is a semi-TD-set of G. Anyway, $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$.

Suppose that $\delta(G') = 1$. Let v_3 be a vertex with $d_{G'}(v_3) = 1$ and $N_{G'}(v_3) = \{v_4\}$. Then $v_3v_1 \in E(G)$ or $v_3v_2 \in E(G)$. Thus $v_3 \notin S_2$ and $N(v_3) = \{v_1, v_2, v_4\}$. Since G is claw-free, $N(v_1) \subseteq \{v, v_2, v_3, v_4\}$ and $N(v_2) \subseteq \{v, v_1, v_3, v_4\}$. Let $G'' = G - \{v, v_1, v_2, v_3, v_4\}$. Then G'' is a connected claw-free graph of order

$n'' = n-5$. When $G'' = C_{n''}$, $3 \leqslant n'' \leqslant 6$ since $k \leqslant 4$. Let $G'' = w_1w_2w_3 \cdots w_{n''}w_1$ and $v_4w_1 \in E(G)$. Note, if $n'' > 3$, then $w_2w_{n''} \notin E(G)$ and $v_4w_2 \in E(G)$ or $v_4w_{n''} \in E(G)$ since G is claw-free. In the case of $n'' \in \{3, 4\}$, $n \geqslant 8$ and $\{v_1, v_4, w_3\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. In the case of $n'' \in \{5, 6\}$, by symmetry, consider $v_4w_2 \in E(G)$. Then $\{v_1, v_4, w_3, w_{n''}\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant 4 < \frac{3}{7}n$, where $n'' \geqslant 10$. When $G'' \in \{K_4, G_1\}$, $n = 9$ and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. When $G'' \in \{G_2, G_3\}$, we can find a minimum semi-TD-set D''' of the graph $G - \{v, v_1, v_2, v_3\}$ containing v_4 with $|D''| \leqslant 4$ by Observation 6(b). Then $D''' \cup \{v_1\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$, where $n = 14$. Hence, we may assume that $G'' \notin \{C_{n''}, K_4, G_1, G_2, G_3\}$.

If $\delta(G'') \geqslant 2$, then G'' is a normal graph. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}(n-5)$ where D'' is a $\gamma_{t2}(G'')$-set. The set $D'' \cup \{v_1, v_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$. Thus, we may assume that $\delta(G'') = 1$.

Let v_5 be a vertex with $d_{G''}(v_5) = 1$ and $N_{G''}(v_5) = \{v_6\}$. Since $N(v_1) \subseteq \{v, v_2, v_3, v_4\}$, $N(v_2) \subseteq \{v, v_1, v_3, v_4\}$ and $N(v_3) = \{v_1, v_2, v_4\}$, $N(v_5) = \{v_4, v_6\}$. Now, $N(v_4) \subseteq \{v_1, v_2, v_3, v_5, v_6\}$, otherwise G has a claw centered at v_4 and it is a contradiction. If $n = 7$, then $v_4v_6 \in E(G)$. Further, $\{v_1, v_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. Thus consider $n \geqslant 8$. Let $X = N(v_6) \setminus \{v_4, v_5\}$. Since G is claw-free, X is a clique of G. Let G''' be the graph obtained from G'' by adding edges between $\{v_5\}$ and X such that $\{v_5\} \cup X$ is a clique in G'''. Then G''' is a connected claw-free graph with $\delta(G''') \geqslant 2$. Note, for any vertex $x \in X$, $d_{G'''}(x) \geqslant 3$. Thus G''' is not a cycle. Since $N_{G'''}[v_5]$ is a clique of G''' and $N_{G'''}[v_5] = N_{G'''}[v_6]$, we have $G''' \notin \{G_1, G_2, G_3\}$. If $G''' = K_4$, then $n = 9$, $\{v_1, v_4, v_6\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Thus we may assume that G''' is a normal graph. Since $N_{G'''}(v_5) = \{v_6\} \cup X$ is a clique of G''', there exists a $\gamma_{t2}(G''')$-set D''' containing vertices of $\{v_6\} \cup X$ by Lemma 4. By the inductive hypothesis, $|D'''| \leqslant \frac{3}{7}(n-5)$. Then $D''' \cup \{v_1, v_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$. □

By Claim 6, we may assume that for any vertex $v \in S_2$, two neighbors of v are not adjacent, called *Assumption* 2. For $k = 1$, let u_1 and u_2 be two neighbors of x_1. For $k \geqslant 2$, we denote by u_1 the neighbor of x_1 not in S_2 and u_2 the neighbor of x_k not in S_2. Note that $d(u_1) \geqslant 3$ and $d(u_2) \geqslant 3$. By Assumption 1, we also get $u_1 \neq u_2$ for $k \geqslant 2$. Let $Y_1 = N(u_1) \setminus \{x_1, u_2\}$ and $Y_2 = N(u_2) \setminus \{x_k, u_1\}$. Since G is claw-free, Y_1 and Y_2 are cliques of G. By Assumption 1, $Y_1 \cap S_2 = \emptyset$ and $Y_2 \cap S_2 = \emptyset$.

Case 2. $k = 4$. We construct G' from G by removing all vertices of $\{u_1, x_1, \ldots, x_4\}$ and adding edges between $\{u_2\}$ and Y_1 such that $\{u_2\} \cup Y_1$ is a clique of G'. Then G' is a connected claw-free graph of order $n' = n - 5$. Next, we divide into two subcases.

Subcase 2.1. $\delta(G') = 1$. Since $Y_1 \cap S_2 = \emptyset$, $d_{G'}(y) \geqslant 2$ for any vertex $y \in Y_1$.

Thus u_2 is the unique vertex of degree 1 in G'. Let $N_{G'}(u_2) = \{u_3\}$. Since $d(u_2) \geqslant 3$, $N(u_2) = \{u_1, x_4, u_3\}$. Then $N(u_1) = \{x_1, u_2, u_3\}$, otherwise G has a claw centered at u_1 and it is a contradiction. Let $Y = N(u_3) \setminus \{u_1, u_2\}$. Since $u_3 \in Y_1$, $d(u_3) \geqslant 3$ and then $Y \neq \emptyset$. As G is claw-free, Y is a clique of G'.

Suppose that $Y \cap S_2 = \emptyset$. Then $G'' = G - \{u_1, x_1, \ldots, x_4, u_2, u_3\}$ is a connected claw-free graph of order $n'' = n - 7$ with $\delta(G'') \geqslant 2$. If $G'' = C_{n''}$, without loss of generality, let $G'' = v_1v_2v_3 \cdots v_{n''}v_1$ and $u_3v_1 \in E(G)$. When $n'' = 3$, $n = 10$ and $\{x_1, x_3, u_3, v_1\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 4 < \frac{3}{7}n$. When $n'' \geqslant 4$, $v_2v_{n''} \notin E(G)$. As G is claw-free, $N(u_3) = \{u_1, u_2, v_1, v_2\}$ or $N(u_3) = \{u_1, u_2, v_1, v_{n''}\}$. By symmetry, consider $N(u_3) = \{u_1, u_2, v_1, v_2\}$. Since $k = 4$, $n'' \leqslant 6$. Now, the structure of G is clear. It is easy to obtain $\gamma_{t2}(G) < \frac{3}{7}n$. Hence, we may assume that $G'' \neq C_{n''}$.

If $G'' \in \{K_4, G_1\}$, then $n = 11$ and $\gamma_{t2}(G) \leqslant 4 < \frac{3}{7}n$. If $G'' \in \{G_2, \ldots, G_6\}$, then we can find a minimum semi-TD-set D''' of $G - \{u_1, x_1, \ldots, x_4, u_2\}$ containing u_3 by Observations 6(b) and 7(b). Note, $|D'''| \leqslant 4$ when $G'' \in \{G_2, G_3\}$, $|D'''| \leqslant 3$ when $G'' = G_4$ and $|D'''| \leqslant 6$ when $G'' \in \{G_5, G_6\}$. Clearly, $D''' \cup \{x_1, x_3\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) < \frac{3}{7}n$. Hence we may assume that G'' is a normal graph and $G'' \notin \{G_4, G_5, G_6\}$. Let D'' be a $\gamma_{t2}(G'')$-set. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}n'' = \frac{3}{7}(n-7)$. Then $D'' \cup \{x_1, x_3, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant |D''| + 3 \leqslant \frac{3}{7}(n-7) + 3 \leqslant \frac{3}{7}n$.

Assume $\gamma_{t2}(G) = \frac{3}{7}n$. Then $\gamma_{t2}(G'') = \frac{3}{7}n''$. According to the inductive hypothesis and $G'' \notin \{G_4, G_5, G_6\}$, $G'' \in \mathcal{G}$, and then $G'' = K_{\frac{n''}{7}} \circ_w H$. Let H_v be the copy of H corresponding to v, where $v \in V(K_{\frac{n''}{7}})$. Let $V(H_v) = \{v, v_1, \ldots, v_6\}$, where $\{v_1, v_6\} \subseteq N(v)$ and $N_{H_v}(v_i) = \{v_{i-1}, v_{i+1}\}$ for $i \in \{2, 3, 4, 5\}$. Without loss of generality, consider $N(u_3) \cap V(H_v) \neq \emptyset$. Combining the definition of $\mathcal{G}$ and $G'' \in \mathcal{G}$, we have $G'' = H_v$ or $G'' - V(H_v) \in \mathcal{G}$. Let $G''' = G'' - V(H_v)$ and D''' be a $\gamma_{t2}(G''')$-set. Thus $|D'''| = \frac{3}{7}|V(G''')| = \frac{3}{7}(n-14)$. If $u_3v_2 \in E(G)$ or $u_3v_3 \in E(G)$, then $D''' \cup \{x_1, x_3, u_3, v_4, v_6\}$ or $D''' \cup \{x_1, x_3, u_3, v_1, v_5\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-14) + 5 < \frac{3}{7}n$, a contradiction. Hence, $u_3v_2 \notin E(G)$ and $u_3v_3 \notin E(G)$. Similarly, $u_3v_5 \notin E(G)$ and $u_3v_4 \notin E(G)$. If $u_3v_1 \in E(G)$, then $u_3v \in E(G)$ since G is claw-free and $u_3v_2 \notin E(G)$. Further, $D''' \cup \{x_1, x_3, u_3, v_3, v_5\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-14) + 5 < \frac{3}{7}n$, a contradiction. Hence $u_3v_1 \notin E(G)$. Similarly, $u_3v_6 \notin E(G)$. Then $N(u_3) \cap V(H_v) = \{v\}$. If $N_{G''}(u_3) \neq V(K_{\frac{n''}{7}})$, then G has a claw centered at v, a contradiction. Hence $N_{G''}(u_3) = V(K_{\frac{n''}{7}})$ and $G \in \mathcal{G}$.

Suppose that $Y \cap S_2 \neq \emptyset$. Let $u_4 \in Y \cap S_2$ and $N(u_4) = \{u_3, u_5\}$. Since G is claw-free, $N(u_3) \subseteq \{u_1, u_2, u_4, u_5\}$. If $n = 9$, then $u_3u_5 \in E(G)$. Further, $\{x_1, x_3, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Thus consider $n \geqslant 10$. Let $Z = N(u_5) \setminus \{u_3, u_4\}$. Since G is claw-free, Z is a clique of G. By Assumption 1, $Z \cap S_2 = \emptyset$. We construct G'' from G by removing all vertices of

$\{u_1, x_1, \ldots, x_4, u_2, u_3\}$ and adding edges between $\{u_4\}$ and Z such that $\{u_4\} \cup Z$ is a clique of G''. Then G'' is a connected claw-free graph of order $n'' = n - 7$ with $\delta(G'') \geqslant 2$. Note, for any vertex $z \in Z$, $d_{G''}(z) \geqslant 4$. Thus G'' is a normal graph and $G'' \notin \{G_4, G_5, G_6\}$. Since $N_{G''}[u_4] = N_{G''}[u_5]$, $G'' \notin \mathcal{G}$. Since $N_{G''}(u_4) = \{u_5\} \cup Z$ is a clique of G'', there exists a $\gamma_{t2}(G'')$-set D'' containing vertices of $\{u_5\} \cup Z$ by Lemma 4. By the inductive hypothesis, $|D''| < \frac{3}{7}(n-7)$. Then $D'' \cup \{x_1, x_3, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) < \frac{3}{7}(n-7) + 3 = \frac{3}{7}n$.

Subcase 2.2. $\delta(G') \geqslant 2$. If G' is a normal graph, then $|D'| \leqslant \frac{3}{7}n'$ by the inductive hypothesis, where D' is a $\gamma_{t2}(G')$-set. When $Y_1 \cap D' \neq \emptyset$, either $D' \cup \{x_1, x_4\}$ is a semi-TD-set of G in the case of $N[u_2] \cap D' \neq \emptyset$ or $D' \cup \{x_2, x_4\}$ is a semi-TD-set of G in the case of $N[u_2] \cap D' = \emptyset$. When $u_2 \in D'$, either $D' \cup \{u_1, x_2\}$ is a semi-TD-set of G in the case of $(\bigcup_{y \in Y_1} N[y]) \cap D' = \emptyset$ or $D' \cup \{u_1, x_3\}$ is a semi-TD-set of G in the case of $(\bigcup_{y \in Y_1} N[y]) \cap D' \neq \emptyset$. When $Y_1 \cap D' = \emptyset$ and $u_2 \notin D'$, $D' \cup \{x_1, x_3\}$ is a semi-TD-set of G. Anyway, $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$. Thus, we may assume that G' is not a normal graph, i.e., $G' \in \{C_{n-5}, K_4, G_1, G_2, G_3\}$.

Consider $d_{G'}(u_2) = 2$. Let $N_{G'}(u_2) = \{u_3, u_4\}$. Clearly, $Y_1 \subseteq \{u_3, u_4\}$. If $u_2u_3 \notin E(G)$, then $u_3 \in Y_1$ and $u_1u_3 \in E(G)$. Since $d(u_2) \geqslant 3$, $u_1u_2 \in E(G)$. Now, G has a claw centered at u_1, a contradiction. Thus $u_2u_3 \in E(G)$. Similarly, $u_2u_4 \in E(G)$. Since G is claw-free, $u_3u_4 \in E(G)$. Hence $G' \in \{C_3, G_1, G_3\}$. If $G' = G_1$, without loss of generality, consider u_2 plays the role of a_3 in G_1. Then $a_4 \in S_2$ and it contradicts Assumption 2. If $G' = G_3$, then u_2 plays the role of c_1 in G_3. Now, $k \geqslant 6$, a contradiction. Thus $G' = C_3$. Further, $\{x_1, x_3, u_2\}$ is a $\gamma_{t2}(G)$-set and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$.

Consider $d_{G'}(u_2) \geqslant 3$. Since $G' \in \{C_{n-5}, K_4, G_1, G_2, G_3\}$, $d_{G'}(u_2) = 3$. Let $N_{G'}(u_2) = \{u_3, u_4, u_5\}$. Clearly, $Y_1 \subseteq \{u_3, u_4, u_5\}$. If $N_{G'}(u_2)$ is not a clique of G', then there exists a vertex in $\{u_3, u_4, u_5\}$, say u_3 such that $u_3 \notin Y_2$ and $u_3 \in Y_1$. That is $u_3u_2 \notin E(G)$ and $u_3u_1 \in E(G)$. Then $u_1u_2 \notin E(G)$, otherwise G has a claw centered at u_1, a contradiction. Since $d(u_2) \geqslant 3$, $N(u_2) = \{x_4, u_4, u_5\}$. As G is claw-free, $u_4u_5 \in E(G)$. Since $u_3 \in Y_1$, $d(u_3) \geqslant 3$ and $d_{G'}(u_3) \geqslant 3$. Thus $G' = G_2$. Without loss of generality, u_2 and u_3 play the roles of b_1 and b_3, respectively. Now, $k \geqslant 5$, a contradiction. Hence $N_{G'}(u_2)$ is a clique of G'. Further $G' = K_4$. Since $d(u_1) \geqslant 3$, $N(u_1) \cap \{u_3, u_4, u_5\} \neq \emptyset$. Let $u_1u_3 \in E(G)$. When $u_2u_3 \in E(G)$, $\{x_2, x_4, u_3\}$ is a $\gamma_{t2}(G)$-set. When $u_2u_3 \notin E(G)$, $u_1u_2 \notin E(G)$ since G is claw-free. Since $d(u_1) \geqslant 3$ and $d(u_2) \geqslant 3$, ($u_1u_4 \in E(G)$ or $u_1u_5 \in E(G)$) and $N(u_2) = \{x_4, u_4, u_5\}$. Then $\{x_2, x_4, u_4\}$ or $\{x_2, x_4, u_5\}$ is a $\gamma_{t2}(G)$-set. Anyway, $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$, where $n = 9$.

Case 3. $k = 3$. We construct G' from G by removing all vertices of $\{u_1, x_1, x_2, x_3, u_2\}$ and adding edges between Y_1 and Y_2 such that $Y_1 \cup Y_2$ is a clique of G'. Then G' is a connected claw-free graph of order $n' = n - 5$.

Suppose that $\delta(G') \geqslant 2$. If G' is a normal graph, then $|D'| \leqslant \frac{3}{7}n'$ by the inductive hypothesis, where D' is a $\gamma_{t2}(G')$-set. Without loss of generality, consider $Y_1 \cap D' \neq \emptyset$ or ($Y_1 \cap D' = \emptyset$ and $Y_2 \cap D' = \emptyset$). When $Y_1 \cap D' \neq \emptyset$, either $D' \cup \{x_2, u_2\}$ is a semi-TD-set of G in the case of $(\bigcup_{y \in Y_2} N[y]) \cap D' = \emptyset$ or $D' \cup \{x_1, u_2\}$ is a semi-TD-set of G in the case of $(\bigcup_{y \in Y_2} N[y]) \cap D' \neq \emptyset$. When $Y_1 \cap D' = \emptyset$ and $Y_2 \cap D' = \emptyset$, $D' \cup \{x_1, x_3\}$ is a semi-TD-set of G. Anyway, $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Hence, we may assume that G' is not a normal graph, i.e., $G' \in \{C_{n-5}, K_4, G_1, G_2, G_3\}$. Let v_1 be a vertex in Y_1. Clearly, $d_{G'}(v_1) = 2$ or 3. Since $Y_1 \cup Y_2$ is a clique of G', $Y_1 \cup Y_2 \subseteq N_{G'}[v_1]$.

Consider $d_{G'}(v_1) = 2$. Then $G' \neq K_4$. Let $N_{G'}(v_1) = \{v_2, v_3\}$. Combining $Y_1 \cup Y_2 \subseteq \{v_1, v_2, v_3\}$ and Assumption 2, $G' \neq G_1$. If $v_1v_2 \notin E(G)$, then $v_1 \notin Y_2$ and $v_1u_2 \notin E(G)$. Further, $d(v_1) = 2$ and it contradicts $Y_1 \cap S_2 = \emptyset$. Hence $v_1v_2 \in E(G)$. Similarly, $v_1v_3 \in E(G)$. If $v_2v_3 \in E(G')$, then $G' \neq G_2$ and $G' \neq G_3$ since $k = 3$. So $G' = C_3$. Further, $\{v_1, x_1, x_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$, where $n = 8$. Thus, we may assume that $v_2v_3 \notin E(G')$. Since G is claw-free, $N(u_1) \cap V(G') = \{v_1, v_2\}$ or $N(u_1) \cap V(G') = \{v_1, v_3\}$. By symmetry, consider $N(u_1) \cap V(G') = \{v_1, v_2\}$. Thus $\{v_1, v_2\} \subseteq Y_1$. Since $Y_1 \cup Y_2 \subseteq N_{G'}[v_1]$ is a clique of G' and $v_2v_3 \notin E(G')$, $Y_1 \cup Y_2 = \{v_1, v_2\}$. Since $d(u_2) \geqslant 3$, $u_2v_1 \in E(G)$ or $u_2v_2 \in E(G)$. Then $u_1u_2 \in E(G)$, otherwise G has a claw centered at v_1 or v_2, a contradiction. When G' is a cycle, $4 \leqslant n' \leqslant 5$ since $k = 3$. Now, the structure of G is clear. It is easy to obtain $\gamma_{t2}(G) < \frac{3}{7}n$. When $G' \in \{G_2, G_3\}$, we can find a minimum semi-TD-set D'' of the graph $G - \{x_1, x_2, x_3, u_2\}$ containing vertex u_1 with $|D''| \leqslant 4$ by Observation 6(b). Then $D'' \cup \{x_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$, where $n = 14$.

Consider $d_{G'}(v_1) = 3$. Let $N_{G'}(v_1) = \{v_2, v_3, v_4\}$. Now, $G' \neq C_{n-5}$. Combining $Y_1 \cup Y_2 \subseteq N_{G'}[v_1]$ and $k = 3$, $G' \notin \{G_2, G_3\}$. Thus $G' \in \{K_4, G_1\}$ and $n = 9$. When $\{v_2, v_3, v_4\} \subseteq N(v_1)$, $\{v_1, x_1, x_3\}$ is a semi-TD-set of G. When $\{v_2, v_3, v_4\} \nsubseteq N(v_1)$, without loss of generality, consider $v_2v_1 \notin E(G)$. Then $v_2 \in Y_2$ and $v_1 \notin Y_2$. Further, $v_1u_2 \notin E(G)$. Since $d(v_1) \geqslant 3$, $N(v_1) = \{u_1, v_3, v_4\}$. As G is claw-free, $u_1u_2 \notin E(G)$. Since $d(u_2) \geqslant 3$, $u_2v_3 \in E(G)$ or $u_2v_4 \in E(G)$. Then $\{x_2, u_2, v_1\}$ is a semi-TD-set of G. Anyway, $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$.

Suppose that $\delta(G') = 1$. Let u_3 be a vertex in $V(G')$ with $d_{G'}(u_3) = 1$ and $N_{G'}(u_3) = \{u_4\}$. Since $\delta(G) \geqslant 2$, $u_3u_1 \in E(G)$ or $u_3u_2 \in E(G)$. Further, $u_3 \in Y_1 \cup Y_2$. Since $Y_1 \cap S_2 = \emptyset$ and $Y_2 \cap S_2 = \emptyset$, $d(u_3) \geqslant 3$ and $N(u_3) = \{u_1, u_2, u_4\}$. Now, $N(u_1) \subseteq \{x_1, u_2, u_3, u_4\}$ and $N(u_2) \subseteq \{x_3, u_1, u_3, u_4\}$, otherwise G has a claw, a contradiction. If $n = 7$, then $d_{G'}(u_4) = 1$. Similar to the case of u_3, $N(u_4) = \{u_1, u_2, u_3\}$. When $u_1u_2 \notin E(G)$, $G = G_4$. When $u_1u_2 \in E(G)$, $\{u_1, x_2\}$ is a $\gamma_{t2}(G)$-set and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. Thus consider $n \geqslant 8$.

Let $Y = N(u_4) \setminus \{u_1, u_2, u_3\}$ and G'' be the graph obtained from G by removing all vertices of $\{u_1, x_1, x_2, x_3, u_2\}$ and adding edges between $\{u_3\}$ and Y such that $\{u_3\} \cup Y$ is a clique of G''. Then G'' is a connected claw-free graph order

$n'' = n-5$ with $\delta(G'') \geqslant 2$. Since $N_{G''}[u_3]$ is a clique of G' and $N_{G''}[u_3] = N_{G''}[u_4]$, $G'' \notin \{G_1, G_2, G_3\}$. Note, for any vertex y of Y, $d_{G''}(y) \geqslant 3$. Thus G' is not a cycle. If $u_1u_2 \notin E(G)$, then $u_1u_4 \in E(G)$ and $u_2u_4 \in E(G)$ since $d(u_1) \geqslant 3$ and $d(u_2) \geqslant 3$. Now, G has a claw centered at u_4, a contradiction. Hence, $u_1u_2 \in E(G)$. Consider $G'' = K_4$. Then $n = 9$ and $\{x_2, u_2, u_4\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Consider G'' is a normal graph. Since $N(u_3) = \{u_4\} \cup Y$ is a clique of G'', there exists a $\gamma_{t2}(G'')$-set D'' containing vertices of $\{u_4\} \cup Y$ according to Lemma 4. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}(n-5)$. Then $D'' \cup \{u_1, x_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Case 4. $k = 2$. Next we consider two subcases: $u_1u_2 \in E(G)$ or $u_1u_2 \notin E(G)$.

Subcase 4.1. $u_1u_2 \in E(G)$. In this subcase, $Y_1 = Y_2$ since G is claw-free. Let $G' = G - \{u_1, x_1, x_2\}$. Then G' is a connected claw-free graph of order $n' = n-3$. Suppose that $\delta(G') \geqslant 2$. Since $N_{G'}[u_2] = \{u_2\} \cup Y_2$ is a clique of G', $G' \neq G_2$ and G' is not a cycle unless $G' = C_3$ in which case $n = 6$ and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. When $G' \in \{K_4, G_1\}$, $n = 7$ and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. When $G = G_3$, u_2 plays the role of c_1 in G_3 and $k \geqslant 6$, a contradiction. When G' is a normal graph, $|D'| \leqslant \frac{3}{7}(n-3)$ by the inductive hypothesis, where D' is a $\gamma_{t2}(G')$-set. In order to dominate u_2, $(\{u_2\} \cup Y_2) \cap D' \neq \emptyset$. Since $Y_1 = Y_2$, $D' \cup \{x_1\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$.

Suppose that $\delta(G') = 1$. Since $Y_1 \cap S_2 = \emptyset$, $d_{G'}(y) \geqslant 3$ for any vertex $y \in Y_1$. Thus u_2 is the unique vertex of degree 1 in G'. Let $N_{G'}(u_2) = \{u_3\}$. Now, $N(u_2) = \{x_2, u_1, u_3\}$. As G is claw-free, $N(u_1) = \{x_1, u_2, u_3\}$. Let $Y = N(u_3) \setminus \{u_1, u_2\}$. Assume $Y \cap S_2 \neq \emptyset$. Let $u_4 \in Y \cap S_2$ and $N(u_4) = \{u_3, u_5\}$. Since G is claw-free, $N(u_3) \subseteq \{u_1, u_2, u_4, u_5\}$. If $n = 7$, then $u_3u_5 \in E(G)$ and $N(u_3) \cap S_2 = \{u_4, u_5\}$, it contradicts Assumption 1. Thus $n \geqslant 8$. Let $Z = N(u_5) \setminus \{u_3, u_4\}$. As G is claw-free, Z is a clique of G. By Assumption 1 and $k = 2$, $Z \cap S_2 = \emptyset$. We construct G'' from G by removing all vertices of $\{u_1, x_1, x_2, u_2, u_3\}$ and adding edges between $\{u_4\}$ and Z such that $\{u_4\} \cup Z$ is a clique of G''. Note, for any vertex $z \in Z$, $d_{G''}(z) \geqslant 4$. Thus G'' is a normal graph of order $n'' = n-5$. Since $N_{G''}(u_4) = \{u_5\} \cup Z$ is a clique of G'', by Lemma 4, we can find a $\gamma_{t2}(G'')$-set D'' containing vertices of $\{u_5\} \cup Z$. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}(n-5)$. The set $D'' \cup \{x_1, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Assume that $Y \cap S_2 = \emptyset$. Then $G'' = G - \{u_1, x_1, x_2, u_2, u_3\}$ is a connected claw-free graph of order $n'' = n-5$ with $\delta(G'') \geqslant 2$. If $G'' = C_{n''}$, without loss of generality, let $G'' = v_1v_2v_3 \cdots v_{n''}v_1$ and $u_3v_1 \in E(G)$. Now, $n'' \geqslant 3$. When $n'' = 3$, we have $n = 8$, $\{x_1, u_3, v_1\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. When $n' \geqslant 4$, $v_2v_{n''} \notin E(G)$. Since G is claw-free, $N(u_3) \cap V(G') = \{v_1, v_2\}$ or $N(u_3) \cap V(G') = \{v_1, v_{n''}\}$. By symmetry, consider $N(u_3) \cap V(G') = \{v_1, v_2\}$. Since $k = 2$, $n'' = 4$ and $n = 9$. The set $\{x_1, u_3, v_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Thus, we may assume that $G'' \neq C_{n''}$. When $G'' \in \{K_4, G_1\}$,

$n = 9$ and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. When $G'' \in \{G_2, G_3\}$, $n = 14$ and we can find a minimum semi-TD-set D''' of $G - \{u_1, x_1, x_2, u_2\}$ containing u_3 with $|D'''| \leqslant 4$ by Observation 6(b). Then $D''' \cup \{x_1\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$. When G'' is a normal graph, by the inductive hypothesis, $|D''| \leqslant \frac{3}{7}(n-5)$, where D'' is a $\gamma_{t2}(G'')$-set. The set $D'' \cup \{x_1, u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Subcase 4.2. $u_1u_2 \notin E(G)$. Let $Y_2 = \{y_1, \ldots, y_l\}$. Since $d(u_2) \geqslant 3$ and $u_1u_2 \notin E(G)$, $l \geqslant 2$. In what follows, we consider two subsubcases.

Subsubcase 4.2.1. There exists a vertex in $Y_1 \cup Y_2$, say y_1 such that $N(y_1) \cap S_2 = \emptyset$. Let G' be the graph obtained from G by removing all vertices of $\{x_2, u_2, y_1\}$ and adding edges between $\{x_1\}$ and $Y_2 \setminus \{y_1\}$ such that $\{x_1\} \cup (Y_2 \setminus \{y_1\})$ is a clique of G'. Since $Y_2 \cap S_2 = \emptyset$ and $N(y_1) \cap S_2 = \emptyset$, G' is a claw-free graph of order $n' = n - 3$ with $\delta(G') \geqslant 2$. Let H_1 be the component of G' containing vertex u_1 and D_{H_1} be a $\gamma_{t2}(H_1)$-set. (Possibly, $G' = H_1$.) When G' is not connected, let H_2 be the other connected component of G' and D_{H_2} a $\gamma_{t2}(H_2)$-set. Clearly, H_1 and H_2 are claw-free graphs with minimum degree at least two.

Consider G' is not connected and $H_2 \in \{C_{|V(H_2)|}, K_4, G_1\}$. If $H_2 = C_{|V(H_2)|}$ and $|V(H_2)| \geqslant 4$, let $H_2 = w_1w_2w_3\cdots w_{|V(H_2)|}w_1$ and $y_1w_1 \in E(G)$. Now, $w_2w_{|V(H_2)|} \notin E(G)$. As G is claw-free, $N(y_1) \cap V(H_2) = \{w_1, w_2\}$ or $N(y_1) \cap V(H_2) = \{w_1, w_{|V(H_2)|}\}$. Then $|V(H_2)| = 4$ since $k = 2$. Without loss of generality, consider $N(y_1) \cap V(H_2) = \{w_1, w_2\}$. Further, $\{w_3, w_4\} \subseteq S_2$ and $\{w_1, w_2\} \cap S_2 = \emptyset$ and $w_1w_2 \in E(G)$. Similar to Subcase 4.1 (i.e., $u_1u_2 \in E(G)$), we have $\gamma_{t2}(G) < \frac{3}{7}n$. Thus we may assume that $H_2 \in \{C_3, K_4, G_1\}$. When $H_2 \in \{C_3, K_4\}$, $G'' = G - V(H_2)$ is a normal graph of order $n'' \leqslant n - 3$. Let D'' be a $\gamma_{t2}(G'')$-set. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}(n-3)$. The set $D'' \cup \{u_3\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$, where u_3 is a vertex in $N(y_1) \cap V(H_2)$. When $H_2 = G_1$, $d(a_3) \geqslant 3$ and $d(a_4) \geqslant 3$ by Assumption 2. Thus $a_3y_1 \in E(G)$, $a_4y_1 \in E(G)$ and G has a claw centered at y_1, a contradiction. Thus, we may assume that either G' is connected, or H_2 is a normal graph, or $H_2 \in \{G_2, G_3\}$.

Suppose that $H_1 \in \{C_{|V(H_1)|}, K_4, G_1, G_2, G_3\}$. Then $d_{H_1}(u_1) = 2$ or 3. Consider $d_{H_1}(u_1) = 2$. Since $d(u_1) \geqslant 3$ and $u_1u_2 \notin E(G)$, $u_1y_1 \in E(G)$. If G' is not connected, then G has a claw centered at y_1, a contradiction. Thus G' is connected and $G' = H_1$. When $G' = C_{n'}$ and $n' \geqslant 4$, $d_{G'}(x_1) = 2$. Further, $Y_2 \setminus \{y_1\} = \{y_2\}$ and $u_1y_2 \notin E(G)$. Let $N(u_1) = \{x_1, y_1, u_3\}$. As G is claw-free, $N(y_1) = \{u_1, u_2, u_3, y_2\}$. Since $k = 2$, $n' \leqslant 6$. Now, the structure of G is clear. It is easy to obtain $\gamma_{t2}(G) < \frac{3}{7}n$. When $G' \in \{C_3, G_1\}$, $n \geqslant 6$ and $\{u_1, u_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) = 2 < \frac{3}{7}n$. When $G' \in \{G_2, G_3\}$, $n = 12$ and we can find a $\gamma_{t2}(G')$-set D' containing vertex x_1 with $|D'| = 4$ by Observation 6(a). Then $D' \cup \{u_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$.

Consider $d_{H_1}(u_1) = 3$. Then $H_1 \neq C_{|V(H_1)|}$. If $H_1 \in \{G_2, G_3\}$, then $k \geqslant 3$, a contradiction. Thus $H_1 \in \{K_4, G_1\}$. Let $N_{H_1}[u_1] = V(H_1) = \{u_1, x_1, y_2, u_3\}$. Since G is claw-free, $u_3y_2 \in E(G)$. When G' is connected, we can think of $D_{H_2} = \emptyset$ and $|D_{H_2}| = \frac{3}{7}|V(H_2)| = 0$. When H_2 is a normal graph, $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$ by the inductive hypothesis. In both cases, $D_{H_2} \cup \{x_2, y_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_2)| + 2 = \frac{3}{7}(n-7) + 2 < \frac{3}{7}n$. When $H_2 \in \{G_2, G_3\}$, $n = 16$ and $|D_{H_2}| = 4$ by Observation 6(a). Clearly, $D_{H_2} \cup \{u_1, u_2\}$ is a semi-TD-set of G. Then $\gamma_{t2}(G) \leqslant 6 < \frac{3}{7}n$.

Suppose that H_1 is a normal graph. By the inductive hypothesis, $|D_{H_1}| \leqslant \frac{3}{7}|V(H_1)|$. If G' is connected or H_2 is a normal graph, then $D_{H_2} = \emptyset$ or $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$ by the inductive hypothesis. When $(Y_2 \setminus \{y_1\}) \cap D_{H_1} \neq \emptyset$, $D_{H_1} \cup D_{H_2} \cup \{x_2\}$ is a semi-TD-set of G. When $(Y_2 \setminus \{y_1\}) \cap D_{H_1} = \emptyset$, $D_{H_1} \cup D_{H_2} \cup \{u_2\}$ is a semi-TD-set of G. Further, $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_1)| + \frac{3}{7}|V(H_2)| + 1 \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$. Thus consider $H_2 \in \{G_2, G_3\}$. Then we can find a minimum semi-TD-set D''' of $G[\{y_1\} \cup V(H_2)]$ containing vertex y_1 with $|D'''| \leqslant 4$ by Observation 6(b). The set $D_{H_1} \cup D''' \cup \{x_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_1)| + 5 = \frac{3}{7}(n-12) + 5 < \frac{3}{7}n$.

Subsubcase 4.2.2. For any vertex y in $Y_1 \cup Y_2$, $N(y) \cap S_2 \neq \emptyset$. Combining $Y_2 \cap S_2 = \emptyset$ and Assumption 1, $|N(y_i) \cap S_2| \leqslant 1$ and further $|N(y_i) \cap S_2| = 1$ for any $i \in \{1, \ldots, l\}$. Let $N(y_i) \cap S_2 = \{y_i'\}$. Combining Y_2 is a clique of G and Assumption 2, $y_i' \neq y_j'$ for any $i \neq j \in \{1, \ldots, l\}$. If $\{y_1', \ldots, y_l'\}$ is not an independent set of G, without loss of generality, consider $y_1'y_2' \in E(G)$. Similar to the Subcase 4.1, we have $\gamma_{t2}(G) < \frac{3}{7}n$, where y_1, y_2, y_1', y_2' play the roles of u_1, u_2, x_1, x_2, respectively. Thus we may assume that $\{y_1', \ldots, y_l'\}$ is an independent set of G. By Assumption 1, $y_i'u_1 \notin E(G)$ and $y_i'u_2 \notin E(G)$. Let $N(y_i') = \{y_i, y_i''\}$. By $k = 2$ and Assumption 1, $y_i'' \neq y_j''$ for any $i \neq j \in \{1, \ldots, l\}$. By Assumption 2, $y_iy_i'' \notin E(G)$. Since G is claw-free, $N(y_i) = \{y_i', u_2\} \cup Y_2$.

We construct G' from G by removing all vertices of $\{x_1, x_2, u_2, y_1, y_1'\}$ and adding edges between $\{y_1''\}$ and $Y_2 \setminus \{y_1\}$ such that $\{y_1''\} \cup (Y_2 \setminus \{y_1\})$ is a clique in G'. Then G' is a claw-free graph of order $n' = n - 5 \geqslant 5$ with $\delta(G') \geqslant 2$. Let H_1 be the component of G' containing vertex u_1 and D_{H_1} a $\gamma_{t2}(H_1)$-set. Possibly, $G' = H_1$. When G' is not connected, let H_2 be the other connected component of G' and D_{H_2} be a $\gamma_{t2}(H_2)$-set. Clearly, H_1 and H_2 are connected claw-free graphs with minimum degree at least two.

As $N_{H_1}[u_1] = Y_1 \cup \{u_1\}$ is a clique of H_1 and $|N_{H_1}[u_1]| \geqslant 3$, $H_1 \neq G_2$ and $H_1 \neq C_{|V(H_1)|}$ unless $H_1 = C_3$. If $H_1 \in \{C_3, K_4, G_1\}$, then $|V(H_1)| \leqslant 4$. Further, since $n' \geqslant 5$, G' is not connected. Since $d(u_1) \geqslant 3$, $|Y_1| \geqslant 2$. Let z_1 and z_2 be two vertices in Y_1. Similarly, there exist two vertices z_1' and z_2' adjacent to z_1 and z_2, respectively. Then $|V(H_1)| \leqslant 5$, a contradiction. If $H_1 = G_3$, then u_1 plays the role of c_1 in G_3. Thus G' is connected, otherwise $k \geqslant 6$, a contradiction. Further, $G' = G_3$. Since $\{y_1''\} \cup (Y_2 \setminus \{y_1\})$ is a clique of G', $|\{y_1''\} \cup (Y_2 \setminus \{y_1\})| = 2$ and

$Y_2 \setminus \{y_1\} = \{y_2\}$ according to the structure of G_3. Since $k = 2$, without loss of generality, y_1'' and y_2 play the roles of c_8 and c_9 in G_3, respectively. Since $N(y_2) = \{y_2', u_2\} \cup Y_2$, $y_1''y_2 \notin E(G)$. Further, $d(c_4) = d(c_6) = d(y_1'') = 2$, it contradicts $k = 2$.

Hence H_1 is a normal graph. By the inductive hypothesis, $|D_{H_1}| \leqslant \frac{3}{7}|V(H_1)|$. Assume that G' is not connected and $H_2 \in \{C_{|V(H_2)|}, K_4, G_1, G_2, G_3\}$. Note, $\{y_1'', y_2, y_2', y_2''\} \subseteq V(H_2)$. Since $d_{H_2}(y_2') = d(y_2') = 2$, $H_2 \neq K_4$. Also $H_2 \neq G_1$, for $y_2y_2'' \notin E(G)$. If H_2 is a cycle, then $|N(y_1'') \cap S_2| = 2$ and it contradicts Assumption 1 or $k = 2$. If $H_2 = G_2$, then y_1'' plays the role of a vertex of $\{b_7, b_8, b_9\}$ in G_2 since $k = 2$. Further, $Y_2 \setminus \{y_1\} = \{y_2\}$. When y_1'' plays the role of b_7 or b_8, without loss of generality, consider y_1'' plays the role of b_7. Since $k = 2$, y_2 plays the role of b_9. Then $d(y_1') = d(b_7) = d(b_5) = 2$, for $y_2y_1'' \notin E(G)$. Further $k \geqslant 3$, a contradiction. When y_1'' plays the role of b_9, without loss of generality, y_2 plays the role of b_7. Then $d(y_1'') = d(b_8) = d(b_6) = 2$, a contradiction. If $H_2 = G_3$, then y_1'' plays the role of a vertex of $\{c_8, c_9\}$ in G_3 since $k = 2$. Thus $d(c_1) = 2$ and it has two adjacent neighbors. It contradicts Assumption 2.

Therefore, G' is connected or H_2 is a normal graph. Then $D_{H_2} = \emptyset$ or $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$ by the inductive hypothesis. When $(Y_2 \setminus \{y_1\}) \cap (D_{H_1} \cup D_{H_2}) \neq \emptyset$, the set $D_{H_1} \cup D_{H_2} \cup \{x_2, y_1'\}$ is a semi-TD-set of G. When $(Y_2 \setminus \{y_1\}) \cap (D_{H_1} \cup D_{H_2}) = \emptyset$, the set $D_{H_1} \cup D_{H_2} \cup \{x_2, y_1\}$ is a semi-TD-set of G. Anyway $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_1)| + \frac{3}{7}|V(H_2)| + 2 = \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$.

Case 5. $k = 1$. By Assumption 2, $u_1u_2 \notin E(G)$. Let $Y_2 = \{y_1, \ldots, y_l\}$. Since $d(u_2) \geqslant 3$, $l \geqslant 2$.

Subcase 5.1. There exists a vertex in $Y_1 \cup Y_2$, say y_1 such that $N(y_1) \cap S_2 = \emptyset$. We construct G' from G by removing all vertices of $\{x_1, u_2, y_1\}$ and adding edges between $\{u_1\}$ and $Y_2 \setminus \{y_1\}$ such that $\{u_1\} \cup (Y_2 \setminus \{y_1\})$ is a clique of G'. Then G' is a claw-free graph of order $n' = n - 3$. If $\delta(G') = 1$, then $d_{G'}(u_1) = 1$ or $d_{G'}(y_2) = 1$. Thus $Y_2 \setminus \{y_1\} = \{y_2\}$. When $d_{G'}(u_1) = 1$, $N(u_1) = \{x_1, y_1, y_2\}$ since $d(u_1) \geqslant 3$. When $d_{G'}(y_2) = 1$, $N(y_2) = \{u_1, u_2, y_1\}$ for $d(y_2) \geqslant 3$. Since $d(u_1) \geqslant 3$ and G is claw-free, we also have $N(u_1) = \{x_1, y_1, y_2\}$. In both cases, since G is claw-free, $n = 5$. It contradicts $n \geqslant 6$. Thus $\delta(G') \geqslant 2$.

Let H_1 be the component of G' containing vertex u_1 and D_{H_1} be a $\gamma_{t2}(H_1)$-set. Possibly, $G' = H_1$. Consider G' is not connected. Let H_2 be the other connected component of G' and D_{H_2} be a $\gamma_{t2}(H_2)$-set. As G is claw-free, $N_{H_2}(y_1)$ is a clique of H_2. Since $k = 1$, $H_2 \notin \{G_2, G_3\}$ and H_2 is not a cycle unless $H_2 = C_3$. When $H_2 \in \{C_3, K_4\}$, $G'' = G - V(H_2)$ is a normal graph of order n''. Let D'' be a $\gamma_{t2}(G'')$-set. By the inductive hypothesis, $|D''| \leqslant \frac{3}{7}n''$. In order to dominate y_1, $N_{G''}[y_1] \cap D'' \neq \emptyset$. Then $D'' \cup \{v\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$, where v is a neighbor of y_1 in H_2. When $H_2 = G_1$, $d(a_3) \geqslant 3$ and $d(a_4) \geqslant 3$ by Assumption 2. Thus $y_1a_3 \in E(G)$, $y_1a_4 \in E(G)$ and G has a claw centered at y_1, a contradiction. Hence we may assume that if

G' is not connected, then H_2 is a normal graph. By the inductive hypothesis, $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$. Note, if G' is connected, we can think of $D_{H_2} = \emptyset$ and $|D_{H_2}| = 0 = \frac{3}{7}|V(H_2)|$. Anyway, $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$.

When $H_1 \in \{K_4, G_1\}$, $D_{H_2} \cup \{u_1, u_2\}$ or $D_{H_2} \cup \{x_1, y_2\}$ is a semi-TD-set of G. Then $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_2)| + 2 = \frac{3}{7}(n-7) + 2 < \frac{3}{7}n$. When $H_1 \in \{G_2, G_3\}$, we can find a $\gamma_{t2}(H_1)$-set D_{H_1} containing u_1 with $|D_{H_1}| = 4$ by Observation 6(a). Then $D_{H_1} \cup D_{H_2} \cup \{u_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 4 + \frac{3}{7}|V(H_2)| + 1 = \frac{3}{7}(n-12) + 5 < \frac{3}{7}n$. When H_1 is a normal graph, by the inductive hypothesis, $|D_{H_1}| \leqslant \frac{3}{7}|V(H_1)|$. Then either $D_{H_1} \cup D_{H_2} \cup \{u_2\}$ is a semi-TD-set of G in the case of $(Y_2 \setminus \{y_1\}) \cap D_{H_1} = \emptyset$ or $D_{H_1} \cup D_{H_2} \cup \{x_1\}$ is a semi-TD-set of G in the case of $(Y_2 \setminus \{y_1\}) \cap D_{H_1} \neq \emptyset$. Further, $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_1)| + \frac{3}{7}|V(H_2)| + 1 = \frac{3}{7}(n-3) + 1 < \frac{3}{7}n$.

Hence, we may assume that H_1 is a cycle. If $|V(H_1)| = 3$, then $D_{H_2} \cup \{u_1, u_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}|V(H_2)| + 2 = \frac{3}{7}(n-6) + 2 < \frac{3}{7}n$. Thus consider $|V(H_1)| \geqslant 4$. Since $\{u_1\} \cup (Y_2 \setminus \{y_1\})$ is a clique of G', $Y_2 \setminus \{y_1\} = \{y_2\}$. Let $N_{H_1}(u_1) = \{y_2, u_3\}$ and $N_{H_1}(u_3) = \{u_1, u_4\}$. As G is claw-free, $u_1y_2 \notin E(G)$. Since $d(u_3) \geqslant 3$, $N(u_1) = \{x_1, u_3, y_1\}$. Since G is claw-free, $N(y_1) = \{u_1, u_2, u_3, y_2\}$. Thus G' is connected, i.e., $G' = H_1 = C_{|V(H_1)|}$. Since $k = 1$, $u_4y_2 \in E(G)$ and $n = 7$. Now, $\{u_1, y_2\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 2 < \frac{3}{7}n$.

Subcase 5.2. For any vertex y in $Y_1 \cup Y_2$, $N(y) \cap S_2 \neq \emptyset$. By Assumption 1, $|N(y) \cap S_2| \leqslant 1$, and then $|N(y) \cap S_2| = 1$, where $y \in Y_1 \cup Y_2$. Let $N(y_i) \cap S_2 = \{y_i'\}$ for any $i \in \{1, \ldots, l\}$. By Assumption 2, $y_i' \neq y_j'$ for any $i \neq j \in \{1, \ldots, l\}$. Again by Assumption 1, $E[\{y_i'\}, \{u_1, u_2\}] = \emptyset$. Since $k = 1$, $\{y_1', \ldots, y_l'\}$ is an independent set of G. Let $N(y_i') = \{y_1, y_i''\}$. According to Assumption 1 and $k = 1$, $y_i'' \neq y_j''$ for any $i \neq j \in \{1, \ldots, l\}$. Again by Assumption 2, $y_iy_i'' \notin E(G)$. As G is claw-free, $N(y_i) = \{u_2, y_i'\} \cup Y_2$.

Let G' be the graph obtained from G by removing all vertices of $\{u_1, x_1, u_2, y_1, y_1'\}$ and adding edges between $\{y_1''\}$ and $Y_2 \setminus \{y_1\}$ such that $\{y_1''\} \cup (Y_2 \setminus \{y_1\})$ is a clique of G'. Then G' is a claw-free graph of order $n' = n - 5$. Since $k = 1$, $d(y_i'') \geq 3$ for any $i \in \{1, \ldots, l\}$. Further, $\delta(G') \geqslant 2$. Let H_1 be the component of G' containing y_1'' and D_{H_1} be a $\gamma_{t2}(H_1)$-set. Possibly, $G' = H_1$.

Suppose that $H_1 \in \{C_{|V(H_1)|}, K_4, G_1, G_2, G_3\}$. Since $d_{H_1}(y_2') = d(y_2') = 2$ and $y_2y_2'' \notin E(G)$, $H_1 \notin \{K_4, G_1\}$. If $d_{H_1}(y_1'') = d_{H_1}(y_2'') = 2$, then $y_1''u_1 \in E(G)$ and $y_2''u_1 \in E(G)$ since $d(y_1'') \geqslant 3$ and $d(y_1'') \geqslant 3$. As G is claw-free, $y_1''y_2'' \in E(G)$ (i.e., $H_1 = C_4$) and $n = 9$. Then $\{x_1, y_1, y_2''\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 3 < \frac{3}{7}n$. Hence, we may assume that $d_{H_1}(y_1'') \geqslant 3$ or $d_{H_1}(y_2'') \geqslant 3$. So $H_1 \in \{G_2, G_3\}$. According to the structure of G_2 and G_3, $y_1''y_2'' \notin E(G)$ and $d_{H_1}(y_1'') = 2$ or $d_{H_1}(y_2'') = 2$. Hence ($d_{H_1}(y_1'') = 3$ and $d_{H_1}(y_2'') = 2$) or ($d_{H_1}(y_1'') = 2$ and $d_{H_1}(y_2'') = 3$).

Consider $d_{H_1}(y_1'') = 3$ and $d_{H_1}(y_2'') = 2$. Let $N_{H_1}(y_2'') = \{y_2', u_3\}$. Since

$d(y_2'') \geqslant 3$, $N(y_2'') = \{y_2', u_3, u_1\}$. As G is claw-free, $N(u_1) = \{x_1, y_2'', u_3\}$ and $n = 14$. Since $k = 1$, $H_1 = G_2$. Without loss of generality, y_1'', y_2, y_2'' play the roles of b_3, b_5, b_9 in G_2, respectively. The set $\{u_2, y_1', b_1, b_6, y_2''\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$. Consider $d_{H_1}(y_1'') = 2$ and $d_{H_1}(y_2'') = 3$. Let $N_{H_1}(y_1'') = \{y_2, u_4\}$. Since $d(y_1'') \geqslant 3$, $N(y_1'') = \{y_1', u_4, u_1\}$. As G is claw-free, $N(u_1) = \{x_1, y_1'', u_4\}$ and $n = 14$. Since $k = 1$, $H_1 = G_2$. Without loss of generality, y_2'', y_2, y_1'' play the roles of b_3, b_7, b_9 in G_2, respectively. Then $\{u_2, b_1, b_5, b_6, y_1''\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant 5 < \frac{3}{7}n$.

Suppose that H_1 is a normal graph. By the inductive hypothesis, $|D_{H_1}| \leqslant \frac{3}{7}|V(H_1)|$. When G' is not connected, $N_{H_1}(u_1) = \emptyset$ and let H_2 be the other connected component of G' and D_{H_2} be a $\gamma_{t2}(H_2)$-set. Let $Y_1 = \{z_1, \ldots, z_p\}$, where $p \geqslant 2$. Similar to the discussion of Y_2, we can let $N(z_1) \cap S_2 = \{z_1'\}$, $N(z_2) \cap S_2 = \{z_2'\}$, $N(z_1') = \{z_1, z_1''\}$ and $N(z_2') = \{z_2, z_2''\}$. Note, $d(z_1'') \geqslant 3$ and $d(z_2'') \geqslant 3$. Thus H_2 is a normal graph. By the inductive hypothesis, $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$. When G' is connected, we can think of $D_{H_2} = \emptyset$ and $|D_{H_2}| = 0 = \frac{3}{7}|V(H_2)|$. Anyway, $|D_{H_2}| \leqslant \frac{3}{7}|V(H_2)|$. If $(Y_2 \setminus \{y_1\}) \cap D_{H_1} \neq \emptyset$, then $D_{H_1} \cup D_{H_2} \cup \{x_1, y_1'\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$. Thus consider $(Y_2 \setminus \{y_1\}) \cap D_{H_1} = \emptyset$. Then $D_{H_1} \cup D_{H_2} \cup \{x_1, y_1\}$ is a semi-TD-set of G and $\gamma_{t2}(G) \leqslant \frac{3}{7}(n-5) + 2 < \frac{3}{7}n$. ■

By Proposition 5, $\gamma_{t2}(C_n) = \lceil \frac{2}{5}n \rceil$. If $n \notin \{3, 4, 6, 8, 11, 13, 16, 18, 23\}$, then $\lceil \frac{2}{5}n \rceil \leqslant \frac{3}{7}n$. Further, $\lceil \frac{2}{5}n \rceil = \frac{3}{7}n$ if and only if $n \in \{7, 14, 21, 28\}$. Combined with Theorem 8, we have the following theorem.

Theorem 9. *If $G \notin \{C_3, C_4, C_6, C_8, C_9, C_{11}, C_{13}, C_{16}, C_{18}, C_{23}, K_4, G_1, G_2, G_3\}$ is a connected claw-free graph of order n with $\delta(G) \geqslant 2$, then $\gamma_{t2}(G) \leqslant \frac{3}{7}n$, with equality if and only if $G \in \{C_7, C_{14}, C_{21}, C_{28}, G_4, G_5, G_6\} \cup \mathcal{G}$.*

4. CONCLUSION

In this paper, we first prove that for any connected claw-free graph $G \neq K_4$ of order n with $\delta(G) \geqslant 3$, $\gamma_{t2}(G) \leqslant \frac{2}{5}n$. Thus we solve Conjecture 1 for the case of claw-free graphs. Note that K_5 is a graph attaining the bound of Theorem 3. Later, we prove that for any connected claw-free graph G of order n with $\delta(G) \geqslant 2$ unless fourteen graphs (ten of which are cycles), $\gamma_{t2}(G) \leqslant \frac{3}{7}n$, and we characterize the (infinite) family of extremal graphs. On the one hand, one future work proceed to consider Conjecture 1. On the other hand, one can consider the following problem.

Problem 10. What is the exact upper bound on $\gamma_{t2}(G)$ for connected claw-free graph G in terms of its order n and minimum degree $\delta(G) \geqslant 4$?

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