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ON WEAKLY TURÁN-GOOD GRAPHS

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Abstract

Given graphs H and F with $\chi(H) < \chi(F)$, we say that H is weakly F-Turán-good if among n-vertex F-free graphs, a $(\chi(F) - 1)$-partite graph contains the most copies of H. Let H be a bipartite graph that contains a complete bipartite subgraph K such that each vertex of H is adjacent to a vertex of K. We show that H is weakly K_3-Turán-good, improving a very recent asymptotic bound due to Grzesik, Győri, Salia and Tompkins. They also showed that for any r there exist graphs that are not weakly K_r-Turán-good. We show that for any non-bipartite F there exist graphs that are not weakly F-Turán-good. We also show examples of graphs that are C_{2k+1}-Turán-good but not $C_{2\ell+1}$-Turán-good for every $k > \ell$.

Keywords: generalized Turán problem, extremal, Turán-good.

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1. Introduction

Given a graph F, another graph G is called *F-free* if G does not contain any copy of F as a non-necessarily induced subgraph. Let $\mathrm{ex}(n, F)$ denote the largest number of edges in n-vertex F-free graphs. Turán [24] proved that $\mathrm{ex}(n, K_{r+1}) = |E(T(n, r))|$, where the *Turán graph* $T(n, r)$ is the complete r-partite graph with each part of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Simonovits [22] proved that for an $(r + 1)$-chromatic F we have $\mathrm{ex}(n, F) = |E(T(n, r))|$ for sufficiently large n if and only if F has a color-critical edge, i.e., an edge whose removal decreases the chromatic number. The Erdős-Stone-Simonovits theorem [6, 7] states that for any $(r + 1)$-chromatic graph F we have $\mathrm{ex}(n, F) = |E(T(n, r))| + o(n^2)$.

Given graphs H and G, let $\mathcal{N}(H, G)$ denote the number of copies of H in G. In *generalized Turán problems*, we deal with $\mathrm{ex}(n, H, F)$, which is the largest

$\mathcal{N}(H,G)$ where G is an n-vertex F-free graph. The first result in this area is due to Zykov [25], who showed that $\mathrm{ex}(n,K_k,K_{r+1}) = \mathcal{N}(K_k,T(n,r))$.

Generalized Turán problems have attracted several researchers since then. One of the main directions of research has been studying when the Turán graph, or more generally a complete $(\chi(F)-1)$-partite graph is extremal. Given a graph F with $\chi(F) = r+1$, we say that H is *F-Turán-good* if $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T(n,r))$ for n sufficiently large, and we say that H is *weakly F-Turán-good* if $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T)$ for n sufficiently large for some complete r-partite n-vertex graph T. Note that for a given H, it is straightforward but complicated to determine which n-vertex complete r-partite graph contains the most copies of H.

Győri, Pach and Simonovits [16] studied K_{r+1}-Turán-good graphs. They showed that if H is a complete k-partite graph with $k \le r$, then H is weakly K_{r+1}-Turán-good. They also constructed several K_{r+1}-Turán-good graphs. In particular, if H is a bipartite graph with a matching containing all but at most one of its vertices, then H is K_3-Turán-good.

Gerbner and Palmer [14] initiated the study of F-Turán-good graphs for non-complete graphs F. They also conjectured that paths P_k are K_{r+1}-Turán-good for any $r \ge 2$, where P_k denote the path on k vertices. This was proved in [12], after partial results in [9, 18, 20, 21].

We say that H is *asymptotically F-Turán-good* if we have $\mathrm{ex}(n,H,F) = (1+o(1))\mathcal{N}(H,T(n,r))$ and we say that H is *asymptotically weakly F-Turán-good* if $\mathrm{ex}(n,H,F) = (1+o(1))\mathcal{N}(H,T)$ for some complete r-partite graph T. Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$. Assume that each vertex $v \in V(H)$ is adjacent to a vertex of $V(K)$. Grzesik, Győri, Salia and Tompkins [15] showed that H is asymptotically weakly K_3-Turán-good. We improve this to an exact result. Moreover, we extend the result to any 3-chromatic graph with a color-critical edge in place of K_3.

Theorem 1.1. *Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$ and assume that each vertex $v \in V(H)$ is adjacent to a vertex of $V(K)$. Let F be a 3-chromatic graph with a color-critical edge. Then H is weakly F-Turán-good.*

Considering the large variety of weakly K_{r+1}-Turán-good graphs, it is a natural idea that maybe all graphs of chromatic number at most r have this property. However, Győri, Pach and Simonovits [16] showed a bipartite graph that is not weakly K_3-Turán-good, moreover, not even asymptotically weakly K_3-Turán-good. Grzesik, Győri, Salia and Tompkins [15] showed for any $r \ge 2$ an r-chromatic graph that is not asymptotically weakly K_{r+1}-Turán-good. Here we extend this.

Proposition 1. *For any non-bipartite F, there exists a graph of chromatic number $\chi(F)-1$ that is not asymptotically weakly F-Turán-good.*

In extremal graph theory, graphs with a color-critical edge often behave similarly to cliques. We believe that this is the case in our setting as well.

Conjecture 1.2. *If F has a color-critical edge and chromatic number $r+1$, and H is weakly K_{r+1}-Turán-good, then H is weakly F-Turán-good.*

This conjecture is supported by the fact that the asymptotic version is true: if H is asymptotically weakly K_{r+1}-Turán-good, then H is asymptotically weakly F-Turán-good. In fact, H is asymptotically weakly F'-Turán-good for any $(r+1)$-chromatic graph F'. This follows from a theorem in [13], stating that $\mathrm{ex}(n,H,F)\leq \mathrm{ex}(n,H,K_{r+1})+o(n^{|V(H)|})$. However, the reverse is not true. In fact, for every $k>2$ we can construct a graph that is asymptotically C_{2k+1}-Turán-good and not asymptotically weakly K_3-Turán-good. We prove more.

A *blow-up* of a graph G is obtained by replacing each vertex v_i with a non-empty independent set V_i, and each edge v_iv_j is replaced by all the possible edges between V_i and V_j. The sets V_i are called blown-up classes. We denote by $G(m)$ the blow-up where each blown-up class has order m. A vertex v of G is *color-critical* if the removal of v decreases the chromatic number.

Theorem 1.3. *Let F be a 3-chromatic graph and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then there is an asymptotically F-Turán-good graph H that is not asymptotically weakly $C_{2\ell+1}$-Turán-good for any $\ell<k$. Furthermore, if F has a color-critical vertex, then H is F-Turán-good.*

2. Proofs

We say that H is *F-Turán-stable* if the following holds. If G is an n-vertex F-free graph with $\mathcal{N}(H,G)\geq \mathrm{ex}(n,H,F)-o(n^{|V(H)|})$, then G can be obtained from $T(n,\chi(F)-1)$ by adding and removing $o(n^2)$ edges. We say that H is *weakly F-Turán-stable* if the following holds. If G is an n-vertex F-free graph with $\mathcal{N}(H,G)\geq \mathrm{ex}(n,H,F)-o(n^{|V(H)|})$, then G can be obtained from a complete $(\chi(F)-1)$-partite graph by adding and removing $o(n^2)$ edges. Note that it is equivalent to the property that G can be turned into a $(\chi(F)-1)$-partite graph G' by removing $o(n^2)$ edges. Indeed, if the second property holds but the first does not, then we need to add $\Omega(n^2)$ edges to turn G' to a complete $(\chi(F)-1)$-partite graph G''. It is easy to see that we removed $o(n^{|V(H)|})$ copies of H and then added $\Omega(n^{|V(H)|})$ copies of H. Indeed, each edge in G'' is clearly in $\Omega(n^{|V(H)|-2})$ copies of H. Therefore, $\mathcal{N}(H,G)\leq \mathcal{N}(H,G'')-\Omega(n^{|V(H)|})$, a contradiction.

The well-known Erdős-Simonovits stability theorem [3, 4, 23] states that K_2 is F-Turán-stable for every F. Ma and Qiu [19] studied such stability in generalized Turán problems first and showed that K_k is F-Turán-stable for every F with chromatic number more than k. Hei, Hou and Liu [18] and later Gerbner [11] studied the connection of such stability and exact result more generally. We will use two results from [11].

Theorem 2.1 (Gerbner [11])**.** (i) *If H is weakly K_r-Turán-stable, then H is weakly F-Turán-stable for any r-chromatic graph F.*

(ii) *If F has a color-critical edge, then weakly F-Turán-stable graphs are also weakly F-Turán-good.*

We will consider the *double star $S_{a,b}$*. It consists of a central edge uv, a leaves joined to u and b leaves joined to v. Győri, Wang and Woolfson [17] proved that $S_{a,b}$ is weakly K_3-Turán-good. Gerbner [10] showed that $S_{a,b}$ is weakly F-Turán-good for any 3-chromatic graph F with a color-critical edge. We show that $S_{a,b}$ is weakly K_3-Turán-stable.

Proposition 2. *$S_{a,b}$ is weakly K_3-Turán-stable.*

Proof. We let $f(x,y)=\binom{x-1}{a}\binom{y-1}{b}+\binom{y-1}{a}\binom{x-1}{b}$ if $a\neq b$ and $f(x,y)=\binom{x-1}{a}\binom{y-1}{b}$ if $a=b$. Győri, Wang and Woolfson [17] showed that in a K_3-free n-vertex graph G with maximum degree $\Delta>n/2$, each edge is the central edge of at most $f(\Delta',n-\Delta')$ copies of $S_{a,b}$ for some $n/2\le\Delta'\le\Delta$. Moreover, G has at most $\Delta(n-\Delta)$ edges. Let F_2 denote the graph consisting of two triangles sharing a vertex. Gerbner [10] showed that for any $\varepsilon'>0$ there exists $\delta'>0$ such that if an F_2-free n-vertex graph G with maximum degree $\Delta\ge n/2$ has at least $\Delta(n-\Delta)-\delta' n^2$ edges, then G can be turned into a bipartite graph by deleting at most $\varepsilon' n^2$ edges. We will pick a small δ'.

Let G be a K_3-free n-vertex graph with at least $\mathrm{ex}(n,S_{a,b},K_3)-\delta n^{a+b+2}$ copies of $S_{a,b}$ and we want to show that G can be turned into a bipartite graph by deleting at most εn^2 edges. We consider two cases based on the maximum degree of G. In each case, we argue that either G has sufficiently many edges to apply a previously established stability result or G has too few edges to be near extremal.

Assume first that G has maximum degree $\Delta>n/2$. If the conclusion does not hold, then G has at most $\Delta(n-\Delta)-\delta' n^2\le\Delta'(n-\Delta')-\delta' n^2$ edges by the previous paragraph. Each edge is the central edge at most $q:=f(\Delta',n-\Delta')$ copies of $S_{a,b}$, thus there are at most $\Delta'(n-\Delta')q-\delta' n^2 q=\mathcal{N}(S_{a,b},K_{\Delta',n-\Delta'})-\delta'\alpha n^{a+b+2}$ copies of $S_{a,b}$ in G for some $\alpha>0$ that depends on a and b, but not on ε. Therefore, picking a sufficiently small δ', we obtain a contradiction with our assumption on G.

Assume now that $\Delta\le n/2$. In this case we will show that G can be turned into $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ by deleting at most εn^2 edges. By the ordinary Erdős-Simonovits stability the conclusion holds unless G has less than $n^2/4 - \delta' n^2$ edges. Observe that each edge is the central edge of at most $f(\Delta,\Delta) \leq f(\lfloor n/2\rfloor, \lfloor n/2\rfloor) \leq f(\lfloor n/2\rfloor, \lceil n/2\rceil) =: q'$ copies of $S_{a,b}$. Therefore, there are at most $n^2q'/4 - \delta' n^2 q' \leq \mathcal{N}(S_{a,b}, K_{\lfloor n/2\rfloor,\lceil n/2\rceil}) - \delta'\alpha n^{a+b+2}$ copies of $S_{a,b}$ in G for some $\alpha > 0$ that does not depend on ε. We obtain a contradiction with our assumption on G as in the previous paragraph. ■

Note that this, combined with Theorem 2.1, gives a simpler proof of the theorem from [10] stating that $S_{a,b}$ is F-Turán-good for every 3-chromatic graph F with a color-critical edge.

Proposition 3. *Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$ and assume that each vertex $v \in V(H)$ is adjacent to a vertex of $V(K)$. Then H is weakly K_3-Turán-stable.*

Proof. Recall that [15] showed that H is asymptotically weakly K_3-Turán-good. We follow their proof. First they showed that it is enough to consider H that consists of K and some pendant edges. Assume that H has $a+1$ vertices in one of its parts and $b+1$ vertices in the other part. Let G be a K_3-free n-vertex graph, let $p(G)$ denote the number of labeled copies of H in G and $q(G)$ denote the number of labeled copies of $S_{a,b}$ in G. It is shown in [15] that $p(G) \leq q(G) + o(n^{a+b+2})$. Observe that in a complete bipartite graph the same ordered sets of $a+b+2$ vertices induce copies of H and $S_{a,b}$, and obviously the ratio of the numbers of labeled and unlabeled copies of a graph depends only on the graph itself. These imply the asymptotic result.

Let us assume now that G contains $\mathrm{ex}(n,H,K_3) - o(n^{a+b+2}) = \mathcal{N}(H,T) - o(n^{a+b+2})$ copies of H, for some n-vertex complete bipartite graph T. Then the number of labeled copies of H also differs by $o(n^{a+b+2})$, i.e., $p(G) = p(T) - o(n^{a+b+2})$. Therefore, we have $q(G) = q(T) - o(n^{a+b+2})$, thus $\mathcal{N}(S_{a,b}, G) = \mathcal{N}(S_{a,b}, T) - o(n^{a+b+2})$. This, combined with Proposition 2, completes the proof. ■

Combined with Theorem 2.1, the above proposition implies Theorem 1.1.

Let us continue with the proof of Proposition 1. Recall that it states that for any non-bipartite F, there exists a graph of chromatic number $\chi(F) - 1$ that is not asymptotically weakly F-Turán-good. Our construction is a slight generalization of the construction in [15], which we describe next, after some necessary definition.

Given a graph G, G^k denotes the graph we obtain by connecting two vertices of G if and only if they are at distance at most k in G. The graph H that is not weakly K_{r+1}-Turán-good in [15] is obtained from P_{2r+2}^{r-1} by replacing the end-vertices of the original path by sufficiently large independent sets. Then they show that a very unbalanced blow-up of C_{2r+1}^{r-1} contains more copies of H than any r-partite graph. On the other hand, any blow-up of C_{2r+1}^{r-1} is K_{r+1}-free, since any set of $r+1$ vertices contains either 2 vertices from the same blown-up class, or 2 vertices that belong to classes that are at distance r in the original C_{2r+1}.

For simplicity, in the following proof, we will count labeled copies of H inside some host graphs. It is easy to see that it only differs by a multiplicative factor (the number of automorphisms of H), independent of the host graph, thus does not affect our result.

***Proof of Proposition* 1.** Let $\chi(F) = r+1$ and $k > |V(H)|$ be an integer that is divisible by r. We are going to use the following graph H. We take P_k^{r-1} and replace the end-vertices of the original path by independent sets of order a, where a is sufficiently large. We will show that a very unbalanced blow-up of C_k^{r-1} contains more copies of H than any r-partite graph.

We show that if $k > |V(F)|$ and r divides k, then any blow-up of C_k^{r-1} is F-free. Indeed, a copy of F would not share any vertices with at least one of the blown-up classes, say with V_k. But the remaining graph is r-colorable (hence F-free), as shown by the following coloring. Let us color the vertices inside each blown-up class by the same color, and color the blown-up classes the following way. We go through the original C_k in a cyclic order, and color V_j by color j modulo r.

Observe that there is a unique r-coloring of any blow-up of P_k^{r-1}. In H, if k is not congruent to 1 modulo r, then the two a-sets corresponding to the end vertices of the original path have different color. Therefore, inside an r-partite n-vertex graph G, those sets are in different parts. It implies that there at most $n^{k-2}\left(\frac{n}{2}\right)^{2a}$ labeled copies of H in G.

Let G' be the blow-up of C_k^{r-1} where each vertex is replaced by $\lfloor \gamma n \rfloor$ vertices except one vertex is replaced by $n-(k-1)\lfloor \gamma n \rfloor$ vertices. Then the number of labeled copies of H in G' is at least $(\gamma n)^{k-2}(n-(k-1)\lfloor \gamma n \rfloor)^{2a} + o(n^{k-2+2a})$. For a given $\gamma < 1/2k$, we can pick a such that $\gamma^{k-2}(1-(k-1)\gamma)^{2a} > \frac{1}{2^{2a}}$, completing the proof. ∎

Let us turn to the proof of Theorem 1.3. We start with a lemma.

Lemma 4. *Let F be a 3-chromatic graph with a color-critical vertex and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then F is the subgraph of a blow-up of C_{2k+1} where one of the blown-up classes has order* 1.

Proof. Let v be a color-critical vertex of F and consider a blow-up G of C_{2k+1} with parts $V_1, \dots, V_{2k+1}$ in cyclic order, such that G contains F. Assume that $v \in V_{2k+1}$. We pick G such a way that V_{2k+1} is as small as possible.

Let us assume that there is another vertex $v' \in V_{2k+1}$. If v' is adjacent only to vertices in V_1, then we can move v' to V_2, thus $|V_{2k+1}|$ decreases, a contradiction. If v' is adjacent only to vertices in V_{2k}, then we can move v' to

V_{2k-1}, a contradiction. Assume that v' has neighbors $v_1 \in V_1$ and $v_{2k} \in V_{2k}$. Let G' denote the bipartite graph we obtain from G by deleting V_{2k+1}, and let U denote the component containing v_1 in G'. Then we move every vertex of U from V_i to V_{2k+1-i} for every i. We repeat this for every remaining neighbor of v' in V_1. At the end, v' has neighbors only in V_{2k}, thus we can move v' to V_{2k-1}, a contradiction. ■

Proposition 5. *If F is a 3-chromatic graph with a color-critical vertex, then for any integers ℓ and $m \geq |V(F)|$ we have that $P_{2\ell}(m)$ is F-Turán-good.*

We remark that the first result concerning F-Turán-good graphs when F does not have a color-critical edge is due to Gerbner and Palmer [14], who showed that C_4 is F_2-Turán-good, where F_2 consists of two triangles sharing a vertex. Gerbner [8] constructed F-Turán-good graphs for every F with a color-critical vertex, but they were always complete $(\chi(F)-1)$-partite graphs. In particular, $K_{m,m} = P_2(m)$ is F-Turán-good. The above proposition gives the first examples of another kind.

Proof. We apply induction on ℓ, the base case $\ell = 1$ was mentioned above. Assume the statement holds for ℓ and prove it for $\ell + 1$. We count the copies of $P_{2\ell+2}(m)$ in an n-vertex F-free graph G the following way. First we pick a copy of $P_{2\ell}(m)$, the number of ways to pick them is maximized when $G = T(n, 2)$ by induction. Then, among the remaining $n - 2\ell m$ vertices, we pick a copy of $P_2(m) = K_{m,m}$. The number of ways to pick it is maximized when there is a $T(n - 2\ell m, 2)$ on the remaining $n - 2\ell m$ vertices, which is achieved when $G = T(n, 2)$.

We show that there are at most two ways to add the copy of $P_2(m)$ to the copy of $P_{2\ell}(m)$ in G to create a copy of $P_{2\ell+2}(m)$. Indeed, we can do that if each vertex of a part of $K_{m,m}$ is adjacent to each vertex of one of the ends of $P_{2\ell}(m)$. It cannot happen with the same end of $P_{2\ell}(m)$ and both parts of $K_{m,m}$, as that would mean G contains $K_{m,m,1}$, which contains F, a contradiction. In $T(n, 2)$, there are always two ways to add a copy of $P_2(m)$ to a copy of $P_{2\ell}(m)$ in $T(n, 2)$ to create a copy of $P_{2\ell+2}(m)$. Therefore, this third factor is also maximized by the Turán graph, completing the proof. ■

Let us denote by $P_{2k+2}(m, a, b)$ the following blow-up of P_{2k+2}. We replace the end vertices v_1 and v_{2k+2} by independent sets V_1 of order a and V_{2k+2} of order b, and we replace the middle vertices $v_2, \ldots, v_{2k+1}$ by independent sets $V_2, \ldots, V_{2k+1}$ of order m.

Proposition 6. *Let F be a 3-chromatic graph and assume that F is contained in a blow-up of C_{2k+1}. If $a \geq b \geq m \geq |V(F)|$ and $a \leq b + 1/2 + \sqrt{2b + 1/4}$, then*

$P_{2k+2}(m,a,b)$ *is asymptotically* F*-Turán-good. Moreover, if* F *has a color-critical vertex and* $a < b + 1/2 + \sqrt{2b + 1/4}$*, then* $P_{2k+2}(m,a,b)$ *is* F*-Turán-good.*

As we have mentioned, $K_{a,b}$ is weakly K_3-Turán-good by a result of Győri, Pach and Simonovits [16], thus only an optimization is needed here. Brown and Sidorenko [2] did this optimization in a slightly different context, and obtained that $T(n,2)$ is asymptotically optimal if $b \geq \binom{a-b}{2}$. Ma and Qiu [19] showed that $K_{a,b}$ is K_3-Turán-good if and only if $a < b + 1/2 + \sqrt{2b + 1/4}$.

Proof. Let H_0 denote the subgraph of $P_{2k+2}(m,a,b)$ obtained by deleting V_1 and V_{2k+2}, i.e., $H_0 = P_{2k}(m)$. Then H_0 has a complete matching, thus is K_3-Turán-good by a result of Győri, Pach and Simonovits [16]. This implies that H_0 is asymptotically F-Turán-good by a result of Gerbner and Palmer [13] mentioned in the introduction. We show that the number of ways to extend H_0 to $P_{2k+2}(m,a,b)$ in G is also asymptotically at most the number of ways to extend H_0 to $P_{2k+2}(m,a,b)$ in the Turán graph.

Let us pick a copy of H_0 in G and let U denote the set of its vertices. Observe that there are at most $m-1$ vertices in G that are not in H_0 and are connected to all the vertices of $U \cap V_2$ and $U \cap V_{2k+1}$, as otherwise G contains $C_{2k+1}(m)$, which contains F. Let x be the number of common neighbors of the vertices of V_2 that are not in U. Then there are at most $n-2km+m-1-x$ common neighbors of the vertices of V_{2k+1} that are not in U. We need to pick a common neighbors of the m vertices in V_2, and b common neighbors of the m vertices in V_{2k+1}, or the other way around, thus there are at most $\binom{x}{a}\binom{n-2km+m-1-x}{b} + \binom{x}{b}\binom{n-2km+m-1-x}{a}$ ways to extend H_0 to $P_{2k+2}(m,a,b)$ if $a \neq b$, and half of this if $a = b$. Observe that this is equal to the number of copies of $K_{a,b}$ in $K_{x,n-2km+m-1-x}$, which is at most the number of copies of $K_{a,b}$ in $T(n-2km+m-1,2)$, using the theorem of Brown and Sidorenko we mentioned. It is easy to see that $\mathcal{N}(K_{a,b}, T(n-2km+m-1,2)) = (1+o(1))\mathcal{N}(K_{a,b}, T(n-2km,2))$. Therefore, the Turán graph asymptotically maximizes the number of ways to extend H_0 to H, completing the proof for general F. Let us assume now that F has a color-critical vertex v. Recall that by Lemma 4, F is contained in a blow-up of C_{2k+1} where one blown-up class has order 1. We use this analogously to the argument before. We pick a copy of H_0 in G, then there are no vertices that are not in H_0 and are connected to all the vertices in V_2 and V_{2k+1}. Let x be the number of common neighbors of V_2 that are not in the copy of H. We need to pick a common neighbors of the m vertices in V_2, and b common neighbors of the m vertices in V_{2k+1}, or the other way around. There are at most $\binom{x}{a}\binom{n-2km}{b} + \binom{x}{b}\binom{n-2km}{a}$ ways to do this if $a \neq b$ and half of this if $a = b$. Observe that this is equal to the number of copies of $K_{a,b}$ in $K_{x,n-2km-x}$, which is at most the number of copies of $K_{a,b}$ in $T(n-2km,2)$ by the theorem of Ma and Qiu we mentioned. This shows that the Turán graph maximizes the number of ways to extend H_0 to H, completing the proof. ∎

Now we are ready to prove Theorem 1.3 which we restate here for convenience.

Theorem. *Let F be a 3-chromatic graph and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then there is an asymptotically F-Turán-good graph H that is not asymptotically weakly $C_{2\ell+1}$-Turán-good for any $\ell < k$. Furthermore, if F has a color-critical vertex, then H is F-Turán-good.*

Proof. Proposition 6 presented an asymptotically F-Turán-good graph $H = P_{2k+2}(m, a, b)$ (which is an F-Turán-good graph if F has a color-critical vertex). We will show that H is not asymptotically weakly $C_{2\ell+1}$-Turán-good. Observe that the a vertices and the b vertices corresponding to the end vertices of the original P_{2k+2} must be in different parts of any bipartite graph. On the other hand, they can be in the same blown-up class of C_{2k+1}, which are $C_{2\ell+1}$-free graphs. From here, the calculation to show that H is not asymptotically weakly F'-Turán-good is the same as the calculation in the proof of Proposition 1, thus we omit the details. ■

3. Concluding Remarks

We have shown examples of graphs that are not asymptotically weakly F-Turán-good, for any non-bipartite graph F. Our examples, just like all the earlier examples for the case F is a clique, are built on the idea of forcing two large sets of vertices into different part in every $(\chi(F)-1)$-partite graph. In particular, the examples themselves are $(\chi(F)-1)$-chromatic. We do not know any examples of graphs that are not F-Turán-good and have chromatic number less than $\chi(F)-1$ in the case F has a color-critical edge.

Some of our results deal with asymptotically weakly F-Turán-good graphs, while other results deal with weakly F-Turán-good graphs. In the case F has a color-critical edge, we are not aware of any graph that is asymptotically weakly F-Turán-good but not weakly F-Turán-good. In fact, the situation is much worse. In each of the cases when we can show that H is not weakly F-Turán-good, i.e., we can find an F-free n-vertex graph that contains more copies of H than any $(\chi(F)-1)$-partite n-vertex graph, then we do not actually know $\mathrm{ex}(n, H, F)$. There are constructions with more copies of H than any $(\chi(F)-1)$-partite n-vertex graph, but we do not know whether they are extremal.

Let us recall that Theorem 1.3 contains the assumption that C_{2k+1} is the longest odd cycle such that the blow-up of it contains F. Then F cannot contain any odd cycle of length less than $2k+1$. Then blow-ups of F do not contain such cycles either. Thus Theorem 1.3 is a special case of the following conjecture.

Conjecture 3.1. *For given $r+1$-chromatic graphs F, F', there exists a graph H that is asymptotically weakly F-Turán-good but not asymptotically weakly F'-*

Turán-good if and only F' is not a subgraph of any blow-up of F. Furthermore, if F has a color-critical vertex, then we can find H that is F-Turán-good.

Note that one of the directions easily follows from known results. Let H be asymptotically weakly F-Turán-good. A result of Alon and Shikhelman [1] states that if F' is a subgraph of a blow-up of F, then $\mathrm{ex}(n, F, F') = o(n^{|V(F)|})$. Combined with the removal lemma [5], this shows that we can delete the copies of F from any F'-free n-vertex graph G by deleting $o(n^2)$ edges, thus $o(n^{|V(H)|})$ copies of H. The resulting graph is F-free, thus contains at most $(1 + o(1))\mathcal{N}(H, T)$ copies of H for some complete r-partite graph T, thus G also contains at most $(1 + o(1))\mathcal{N}(H, T)$ copies of H, showing that H is asymptotically weakly F'-Turán-good.

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