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ON WEAKLY TURÁN-GOOD GRAPHS

Dániel Gerbner

Alfréd Rényi Institute of Mathematics e-mail: gerbner.daniel@renyi.hu

Abstract

Given graphs H and F with $\chi(H) < \chi(F)$, we say that H is weakly F-Turán-good if among n-vertex F-free graphs, a $(\chi(F) - 1)$ -partite graph contains the most copies of H. Let H be a bipartite graph that contains a complete bipartite subgraph K such that each vertex of H is adjacent to a vertex of K. We show that H is weakly K_3 -Turán-good, improving a very recent asymptotic bound due to Grzesik, Győri, Salia and Tompkins. They also showed that for any r there exist graphs that are not weakly K_r -Turán-good. We show that for any non-bipartite F there exist graphs that are C_{2k+1} -Turán-good but not $C_{2\ell+1}$ -Turán-good for every $k > \ell$.

Keywords: generalized Turán problem, extremal, Turán-good.

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1. INTRODUCTION

Given a graph F, another graph G is called F-free if G does not contain any copy of F as a non-necessarily induced subgraph. Let ex(n, F) denote the largest number of edges in n-vertex F-free graphs. Turán [24] proved that $ex(n, K_{r+1}) =$ |E(T(n,r))|, where the Turán graph T(n,r) is the complete r-partite graph with each part of order $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Simonovits [22] proved that for an (r + 1)chromatic F we have ex(n, F) = |E(T(n, r))| for sufficiently large n if and only if F has a color-critical edge, i.e., an edge whose removal decreases the chromatic number. The Erdős-Stone-Simonovits theorem [6, 7] states that for any (r + 1)chromatic graph F we have $ex(n, F) = |E(T(n, r))| + o(n^2)$.

Given graphs H and G, let $\mathcal{N}(H,G)$ denote the number of copies of H in G. In generalized Turán problems, we deal with ex(n, H, F), which is the largest

 $\mathcal{N}(H,G)$ where G is an *n*-vertex F-free graph. The first result in this area is due to Zykov [25], who showed that $ex(n, K_k, K_{r+1}) = \mathcal{N}(K_k, T(n, r))$.

Generalized Turán problems have attracted several researchers since then. One of the main directions of research has been studying when the Turán graph, or more generally a complete $(\chi(F) - 1)$ -partite graph is extremal. Given a graph F with $\chi(F) = r + 1$, we say that H is F-Turán-good if ex(n, H, F) = $\mathcal{N}(H, T(n, r))$ for n sufficiently large, and we say that H is weakly F-Turán-good if $ex(n, H, F) = \mathcal{N}(H, T)$ for n sufficiently large for some complete r-partite nvertex graph T. Note that for a given H, it is straightforward but complicated to determine which n-vertex complete r-partite graph contains the most copies of H.

Győri, Pach and Simonovits [16] studied K_{r+1} -Turán-good graphs. They showed that if H is a complete k-partite graph with $k \leq r$, then H is weakly K_{r+1} -Turán-good. They also constructed several K_{r+1} -Turán-good graphs. In particular, if H is a bipartite graph with a matching containing all but at most one of its vertices, then H is K_3 -Turán-good.

Gerbner and Palmer [14] initiated the study of F-Turán-good graphs for noncomplete graphs F. They also conjectured that paths P_k are K_{r+1} -Turán-good for any $r \ge 2$, where P_k denote the path on k vertices. This was proved in [12], after partial results in [9, 18, 20, 21].

We say that H is asymptotically F-Turán-good if we have $ex(n, H, F) = (1 + o(1))\mathcal{N}(H, T(n, r))$ and we say that H is asymptotically weakly F-Turángood if $ex(n, H, F) = (1 + o(1))\mathcal{N}(H, T)$ for some complete r-partite graph T. Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$. Assume that each vertex $v \in V(H)$ is adjacent to a vertex of V(K). Grzesik, Győri, Salia and Tompkins [15] showed that H is asymptotically weakly K_3 -Turán-good. We improve this to an exact result. Moreover, we extend the result to any 3-chromatic graph with a color-critical edge in place of K_3 .

Theorem 1.1. Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$ and assume that each vertex $v \in V(H)$ is adjacent to a vertex of V(K). Let F be a 3-chromatic graph with a color-critical edge. Then H is weakly F-Turán-good.

Considering the large variety of weakly K_{r+1} -Turán-good graphs, it is a natural idea that maybe all graphs of chromatic number at most r have this property. However, Győri, Pach and Simonovits [16] showed a bipartite graph that is not weakly K_3 -Turán-good, moreover, not even asymptotically weakly K_3 -Turán-good. Grzesik, Győri, Salia and Tompkins [15] showed for any $r \ge 2$ an r-chromatic graph that is not asymptotically weakly K_{r+1} -Turán-good. Here we extend this. **Proposition 1.** For any non-bipartite F, there exists a graph of chromatic number $\chi(F) - 1$ that is not asymptotically weakly F-Turán-good.

In extremal graph theory, graphs with a color-critical edge often behave similarly to cliques. We believe that this is the case in our setting as well.

Conjecture 1.2. If F has a color-critical edge and chromatic number r + 1, and H is weakly K_{r+1} -Turán-good, then H is weakly F-Turán-good.

This conjecture is supported by the fact that the asymptotic version is true: if H is asymptotically weakly K_{r+1} -Turán-good, then H is asymptotically weakly F-Turán-good. In fact, H is asymptotically weakly F'-Turán-good for any (r + 1)-chromatic graph F'. This follows from a theorem in [13], stating that $ex(n, H, F) \leq ex(n, H, K_{r+1}) + o(n^{|V(H)|})$. However, the reverse is not true. In fact, for every k > 2 we can construct a graph that is asymptotically C_{2k+1} -Turán-good and not asymptotically weakly K_3 -Turán-good. We prove more.

A blow-up of a graph G is obtained by replacing each vertex v_i with a nonempty independent set V_i , and each edge $v_i v_j$ is replaced by all the possible edges between V_i and V_j . The sets V_i are called blown-up classes. We denote by G(m) the blow-up where each blown-up class has order m. A vertex v of G is color-critical if the removal of v decreases the chromatic number.

Theorem 1.3. Let F be a 3-chromatic graph and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then there is an asymptotically F-Turán-good graph H that is not asymptotically weakly $C_{2\ell+1}$ -Turán-good for any $\ell < k$. Furthermore, if F has a color-critical vertex, then H is F-Turán-good.

2. Proofs

We say that H is F-Turán-stable if the following holds. If G is an n-vertex F-free graph with $\mathcal{N}(H,G) \geq \exp(n,H,F) - o(n^{|V(H)|})$, then G can be obtained from $T(n,\chi(F)-1)$ by adding and removing $o(n^2)$ edges. We say that H is weakly F-Turán-stable if the following holds. If G is an n-vertex F-free graph with $\mathcal{N}(H,G) \geq \exp(n,H,F) - o(n^{|V(H)|})$, then G can be obtained from a complete $(\chi(F)-1)$ -partite graph by adding and removing $o(n^2)$ edges. Note that it is equivalent to the property that G can be turned into a $(\chi(F)-1)$ -partite graph G' by removing $o(n^2)$ edges. Indeed, if the second property holds but the first does not, then we need to add $\Omega(n^2)$ edges to turn G' to a complete $(\chi(F)-1)$ -partite graph G''. It is easy to see that we removed $o(n^{|V(H)|})$ copies of H and then added $\Omega(n^{|V(H)|})$ copies of H. Indeed, each edge in G'' is clearly in $\Omega(n^{|V(H)|-2})$ copies of H. Therefore, $\mathcal{N}(H,G) \leq \mathcal{N}(H,G'') - \Omega(n^{|V(H)|})$, a contradiction.

The well-known Erdős-Simonovits stability theorem [3, 4, 23] states that K_2 is *F*-Turán-stable for every *F*. Ma and Qiu [19] studied such stability in

generalized Turán problems first and showed that K_k is *F*-Turán-stable for every *F* with chromatic number more than *k*. Hei, Hou and Liu [18] and later Gerbner [11] studied the connection of such stability and exact result more generally. We will use two results from [11].

Theorem 2.1 (Gerbner [11]). (i) If H is weakly K_r -Turán-stable, then H is weakly F-Turán-stable for any r-chromatic graph F.

 (ii) If F has a color-critical edge, then weakly F-Turán-stable graphs are also weakly F-Turán-good.

We will consider the *double star* $S_{a,b}$. It consists of a central edge uv, a leaves joined to u and b leaves joined to v. Győri, Wang and Woolfson [17] proved that $S_{a,b}$ is weakly K_3 -Turán-good. Gerbner [10] showed that $S_{a,b}$ is weakly F-Turángood for any 3-chromatic graph F with a color-critical edge. We show that $S_{a,b}$ is weakly K_3 -Turán-stable.

Proposition 2. $S_{a,b}$ is weakly K_3 -Turán-stable.

Proof. We let $f(x, y) = \binom{x-1}{a}\binom{y-1}{b} + \binom{y-1}{a}\binom{x-1}{b}$ if $a \neq b$ and $f(x, y) = \binom{x-1}{a}\binom{y-1}{b}$ if a = b. Győri, Wang and Woolfson [17] showed that in a K_3 -free *n*-vertex graph G with maximum degree $\Delta > n/2$, each edge is the central edge of at most $f(\Delta', n - \Delta')$ copies of $S_{a,b}$ for some $n/2 \leq \Delta' \leq \Delta$. Moreover, G has at most $\Delta(n - \Delta)$ edges. Let F_2 denote the graph consisting of two triangles sharing a vertex. Gerbner [10] showed that for any $\varepsilon' > 0$ there exists $\delta' > 0$ such that if an F_2 -free *n*-vertex graph G with maximum degree $\Delta \geq n/2$ has at least $\Delta(n - \Delta) - \delta' n^2$ edges, then G can be turned into a bipartite graph by deleting at most $\varepsilon' n^2$ edges. We will pick a small δ' .

Let G be a K_3 -free n-vertex graph with at least $ex(n, S_{a,b}, K_3) - \delta n^{a+b+2}$ copies of $S_{a,b}$ and we want to show that G can be turned into a bipartite graph by deleting at most εn^2 edges. We consider two cases based on the maximum degree of G. In each case, we argue that either G has sufficiently many edges to apply a previously established stability result or G has too few edges to be near extremal.

Assume first that G has maximum degree $\Delta > n/2$. If the conclusion does not hold, then G has at most $\Delta(n-\Delta)-\delta'n^2 \leq \Delta'(n-\Delta')-\delta'n^2$ edges by the previous paragraph. Each edge is the central edge at most $q := f(\Delta', n-\Delta')$ copies of $S_{a,b}$, thus there are at most $\Delta'(n-\Delta')q-\delta'n^2q = \mathcal{N}(S_{a,b}, K_{\Delta',n-\Delta'})-\delta'\alpha n^{a+b+2}$ copies of $S_{a,b}$ in G for some $\alpha > 0$ that depends on a and b, but not on ε . Therefore, picking a sufficiently small δ' , we obtain a contradiction with our assumption on G.

Assume now that $\Delta \leq n/2$. In this case we will show that G can be turned into $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ by deleting at most εn^2 edges. By the ordinary Erdős-Simonovits stability the conclusion holds unless G has less than $n^2/4 - \delta' n^2$ edges. Observe that each edge is the central edge of at most $f(\Delta, \Delta) \leq f(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor) \leq f(\lfloor n/2 \rfloor, \lfloor n/2 \rfloor) =: q'$ copies of $S_{a,b}$. Therefore, there are at most $n^2q'/4 - \delta' n^2q' \leq \mathcal{N}(S_{a,b}, K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) - \delta' \alpha n^{a+b+2}$ copies of $S_{a,b}$ in G for some $\alpha > 0$ that does not depend on ε . We obtain a contradiction with our assumption on G as in the previous paragraph.

Note that this, combined with Theorem 2.1, gives a simpler proof of the theorem from [10] stating that $S_{a,b}$ is *F*-Turán-good for every 3-chromatic graph *F* with a color-critical edge.

Proposition 3. Let H be a bipartite graph containing a subgraph K isomorphic to $K_{s,t}$ and assume that each vertex $v \in V(H)$ is adjacent to a vertex of V(K). Then H is weakly K_3 -Turán-stable.

Proof. Recall that [15] showed that H is asymptotically weakly K_3 -Turán-good. We follow their proof. First they showed that it is enough to consider H that consists of K and some pendant edges. Assume that H has a+1 vertices in one of its parts and b+1 vertices in the other part. Let G be a K_3 -free n-vertex graph, let p(G) denote the number of labeled copies of H in G and q(G) denote the number of labeled copies of $S_{a,b}$ in G. It is shown in [15] that $p(G) \leq q(G) + o(n^{a+b+2})$. Observe that in a complete bipartite graph the same ordered sets of a + b + 2vertices induce copies of H and $S_{a,b}$, and obviously the ratio of the numbers of labeled and unlabeled copies of a graph depends only on the graph itself. These imply the asymptotic result.

Let us assume now that G contains $ex(n, H, K_3) - o(n^{a+b+2}) = \mathcal{N}(H, T) - o(n^{a+b+2})$ copies of H, for some *n*-vertex complete bipartite graph T. Then the number of labeled copies of H also differs by $o(n^{a+b+2})$, i.e., $p(G) = p(T) - o(n^{a+b+2})$. Therefore, we have $q(G) = q(T) - o(n^{a+b+2})$, thus $\mathcal{N}(S_{a,b}, G) = \mathcal{N}(S_{a,b}, T) - o(n^{a+b+2})$. This, combined with Proposition 2, completes the proof.

Combined with Theorem 2.1, the above proposition implies Theorem 1.1.

Let us continue with the proof of Proposition 1. Recall that it states that for any non-bipartite F, there exists a graph of chromatic number $\chi(F) - 1$ that is not asymptotically weakly F-Turán-good. Our construction is a slight generalization of the construction in [15], which we describe next, after some necessary definition.

Given a graph G, G^k denotes the graph we obtain by connecting two vertices of G if and only if they are at distance at most k in G. The graph H that is not weakly K_{r+1} -Turán-good in [15] is obtained from P_{2r+2}^{r-1} by replacing the end-vertices of the original path by sufficiently large independent sets. Then they show that a very unbalanced blow-up of C_{2r+1}^{r-1} contains more copies of H than any *r*-partite graph. On the other hand, any blow-up of C_{2r+1}^{r-1} is K_{r+1} -free, since any set of r+1 vertices contains either 2 vertices from the same blown-up class, or 2 vertices that belong to classes that are at distance r in the original C_{2r+1} .

For simplicity, in the following proof, we will count labeled copies of H inside some host graphs. It is easy to see that it only differs by a multiplicative factor (the number of automorphisms of H), independent of the host graph, thus does not affect our result.

Proof of Proposition 1. Let $\chi(F) = r + 1$ and k > |V(H)| be an integer that is divisible by r. We are going to use the following graph H. We take P_k^{r-1} and replace the end-vertices of the original path by independent sets of order a, where a is sufficiently large. We will show that a very unbalanced blow-up of C_k^{r-1} contains more copies of H than any r-partite graph.

We show that if k > |V(F)| and r divides k, then any blow-up of C_k^{r-1} is F-free. Indeed, a copy of F would not share any vertices with at least one of the blown-up classes, say with V_k . But the remaining graph is r-colorable (hence F-free), as shown by the following coloring. Let us color the vertices inside each blown-up class by the same color, and color the blown-up classes the following way. We go through the original C_k in a cyclic order, and color V_j by color j modulo r.

Observe that there is a unique *r*-coloring of any blow-up of P_k^{r-1} . In *H*, if k is not congruent to 1 modulo *r*, then the two *a*-sets corresponding to the end vertices of the original path have different color. Therefore, inside an *r*-partite *n*-vertex graph *G*, those sets are in different parts. It implies that there at most $n^{k-2} \left(\frac{n}{2}\right)^{2a}$ labeled copies of *H* in *G*.

Let G' be the blow-up of C_k^{r-1} where each vertex is replaced by $\lfloor \gamma n \rfloor$ vertices except one vertex is replaced by $n - (k-1)\lfloor \gamma n \rfloor$ vertices. Then the number of labeled copies of H in G' is at least $(\gamma n)^{k-2}(n - (k-1)\lfloor \gamma n \rfloor)^{2a} + o(n^{k-2+2a})$. For a given $\gamma < 1/2k$, we can pick a such that $\gamma^{k-2}(1 - (k-1)\gamma)^{2a} > \frac{1}{2^{2a}}$, completing the proof.

Let us turn to the proof of Theorem 1.3. We start with a lemma.

Lemma 4. Let F be a 3-chromatic graph with a color-critical vertex and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then F is the subgraph of a blow-up of C_{2k+1} where one of the blown-up classes has order 1.

Proof. Let v be a color-critical vertex of F and consider a blow-up G of C_{2k+1} with parts V_1, \ldots, V_{2k+1} in cyclic order, such that G contains F. Assume that $v \in V_{2k+1}$. We pick G such a way that V_{2k+1} is as small as possible.

Let us assume that there is another vertex $v' \in V_{2k+1}$. If v' is adjacent only to vertices in V_1 , then we can move v' to V_2 , thus $|V_{2k+1}|$ decreases, a contradiction. If v' is adjacent only to vertices in V_{2k} , then we can move v' to V_{2k-1} , a contradiction. Assume that v' has neighbors $v_1 \in V_1$ and $v_{2k} \in V_{2k}$. Let G' denote the bipartite graph we obtain from G by deleting V_{2k+1} , and let U denote the component containing v_1 in G'. Then we move every vertex of U from V_i to V_{2k+1-i} for every i. We repeat this for every remaining neighbor of v' in V_1 . At the end, v' has neighbors only in V_{2k} , thus we can move v' to V_{2k-1} , a contradiction.

Proposition 5. If F is a 3-chromatic graph with a color-critical vertex, then for any integers ℓ and $m \ge |V(F)|$ we have that $P_{2\ell}(m)$ is F-Turán-good.

We remark that the first result concerning F-Turán-good graphs when F does not have a color-critical edge is due to Gerbner and Palmer [14], who showed that C_4 is F_2 -Turán-good, where F_2 consists of two triangles sharing a vertex. Gerbner [8] constructed F-Turán-good graphs for every F with a color-critical vertex, but they were always complete $(\chi(F) - 1)$ -partite graphs. In particular, $K_{m,m} = P_2(m)$ is F-Turán-good. The above proposition gives the first examples of another kind.

Proof. We apply induction on ℓ , the base case $\ell = 1$ was mentioned above. Assume the statement holds for ℓ and prove it for $\ell + 1$. We count the copies of $P_{2\ell+2}(m)$ in an *n*-vertex *F*-free graph *G* the following way. First we pick a copy of $P_{2\ell}(m)$, the number of ways to pick them is maximized when G = T(n, 2) by induction. Then, among the remaining $n - 2\ell m$ vertices, we pick a copy of $P_2(m) = K_{m,m}$. The number of ways to pick it is maximized when there is a $T(n - 2\ell m, 2)$ on the remaining $n - 2\ell m$ vertices, which is achieved when G = T(n, 2).

We show that there are at most two ways to add the copy of $P_{2\ell}(m)$ to the copy of $P_{2\ell}(m)$ in G to create a copy of $P_{2\ell+2}(m)$. Indeed, we can do that if each vertex of a part of $K_{m,m}$ is adjacent to each vertex of one of the ends of $P_{2\ell}(m)$. It cannot happen with the same end of $P_{2\ell}(m)$ and both parts of $K_{m,m}$, as that would mean G contains $K_{m,m,1}$, which contains F, a contradiction. In T(n, 2), there are always two ways to add a copy of $P_{2\ell}(m)$ to a copy of $P_{2\ell}(m)$ in T(n, 2) to create a copy of $P_{2\ell+2}(m)$. Therefore, this third factor is also maximized by the Turán graph, completing the proof.

Let us denote by $P_{2k+2}(m, a, b)$ the following blow-up of P_{2k+2} . We replace the end vertices v_1 and v_{2k+2} by independent sets V_1 of order a and V_{2k+2} of order b, and we replace the middle vertices v_2, \ldots, v_{2k+1} by independent sets V_2, \ldots, V_{2k+1} of order m.

Proposition 6. Let F be a 3-chromatic graph and assume that F is contained in a blow-up of C_{2k+1} . If $a \ge b \ge m \ge |V(F)|$ and $a \le b + 1/2 + \sqrt{2b + 1/4}$, then $P_{2k+2}(m, a, b)$ is asymptotically *F*-Turán-good. Moreover, if *F* has a color-critical vertex and $a < b + 1/2 + \sqrt{2b + 1/4}$, then $P_{2k+2}(m, a, b)$ is *F*-Turán-good.

As we have mentioned, $K_{a,b}$ is weakly K_3 -Turán-good by a result of Győri, Pach and Simonovits [16], thus only an optimization is needed here. Brown and Sidorenko [2] did this optimization in a slightly different context, and obtained that T(n, 2) is asymptotically optimal if $b \ge {a-b \choose 2}$. Ma and Qiu [19] showed that $K_{a,b}$ is K_3 -Turán-good if and only if $a < b + 1/2 + \sqrt{2b + 1/4}$.

Proof. Let H_0 denote the subgraph of $P_{2k+2}(m, a, b)$ obtained by deleting V_1 and V_{2k+2} , i.e., $H_0 = P_{2k}(m)$. Then H_0 has a complete matching, thus is K_3 -Turán-good by a result of Győri, Pach and Simonovits [16]. This implies that H_0 is asymptotically *F*-Turán-good by a result of Gerbner and Palmer [13] mentioned in the introduction. We show that the number of ways to extend H_0 to $P_{2k+2}(m, a, b)$ in *G* is also asymptotically at most the number of ways to extend H_0 to $P_{2k+2}(m, a, b)$ in the Turán graph.

Let us pick a copy of H_0 in G and let U denote the set of its vertices. Observe that there are at most m-1 vertices in G that are not in H_0 and are connected to all the vertices of $U \cap V_2$ and $U \cap V_{2k+1}$, as otherwise G contains $C_{2k+1}(m)$, which contains F. Let x be the number of common neighbors of the vertices of V_2 that are not in U. Then there are at most n-2km+m-1-x common neighbors of the vertices of V_{2k+1} that are not in U. We need to pick a common neighbors of the *m* vertices in V_2 , and *b* common neighbors of the *m* vertices in V_{2k+1} , or the other way around, thus there are at most $\binom{x}{a}\binom{n-2km+m-1-x}{b} + \binom{x}{b}\binom{n-2km+m-1-x}{a}$ ways to extend H_0 to $P_{2k+2}(m, a, b)$ if $a \neq b$, and half of this if a = b. Observe that this is equal to the number of copies of $K_{a,b}$ in $K_{x,n-2km+m-1-x}$, which is at most the number of copies of $K_{a,b}$ in T(n-2km+m-1,2), using the theorem of Brown and Sidorenko we mentioned. It is easy to see that $\mathcal{N}(K_{a,b}, T(n-2km+m-1,2)) =$ $(1 + o(1))\mathcal{N}(K_{a,b}, T(n - 2km, 2))$. Therefore, the Turán graph asymptotically maximizes the number of ways to extend H_0 to H, completing the proof for general F. Let us assume now that F has a color-critical vertex v. Recall that by Lemma 4, F is contained in a blow-up of C_{2k+1} where one blown-up class has order 1. We use this analogously to the argument before. We pick a copy of H_0 in G, then there are no vertices that are not in H_0 and are connected to all the vertices in V_2 and V_{2k+1} . Let x be the number of common neighbors of V_2 that are not in the copy of H. We need to pick a common neighbors of the mvertices in V_2 , and b common neighbors of the m vertices in V_{2k+1} , or the other way around. There are at most $\binom{x}{a}\binom{n-2km}{b} + \binom{x}{b}\binom{n-2km}{a}$ ways to do this if $a \neq b$ and half of this if a = b. Observe that this is equal to the number of copies of $K_{a,b}$ in $K_{x,n-2km-x}$, which is at most the number of copies of $K_{a,b}$ in T(n-2km,2)by the theorem of Ma and Qiu we mentioned. This shows that the Turán graph maximizes the number of ways to extend H_0 to H, completing the proof.

Now we are ready to prove Theorem 1.3 which we restate here for convenience.

Theorem. Let F be a 3-chromatic graph and let C_{2k+1} be the longest odd cycle such that a blow-up of it contains F. Then there is an asymptotically F-Turángood graph H that is not asymptotically weakly $C_{2\ell+1}$ -Turán-good for any $\ell < k$. Furthermore, if F has a color-critical vertex, then H is F-Turán-good.

Proof. Proposition 6 presented an asymptotically F-Turán-good graph $H = P_{2k+2}(m, a, b)$ (which is an F-Turán-good graph if F has a color-critical vertex). We will show that H is not asymptotically weakly $C_{2\ell+1}$ -Turán-good. Observe that the a vertices and the b vertices corresponding to the end vertices of the original P_{2k+2} must be in different parts of any bipartite graph. On the other hand, they can be in the same blown-up class of C_{2k+1} , which are $C_{2\ell+1}$ -free graphs. From here, the calculation to show that H is not asymptotically weakly F'-Turán-good is the same as the calculation in the proof of Proposition 1, thus we omit the details.

3. Concluding Remarks

We have shown examples of graphs that are not asymptotically weakly F-Turángood, for any non-bipartite graph F. Our examples, just like all the earlier examples for the case F is a clique, are built on the idea of forcing two large sets of vertices into different part in every $(\chi(F) - 1)$ -partite graph. In particular, the examples themselves are $(\chi(F) - 1)$ -chromatic. We do not know any examples of graphs that are not F-Turán-good and have chromatic number less than $\chi(F) - 1$ in the case F has a color-critical edge.

Some of our results deal with asymptotically weakly F-Turán-good graphs, while other results deal with weakly F-Turán-good graphs. In the case F has a color-critical edge, we are not aware of any graph that is asymptotically weakly F-Turán-good but not weakly F-Turán-good. In fact, the situation is much worse. In each of the cases when we can show that H is not weakly F-Turán-good, i.e., we can find an F-free n-vertex graph that contains more copies of H than any $(\chi(F) - 1)$ -partite n-vertex graph, then we do not actually know ex(n, H, F). There are constructions with more copies of H than any $(\chi(F) - 1)$ -partite nvertex graph, but we do not know whether they are extremal.

Let us recall that Theorem 1.3 contains the assumption that C_{2k+1} is the longest odd cycle such that the blow-up of it contains F. Then F cannot contain any odd cycle of length less than 2k+1. Then blow-ups of F do not contain such cycles either. Thus Theorem 1.3 is a special case of the following conjecture.

Conjecture 3.1. For given r + 1-chromatic graphs F, F', there exists a graph H that is asymptotically weakly F-Turán-good but not asymptotically weakly F'-

Turán-good if and only F' is not a subgraph of any blow-up of F. Furthermore, if F has a color-critical vertex, then we can find H that is F-Turán-good.

Note that one of the directions easily follows from known results. Let H be asymptotically weakly F-Turán-good. A result of Alon and Shikhelman [1] states that if F' is a subgraph of a blow-up of F, then $ex(n, F, F') = o(n^{|V(F)|})$. Combined with the removal lemma [5], this shows that we can delete the copies of Ffrom any F'-free n-vertex graph G by deleting $o(n^2)$ edges, thus $o(n^{|V(H)|})$ copies of H. The resulting graph is F-free, thus contains at most $(1 + o(1))\mathcal{N}(H, T)$ copies of H for some complete r-partite graph T, thus G also contains at most $(1 + o(1))\mathcal{N}(H, T)$ copies of H, showing that H is asymptotically weakly F'-Turán-good.

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