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CONNECTED COALITIONS IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph, and define a connected coalition as a pair of disjoint vertex sets U_1 and U_2 such that $U_1 \cup U_2$ forms a connected dominating set, but neither U_1 nor U_2 individually forms a connected dominating set. A connected coalition partition of G is a partition $\Phi = \{U_1, U_2, \ldots, U_k\}$ of the vertices such that each set $U_i \in \Phi$ either consists of only a single vertex with degree $n-1$, or forms a connected coalition with another set $U_j \in \Phi$ that is not a connected dominating set. The connected coalition number $CC(G)$ is defined as the largest possible size of a connected coalition partition for G. The objective of this study is to initiate an examination into the notion of connected coalitions in graphs and present essential findings. More precisely, we provide a thorough characterization of all graphs possessing a connected coalition partition. Moreover, we establish that, for any graph G with order n, a minimum degree of 1, and no full vertex, the condition $CC(G) < n$ holds. In addition, we prove that any tree T achieves

$CC(T) = 2$. Lastly, we propose two polynomial-time algorithms that determine whether a given connected graph G of order n satisfies $CC(G) = n$ or $CC(G) = n - 1$.

Keywords: coalition, coalition partition, polynomial-time algorithm, corona product.

2020 Mathematics Subject Classification: 05C69, 05C85.

1. Introduction

Consider a simple graph $G = (V, E)$, where V represents the vertex set and E the edge set. The *open neighborhood* of a vertex v in V is defined as the set of adjacent vertices, denoted by $N(v)$, while the *closed neighborhood*, represented by $N[v]$, includes v itself. The *degree* of a vertex v, denoted by $deg(v)$, is the number of vertices in its open neighborhood. A vertex v in G is referred to as a *pendant vertex* if it has only one adjacent vertex in its open neighborhood, which is called its *support vertex*, represented by $spo(v)$. An edge is considered *pendant* if one of its endpoints is a pendant vertex. In a tree T, a vertex with degree one is called a *leaf*, while the adjacent vertex is referred to as the *support vertex*. The set of leaves in T is denoted by $L(T)$, and its size by $l(T)$. A vertex with degree $n-1$ in a graph G with n vertices is called a *full* or *universal vertex*, while a vertex with degree 0 is an *isolate*. The minimum and maximum degrees of G are represented by $\delta(G)$ and $\Delta(G)$, respectively. A subset $V_i \subseteq V$ is called *one-element* if it has only one element, and *more-element* if it has more than one element.

Within the context of a graph G, a subset S of vertices is classified as a *dominating set* if for each vertex in the complement set $V \setminus S$, there exists at least one vertex in S that is adjacent to it. A dominating set S is designated as connected if the subgraph induced by the vertices in S is connected. The minimum size of a connected dominating set in G is denoted as the *connected domination number*, represented by $\gamma_c(G)$ [6]. This concept was initially introduced in 1979 by Sampathkumar and Walikar, with the guidance of Hedetniemi [14]. Connected domination has been of substantial interest in recent years, particularly in the Wireless Sensor Networks domain, due to its crucial applications [11–13]. For further in-depth information on this subject, readers are referred to relevant literature.

A *domatic partition* refers to a partition of a vertices of the graph into dominating sets. Similarly, a *connected domatic partition* is a partition into connected dominating sets. The *domatic number* $d(G)$ of a graph G is the size of a domatic partition with the largest size. On the other hand, the maximum size of a connected domatic partition is denoted by $d_c(G)$ and is known as the *connected domatic number*. The domatic number was first introduced by Cockayne and

Hedetniemi in their seminal paper [5]. Zelinka later introduced the concept of connected domatic number in [17]. Further information on these concepts can be found in authoritative sources such as [7, 15–17].

In the seminal work [8], the concept of coalitions and coalition partitions was first introduced and subsequently explored in the field of graph theory, as evidenced by notable contributions such as [2, 4, 9, 10]. Although initially based on general graph properties, the focus of these investigations has primarily been on their relationship to the fundamental concept of dominating sets. A *coalition* in a graph G is defined as the union of two disjoint sets of vertices U_1 and U_2, such that neither U_1 nor U_2 individually dominates G, but their union does. The sets U_1 and U_2 are referred to as *coalition partners*. On the other hand, a *coalition partition* $\Phi = \{U_1, U_2, \ldots, U_k\}$ of G is a partition of its vertex set where each U_i in Φ is either a one-element dominating set of G or a non-dominating set that forms a coalition with another non-dominating set $U_j \in \Phi$. The *coalition number* $C(G)$ of a graph G is the maximum number of sets that can be present in a coalition partition of G.

For every coalition partition Φ of a graph G, there is a corresponding graph called the *coalition graph* of G with respect to Φ, denoted as $CG(G, \Phi)$. The vertices of this graph correspond one-to-one with the sets of Φ, and two vertices are adjacent in $CG(G, \Phi)$ if and only if their corresponding sets form a coalition. The study of coalition graphs, particularly for paths, cycles, and trees, was conducted in [9]. The concept of total coalition was introduced and explored in [1], while the coalition parameter for cubic graphs of order at most 10 was investigated in [2].

According to Section 4 of reference [8], there are open problems and areas for future research which suggest exploring connected coalition partition. Inspired by this, our focus is on the examination of connected coalitions and their partitions.

In Section 2, we define and discuss some properties of connected coalitions. In Section 3, we determine the connected coalition number of graphs with at least one pendant edge. Furthermore, we consider the connected coalition of trees in Section 4. In Section 5, we present two polynomial-time algorithms that take a graph G with n vertices and determine whether $CC(G) = n$ or $CC(G) = n - 1$. Finally, we present some open problems for future works in Section 6.

2. Introduction to Connected Coalition

Definition 1 (Connected coalition)**.** For a graph G with vertex set V, two sets $U_1, U_2 \subseteq V$ form a connected coalition, if neither U_1 nor U_2 is a connected dominating set but $U_1 \cup U_2$ is a connected dominating set in G.

Definition 2 (Connected coalition partition)**.** Let G be a graph. A connected coalition partition of G is a partition $\Phi = \{U_1, U_2, \ldots, U_k\}$ of the vertex set of G

such that each set U_i in Φ is either a connected dominating set comprising a single vertex with degree $n-1$, or forms a connected coalition with another set $U_j \in \Phi$ that is not a connected dominating set. We define the connected coalition number of G, denoted by $CC(G)$, as the maximum cardinality of a connected coalition partition in G. A partition of G into $CC(G)$ connected sets is referred to as a $CC(G)$-partition.

Considering the graph G should be connected, we have the following trivial observation.

Observation 3. *For any disconnected graph G of order $n \geq 2$, we have $CC(G) = 0$.*

We can use the following result to describe the graphs G for which $CC(G)=1$.

Lemma 4. *For any graph G, $CC(G) = 1$ if and only if $G = K_1$.*

Proof. If $CC(G) = 1$, then $\{V\}$ is a $CC(G)$-partition. By Definition 2, we must have $|V| = 1$. So, it is clear that $G = K_1$. Conversely, if $G = K_1$, clearly we have $CC(G) = 1$. ■

Now, we prove the following lemma.

Lemma 5. *If G is a connected graph of order $n > 1$ with no full vertex, then $CC(G) \geq 2d_c(G)$.*

Proof. Let G be a graph that does not have any full vertices and has a connected domatic partition $\mathcal{C} = \{C_1, C_2, \dots, C_s\}$, where s is equal to the connected domatic number $d_c(G)$. Since G does not contain any universal vertices, all the sets C_i are not one-element sets. We assume that the sets $\{C_1, C_2, \dots, C_{s-1}\}$ are minimal connected dominating sets, and if any set C_i is not minimal and connected, we find a subset $C'_i \subseteq C_i$ that is a minimal connected dominating set, and add the remaining vertices to C_s. It is important to note that if we partition a more-element, minimal connected dominating set into two non-empty sets, we create two non-connected dominating sets that form a connected coalition when combined. As a result, we divide each more-element set C_i into two sets, namely $C_{i,1}$ and $C_{i,2}$, that together form a connected coalition. This gives us a new partition $\mathcal{C}' = \{C_{1,1}, C_{1,2}, C_{2,1}, C_{2,2}, \dots, C_{s-1,1}, C_{s-1,2}\}$ consisting of non-connected dominating sets that pair with another non-connected dominating set in $\mathcal{C}'$ to form a coalition. We then examine the connected dominating set C_s. If C_s is a minimal connected dominating set, we also divide it into two non-connected dominating sets, add these sets to $\mathcal{C}'$, and obtain a connected coalition partition of cardinality at least $2s$. Then, since $s = d_c(G)$, $CC(G) \geq 2d_c(G)$.

If C_s is not a minimal connected dominating set, we aim to identify a subset $C'_s \subseteq C_s$ that fulfills this condition. We then partition C'_s into two non-connected dominating sets, which together form a connected coalition. Afterwards, we define C''_s as the complement of C'_s in C_s, and we append $C'_{s,1}$ and $C'_{s,2}$ to the collection $\mathcal{C}'$. If C''_s can merge with any non-connected dominating set to form a connected coalition, we can obtain a connected coalition partition of G with a cardinality of at least at least $2s+1$ by adding C''_s to $\mathcal{C}'$. Then, $CC(G) \geq 2d_c(G)+1$. However, if C''_s cannot form a connected coalition with any set in $\mathcal{C}'$, we eliminate $C'_{s,2}$ from $\mathcal{C}'$ and add the set $C'_{s,2} \cup C''_s$ to $\mathcal{C}'$. This leads to a connected coalition partition of a cardinality at least $2s$. Then, $CC(G) \geq 2d_c(G)$.

Due to the above arguments, we easily conclude that $CC(G) \geq 2d_c(G)$. This completes the proof. ∎

It is remarkable that for any graph G, $d_c(G) \geq 1$. Based on Lemma 5, we have the following result.

Theorem 6. *If G is a connected graph of order $n > 1$ with no full vertex, then $CC(G) \geq 2$.*

By Theorem 6, we immediately conclude the following result.

Corollary 7. *If G is a connected graph with $CC(G) < 2$, then G contains at least one full vertex.*

The primary objective of this research is to examine the plausibility of a connected c-partition's existence in a given graph G. To achieve this goal, we introduce a family of graphs, denoted as $\mathcal{F}$, as follows. For any two graphs G and H, their join $G+H$ is defined as a graph formed by linking every vertex of G to every vertex of H using disjoint copies of G and H. The subsequent definition is stated below.

Definition 8. A family $\mathcal{F}$ of graphs is constructed as follows.

Step 1. We add all disconnected graphs G of order $n \geq 2$ into $\mathcal{F}$.

Step 2. For any graph $G \in \mathcal{F}$, we add $G + K_1$ into $\mathcal{F}$.

It is remarkable that the family $\mathcal{F}$ contains both many disconnected graphs and many connected graphs. For instance, Figure 1 shows the connected graph $(2K_2 + K_1) + K_1$ belonging to $\mathcal{F}$. As another example, consider the friendship graphs F_n which is a graph with $2n+1$ vertices and $3n$ edges, formed by the join of $K_1 + nK_2$. Based on Definition 8, we have $F_n \in \mathcal{F}$.

Now, we prove the following Lemma.

Lemma 9. *For any graph G, if $G \in \mathcal{F}$, then $CC(G) = 0$.*

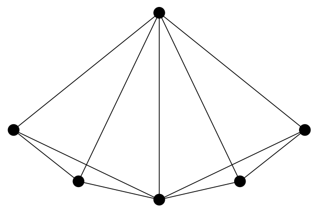

Figure 1. The connected graph $(2K_2 + K_1) + K_1$ belonging to $\mathcal{F}$.

Proof. Using induction on the number of full vertices of G, we prove that $CC(G) = 0$. For the base step, if G has no full vertex, then since $G \in \mathcal{F}$, G is a disconnected graph of order $n \geq 2$. By Observation 3, we have $CC(G) = 0$. As the inductive hypothesis, suppose that for any graph $H \in \mathcal{F}$ such that the number of its full vertices is less than the number of full vertices of G, it holds that $CC(H) = 0$. For induction step, suppose that $G = H + K_1$, where $H \in \mathcal{F}$. Let u be the vertex of K_1. By induction hypothesis, we have $CC(H) = 0$. Now, if $CC(G) \neq 0$, then by applying Lemma 4 since $G \neq K_1$, it follows that $CC(G) \geq 2$. Since u is the full vertex of G, the set $\{u\}$ belongs to any $CC(G)$-partition Φ. Now, by removing $\{u\}$ of Φ, we obtain a connected coalition partition for H with $CC(H) \geq 1$, which is a contradiction. Thus, $CC(G) = 0$. ■

The next theorem shows a necessary and sufficient condition for the existence of a connected coalition partition of a graph G.

Theorem 10. *For any graph G, $CC(G) = 0$ if and only if $G \in \mathcal{F}$.*

Proof. Based on Corollary 7, under the assumption that $CC(G) < 2$, it follows that G must possess at least one universal vertex. Now, if $G \in \mathcal{F}$, by Lemma 9, we have $CC(G) = 0$, Conversely, suppose that $CC(G) = 0$. To prove $G \in \mathcal{F}$, we use the induction on the number of full vertices of G. For the base step, we assume that G contains exactly one full vertex u. Now, consider the graph $G' = G[V \backslash \{u\}]$. If G' is a connected graph, then, since G' does not have any isolated vertices, by applying Theorem 6, we can conclude that $CC(G') \geq 2$. Hence, using a $CC(G')$-partition and the one-element set $\{u\}$, we can construct a $CC(G)$-partition with $CC(G) \geq 3$, which is a contradiction. Hence, G' must be disconnected. Hence, by the definition of $\mathcal{F}$, we have $G \in \mathcal{F}$.

For induction hypothesis, we assume that if G' is a connected graph with $CC(G') = 0$ such that the number of its full vertices is less than G, then $G' \in \mathcal{F}$.

Now, we prove the induction step. Let u be the full vertex of G. Consider the graph $G' = G[V \backslash \{u\}]$. Now, we have two cases.

Case 1. G' is disconnected. Then, by the definition of $\mathcal{F}$, $G \in \mathcal{F}$.

Case 2. G' is connected. Because $CC(G) = 0$, we can conclude that $CC(G') = 0$ as well. Otherwise, following similar reasoning to the previous arguments, we could use a $CC(G')$-partition and the one-element set u to create a $CC(G)$-partition with $CC(G) \geq 3$, which would contradict the initial assumption. Now, since G' is connected and has $CC(G') = 0$, the application of Corollary 7 implies that G' must have at least one full vertex. It is clear that the number of full vertices of G' is less than the number of full vertices of G. Then, by induction hypothesis, $G' \in \mathcal{F}$. Hence, by the definition of $\mathcal{F}$, we can see $G \in \mathcal{F}$. This completes the proof. ∎

By Theorem 10 and Lemma 4, we conclude the following result.

Corollary 11. *If $G \notin \mathcal{F}$ is a connected graph, then $1 \leq CC(G) \leq n$.*

As previously demonstrated, the lower bound of Corollary 11 is achieved by the graph K_1, while the upper bound is reached by complete graphs K_n and complete bipartite graphs $K_{r,s}$ where $2 \leq r \leq s$ and $r + s = n$.

Corollary 12. *If G is a connected graph of order n and with k universal vertices such that $G \notin \mathcal{F}$ and $G \neq K_n$, then $CC(G) \geq k + 2 \geq 3$.*

3. Graphs with Pendant Edges

In the this section, we will discuss about the connected coalition number of graphs with $\delta(G) = 1$. First we have the following results.

Lemma 13. *For a connected graph G, assume that Φ is a $CC(G)$-partition. Let x be a pendant vertex and $y = spo(x)$. Let $A \in \Phi$ with $y \in A$. If any two sets $C, D \in \Phi$ form a connected coalition, then $C = A$ or $D = A$.*

Proof. Suppose on contrary that $C \neq A$ and $D \neq A$. Since C and D form a connected coalition, then $C \cup D$ is a connected dominating set. If $C \cup D$ has no neighbor of x, then x is not dominated by $C \cup D$. Hence, C and D do not form a connected coalition, which is a contradiction. So, $C = A$ or $D = A$. ∎

Lemma 14. *Let $G = (V, E)$ be a connected graph with no full vertex and with $\delta(G) = 1$ and $CC(G) \geq 3$. Let x be a pendant vertex and $y = spo(x)$. Let Φ be a $CC(G)$-partition. If $A \in \Phi$ with $y \in A$, then for any pendant vertex $w \in V$, it holds that $spo(w) \in A$.*

Proof. Assuming G is a graph and w is a pendant vertex of G, let z denote its pendant neighbor. If z is an element of set A, then the task is complete. Otherwise, we proceed by selecting a set $B \in \Phi$ that contains z. Suppose x is not a member of set A. Then, there must exist a set $X \in \Phi$ that contains x. According to Lemma 13, sets X and A form a connected coalition. However, since z belongs to B and in accordance with Lemma 13, sets X and A cannot form a connected coalition, leading to a contradiction. Consequently, we deduce that x is an element of A. Additionally, Lemma 13 asserts that if sets C and D form a connected coalition, and z is an element of set B, then either C is equivalent to B or D is equivalent to B. Furthermore, Lemma 13 indicates that since y belongs to set A, either C is equivalent to A or D is equivalent to A if sets C and D form a connected coalition. By extension, it follows that $CC(G)$ cannot be greater than or equal to three, and in fact, it must be two. This is a contradiction since we initially assumed that $CC(G) \geq 3$. Thus, it must be the case that z is a member of set A. ■

We recall the definition of corona product of graphs. The corona product of two graphs H_1 and H_2, denoted by $H_1 \circ H_2$, is defined as the graph obtained by taking one copy of H_1 and $|V(H_1)|$ copies of H_2 and joining the i-th vertex of H_1 to every vertex in the i-th copy of H_2. In the following, we compute the connected coalition number of connected graphs of the form $H \circ K_1$. To aid our discussion, we state and prove the following theorem.

Theorem 15. *If G is a connected graph of the form $H \circ K_1$, then $CC(G) = 2$.*

Proof. Let Φ be a $CC(G)$-partition of G. By Theorem 6, we have $CC(G) \geq 2$. It suffices to prove that $CC(G) \leq 2$. Suppose on the contrary that $CC(G) \geq 3$. Let x be a pendant vertex and $y = spo(x)$. Let Φ be a $CC(G)$-partition. Let $A \in \Phi$ with $y \in A$. Then, by Lemma 14, for any pendant vertex $w \in V$, it holds that $spo(w) \in A$. Hence, all vertices v of G with $deg(v) \geq 2$ lie in A. Then, A is a connected dominating set of G, which is a contradiction. Hence, $CC(G) \leq 2$, and since $CC(G) \geq 2$, we have $CC(G) = 2$. ■

We close this section with the following result.

Theorem 16. *If G is a connected graph of order n with $\delta(G) = 1$ and with no universal vertex, then $CC(G) < n$.*

Proof. Assume Φ is a $CC(G)$-partition of G into connected coalitions and v is a leaf vertex of G with neighbor u. Suppose that $CC(G) = n$, which implies that Φ is a partition where every set $U_i \in \Phi$ for $1 \leq i \leq n$ contains only one vertex. Thus, Φ must include $\{v\}$ and $\{u\}$. If $\{v\}$ and $\{u\}$ form a connected coalition, then u is adjacent to all other vertices, making it a universal vertex, which contradicts the assumption that G has no universal vertex. It is impossible for $\{v\}$ and $\{u\}$ to not form a connected coalition, as u is the only neighbor of v. Therefore, $CC(G) < n$. ■

4. Trees

In this section, we determine the connected coalition number for trees. First we have the following theorem.

Theorem 17. *For any tree T of order n with no full vertex, we have $CC(T) = 2$.*

Proof. By Theorem 6, we have $CC(T) \geq 2$. It suffices to prove that $CC(T) \leq 2$. Suppose on the contrary that $CC(T) \geq 3$. Now we may assume that a and b are two vertices of T such that a is a leaf and b is a support vertex of a. Let Φ be a $CC(T)$-partition, and suppose that $V_1 \in \Phi$ with $b \in V_1$. Since $CC(T) \geq 3$, without loss of generality, assume that $V_2, V_3 \in \Phi$ are two distinct sets such that $V_2 \neq V_1$ and $V_3 \neq V_1$. By Lemma 13, each of V_2 and V_3 form a connected coalition with V_1, however, V_2 and V_3 do not form a connected coalition. Now, we consider the following cases.

Case 1. $T[V_1]$ is connected. By Definition 2, V_1 is not a dominating set. Then, there exists a vertex $u \notin V_1$ with no neighbors in V_1. Hence, if any set $A \in \Phi$ is in connected coalition with V_1, then $A \cap N[u] \neq \emptyset$. Assume, without loss of generality, that $u \in V_3$. Let $u_1 \in N(u)$ and assume, without loss of generality, that $u_1 \in V_2$. Since $T[V_1 \cup V_2]$ is connected, there is a path P_{u_1x} between u_1 and x for some vertex $x \in V_1$. Note that all vertices on P_{u_1x} are inside $V_1 \cup V_2$. Also, since $T[V_1 \cup V_3]$ is connected, there is a path Q_{yu} between y and u for some vertex $y \in V_1$. Note that all vertices on Q_{yu} are inside $V_1 \cup V_3$ (see Figure 2). Since $T[V_1]$ is connected, there is a path R_{xy} between x and y inside V_1. Since $u_1 \in N(u)$, there is a cycle $uu_1P_{u_1x}R_{xy}Q_{yu}$ in T, which is a contradiction.

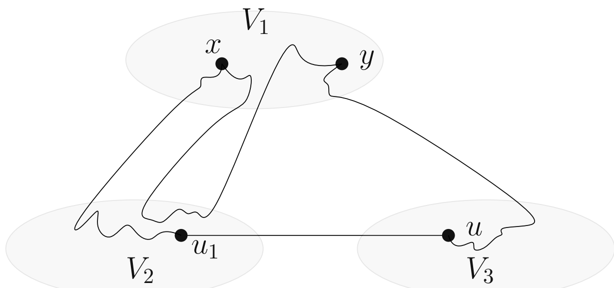

Figure 2. The case that $T[V_1]$ is connected.

Case 2. $T[V_1]$ is not connected. Assume that $x, y \in V_1$ such that there is no path between them in $G[V_1]$. Since $G[V_1 \cup V_2]$ is connected, there is a path $P_{x,y}$ between x and y that lies in $G[V_1 \cup V_2]$. Also, since $G[V_1 \cup V_3]$ is connected, there is a path $Q_{x,y}$ between x and y that lies in $G[V_1 \cup V_3]$. Hence, it is clear that there are two paths between x and y in T, which is a contradiction. ■

$$\mathcal{E} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Figure 3. The edge-dominated matrix $\mathcal{E}$ for C_6.

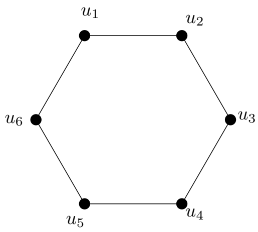

Figure 4. C_6.

As an immediate consequence of Theorem 17, we have the following result for the paths.

Corollary 18. *For any path P_n of order n, where $n \neq 3$, we have $CC(P_n) = 2$.*

5. Graphs G with $CC(G) = n$ and $CC(G) = n - 1$

For a given graph G, computing $CC(G)$ seems to be an NP-hard problem, and therefore, computing $CC(G)$ for a class of graphs in polynomial time seems to be interesting. In this section, we present two polynomial-time algorithms that for a given connected graph G of order n determine whether it holds $CC(G) = n$ or $CC(G) = n - 1$. For the sake of simplicity, we assume that G has no full vertex.

5.1. Graphs with $CC(G) = n$

Let e_{pq} be an edge of G with two end vertices p and q. A vertex $x \in V$ is called *edge-dominated* by the edge e_{pq}, if x is adjacent to p or q. Now, we define the *edge-domination matrix* $\mathcal{E}_{m\times n}$ with m rows and n columns on the graph G, where m is number of the edges of G. The definition is as follows.

$$\mathcal{E}(e_{pq}, x) = \begin{cases} 1 & \text{if the vertex } x \text{ is edge-dominated by the edge } e_{pq}, \\ 0 & \text{otherwise.} \end{cases}$$

For example, the matrix $\mathcal{E}$ depicted in Figure 3 is the edge-dominated matrix of the graph C_6 depicted in Figure 4.

For the graph G, the incidence matrix $\mathbf{VE}_{m\times n}$ with m rows and n columns is defined as follows.

$$\mathbf{VE}(e, x) = \begin{cases} 1 & \text{if the vertex } x \text{ is incident to the edge } e, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we prove the following theorem.

Theorem 19. *For any connected graph G of order n and with no full vertex, $CC(G) = n$ if and only if for any vertex $x \in V$, there is an edge e with $\mathbf{VE}(e, x) = 1$ such that*

$$\sum_{v \in V} \mathcal{E}(e, v) = n.$$

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the vertices of G. Suppose that $CC(G) = n$. Then, there is a $CC(G)$-partition $\Phi = \{\{v_1\}, \ldots, \{v_n\}\}$ in such a way that any set $\{v_i\}$ has a connected coalition partner $\{v_j\}$ where $j \neq i$. Let $x \in V$ be an arbitrary vertex. Since $\{x\} \in \Phi$, there is a vertex $u \in V$ such that $\{u\} \in \Phi$ forms a connected coalition with $\{x\}$. By the definition, $\{x, u\}$ is a connected dominating set. Then, $e = (x, u)$ is an edge of G, and all vertices of G is dominated by $\{x, u\}$. Therefore, $\mathbf{VE}(e, x) = 1$ and $\mathcal{E}(e, v) = 1$ for any vertex $v \in V$. Hence, $\sum_{v \in V} \mathcal{E}(e, v) = n$. The proof of the converse, is straightforward. ■

Now, we will describe the algorithm. The algorithm first computes the matrices $\mathcal{E}$ and $\mathbf{VE}$ for the graph G. Then, for all vertices $x \in V$ the following operations is applied. For all edges e, the algorithm checks whether $\mathbf{VE}(e, x) = 1$. If $\mathbf{VE}(e, x) = 1$ and $\sum_{v \in V} \mathcal{E}(e, v) = n$, then we consider $f = 1$, and the algorithm checks another vertex of V. In the algorithm, we used two variables f and $flag$ to determine which vertices satisfy the conditions of Theorem 19. For more details, see Algorithm 1.

Now, we compute the time complexity of algorithm CHECKCCG$n(G, V, E)$. It is clear that the computations of the matrices $\mathcal{E}$ and $\mathbf{VE}$ take $O(mn)$ times. Then, since we have three **foreach** loops, the algorithm implies that the overall running time of three loops is $O(n^2m)$. Hence, the overall running time of the algorithm is $O(n^2m) + O(nm) = O(n^2m)$. Since $m \in O(n^2)$, then the time complexity of the algorithm is $O(n^4)$. Hence, we have the following theorem.

Theorem 20. *The worst-case time complexity of algorithm* CHECKCCG$n(G, V, E)$ *is* $O(n^4)$.

5.2. Graphs with $CC(G) = n - 1$

Let $p = (a, b, c)$ be a triple of vertices a, b and c. A vertex $x \in V$ is called *three-vertex-dominated* by p, if x is dominated by $\{a, b, c\}$. Now, we define three-vertex-dominated matrix $\mathcal{H}$ as follows.

$$\mathcal{H}(\{a, b, c\}, x) = \begin{cases} 1 & \text{if the vertex } x \text{ is dominated by } \{a, b, c\}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we prove the following theorem.

Theorem 21. *For any connected graph G of order n and with no full vertex, $CC(G) = n - 1$ if and only if there are two vertices $u, v \in V$ such that for any vertex $x \in V \backslash \{u, v\}$,*

Algorithm 1: CheckCCG$n(G, V, E)$

```
input: A connected graph G with no full vertex, and with vertex set V and the edge set E.
Computes the martices ℰ and VE;
f = 0;
s := 0;
foreach x ∈ V do
    foreach e ∈ E do
        if VE(e, x) == 1 then
            foreach v ∈ V do
                s = s + ℰ(e, v);
            end
            if s == n then
                f = 1;
                break;
            end
        end
    end
    if f = 0 then
        flag := 0;
        break;
    end
    else
        flag := 1;
        f = 0;
    end
end
if flag = 1 then
    return yes;
end
else
    return no;
end
```

1. *there is an edge* $e = (p, q)$ *with* $p, q \notin \{u, v\}$ *and* $\mathbf{VE}(e, x) = 1$ *such that* $\sum_{v \in V} \mathcal{E}(e, v) = n$, *or*
2. $G[x, u, v]$ *is connected and* $\sum_{w \in V} \mathcal{H}(\{x, u, v\}, w) = n$,

and there is a vertex $y \in V \backslash \{u, v\}$ *such that* $G[y, u, v]$ *is connected and*

$$\sum_{w \in V} \mathcal{H}(\{y, u, v\}, w) = n.$$

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the vertices of G. Suppose that $CC(G) = n - 1$. Then, there is a $CC(G)$-partition $\Phi = \{\{w_1\}, \ldots, \{w_{n-2}\}, \{u, v\}\}$. Let $C \in \Phi$ be an arbitrary set. Suppose that $C = \{x\}$ is one-element. If C forms a connected coalition with a set $\{a\} \in \Phi$, then by the definition, $\{x, a\}$ is a connected dominating set. Then, $e = (x, a)$ is an edge of G, and all vertices of G is dominated by $\{x, a\}$. Therefore, $\mathbf{VE}(e, x) = 1$ and $\mathcal{E}(e, v) = 1$ for any vertex $v \in V$. Hence, $\sum_{v \in V} \mathcal{E}(e, v) = n$. Now, if C forms a connected coalition with the set $\{u, v\} \in \Phi$, then by the definition, $\{x, u, v\}$ is a connected dominating set.

Therefore, $G[x,u,v]$ is connected, and $\sum_{w\in V}\mathcal{H}(\{x,u,v\},w)=n$.

Now, suppose that $C=\{u,v\}$. Then, there is a vertex $\{y\}\in\Phi$ that forms a connected coalition with C. Therefore, by the definition, $\{y,u,v\}$ is a connected dominating set. Then, $G[y,u,v]$ is connected, and $\sum_{w\in V}\mathcal{H}(\{y,u,v\},w)=n$. The proof of the converse, is straightforward. ∎

Now, our second algorithm depicted in Algorithm 2. The algorithm is based on Theorem 21.

Algorithm 2: CHECKCCG2(G,V,E)

```
input: A connected graph G with no full vertex, and with vertex set V and the edge set E.
Computes the martices H, E, and VE;
f = 0;
s := 0;
foreach u ∈ V do
    foreach v ∈ V with u ≠ v do
        foreach x ∈ V\{u,v} do
            foreach e ∈ E with VE(e,x) = 1 do
                if G[{x,u,v}] is connected and ∑_{w∈V} H({x,u,v},w) = n, or
                  ∑_{w∈V} E(e,w) = n then
                    f = 1;
                    break;
                end
            end
            if f = 0 then
                flag := 0;
                break;
            end
            else
                flag = 1;
                f = 0;
            end
        end
        if flag = 1 then
            return yes;
        end
    end
    if flag = 1 then
        return yes;
    end
end
if flag = 1 then
    return yes;
end
else
    return no;
end
```

It is not hard to see that algorithm CHECKCCG2(G,V,E) has four **foreach** loops and two summations. Then, the overall running time of the algorithm is $O(n^6)$. Now, we have the following result.

Theorem 22. *The worst-case time complexity of algorithm* CheckCCG2(G, V, E) *is* $O(n^6)$.

6. Conclusion and Future Works

This paper presents the notion of connected coalition in graphs and investigates several properties of the connected coalition number. We characterized all graphs that have a connected coalition partition. We have shown that for any graph G with $\delta(G) = 1$ and with no full vertex, $CC(G) \leq n - 1$. Also we proved that for any tree T, $CC(T) = 2$. Finally, we have presented two polynomial-time algorithms that take a graph G with n vertices and determine whether $CC(G) = n$ or $CC(G) = n - 1$.

There are many open problems in study of the connected coalition number of a graph that we state and close the paper with some of them.

1. What is the connected coalition number of graph operations, such as corona, Cartesian, join, lexicographic, and so on?
2. What is the connected coalition number of natural and fractional powers of a graph (see e.g. [3])?
3. What are the effects on $CC(G)$ when G is modified by operations on vertex and edge of G?
4. Similar to the coalition graph of G, it is natural to define and study the connected coalition graph of G for connected coalition partition Φ, which can be denoted by $CCG(G, \Phi)$, and is defined as follows. Corresponding to any connected coalition partition $\Phi = \{V_1, V_2, \ldots, V_k\}$ in a graph G, a *connected coalition graph* $CCG(G, \Phi)$ is associated in which there is a one-to-one correspondence between the vertices of $CCG(G, \Phi)$ and the sets $V_1, V_2, \ldots, V_k$ of Φ, and two vertices of $CCG(G, \Phi)$ are adjacent if and only if their corresponding sets in Φ form a connected coalition.

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