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THE TURÁN NUMBER OF THREE DISJOINT PATHS

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Abstract

The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. Let P_k denote the path on k vertices and let $\bigcup_{i=1}^{m} P_{k_i}$ denote the disjoint union of P_{k_i} for $1 \leq i \leq m$; in particular, write $\bigcup_{i=1}^{m} P_{k_i} = mP_k$ if $k_i = k$ for all $1 \leq i \leq m$. Yuan and Zhang determined $\mathrm{ex}(n, \bigcup_{i=1}^{m} P_{k_i})$ for all integers n if at most one of $k_1, \ldots, k_m$ is odd. Much less is known for all integers n if at least two of $k_1, \ldots, k_m$ are odd. Partial results such as $\mathrm{ex}(n, mP_3)$, $\mathrm{ex}(n, P_3 \cup P_{2\ell+1})$, $(n, 2P_5)$, $\mathrm{ex}(n, 2P_7)$ and $\mathrm{ex}(n, 3P_5)$ have been established by several researchers. In this paper, we develop new functions and determine $\mathrm{ex}(n, 3P_7)$ and $\mathrm{ex}(n, 2P_3 \cup P_{2\ell+1})$ for all integers n. We also characterize all the extremal graphs. Both results contribute to a conjecture of Yuan and Zhang.

Keywords: Turán number, disjoint paths, extremal graph.

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1. Introduction

The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in an n-vertex graph that does not contain H as a subgraph. The study of Turán numbers plays a central role in extremal graph theory. One of the best known results in this area is the Erdős-Gallai Theorem [6] about the path P_k on k vertices.

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Theorem 1 (Erdős and Gallai [6]). *Let $n \geq k \geq 2$ and G be a graph with n vertices. If G does not contain P_k as a subgraph, then $e(G) \leq (k-2)n/2$ with equality if and only if n is divisible by $k-1$ and G is isomorphic to the disjoint union of K_{k-1}.*

Faudree and Schelp [7] further extended this theorem and determined $\mathrm{ex}(n,P_k)$ for all integers n and k, and characterized all the extremal graphs. Given two graphs G_1 and G_2, denote by $G_1 \cup G_2$ the disjoint union of G_1 and G_2, and by kG_1 the disjoint union of k copies of G_1. Let K_n denote the complete graph on n vertices.

Theorem 2 (Faudree and Schelp [7]). *Let $n = t(k-1) + r$ for some integers $t \geq 0$ and $0 \leq r \leq k-2$. Then*

$$\mathrm{ex}(n, P_k) = t\binom{k-1}{2} + \binom{r}{2}.$$

Moreover, the extremal graphs are characterized.

Remark. If k is odd, then the extremal graph in Theorem 2 is isomorphic to $tK_{k-1} \cup K_r$.

Let $\mathrm{ex}_{\mathrm{con}}(n, H)$ denote the maximum number of edges in an n-vertex connected graph that does not contain H as a subgraph. Kopylov [13] and Balister, Győri, Lehel and Schelp [1] determined $\mathrm{ex}_{\mathrm{con}}(n, P_k)$ and characterized all the extremal graphs. For two graphs G_1 and G_2, let $G_1 + G_2$ be the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2. Denote by $\overline{G}$ the complement graph of G.

Theorem 3 (Kopylov [13]; Balister, Győri, Lehel and Schelp [1]). *Let $n \geq k \geq 4$. Then*

$$\begin{aligned}\mathrm{ex}_{\mathrm{con}}(n, P_k) = \max\Bigg\{&\binom{k-2}{2} + (n-k+2), \binom{\lfloor \frac{k}{2} \rfloor - 1}{2} \\ &+ \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)\left(n - \left\lfloor \frac{k}{2} \right\rfloor + 1\right) + c_k\Bigg\},\end{aligned}$$

where $c_k = 1$ if k is odd and $c_k = 0$ if k is even. Moreover, the extremal graphs are characterized.

Remark. If k is odd, then the extremal graph in Theorem 3 is isomorphic to $K_1 + \left(K_{k-3} \cup \overline{K_{n-k+2}}\right)$ for $n \leq (5k-7)/4$ and $K_{\lfloor k/2 \rfloor - 1} + \left(K_2 \cup \overline{K_{n-\lfloor (k+1)/2 \rfloor}}\right)$ for $n \geq (5k-7)/4$.

Let $\bigcup_{i=1}^{m} P_{k_i}$ denote the disjoint union of P_{k_i} for $1 \leq i \leq m$; in particular, write $\bigcup_{i=1}^{m} P_{k_i} = mP_k$ if $k_i = k$ for all $1 \leq i \leq m$. Erdős and Gallai [6] determined $\mathrm{ex}(n, mP_2)$ for all integers n and m. Bushaw and Kettle [3] determined

$\mathrm{ex}(n, mP_k)$ for sufficiently large n. Later, Lidický, Liu and Palmer [15] determined $\mathrm{ex}(n, \bigcup_{i=1}^{m} P_{k_i})$ for sufficiently large n. However, for small n, much less is known for $\mathrm{ex}(n, \bigcup_{i=1}^{m} P_{k_i})$. Gorgol [11] first determined $\mathrm{ex}(n, 2P_3)$ and $\mathrm{ex}(n, 3P_3)$ for all integers n. Since then, Campos and Lopes [5], independently, Yuan and Zhang [17], determined $\mathrm{ex}(n, mP_3)$ for all integers n and m. Recently, Yuan and Zhang [18] made a big step and determined $\mathrm{ex}(n, \bigcup_{i=1}^{m} P_{k_i})$ for all integers n when at most one of $k_1, \ldots, k_m$ is odd.

Definition (Yuan and Zhang [18])**.** Let $n \geq m \geq \ell \geq 2$ be three integers and $n = (m-1) + t(\ell - 1) + r$ with $t \geq 0$ and $0 \leq r < \ell - 1$. Define

$$\phi\,(n, m, \ell) = \binom{m-1}{2} + t\binom{\ell-1}{2} + \binom{r}{2}$$

and

$$\psi(n, m) = \binom{\lfloor \frac{m}{2} \rfloor - 1}{2} + \left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)\left(n - \left\lfloor \frac{m}{2} \right\rfloor + 1\right).$$

We mention that $\mathrm{ex}(n, H) = \binom{n}{2}$ for any H on more than n vertices and K_n is the unique extremal graph. It follows that we may assume $n \geq |V(H)|$ when it comes to $\mathrm{ex}(n, H)$.

Theorem 4 (Yuan and Zhang [18])**.** *Let $k_1 \geq \cdots \geq k_m \geq 2$ and $n \geq \sum_{i=1}^{m} k_i$. If at most one of $k_1, \ldots, k_m$ is odd, then*

$$\begin{aligned} &\mathrm{ex}\left(n, \bigcup_{i=1}^{m} P_{k_i}\right) \\ &= \max\left\{\phi(n, k_1, k_1), \phi(n, k_1 + k_2, k_2), \ldots, \phi(n, \sum_{i=1}^{m} k_i, k_m), \psi\left(n, \sum_{i=1}^{m} k_i\right)\right\}. \end{aligned}$$

Moreover, if $k_1, \ldots, k_m$ are all even, then the extremal graphs are characterized.

If at least two of $k_1, \ldots, k_m$ are odd, then there are only few partial results for all integers n. Bielak and Kieliszek [2], independently, Yuan and Zhang [18] determined $\mathrm{ex}(n, 2P_5)$; Lan, Qin and Shi [14] determined $\mathrm{ex}(n, 2P_7)$; and recently, Feng and Hu [8] determined $\mathrm{ex}(n, 3P_5)$. Let M_n be the graph consisting of $\lfloor n/2 \rfloor$ independent edges and one possible isolated vertex. Yuan and Zhang [18] also established the following theorem.

Theorem 5 (Yuan and Zhang [18])**.** *Let $n \geq 2\ell + 4$ with $\ell \geq 2$. Then*

$$\mathrm{ex}(n, P_3 \cup P_{2\ell+1}) = \max\{\phi(n, 2\ell+1, 2\ell+1), \phi(n, 2\ell+4, 3), \psi(n, 2\ell+3) + 1\}.$$

Moreover, the extremal graph is isomorphic to either $tK_{2\ell} \cup K_r$, $K_{2\ell+3} \cup M_{n-2\ell-3}$ or $K_\ell + \left(K_2 \cup \overline{K_{n-\ell-2}}\right)$, where $n = t\,(2\ell) + r$ with $0 \leq r \leq 2\ell - 1$.

The authors also suggested a general conjecture on disjoint union of paths as follows.

Conjecture 6 (Yuan and Zhang [18])**.** *Let $k_1 \geq \cdots \geq k_m \geq 2$ and $n \geq \sum_{i=1}^m k_i$. If at least one of $k_1, \ldots, k_m$ is not three, then*

$$\mathrm{ex}\left(n, \bigcup_{i=1}^m P_{k_i}\right)$$
$$= \max\left\{\phi(n,k_1,k_1), \phi(n,k_1+k_2,k_2), \ldots, \phi(n,\sum_{i=1}^m k_i, k_m), \psi\left(n, 2\sum_{i=1}^m \left\lfloor \frac{k_i}{2} \right\rfloor\right) + c\right\},$$

where $c = 1$ if all of $k_1, \ldots, k_m$ are odd, and $c = 0$ otherwise.

For more results related to the Turán problem of graphs and hypergraphs on paths, forests or cycles, we refer the reader to [4, 9, 10, 12, 16]. In this paper, we consider the Turán number of three disjoint paths on odd number of vertices and prove the following results, which partially confirm the conjecture of Yuan and Zhang.

Theorem 7. *Let $n \geq 21$ be an integer. Then*

$$\mathrm{ex}(n, 3P_7) = \begin{cases} \phi(n, 21, 7), & \text{if } n \leq 31, \\ 8n - 35, & \text{if } n \geq 32. \end{cases}$$

Moreover, the extremal graph is isomorphic to $K_{20} \cup K_{n-20}$ for $21 \leq n \leq 26$, $K_{20} \cup K_6 \cup K_{n-26}$ for $27 \leq n \leq 31$ and $K_8 + \left(K_2 \cup \overline{K_{n-10}}\right)$ for $n \geq 32$.

Theorem 8. *Let $n \geq 2\ell + 7$ with $\ell \geq 2$. Then*

$$\mathrm{ex}(n, 2P_3 \cup P_{2\ell+1}) = \max\{\phi(n, 2\ell+1, 2\ell+1), \phi(n, 2\ell+7, 3), \psi(n, 2\ell+4) + 1\}.$$

Moreover, the extremal graph is isomorphic to one of the graphs $tK_{2\ell} \cup K_r$, $K_{2\ell+6} \cup M_{n-2\ell-6}$ and $K_{\ell+1} + \left(K_2 \cup \overline{K_{n-\ell-3}}\right)$, where $n = t\,(2\ell) + r$ with $0 \leq r \leq 2\ell - 1$.

This paper is organized as follows. In the remainder of this section, we describe notation and terminology used in our proof. In Section 2, we develop new functions and prove Theorem 7. In Section 3, we give a proof of Theorem 8.

Notation. Let G be a simple graph. For a subset $S \subset V(G)$, let $G[S]$ be the subgraph of G induced by S and $G-S = G[V(G)\setminus S]$. For each $v \in V(G)$, denote $N_S(v)$ the set of neighbors of v contained in S and $d_S(v)$ the number of edges between v and $S \setminus \{v\}$. When $S = V(G)$, we simplify $N_{V(G)}(v)$ and $d_{V(G)}(v)$ as $N_G(v)$ and $d_G(v)$, respectively. For any $S, T \subseteq V(G)$, let $e_G(S)$ denote the number of edges of G with both ends in S, and $e_G(S,T)$ denote the number of edges of G with one end in S and the other end in T. We will drop the reference to G when there is no danger of confusion. For $x, y \in V(G)$, we call that x *hits* y if $xy \in E(G)$ and x *misses* y if $xy \notin E(G)$. A graph is *H-free* if it contains no copy of H as a subgraph. Usually, we denote C_k a cycle of length k and write $[k] := \{1, \dots, k\}$.

2. Turán Number of the Graph $3P_7$

2.1. Longest cycles in graphs

In this subsection, we mainly present some useful lemmas about longest cycles in graphs. First, we shall use the following upper bound, proved by Erdős and Gallai [6], on the maximum number of edges in graphs without long cycles.

Lemma 9 (Erdős and Gallai [6]). *Let G be a graph with n vertices. If G does not contain any cycles of length more than ℓ, then $e(G) \leq \ell(n-1)/2$, where the equality holds if and only if $n-1$ is divisible by $\ell - 1$.*

We also give a simple proposition on longest cycles, which is used frequently throughout Section 2. The proof details are omitted.

Proposition 10. *Let G be a graph and $C_\ell = x_0x_1 \cdots x_{\ell-1}$ be a longest cycle in G. For any $x \in V(G)\backslash V(C_\ell)$ and $0 \leq i \neq j \leq \ell - 1$, if $xx_i, xx_j \in E(G)$, then $|i-j| > 1$ and $x_{i+1}x_{j+1}, x_{i-1}x_{j-1} \notin E(G)$, where we take all the subscripts modulo ℓ.*

Let G be a graph and $C_\ell = x_0x_1 \cdots x_{\ell-1}$ ($\ell > 3$) be a longest cycle in G. Define

$$V^* = \{v \in V(G)\backslash V(C_\ell) : d_{C_\ell}(v) \geq 1\}.$$

Choose some vertex $f \in V^*$ such that $d_{C_\ell}(f)$ is maximum. Suppose that $N_{C_\ell}(f) = \{x_{i_1}, \dots, x_{i_s}\}$ with $0 \leq i_1 < \cdots < i_s \leq \ell - 1$. For any $j \in [s]$, let t_j denote the number of vertices in C_ℓ between x_{i_j} and $x_{i_{j+1}}$, and $s_1 = |\{j \in [s] : t_j = 1\}|$. We establish a useful lemma on the number of edges in $G[V(C_\ell)]$ in terms of s and s_1.

Lemma 11. *Let $C_\ell = x_0x_1 \cdots x_{\ell-1}$ be a longest cycle in G. Then*

$$e(G[V(C_\ell)]) \leq f_\ell(s, s_1) := \binom{\ell}{2} - \frac{-s_1^2 + (2s - \ell + 1)s_1 + (s-1)\ell}{2}.$$

Proof. For any $j \in [s]$ with $t_j = 1$, it follows from Proposition 10 that

$$d_{C_\ell}(x_{i_j+1}) \leq (\ell - 1) - s - (s - s_1 - 1) = \ell - 2s + s_1.$$

For any $j \in [s]$ with $t_j \geq 2$, we claim that

$$d_{C_\ell}(x_{i_j+1}) + d_{C_\ell}(x_{i_{j+1}-1}) \leq \ell - 1 + t_j.$$

This follows from the fact that at most one of $x_p x_{i_j+1}$ and $x_{p+1}x_{i_{j+1}-1}$ belongs to $E(G)$ for any $p \geq i_{j+1}$ or $p \leq i_j - 1$. Otherwise, $f x_{i_j} x_{i_j-1} \cdots x_{p+1} x_{i_{j+1}-1} x_{i_{j+1}-2} \cdots x_{i_j+1} x_p \cdots x_{i_{j+1}} f$ is a $C_{\ell+1}$ in G, a contradiction. Note that $\sum_{t_j \geq 2} t_j = \ell - s - s_1$. Thus, we have

$$\begin{aligned} e(G[V(C_\ell)]) &\leq \binom{\ell}{2} - \frac{1}{2} \sum_{t_j=1} \left(\ell - 1 - d_{C_\ell}(x_{i_j+1})\right) \\ &\quad - \frac{1}{2} \sum_{t_j \geq 2} \left(2\ell - 2 - d_{C_\ell}(x_{i_j+1}) - d_{C_\ell}(x_{i_{j+1}-1})\right) \\ &\leq \binom{\ell}{2} - \frac{s_1 \left(2s - s_1 - 1\right) + \sum_{t_j \geq 2} \left(\ell - 1 - t_j\right)}{2} = f_\ell \left(s, s_1\right). \end{aligned}$$

This completes the proof. ■

Remark. For any fixed integer $s \geq 0$, $f_\ell(s, s_1)$ is increasing with respect to s_1. Note that $s_1 \leq s$ and $2s \leq \ell$. If $2s = \ell$, then $f_\ell(s, s_1) = f_\ell(s, s) = \frac{s(3s-1)}{2}$ as $s_1 = s$; if $2s < \ell$, then $f_\ell(s, s_1) \leq f_\ell(s, s-1) = \binom{\ell}{2} - \frac{(s-1)(s+2)}{2}$ as $s_1 \leq s - 1$ in this situation. The exact values of $f_\ell(s, s_1)$ for $15 \leq \ell \leq 19$ can be found in Appendix A.

Lemma 12. *Let $C_\ell = x_0 x_1 \cdots x_{\ell-1}$ be a longest cycle in G and $P_k = f_1 f_2 \cdots f_k$ be any path in $G - C_\ell$ such that $\ell > 2k$ and $N_{C_\ell}(f_1) \neq \emptyset$. Then $d_{C_\ell}(f_i) \leq \lfloor \ell/2 \rfloor + 1 - i$ for each $i \in [k]$. In particular, if $d_{C_\ell}(f_1) = \lfloor \ell/2 \rfloor$, then $d_{C_\ell}(f_i) = 0$ for each $i \geq 2$.*

Proof. Suppose that $x_0 f_1 \in E(G)$. Since C_ℓ is a longest cycle in G, we have $x_j f_i \notin E(G)$ for any $i \in [k]$ and $j \in [i] \cup \{\ell - i, \ldots, \ell - 1\}$. It follows that $d_{C_\ell}(f_i) \leq \lceil (\ell - 2i - 1)/2 \rceil + 1 = \lfloor \ell/2 \rfloor + 1 - i$ as $\ell > 2k$. The second part is clearly true by the maximality of ℓ. Thus, we complete the proof. ■

2.2. Proof of Theorem 7

In this subsection, we give a proof of Theorem 7. Throughout this proof, we may assume that $n \geq 21$. Let $G^* = K_{20} \cup K_{n-20}$ for $21 \leq n \leq 26$, $G^* = K_{20} \cup K_6 \cup K_{n-26}$ for $27 \leq n \leq 31$ and $G^* = K_8 + \left(K_2 \cup \overline{K_{n-10}}\right)$ for $n \geq 32$ (see Table 1). It is easy to see that G^* is $3P_7$-free.

Let G be any $3P_7$-free graph with n vertices and $e(G) \geq e(G^*)$. Let $C_\ell = x_0 x_1 \cdots x_{\ell-1}$ be a longest cycle in G and write $F = G - C_\ell$. It follows easily that $\ell \leq 20$ as G is $3P_7$-free. We give all the possible values of ℓ for different n as shown in Table 1; otherwise, $e(G) < e(G^*)$ by Lemma 9, a contradiction.

n	G^*	$e(G^*)$	ℓ
21	$K_{20} \cup K_1$	190	20
22	$K_{20} \cup K_2$	191	19,20
23	$K_{20} \cup K_3$	193	18,19,20
24	$K_{20} \cup K_4$	196	18,19,20
25	$K_{20} \cup K_5$	200	17,18,19,20
26	$K_{20} \cup K_6$	205	17,18,19,20
27	$K_{20} \cup K_6 \cup K_1$	205	16,17,18,19,20
28	$K_{20} \cup K_6 \cup K_2$	206	16,17,18,19,20
29	$K_{20} \cup K_6 \cup K_3$	208	15,16,17,18,19,20
30	$K_{20} \cup K_6 \cup K_4$	211	15,16,17,18,19,20
31	$K_{20} \cup K_6 \cup K_5$	215	15,16,17,18,19,20
≥ 32	$K_8 + \left(K_2 \cup \overline{K_{n-10}}\right)$	$8n - 35$	15,16,17,18,19,20

Table 1. G^* and all the possible values of ℓ for different n.

For any couple of (a, b) as shown in Table 2, it is easy to check that $an + b < e(G^*)$ for all integers $n \geq 21$. In what follows, we show that G is isomorphic to G^*, or we have $e(G) \leq an + b$ for some couple of (a, b) as shown in Table 2. This leads to a contradiction. We proceed our proof by showing the following series of claims.

a	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	7.5	8
b	135	120	105	90	75	59	44	28	12	-4	-20	-36

Table 2. a and b.

Claim 13. *G is isomorphic to G^* for $\ell = 20$ and $n \leq 31$; Otherwise, $15 \leq \ell \leq 19$ and $\max_{v \in V(F)} d_{C_\ell}(v) \geq 3$.*

Proof. If $\ell = 20$, then $d_{C_\ell}(v) = 0$ for each $v \in V(F)$. It follows that $e(G) \leq \binom{20}{2} + \mathrm{ex}(n-20, P_7)$. Since $\mathrm{ex}(n-20, P_7) \leq 5(n-20)/2$ by Theorem 1, this is a contradiction for $n \geq 32$. For $21 \leq n \leq 31$, we have $e(G) = e(G^*) = \binom{20}{2} + \mathrm{ex}(n-20, P_7)$, implying that G is isomorphic to G^*, as desired.

If $d_{C_\ell}(v) \leq 2$ for any $v \in V(F)$, then $e(G) \leq \binom{\ell}{2} + 2(n-\ell) + 5(n-\ell)/2 = 4.5n + \ell(\ell-10)/2$. It follows easily that $e(G) \leq 4.5n + \ell(\ell-10)/2 \leq 4.5n + 72 < e(G^*)$ for all $15 \leq \ell \leq 18$ by Table 2, a contradiction. Thus, it suffices to check the case $\ell = 19$. Since G is $3P_7$-free, we know that the set I consisting of all the vertices that have neighbours in C_{19} is an independent set in F. This yields that $e(G) \leq \binom{19}{2} + 2|I| + 5(n - |I| - 19)/2 \leq 2.5n + 123.5 < e(G^*)$ by Table 2, a contradiction. $\square$

Suppose that $\hat{P}_k = f_1 \cdots f_k$ is a longest path in F such that $N_{C_\ell}(f_1) \neq \emptyset$. Clearly, we have $k + \ell \leq 20$ as G is $3P_7$-free. Without loss of generality, we may assume that $x_0 f_1 \in E(G)$. In the following proofs, we will often use the maximality of ℓ, k and the fact that G is $3P_7$-free.

Claim 14. *If $k = 1$, then $e(G) \leq e(G^*)$. The equality holds if and only if G is isomorphic to G^* for $n \geq 32$.*

Proof. Let f be a vertex in F such that $d_{C_\ell}(f)$ is maximum. Since $k = 1$, we may choose $f_1 = f$. Let V^*, s and s_1 be defined as those in Subsection 2.1. By the maximality of k, we know that V^* is an independent set in G and $e(V^*, V(F)\backslash V^*) = 0$. It follows that

$$\begin{aligned} e(G) &\leq e(G[V(C_\ell)]) + \sum_{v \in V^*} e(v, V(C_\ell)) + \mathrm{ex}(n - \ell - |V^*|, P_7) \\ &\leq f_\ell(s, s_1) + s|V^*| + 5(n - \ell - |V^*|)/2 \leq f_\ell(s, s_1) + s(n - \ell), \qquad (1) \end{aligned}$$

where the last inequality follows from the fact $s \geq 3$ by Claim 13. Recall that $f_\ell(s, s_1) \leq f_\ell(s, s-1)$ for $2s < \ell$. Clearly, $f_\ell(s, s-1) + s(n - \ell)$ is monotonically increasing with respect to s for $s \leq n - \ell$. Thus, by Lemma 11, we have

$$\begin{aligned} e(G) &\leq f_\ell(s, s-1) + s(n - \ell) \\ &\leq f_\ell(\lfloor \ell/2 \rfloor, \lfloor \ell/2 \rfloor - 1) + \lfloor \ell/2 \rfloor (n - \ell) \qquad (2) \\ &\leq f_\ell(\lfloor \ell/2 \rfloor, \lfloor \ell/2 \rfloor) + \lfloor \ell/2 \rfloor (n - \ell). \qquad (3) \end{aligned}$$

We use (2) if $2s < \ell$, otherwise we use (3) in the coming inequalities. Therefore, we obtain the following: (i) $e(G) \leq 7n - 27$ for $\ell = 15$ and $e(G) \leq 8n - 36$ for $\ell = 16$, (ii) $e(G) \leq 7n - 10$ for $\ell = 17$ and $s \leq 7$, (iii) $e(G) \leq 5n + 49$ for $\ell = 18$ and $s \leq 5$, and (iv) $e(G) \leq 3n + 109$ for $\ell = 19$ and $s \leq 3$. In particular, (v) $e(G) \leq 4n + 86$ for $\ell = 19$ and $s = 4$; (vi) $e(G) \leq 6n + 25$ for $\ell = 18$ and $s = 6$. Thus, we have $e(G) < e(G^*)$ for any of the above cases. In what follows, we consider the remaining cases when $17 \leq \ell \leq 19$.

Case 1. $\ell = 19$ and $5 \leq s \leq 9$. Note that $N_{C_{19}}(v) \subseteq \{x_0, x_2, x_3, x_5, x_7, x_9, x_{10}, x_{12}, x_{14}, x_{16}, x_{17}\}$ for any $v \in V^*\backslash\{f\}$ as $x_0 f \in E(G)$ and G is $3P_7$-free. This implies that $d_{C_{19}}(v) \leq 8$ for any $v \in V^*\backslash\{f\}$ as at most one of x_i and x_{i+1} belongs to $N_{C_{19}}(v)$. Recall that $N_{C_\ell}(f) = \{x_{i_1}, \ldots, x_{i_s}\}$ with $0 \leq i_1 < \cdots < i_s \leq \ell - 1$, and t_j denotes the number of vertices in C_ℓ between x_{i_j} and $x_{i_{j+1}}$ for any $j \in [s]$. If $t_{j-1} > 1$ and $t_{j+1} > 1$ for any $j \in [s]$ with $t_j = 1$, then $s_1 \leq \lfloor s/2 \rfloor$. Thus

$$\begin{aligned} e(G) &\leq f_{19}(s, s_1) + s + \min\{8, s\}(|V^*| - 1) + 5(n - 19 - |V^*|)/2 \\ &\leq \max\{5n + 48, 6n + 23, 7n - 8.5, 8n - 35.5\} < e(G^*). \end{aligned}$$

As a consequence, there exists $j \in [s]$ such that $t_j = t_{j+1} = 1$. This means that $\{x_{i_j}, x_{i_j+2}, x_{i_j+4}\} \subseteq N_{C_{19}}(f)$. Without loss of generality, we might assume that $\{x_0, x_{17}, x_{15}\} \subseteq N_{C_{19}}(f)$, then we have $N_{C_{19}}(v) \subseteq \{x_3, x_5, x_{10}, x_{12}\}$ for any $v \in V^*\backslash\{f\}$ since G is $3P_7$-free. Thus, we conclude that $d := \max_{v\in V^*\backslash\{f\}} d_{C_\ell}(v) \leq 4$. It follows that

$$e(G) \leq f_{19}(s, s_1) + s + d(|V^*| - 1) + 5(n - 19 - |V^*|)/2.$$

Choose $s_1 = s-1$ and it is easy to check that $e(G) \leq 4n+84.5 < e(G^*)$ according to Table 2.

Case 2. $\ell = 18$ and $7 \leq s \leq 9$. If $s \in \{7, 8\}$, then we may assume that $s_1 \geq 6$ and $|\{v \in V^* : d_{C_{18}}(v) = s\}| \geq 2$; otherwise,

$$\begin{aligned} e(G) &\leq \max\{f_\ell(s, 5) + s(n - 18), f_\ell(s, s - 1) + s + (s - 1)(n - 19)\} \\ &\leq \max\{6n + 19, 7n - 7, 8n - 39\} < e(G^*). \end{aligned}$$

Let $d_{C_{18}}(f') = d_{C_{18}}(f) = s$ for some $f' \in V^*\backslash\{f\}$, $E_1 = \bigcup_{x_i,x_j\in N_{C_{18}}(f)}\{x_{i+1}x_{j+1}, x_{i-1}x_{j-1}\}$ and $E_2 = \bigcup_{x_i,x_j\in N_{C_{18}}(f')}\{x_{i+1}x_{j+1}, x_{i-1}x_{j-1}\}$. Clearly, $E_i \cap E(G) = \emptyset$ for any $i \in \{1, 2\}$ by Proposition 10. We first consider the case $s = 7$. We may assume that $N_{C_{18}}(f) = \{x_0, x_2, x_4, x_6, x_8, x_{10}, x_{12}\}$ as $s_1 \geq 6$. If $N_{C_{18}}(f') = N_{C_{18}}(f)$, then $N_{C_{18}}(v) = \{x_{15}\}$ for any $v \in V_1\backslash\{f, f'\}$ as G is $3P_7$-free. This means that $e(G) \leq f_{18}(7, 6) + 7 \times 2 + 5(n - 20)/2 = 2.5n + 90 < e(G^*)$. Suppose that $N_{C_{18}}(f') = \{x_{j_0}, x_{i_1}, \ldots, x_{i_6}\}$ with $x_{j_0} \notin N_{C_{18}}(f)$. Therefore, at least one of x_{j_0+1} and x_{j_0-1} is not associated with the vertex pairs in E_1, say x_{j_0+1}. This implies $x_{j_0+1}x_{j_i+1} \in E_2\backslash E_1$ for each $i \in [6]$. It follows that $|E_1| \geq 27$ and $|E_2\backslash E_1| \geq 6$. Thus $e(G) \leq \binom{18}{2} - |E_1| - |E_2\backslash E_1| + 7(n - 18) = 7n - 6 < e(G^*)$. Now, we consider the case $s = 8$. By symmetry, it is easy to check that $N_{C_{18}}(f)$ is one of the following five sets $M = X^* \cup \{x_8, x_{10}, x_{12}, x_{14}\}$, $A = X^* \cup \{x_8, x_{10}, x_{12}, x_{15}\}$, $B = X^* \cup \{x_8, x_{10}, x_{13}, x_{15}\}$, $C = X^* \cup \{x_8, x_{11}, x_{13}, x_{15}\}$, $D = X^* \cup \{x_9, x_{11}, x_{13}, x_{15}\}$, where $X^* = \{x_0, x_2, x_4, x_6\}$. If $N_{C_{18}}(f') = N_{C_{18}}(f) \in \{M, A, B\}$, then $V^*\backslash\{f, f'\} = \emptyset$. If $N_{C_{18}}(f') = N_{C_{18}}(f) \in \{C, D\}$, then $N_{C_{19}}(v) \subseteq \{x_2, x_4, x_{11}, x_{13}\}$ for any $v \in V^*\backslash\{f, f'\}$. Thus $e(G) \leq f_{18}(8, 7)+8\times2+4(n-20) = 4n+54 < e(G^*)$. Suppose that $N_{C_{18}}(f') = \{x_{j_0}, x_{i_1}, \ldots, x_{i_6}\}$ with $x_{j_0} \notin N_{C_{18}}(f)$. If $N_{C_{18}}(f) = M$, then $j_0 \neq 16$; otherwise, $N_{C_{18}}(v) \subseteq \{x_0, x_{16}, x_{14}\}$ for any $v \in V^*\backslash\{f, f'\}$ and $e(G) \leq \binom{18}{2} + 2s + 3(|V^*| - 2) + 5(n - 18 - |V^*|)/2 < e(G^*)$. Thus, we have $|E_1| \geq 35$ and $|E_2\backslash E_1| \geq 14$. If $N_{C_{18}}(f) \in \{A, B, C, D\}$, then $|E_1| \geq 41$ and $|E_2\backslash E_1| \geq 7$. In either case, we conclude that $e(G) \leq \binom{18}{2} - |E_1| - |E_2\backslash E_1| + 8(n - 18) = 8n - 39 < e(G^*)$.

If $s = 9$, then we may assume that $N_{C_{18}}(f) = \{x_0, x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_{16}\}$. Note that if $v_1x_i \in E(G)$ for some $v_1 \in V^*\backslash\{f\}$ and some $0 \leq i \leq \ell - 1$, then $N_{C_{18}}(v) \subseteq \{x_a, x_b, x_i\}$ with $i + 2 \equiv a \pmod{18}$ and $i - 2 \equiv b \pmod{18}$ for any $v \in V^*\backslash\{f, v_1\}$. Thus $e(G) \leq f_{18}(9, 9) + 9 \times 2 + 3(n - 20) = 3n + 75 < e(G^*)$.

Case 3. $\ell = 17$ and $s = 8$. It follows from (1) that $e(G) \leq f_{17}(8,7) + 8(n-17) = 8n-35$, implying $e(G) < e(G^*)$ for any $21 \leq n \leq 31$. Thus, we have $e(G) = e(G^*) = 8n-35$ with $n \geq 32$. This means that (i) $s_1 = 7$, (ii) $e(G[C_{17}]) = f_{17}(8,7) = 101$ and $|V^*| = n-17$, and (iii) $d_{C_{17}}(v) = 8$ for each $v \in V^*$. Since $x_0 f_1 \in E(G)$ and $d_{C_{17}}(f_1) = 8$, we may assume that $N_{C_{17}}(f_1) = \{x_0, x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}\}$. In view of the proof of Lemma 11, we conclude that $d_{C_{17}}(x_i) = \ell - 2s + s_1 = 8$ for any $i \in I_1 := \{1,3,5,7,9,11,13\}$ and $d_{C_{17}}(x_{15}) + d_{C_{17}}(x_{16}) = \ell - 1 + 2 = 18$. Note also that $x_i x_j \notin E(G)$, except for $x_{15}x_{16} \in E(G)$, for any $i, j \in I_1 \cup \{15,16\}$ by Proposition 10. This means that $N(x_{15}) = N_{C_{17}}(f_1) \cup \{x_{16}\}$, $N(x_{16}) = N_{C_{17}}(f_1) \cup \{x_{15}\}$ and $N(x_i) = N_{C_{17}}(f_1)$ for any $i \in I_1$. Let $I_2 := \{0,2,4,6,8,10,12,14\}$. Clearly,

$$\begin{aligned}\sum_{i \in I_2} d_{C_{17}}(x_i) &= 2e(G[C_{17}]) - \sum_{i \in I_1} d_{C_{17}}(x_i) - (d_{C_{17}}(x_{15}) + d_{C_{17}}(x_{16})) \\ &= 202 - 7 \times 8 - 18 = 128.\end{aligned}$$

This together with the fact that $d_{C_{17}}(x_i) \leq 16$ for each $i \in I_2$ implies that $N_{C_{17}}(x_i) = V(C_{17}) \backslash \{x_i\}$ for each $i \in I_2$. Thus, $N_{C_{17}}(f_1)$ forms a copy of K_8 in G and $G[V(C_{17})] = K_8 + (K_2 \cup \overline{K_7})$. Moreover, it is easy to check that $N_{C_{17}}(v) = N_{C_{17}}(f_1)$ for each $v \in V^*$; otherwise, we obtain a cycle of length larger than 17. Consequently, we obtain $G \cong G^*$ for $n \geq 32$, as desired. □

Let V_1 be the set of all vertices in $F_1 = F - \hat{P}_k$ that have neighbours in C_ℓ; let V_2 be the set of all vertices in $V(F_1) \backslash V_1$ that have neighbours in $V_1 \cup V(\hat{P}_k)$; let V_3 be the set of all vertices in $V(F_1) \backslash (V_1 \cup V_2)$ that have neighbours in V_2; and let $V_0 = V(F_1) \backslash (V_1 \cup V_2 \cup V_3)$. Thus

$$\begin{aligned} e(G) = {} & e(G[V(C_\ell)]) + \Big(e(V(\hat{P}_k), V(C_\ell)) + e(G[V(\hat{P}_k)])\Big) \\ & + \Big(e(V_1, V(C_\ell) \cup V(\hat{P}_k)) + e(G[V_1])\Big) + \Big(e(V_2, V_1 \cup V(\hat{P}_k)) + e(G[V_2])\Big) \\ & + \Big(e(V_3, V_2 \cup V_0) + e(G[V_3])\Big) + e(G[V_0]). \end{aligned} \tag{4}$$

By Lemma 12, we have

$$e(V(\hat{P}_k), V(C_\ell)) + e(G[V(\hat{P}_k)]) \leq \sum_{i=1}^{k} d_{C_\ell}(f_i) + \binom{k}{2} \leq k\lfloor \ell/2 \rfloor - 1.$$

Let $s := \max_{v \in V_1} d_{C_\ell}(v)$ and s_1 be defined as that in Subsection 2.1, where V_1 plays the role of V^* in the following proof. We may assume that $s \geq 2$; otherwise,

$$\begin{aligned}e(G) &= e(G[V(C_\ell)]) + e(V_1, V(C_\ell)) + e(V(\hat{P}_k), V(C_\ell)) + e(F)\\ &\le \binom{\ell}{2} + (n-\ell-k) + k\lfloor \ell/2\rfloor - \binom{k}{2} + \frac{5}{2}(n-\ell)\\ &\le \frac{7}{2}n + \frac{\ell+k-8}{2}\ell - \binom{k+1}{2},\end{aligned}$$

implying that $e(G) \le 7n/2 + 6\ell - 3 \le 7n/2 + 105 < e(G^*)$ as $\ell + k \le 20$ and $k \ge 2$, a contradiction.

Claim 15. *For any $k \ge 2$, we have $e(G) < e(G^*)$.*

Proof. We proceed our proof by considering the following several cases in terms of k. We mention that $2 \le s \le \lfloor \ell/2\rfloor$.

Case 1. $k = 2$ and $\ell \in \{15, 16, 17, 18\}$. By the maximality of k, we obtain that (i) $G[V_1]$ is P_3-free and $e(V_1, \{f_1, f_2\}) = 0$; (ii) V_2 is an independent set and $V_3 = \emptyset$; (iii) $d_{V_1\cup\{f_1\}}(v) \le 1$ and $vf_2 \notin E(G)$ for each $v \in V_2$ and $e(V_0, V(F)\backslash V_0) = 0$. This together with Lemma 12 (for $k = 2$) implies that $e(V_1, V(C_\ell \cup \hat{P}_k)) + e(G[V_1]) \le \min\{s + 1/2, \lfloor \ell/2\rfloor\}|V_1|$. Thus, by (4)

$$\begin{aligned}e(G) &\le f_\ell(s, s_1) + 2\lfloor \ell/2\rfloor - 1 + \min\{s + 1/2, \lfloor \ell/2\rfloor\}|V_1| + |V_2| + 5|V_0|/2\\ &\le f_\ell(s, s_1) + 2\lfloor \ell/2\rfloor - 1 + \min\{s + 1/2, \lfloor \ell/2\rfloor\}(n - \ell - 2).\end{aligned}\tag{5}$$

We first consider the case $\ell \le 17$. If $s = 8$, then $\ell \in \{16, 17\}$ and by the remark after Lemma 11,

$$\begin{aligned}e(G) &\le f_\ell(s, s_1) + 8(n-\ell) - 1\\ &\le \max\left\{\frac{s(3s-1)}{2} + 8(n-16) - 1, \binom{17}{2} - \frac{(s-1)(s+2)}{2} + 8(n-17) - 1\right\}\\ &= 8n - 36 < e(G^*).\end{aligned}$$

If $s \le 7$ by (5), then

$$e(G) \le \binom{\ell}{2} - \frac{(s-1)(s+2)}{2} + \ell - 1 + \left(s + \frac{1}{2}\right)(n - \ell - 2).$$

Note that $n \ge 25$ by Table 1. This implies that the last function is monotonically nondecreasing with respect to the integer $s \le 6$. Thus, $e(G) \le 6.5n + \ell^2/2 - 6\ell - 34 \le 6.5n < e(G^*)$ for $s \le 6$, and $e(G) \le 7.5n + \ell^2/2 - 7\ell - 43 \le 7.5n - 26 < e(G^*)$ for $s = 7$.

Now, we consider the case $\ell = 18$. If $s = 5$, then $e(G) \le 5.5n + 43 < e(G^*)$ by (5). If $s \le 4$, then

$$e(G) \le \binom{18}{2} - \frac{(s-1)(s+2)}{2} + 17 + \left(s + \frac{1}{2}\right)(n-20) \le 171 - \frac{3}{2}s + \left(s + \frac{1}{2}\right)(n-20)$$

as $s > 1$. Note that $n \geq 23$ by Table 1. This implies that the last function is monotonically increasing with respect to s. Thus, $e(G) \leq 4.5n+75 < e(G^*)$. It suffices to assume that $s \geq 6$. Let A_1 be the set of all isolated vertices in V_1 and $A_2 = V_1 \backslash A_1$. Since G is $3P_7$-free, we have $N_{C_\ell}(v) \subseteq \{x_0, x_2, x_4, x_7, x_9, x_{11}, x_{14}, x_{16}\}$ for any $v \in A_1$ and $N_{C_\ell}(v) \subseteq \{x_0, x_4, x_7, x_{11}, x_{14}\}$ for any $v \in A_2$. This implies that $s \leq 8$, $s_1 \leq s-2$ and $e(V_1, V(C_\ell \cup \hat{P}_k)) + e(G[V_1]) \leq s|V_1| - (s-5.5)|A_2| \leq s|V_1|$. Thus

$$\begin{aligned} e(G) &\leq f_\ell(s, s_1) + 2\lfloor \ell/2 \rfloor - 1 + s(n-\ell-2) \leq f(s, s-2) + 17 + s(n-20) \\ &= \binom{18}{2} - \frac{s^2+s+12}{2} + 17 + s(n-20). \end{aligned}$$

It follows that $e(G) \leq 7n-4 < e(G^*)$ for $s = 7$ and $e(G) \leq 6n+23 < e(G^*)$ for $s = 6$. In what follows, we check the case $s = 8$ more carefully. We aim to show that $d_{C_\ell}(f_1) + d_{C_\ell}(f_2) \leq 12$, implying that $e(G) \leq f_{18}(8,6) + 12 + 1 + 8(n-20) = 8n - 36 < e(G^*)$. This is clearly true if $d_{C_\ell}(f_1) = 9$ by Lemma 12 (for $k = 2$). Suppose that $d_{C_\ell}(f_1) \leq 8$. Note that there exists $v \in V_1$ $N_{C_\ell}(v) = \{x_0, x_2, x_4, x_7, x_9, x_{11}, x_{14}, x_{16}\}$ as $s = 8$. This together with the maximality of ℓ implies that $N_{C_\ell}(f_2) \subseteq \{x_0, x_4, x_5, x_7, x_9, x_{11}, x_{12}, x_{14}\}$ as $x_0 f_1 \in E(G)$, yielding that $d_{C_\ell}(f_2) \leq 6$. Note also that $d_{C_\ell}(f_1) \leq 6$ providing that $x_0 f_2 \in E(G)$ by the symmetry of f_1 and f_2, as desired. Thus, we may assume that and $x_0 \notin N_{C_\ell}(f_2)$ and $d_{C_\ell}(f_2) \leq 5$. Clearly, $d_{C_\ell}(f_1) + d_{C_\ell}(f_2) \leq 8 + 4 = 12$ if $d_{C_\ell}(f_2) \leq 4$. It suffices to check that $d_{C_\ell}(f_1) \leq 7$ if $d_{C_\ell}(f_2) = 5$. This is definitely true by the maximality of ℓ as $N_{C_\ell}(f_2) \subseteq \{x_4, x_5, x_7, x_9, x_{11}, x_{12}, x_{14}\}$, as required.

Case 2. $k = 3$ and $\ell \in \{15, 16, 17\}$. By the maximality of k, there is no P_4 in F_1 whose endpoints hit C_ℓ for any $\ell \in \{15, 16, 17\}$. Thus, we obtain that (i) both $G[V_1]$ is P_4-free and $G[V_2]$ is P_3-free, (ii) V_3 is an independent set in G, and (iii) $e(u, V_2) \leq 1$ for each $u \in V_3$ and $e(V_0, V_3) = 0$.

Let A_1 be the set of all isolated vertices in V_1 and $A_2 = V_1 \backslash A_1$. Note that $e(v, \hat{P}_k) \leq 1$ for each $v \in A_1$ and $e(A_2, \hat{P}_k) = 0$ by the maximality of k. Let $A_{11} = \{v \in A_1 : e(v, \hat{P}_k) = 1\}$ and $A_{12} = A_1 \backslash A_{11}$. Clearly, $e(V_1, V(\hat{P}_k)) = |A_{11}|$ and $e(G[V_1]) \leq |A_2|$ as $G[V_1]$ is P_4-free. Thus,

$$e(V_1, V(C_\ell) \cup V(\hat{P}_k)) + e(G[V_1]) \leq s|V_1| + |A_{11}| + |A_2|.$$

We also define $B_1 = \{u \in V_2 : d_{V_1}(u) \geq 3\}$. For any $u \in B_1$, it is easy to see that $N_{V_1}(u) \subset A_{12}$ and $N_{V_1}(u) \cap N_{V_1}(u') = \emptyset$ for any $u' \in B_1 \backslash \{u\}$; otherwise, we have a P_4 starting at V_1, a contradiction. Let $A'_{12} = \bigcup_{u \in B_1} N_{V_1}(u)$. It follows that $e(B_1, V_1) = \sum_{u \in B_1} d_{V_1}(u) = |A'_{12}| \leq |A_{12}|$. Note also that $e(u, \hat{P}_k) \leq 1$ for each $v \in V_2$. Define $B_0 = \{u \in V_2 : e(u, \hat{P}_k) = 1\}$. Clearly, $e(V_2, V(\hat{P}_k)) = |B_0|$. By the maximality of k, we deduce that $e(B_0, V_1) = 0$. It follows that $e(V_2, V_1) \leq$

$2(|V_2| - |B_1| - |B_0|) + e(B_1, V_1) \leq 2|V_2| - |B_0| + |A'_{12}|$. Thus, we conclude that

$e(V_2, V_1 \cup V(\hat{P}_k)) + e(G[V_2]) \leq (2|V_2| - |B_0| + |A'_{12}|) + |B_0| + |V_2|/2 \leq 5|V_2|/2 + |A'_{12}|.$

Note that $|A_{11}| + |A_2| + |A'_{12}| \leq |V_1|$. It follows from (4) that

$$\begin{aligned} e(G) &\leq f_\ell(s, s_1) + 3\lfloor \ell/2 \rfloor - 1 + (s|V_1| + |A_{11}| + |A_2|) \\ &\quad + (5|V_2|/2 + |A'_{12}|) + |V_3| + 5|V_0|/2 \\ &\leq f_\ell(s, s_1) + 3\lfloor \ell/2 \rfloor - 1 + (s+1)(n - \ell - 3). \end{aligned}$$

We first consider the case $\ell = 17$. Clearly, $e(G) \leq f_{17}(5,4) + 23 + 6(n-20) = 6n + 25 < e(G^*)$ for any $s \leq 5$. Suppose that $s \geq 6$. Since G is $3P_7$-free, $N_{C_\ell}(v) \subseteq \{x_0, x_3, x_5, x_7, x_{10}, x_{12}, x_{14}\}$ for any $v \in V_1$. This implies that $s \in \{6, 7\}$ and $s_1 \leq 4$. Thus, $e(G) \leq 7n - 7.5 < e(G^*)$ for $s = 6$ and $e(G) \leq 8n - 40 < e(G^*)$ for $s = 7$. Now, we consider the case $\ell \in \{15, 16\}$. Suppose that $s \leq 7$. Recall that $n \geq 27$ by Table 1. It follows that the function $f(s, s-1) + 3\lfloor \ell/2 \rfloor - 1 + (s+1)(n - \ell - 3)$ is monotonically increasing with respect to s. Thus, $e(G) \leq 8n + \ell^2/2 - 7\ell - 52 \leq 8n - 36 < e(G^*)$ for $s \leq 7$. Suppose that $s = 8$ and $\ell = 16$. We bound $e(V_1, V(C_\ell))$ more carefully. For any $v \in V_1$ with $d_{F_1}(v) \geq 1$, we have $N_{C_\ell}(v) \subseteq \{x_0, x_2, x_3, x_6, x_7, x_9, x_{10}, x_{13}, x_{14}\}$ for $\ell = 16$. The maximality of ℓ implies that $d_{C_{16}}(v) \leq 5$ for any $v \in A_2 \cup A'_{12}$. Note also that $d_{C_{16}}(v) \leq 6$ for any $v \in A_{11}$ by Lemma 12. This implies that

$$\begin{aligned} e(V_1, V(C_{16})) &\leq 8(|A_{12}| - |A'_{12}|) + 6|A_{11}| + 5(|A_2| + |A'_{12}|) \\ &= 8|V_1| - 3|A_2| - 2|A_{11}| - 3|A'_{12}|. \end{aligned}$$

It follows that

$$\begin{aligned} e(G) &\leq f_\ell(s, s_1) + 3\lfloor \ell/2 \rfloor - 1 + 8|V_1| - 2|A_2| - |A_{11}| - 2|A'_{12}| \\ &\quad + 5|V_2|/2 + |V_3| + 5|V_0|/2 \\ &\leq f_\ell(8, 8) + 23 + 8(n - 19) = 8n - 37 < e(G^*). \end{aligned}$$

Case 3. $k = 4$ and $\ell \in \{15, 16\}$. Since G is $3P_7$-free, F_1 does not contain P_3 whose endpoints hit $C_\ell \cup \hat{P}_k$. Thus we have (i) both $G[V_1]$ and $G[V_2]$ are P_3-free, (ii) V_3 is an independent set in G, and (iii) $e(V_0, V_3) = 0$. By the maximality of k, for any $v \in V_1$, we know $e(v, V(\hat{P}_4)) = 0$ if $d_{V_1}(v) \geq 1$; otherwise, $e(v, V(\hat{P}_4)) \leq 1$. It follows that

$$\sum_{v \in V_1} e(v, V(C_\ell \cup \hat{P}_4)) + e(G[V_1]) \leq (s+1)|V_1|.$$

Similarly, we also have $e(u, V(\hat{P}_4)) \leq 2$, $e(u, V_1) \leq 1$ and $e(u, V_3) \leq 1$ for each $u \in V_2$. It follows from (4) that Note that $N_{C_{16}}(v) \subseteq \{x_0, x_2, x_4, x_5, x_7, x_9, x_{11}, x_{12}, x_{14}\}$ for any $v \in V_1$ and $\ell = 16$. This implies that $d_{C_{16}}(v) \leq 7$ for any $v \in V_1$. Thus, $s \leq 7$ for any $\ell \in \{15, 16\}$. By (4)

$$\begin{aligned} e(G) &\leq f_\ell(s, s_1) + k\lfloor \ell/2 \rfloor - 1 + (s+1)|V_1| + 3|V_2| + |V_3| + 5|V_0|/2 \\ &\leq f_\ell(s, s-1) + 4\lfloor \ell/2 \rfloor - 1 + (s+1)(n - \ell - 4). \end{aligned}$$

Note that $n \geq 27$ by Table 1. This implies that the last function is monotonically nondecreasing with respect to the integer $s \leq 7$. Thus, $e(G) \leq 8n + \ell^2/2 - 13\ell/2 - 60 \leq 8n - 36 < e(G^*)$.

Case 4. $k = 5$ and $\ell = 15$. Since G is $3P_7$-free, we have (i) both $G[V_1]$ and $G[V_2]$ are P_3-free, (ii) V_3 is an independent set in G, and (iii) $e(v, V_1) \leq 1$ for each $v \in V_2$, $e(u, V_2) \leq 1$ for each $u \in V_3$ and $e(V_0, V_3) = 0$. Moreover, $e(v, V(\hat{P}_k)) \leq 2$ for any $v \in V_1 \cup V_2$ by the maximality of k. Since G is $3P_7$-free and $x_0 f_1 \in E(G)$, it is easy to see that $N_{C_{15}}(v) \subseteq \{x_3, x_4, x_5, x_{10}, x_{11}, x_{12}\}$ for any $v \in V_1$. This implies that $s \leq 4$. Thus

$$\begin{aligned} e(G) &\leq f_\ell(s, s-1) + k\lfloor \ell/2 \rfloor + (s + 5/2)|V_1| + 7|V_2|/2 + |V_3| + 5|V_0|/2 \\ &\leq \binom{15}{2} - \frac{(s-1)(s+2)}{2} + (s + 5/2)(n - 20) \leq 6.5n + 1 < e(G^*). \end{aligned}$$

This completes the proof of this claim. □

In view of Claims 13, 14 and 15, we conclude that G is isomorphic to G^* if (i) $\ell = 20$ and $n \leq 31$, or (ii) $\ell = 17$, $k = 1$ and $n \geq 32$; otherwise, we have $e(G) < e(G^*)$, a contradiction. Thus, we complete the proof of Theorem 7. ■

3. Turán Number of the Graph $2P_3 \cup P_{2\ell+1}$

3.1. Lemmas

In this subsection, we give two lemmas used frequently in our proof of Theorem 8.

Lemma 16. *For an integer $\ell \geq 2$, let G be a graph on n vertices that does not contain a copy of $P_\ell \cup P_3$ and $e(G) \geq \phi(n, \ell+3, 3)$. If G contains a copy of either $C_{\ell+1}$ or $C_{\ell+2}$, then G is isomorphic to $K_{\ell+2} \cup M_{n-\ell-2}$.*

Proof. Suppose that G does not contain a copy of $P_3 \cup P_\ell$ and $e(G) \geq \phi(n, \ell + 3, 3)$. If G contains a copy of $C_{\ell+2}$, then each vertex in $G - C_{\ell+2}$ cannot hit any vertex in $C_{\ell+2}$, and $G - C_{\ell+2}$ consists of independent edges and isolated vertices, which implies that G is isomorphic to $K_{\ell+2} \cup M_{n-\ell-2}$. If G contains a copy of $C_{\ell+1}$

and does not contain a copy of $C_{\ell+2}$, then there is at most one edge between any two consecutive vertices on $C_{\ell+1}$ and $G-C_{\ell+1}$. In addition, $G-C_{\ell+1}$ consists of independent edges and isolated vertices. Hence $e(G) \leq \binom{\ell+1}{2} + \lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{n-\ell-1}{2} \rfloor < \binom{\ell+2}{2} + \lfloor \frac{n-\ell-2}{2} \rfloor = \phi(n, \ell+3, 3)$, a contradiction. ■

Lemma 17. *For an integer $\ell \geq 2$, let G be a graph that does not contain a copy of $P_\ell \cup P_3$. Suppose that G contains a path $P = x_1 \cdots x_{\ell+2}$ on $\ell+2$ vertices and $s = \max\{d_P(v) : v \in V(G)\backslash V(P)\}$. Then* (1) *$d_P(x_1) \leq \ell+1-2s$,* (2) *there exist $2s$ distinct vertices $x_{\beta_\tau}, x_{\gamma_\tau}$ with $\tau \in [s]$ in P such that $d_P(x_{\beta_\tau}) + d_P(x_{\gamma_\tau}) \leq \ell+2$ with $\gamma_\tau > \beta_\tau \geq 2$, and* (3) *$e(P) \leq \binom{\ell+2}{2} - \lceil \frac{s\ell+2s}{2} \rceil$. Moreover,* (4) *if G also does not contain $C_{\ell+1}$ and $C_{\ell+2}$, then $e(P) \leq \binom{\ell+2}{2} - \lceil \frac{s\ell+\ell+1}{2} \rceil$.*

Proof. Let $x \in V(G)\backslash V(P)$ be such that $N_P(x) = \{x_{i_1}, \ldots, x_{i_s}\}$ with $i_1 < \cdots < i_s$. Note that G does not contain a copy of $P_\ell \cup P_3$. It follows that (i) x misses $x_1, x_2, x_{\ell+1}, x_{\ell+2}$; (ii) x hits at most one of x_j and x_{j+1} for $j \in [\ell+1]$, and (iii) x hits at most one of x_j and x_{j+4} for $j \in [\ell-2]$. Thus, we have $3 \leq i_\alpha < i_{\alpha+1} - 1 \leq \ell-1$ and $i_{\alpha+1} - i_\alpha \neq 4$ for any $\alpha \in [s-1]$.

(1) Note that x_1 misses $x_{i_\alpha+1}$ and $x_{i_\alpha+2}$ for each $\alpha \in [s]$. Otherwise, we have a copy of $P_3 \cup P_\ell$ by choosing $xx_{i_\alpha}x_{i_\alpha-1}$ and $x_{i_\alpha-2} \cdots x_1x_{i_\alpha+1} \cdots x_{\ell+2}$ for $x_1x_{i_\alpha+1} \in E(G)$, or choosing $xx_{i_\alpha}x_{i_\alpha+1}$ and $x_{i_\alpha-1} \cdots x_1x_{i_\alpha+2} \cdots x_{\ell+2}$ for $x_1x_{i_\alpha+1} \in E(G)$, a contradiction. In addition, we know that $i_\alpha + 2 < i_{\alpha+1} + 1$ for each $\alpha \in [s-1]$. Thus $d_P(x_1) \leq \ell+1-2s$.

(2) We first show that either $i_\alpha - 2, i_\alpha + 1, i_{\alpha+1} - 2, i_{\alpha+1} + 1$ or $i_\alpha - 2, i_\alpha + 1, i_{\alpha+1} - 1, i_{\alpha+1} + 2$ are four distinct numbers. In fact, if $i_{\alpha+1} = i_\alpha + 2$, then $i_\alpha - 2 < i_{\alpha+1} - 2 < i_\alpha + 1 < i_{\alpha+1} + 1$; if $i_{\alpha+1} \geq i_\alpha + 3$, then $i_\alpha - 2 < i_\alpha + 1 < i_{\alpha+1} - 1 < i_{\alpha+1} + 2$. Similarly, $i_\alpha - 1, i_\alpha + 2, i_{\alpha+1} - 2, i_{\alpha+1} + 1$ are four distinct numbers. In fact, if $2 \leq i_{\alpha+1} - i_\alpha \leq 3$, then $i_\alpha - 1 < i_{\alpha+1} - 2 < i_\alpha + 2 < i_{\alpha+1} + 1$; if $i_{\alpha+1} - i_\alpha \geq 4$, then $i_\alpha - 1 < i_\alpha + 2 < i_{\alpha+1} - 2 < i_{\alpha+1} + 1$ as $i_{\alpha+1} - i_\alpha \neq 4$. Hence, there exist $t_1, \ldots, t_s \in \{1, 2\}$ with $t_1 = 1$ such that $i_1 - t_1, i_1 + 3 - t_1, i_2 - t_2, i_2 + 3 - t_2, \ldots, i_s - t_s, i_s + 3 - t_s$ are $2s$ distinct numbers.

Now, we claim that $d_P(x_{i_\alpha-1}) + d_P(x_{i_\alpha+2}) \leq \ell+2$ and $d_P(x_{i_\alpha-2}) + d_P(x_{i_\alpha+1}) \leq \ell+2$ for $\alpha \in [s]$. In fact, if $x_{i_\alpha-1}$ hits a vertex x_j for $j < i_\alpha - 1$, then $x_{i_\alpha+2}$ must miss x_{j+1}; and if $x_{i_\alpha-1}$ hits a vertex x_j for $j > i_\alpha + 2$, then $x_{i_\alpha+2}$ must miss x_{j+1}. Otherwise, G contains a copy of $P_\ell \cup P_3$, a contradiction. In addition, $x_{i_\alpha-1}$ misses $x_{i_\alpha+2}$ and $x_{\ell+2}$, and $x_{i_\alpha+2}$ misses x_1. Hence, $d_P(x_{i_\alpha+2}) \leq \ell+1-(d_P(x_{i_\alpha-1})-2)-1$, implying the first inequality of our claim. A similar argument as above shows that the second inequality of our claim also holds.

For each $\tau \in [s]$, let $\beta_\tau = i_\tau - t_\tau$ and $\gamma_\tau = i_\tau + 3 - t_\tau$. Due to the above arguments, we have $2s$ distinct vertices $x_{\beta_\tau}, x_{\gamma_\tau}$ with $\tau \in [s]$ in P such that $d_P(x_{\beta_\tau}) + d_P(x_{\gamma_\tau}) \leq \ell+2$ and $\gamma_\tau > \beta_\tau \geq 2$.

(3) Note that $2e(P) = \sum_{i\in[\ell+2]} d_P(x_i) \leq (\ell+1-2s) + s(\ell+2) + (\ell+2-2s-1)(\ell+1)$ in view of (1) and (2). It follows that $e(P) \leq \binom{\ell+2}{2} - \lceil \frac{s\ell+2s}{2} \rceil$.

(4) It is easy to see that $d_P(x_1) + d_P(x_{\ell+1}) \leq \ell$ and $d_P(x_1) + d_P(x_{\ell+2}) \leq \ell+1$. Otherwise, it follows from the proof of Dirac's theorem on Hamiltonian cycles that G must contain a copy of either $C_{\ell+1}$ or $C_{\ell+2}$, a contradiction. Note that $\beta_1 > 1$ and either $x_{\gamma_s} \neq x_{\ell+1}$ or $x_{\gamma_s} \neq x_{\ell+2}$. It follows from (2) that $2e(P) = \sum_{i\in[\ell+2]} d_P(x_i) \leq (\ell+1) + s(\ell+2) + (\ell+2-2s-2)(\ell+1)$, implying that $e(P) \leq \binom{\ell+2}{2} - \lceil \frac{s\ell+\ell+1}{2} \rceil$. ∎

3.2. Proof of Theorem 8

In this subsection, we prove Theorem 8. For any integer $\ell \geq 2$, let G be a graph containing no $2P_3 \cup P_{2\ell+1}$, and

$$e(G) \geq \max\{\phi(n, 2\ell+1, 2\ell+1), \phi(n, 2\ell+7, 3), \psi(n, 2\ell+4)+1\}.$$

If G is $P_{2\ell+1}$-free, then $e(G) = \phi(n, 2\ell+1, 2\ell+1)$ and $G \cong tK_{2\ell} \cup K_r$ by Theorem 2, where $n = t(2\ell) + r$ and $0 \leq r < 2\ell$. Thus, we may assume that G contains $P_{2\ell+1}$ and

$$\begin{aligned} e(G) &\geq \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4)+1\} \\ &= \max\left\{\binom{2\ell+6}{2} + \left\lfloor \frac{n-2\ell-6}{2} \right\rfloor, \binom{\ell+1}{2} + (\ell+1)(n-\ell-1)+1\right\}. \end{aligned}$$

By Theorem 5, we know that G contains $P_3 \cup P_{2\ell+1}$. Note that G is $P_3 \cup P_{2\ell+4}$-free. By Lemma 16, if there exists a copy of $C_{2\ell+6}$ or $C_{2\ell+5}$ in G, then G is isomorphic to $K_{2\ell+6} \cup M_{n-2\ell-6}$. In what follows, we may assume that G does not contain $C_{2\ell+6}$ and $C_{2\ell+5}$.

Claim 18. *G contains no $P_{2\ell+6}$.*

Proof. Suppose that G contains a path $P = x_1x_2\cdots x_{2\ell+6}$ on $2\ell+6$ vertices, and $Y = V(G)\backslash V(P)$. Let $s = \max\{d_P(v) : v \in V(Y)\}$. Note that each $v \in Y$ misses $\{x_1, x_2, x_4, x_5, x_7, x_{2\ell}, x_{2\ell+2}, x_{2\ell+3}, x_{2\ell+5}, x_{2\ell+6}\}$ as G is $2P_3 \cup P_{2\ell+1}$-free. This implies that $s \leq \ell$ for any $\ell \geq 2$. Recall that G is also $P_3 \cup P_{2\ell+4}$-free containing no $C_{2\ell+6}$ and $C_{2\ell+5}$. By Lemma 17(4), we have $e(P) \leq \binom{2\ell+6}{2} - \left\lceil \frac{s(2\ell+4)+2\ell+5}{2} \right\rceil$. Since $G[Y]$ is P_3-free and $e(G) = e(P) + e(P, Y) + e(Y)$, we conclude that

$$(6)\quad e(G) \leq \binom{2\ell+6}{2} - \left\lceil \frac{s\,(2\ell+4)+2\ell+5}{2} \right\rceil + s\,(n-2\ell-6) + \left\lfloor \frac{n-2\ell-6}{2} \right\rfloor.$$

If $n \leq 3\ell+9$, we have $-\lceil (s\,(2\ell+4)+2\ell+5)/2 \rceil + s\,(n-2\ell-6) \leq -(\ell+2)s-\ell-3+(\ell+3)s < 0$, then $e\,(G) < \binom{2\ell+6}{2} + \lfloor \frac{n-2\ell-6}{2} \rfloor = \phi(n, 2\ell+7, 3)$; If $n \geq 3\ell+9$, then right side of the inequality (4) is expanded at most $2\ell^2+9\ell+9+s(n-3\ell-8)+n/2 \leq 2\ell^2+9\ell+9+\ell(n-3\ell-8)+n/2$. So we can verify $e(G) < \binom{\ell+1}{2}+(\ell+1)(n-\ell-1)+1 = \psi(n, 2\ell+4)+1$ for $\ell \geq 5$. This leads to a contradiction in either case. In what follows, we get a contradiction by checking more carefully for the remaining values of ℓ.

If $\ell = 4$, then $N_P(y) \subseteq \{x_3, x_6, x_9, x_{12}\}$ for each $y \in Y$ as G is $2P_3 \cup P_{2\ell+1}$-free. In addition, it is easy to check that (6) also holds unless $s = 4$ and there exist at least two vertices $y_1, y_2 \in Y$ satisfying $N_P(y_i) = \{x_3, x_6, x_9, x_{12}\}$ for $i \in [2]$. Thus, x_2 misses any vertex in $\{x_4, x_5, x_7, x_8, x_{10}, x_{11}, x_{13}, x_{14}\}$ and x_5 misses any vertex in $\{x_1, x_2, x_7, x_8, x_{10}, x_{11}, x_{13}, x_{14}\}$ as G is $2P_3 \cup P_{2\ell+1}$-free, implying that $d_P(x_2) + d_P(x_5) \leq 12 = (2\ell+6) - 2$. Note that $x_{\beta_1} = x_2$ and $x_{\gamma_1} = x_5$ by using Lemma 17(2) with $s = 4$. According to Lemma 17(4), we have

$$e(G) \leq \binom{2\ell+6}{2} - \left\lceil \frac{s\,(2\ell+4)+2\ell+5}{2} \right\rceil - 1 + s\,(n-2\ell-6) + \left\lfloor \frac{n-2\ell-6}{2} \right\rfloor.$$

Thus, $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 22$ and $e(G) < \binom{\ell+1}{2} + (\ell+1)(n-\ell-1)+1$ for $n \geq 22$, a contradiction.

If $\ell = 3$, then $N_P(y) \subseteq \{x_3, x_{10}\}$ for each $y \in Y$. Clearly, $s \leq 2$. In view of (6), $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 20$ and $e(G) < \psi(n, 2\ell+4)+1$ for $n \geq 20$, a contradiction.

If $\ell = 2$, then $N_P(y) \subseteq \{x_3, x_8\}$ for each $y \in Y$, and $d_P(y') = 0$ for each $y' \in Y$ with $d_Y(y') > 0$. It follows that $s \leq 2$ and

$$e(G) \leq \binom{2\ell+6}{2} - \left\lceil \frac{s\,(2\ell+4)+2\ell+5}{2} \right\rceil + s\,(n-2\ell-6).$$

Thus, $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 19$ and $e(G) < \binom{\ell+1}{2} + (\ell+1)(n-\ell-1)+1$ for otherwise, a contradiction. □

Claim 19. *G contains a copy of $P_{2\ell+5}$.*

Proof. Suppose that G is $P_{2\ell+5}$-free. We claim that G is connected. Otherwise, if one of the components, say C with n_1 vertices, contains $P_3 \cup P_{2\ell+1}$ and other components are disjoint edges or isolated vertices, then by Theorem 3

$$\begin{aligned}
e(G) &= e(C) + e(G-C) \leq \mathrm{ex}_{\mathrm{con}}(n_1, P_{2\ell+5}) + \left\lfloor \frac{n-n_1}{2} \right\rfloor \\
&\leq \max\left\{ \binom{2\ell+3}{2} + n_1 - 2\ell - 3, \binom{\ell+1}{2} + (\ell+1)(n_1-\ell-1)+1 \right\} \\
&\quad + \left\lfloor \frac{n-n_1}{2} \right\rfloor < \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4)+1\},
\end{aligned}$$

a contradiction; if one of the components, say C with n_1 vertices, contains $P_{2\ell+1}$ but is $P_3 \cup P_{2\ell+1}$-free and other components are $2P_3$-free, which means $e(C) \leq \mathrm{ex}(n_1, P_3 \cup P_{2\ell+1})$ and $e(G-C) \leq \mathrm{ex}(n-n_1, 2P_3)$, then by Theorem 5,

$$\begin{aligned} e(G) &= e(C) + e(G-C) \leq \mathrm{ex}(n_1, P_3 \cup P_{2\ell+1}) + \mathrm{ex}(n-n_1, 2P_3) \\ &= \max\left\{\binom{2\ell+3}{2} + \left\lfloor \frac{n_1 - 2\ell - 3}{2} \right\rfloor, \binom{\ell}{2} + \ell(n_1 - \ell) + 1\right\} \\ &+ \max\left\{\binom{5}{2} + \left\lfloor \frac{n - n_1 - 5}{2} \right\rfloor, n - n_1\right\} \\ &< \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4) + 1\}, \end{aligned}$$

a contradiction. Hence, G is connected. Since G is $P_{2\ell+5}$-free and connected, it follows from Theorem 3 that

$$\begin{aligned} e(G) &\leq \mathrm{ex}_{\mathrm{con}}(n, P_{2\ell+5}) \\ &= \max\left\{\binom{2\ell+3}{2} + n - 2\ell - 3, \binom{\ell+1}{2} + (\ell+1)(n-\ell-1) + 1\right\} \\ &\leq \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4) + 1\}, \end{aligned}$$

where the last inequality follows from $\binom{\ell+1}{2} + (\ell+1)(n-\ell-1) + 1 = \psi(n, 2\ell+4)+1$, $\binom{2\ell+3}{2} + n - 2\ell - 3 \leq \psi(n, 2\ell+4)+1$ for $n \geq \frac{5\ell+9}{2}$, and $\binom{2\ell+3}{2} + n - 2\ell - 3 < \phi(n, 2\ell+7, 3)$ for $n < \frac{5\ell+9}{2}$. Recall that $e(G) \geq \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4)+1\}$. This implies that

$$\begin{aligned} e(G) &= \max\left\{\binom{2\ell+3}{2} + n - 2\ell - 3, \psi(n, 2\ell+4) + 1\right\} \\ &= \max\{\phi(n, 2\ell+7, 3), \psi(n, 2\ell+4) + 1\}. \end{aligned}$$

Thus, we have $e(G) = \psi(n, 2\ell+4)+1$ and $n \geq \frac{5\ell+9}{2}$ in view of $\psi(n, 2\ell+4)+1 < \binom{2\ell+3}{2} + n - 2\ell - 3 < \phi(n, 2\ell+7, 3)$ for $n < \frac{5\ell+9}{2}$. By Theorem 3, we know that G is isomorphic to $K_{\ell+1} + \left(K_2 \cup \overline{K_{n-\ell-3}}\right)$. □

Let $P = x_1x_2 \cdots x_{2\ell+5}$ be a path on $2\ell+5$ vertices in G, and $Y = V(G)\backslash V(P)$. Choose $y^* \in Y$ such that $d_P(y^*)$ is maximum, and let $N_P(y^*) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$. Clearly, $s \leq \ell+2$ as P is a longest path in G by Claim 18. If $s = \ell+2$, then $N_P(y^*) = \{x_2, x_4, \ldots, x_{2\ell+4}\}$. It follows that $N_P(v) = \emptyset$ for any $v \in Y\backslash\{y^*\}$ as G is $2P_3 \cup P_{2\ell+1}$-free. Then $e(G) \leq \binom{2\ell+5}{2} + (\ell+2) + \left\lfloor \frac{n-2\ell-5}{2} \right\rfloor < \binom{2\ell+6}{2} + \left\lfloor \frac{n-2\ell-6}{2} \right\rfloor$, a contradiction. In addition, if $s = 0$, then $e(G) = e(P) + e(Y) \leq \binom{2\ell+5}{2} + \left\lfloor \frac{n-2\ell-5}{2} \right\rfloor$, a contradiction. Hence, $1 \leq s \leq \ell+1$.

Claim 20. *For each $v \in Y$, we have $x_2v, x_{2\ell+4}v \notin E(G)$.*

Proof. Suppose that there exists $v_0 \in Y$ such that $x_2v_0 \in E(G)$ or $x_{2\ell+4}v_0 \in E(G)$, say $x_{2\ell+4}v_0 \in E(G)$. Let $P' = x_1x_2\cdots x_{2\ell+3}$ and $G' := G[V(P') \cup (Y\backslash\{v_0\})]$. Since G is $2P_3 \cup P_{2\ell+1}$-free, we know that G' is $P_3 \cup P_{2\ell+1}$-free. It follows that $s_1 := \max_{v\in Y\backslash\{v_0\}} d_{P'}(v) \le \ell - 1$ for any $v \in Y\backslash\{v_0\}$.

If $N_Y(x_{2\ell+4}) = \{v_0\}$, then $e(G') \ge e(G) - s - 1 - 2(2\ell+3) > \phi(n-3, 2\ell+4, 3)$ as $N_Y(x_{2\ell+5}) = \emptyset$. This implies that G' does not contain $C_{2\ell+2}$ and $C_{2\ell+3}$ as subgraphs by Lemma 16. It follows from Lemma 17(4) that $e(P') \le \binom{2\ell+3}{2} - \left\lceil\frac{s_1(2\ell+1)+2\ell+2}{2}\right\rceil$ and $e(P) \le e(P') + 1 + 2(2\ell+3) \le \binom{2\ell+5}{2} - \left\lceil\frac{s_1(2\ell+1)+2\ell+2}{2}\right\rceil$. Hence

$$e(G) \le \binom{2\ell+5}{2} - \left\lceil\frac{s_1(2\ell+1)+2\ell+2}{2}\right\rceil + s + s_1(n-2\ell-6) + \left\lfloor\frac{n-2\ell-6}{2}\right\rfloor,$$

yielding that $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 3\ell + \frac{17}{2} + \frac{7}{\ell-1}$ and $e(G) < \psi(n, 2\ell+4)+1$ for otherwise, a contradiction.

Suppose that there exists $v_1 \in N_Y(x_{2\ell+4})\backslash\{v_0\}$. It follows that $G[V(P') \cup \{x_{2\ell+5}\}]$ is $P_3 \cup P_{2\ell+1}$-free. This implies that $d_{P'}(x_{2\ell+5}) \le \ell - 1$. Since G' is also $P_3 \cup P_{2\ell+1}$-free, we have $e(P') \le \binom{2\ell+3}{2} - \left\lceil\frac{s_1(2\ell+1)+2s_1}{2}\right\rceil$ by Lemma 17(3) and $e(P) \le e(P') + 1 + (2\ell+3) + (\ell-1) \le \binom{2\ell+5}{2} - \left\lceil\frac{s_1(2\ell+1)+2s_1}{2}\right\rceil - \ell - 4$. Note also that $vx_{2\ell+4} \notin E(G)$ for each $v \in Y$ with $d_Y(v) > 0$ as P is a longest path in G. Hence

$$e(G) \le \binom{2\ell+5}{2} - \left\lceil\frac{s_1(2\ell+1)+2s_1}{2}\right\rceil - \ell - 4 + s + (s_1+1)(n-2\ell-6),$$

implying that $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 3\ell + 9 + \frac{15}{2\ell-1}$ and $e(G) < \psi(n, 2\ell+4)+1$ for otherwise, a contradiction. $\square$

If $s = \ell + 1$, then $N_P(y^*) = \{x_3, x_5, \ldots, x_{2\ell+3}\}$ by Claim 20. Since G is $2P_3 \cup P_{2\ell+1}$-free, for each $v \in Y\backslash\{y^*\}$, we have $N_P(v) \subseteq \{x_3, x_{2\ell+3}\}$ if $\ell \ge 3$ and $N_P(v) \subseteq \{x_3, x_5, x_7\}$ if $\ell = 2$. Let $s_1 := \max_{v\in Y\backslash\{y^*\}} d_P(v)$. Clearly, $s_1 \le 2$ if $\ell \ge 3$ and $s_1 \le 3$ if $\ell = 2$. We may assume that $N_P(v_1) \ne \emptyset$ for some $v_1 \in Y\backslash\{y^*\}$; otherwise, $e(G) \le \binom{2\ell+5}{2} + (\ell+1) + \left\lfloor\frac{n-2\ell-5}{2}\right\rfloor < \binom{2\ell+6}{2} + \left\lfloor\frac{n-2\ell-6}{2}\right\rfloor$. Suppose that $x_i \in N_P(v_1)$ for some $i \in \{3, 2\ell+3\}$ if $\ell \ge 3$ and $i \in \{3, 5, 7\}$ if $\ell = 2$. Since $v_1x_iy^*$ forms a copy of P_3, we assert that $x_{i-1}x_p$ and $x_{i+1}x_{p+1}$ cannot coexist in G for $p > i+1$ or $p < i-1$; otherwise, $x_1\cdots x_{i-1}x_p\cdots x_{i+1}x_{p+1}\cdots x_{2\ell+5}$ for $p > i+1$ or $x_1\cdots x_px_{i-1}\cdots x_{p+1}\cdots x_{i+1}\cdots x_{2\ell+5}$ for $p < i-1$ contains $P_3 \cup P_{2\ell+1}$ as a subgraph, a contradiction. An argument similar to Lemma 17(2) implies that $d_P(x_{i-1}) + d_P(x_{i+1}) \le 2\ell + 5$. It follows that we can find s_1 pairs of vertices (x_{a_j}, x_{b_j}) satisfying $d_P(x_{a_j}) + d_P(x_{b_j}) \le 2\ell + 5$ where $j \in [s_1]$ and $a_j, b_j \ne 2\ell + 5$. Note also that G does not contain $C_{2\ell+5}$ as a subgraph. Thus, we conclude that

$e(P) \le \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+3)+2\ell+4}{2} \right\rceil$ as a similar argument to Lemma 17(4), yielding that

$$e(G) \le \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+3)+2\ell+4}{2} \right\rceil + s + s_1\left(n-2\ell-6\right) + \left\lfloor \frac{n-2\ell-5}{2} \right\rfloor.$$

If $s_1 \le 2$, then $e\left(G\right) < \phi(n, 2\ell+7, 3)$ for $n < 4\ell+10$ and $e\left(G\right) < \psi(n, 2\ell+4)+1$ for otherwise. In particular, for $\ell = 2$ and $s_1 = 3$, we have $N_P(x_2) \subseteq \{x_1, x_3, x_5, x_8\}$ and $N_P(x_4) \subseteq \{x_3, x_5, x_7\}$ as G is $2P_3 \cup P_{2\ell+1}$-free. Meanwhile, we can get $N_P(x_6) \subseteq \{x_3, x_5, x_7\}$ and $N_P(x_8) \subseteq \{x_2, x_5, x_7, x_9\}$ by symmetry. In addition, $N_P(x_1) \subseteq \{x_2, x_3, x_5\}$ and $N_P(x_9) \subseteq \{x_8, x_7, x_5\}$ by symmetry. It follows that

$$e(G) \le \binom{9}{2} - 2\times 9 - 10 + 3 + 3\left(n - 2\times 2 - 6\right) + \left\lfloor \frac{n-2\times 2-6}{2} \right\rfloor.$$

Thus $e\left(G\right) < \phi(n, 2\ell+7, 3)$ for $n < 22$ and $e\left(G\right) < \psi(n, 2\ell+4)+1$ for $n \ge 22$.

In what follows, we may assume that $d_P(y^*) = s \le \ell$. Suppose that neither $G[V(P)\backslash\{x_i, x_{i+1}\}]$ nor $G[V(P)\backslash\{x_i, x_{i-1}\}]$ contains a copy of $P_{2\ell+3}$ for any $x_i \in N_P(y^*)$. It follows that $d_P(x_{i-2}) + d_P(x_{i+1}) \le 2\ell+3$ and $d_P(x_{i-1}) + d_P(x_{i+2}) \le 2\ell+3$ for any $x_i \in N_P(y^*)$. Note also that $d_P(x_1) \le 2\ell+2-2s$ as x_1 misses x_{i+1}, x_{i+2} for any $x_i \in N_P(y^*)$. An argument similar to Lemma 17(3) implies that $e(P) \le \binom{2\ell+5}{2} - \left\lceil \frac{s(2\ell+3)+2s}{2} \right\rceil$. Hence

$$e(G) \le \binom{2\ell+5}{2} - \left\lceil \frac{s(2\ell+3)+2s}{2} \right\rceil + s(n-2\ell-5) + \left\lfloor \frac{n-2\ell-5}{2} \right\rfloor.$$

It follows that $e\left(G\right) < \phi(n, 2\ell+7, 3)$ for $n < 3\ell+\frac{19}{2}+\frac{4}{\ell}$ and $e\left(G\right) < \psi(n, 2\ell+4)+1$ for otherwise, a contradiction.

Now, we assume that there exists $x_i \in N_P(y^*)$ such that either $G[V(P)\backslash\{x_i, x_{i+1}\}]$ or $G[V(P)\backslash\{x_i, x_{i-1}\}]$ contains a copy of $P_{2\ell+3}$, say $G[V(P)\backslash\{x_i, x_{i-1}\}]$ contains $P_{2\ell+3}$ as a subgraph, denoted by P'. Let $s_1 := \max_{v\in Y\backslash\{y^*\}} d_{P'}(v)$. Obviously, $G[V(P')\cup\{v\}]$ is $P_3 \cup P_{2\ell+1}$-free for any $v \in Y\backslash\{y^*\}$ as G is $2P_3 \cup P_{2\ell+1}$-free, implying that $s_1 \le \ell-1$ for $\ell \ge 3$ and $s_1 \le \ell$ for $\ell = 2$. Let $G' := G[V(P')\cup(Y\backslash\{y^*\})]$. Note that G' is $P_3 \cup P_{2\ell+1}$-free as $y^*x_ix_{i-1}$ forms a P_3. Since P' is a path of length $2\ell+3$, it follows from Lemma 17(3) that $e(P') \le \binom{2\ell+3}{2} - \left\lceil \frac{s_1(2\ell+1)+2s_1}{2} \right\rceil$.

Suppose that there are two vertices $v_1, v_2 \in Y$ (note that v_1 or v_2 is not necessarily y^*) and $x \in \{x_i, x_{i-1}\}$ such that $\{x, v_1, v_2\}$ forms a copy of P_3. It follows that $d_{P'}(x_{i-1}) \le \ell-1$ (if $x = x_i$) or $d_{P'}(x_i) \le \ell-1$ (if $x = x_{i-1}$) as $G[V(P)\backslash\{x\}]$ is $P_3 \cup P_{2\ell+1}$-free. This implies that $e(P) \le e(P') + 1 + (2\ell+3) +$

$(\ell-1) \leq \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+1)+2s_1}{2} \right\rceil - \ell - 4$. Thus

$$e(G) \leq \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+1)+2s_1}{2} \right\rceil - \ell - 4 + s + (s_1+1)(n-2\ell-6) + \left\lfloor \frac{n-2\ell-5}{2} \right\rfloor.$$

If $\ell \geq 2$ and $s_1 \leq \ell - 1$, then $e\,(G) < \phi(n, 2\ell+7, 3)$ for $n < 3\ell + \frac{17}{2} + \frac{13}{2\ell}$ and $e\,(G) < \psi(n, 2\ell+4) + 1$ for otherwise, a contradiction. If $\ell = 2$ and $s_1 = \ell$, then $e(P) \leq 23$ and $e(G) \leq 23+2+2(n-10)+\left\lfloor \frac{n-9}{2} \right\rfloor$. One can easily check that $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 22$ and $e(G) < \psi(n, 2\ell+4)+1$ for $n \geq 22$, a contradiction. Hence, we conclude that $d_Y(y^*) = 0$, $d_{Y\backslash\{y^*\}}(x_i) = 0$ and $d_{Y\backslash\{y^*\}}(x_{i-1}) \leq 1$ as $x_i y^* \in E(G)$. Note that $e(P) \leq e(P') + 1 + 2(2\ell+3) \leq \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+1)+2s_1}{2} \right\rceil$. It follows that

$$e(G) \leq \binom{2\ell+5}{2} - \left\lceil \frac{s_1(2\ell+1)+2s_1}{2} \right\rceil + 2s + s_1(n-2\ell-7) + \left\lfloor \frac{n-2\ell-6}{2} \right\rfloor.$$

If $s_1 \leq \ell-1$, then $e\,(G) < \phi(n, 2\ell+7, 3)$ for $n < 3\ell+\frac{17}{2}+\frac{5}{\ell-1}$ and $e\,(G) < \psi(n, 2\ell+4)+1$ for otherwise, a contradiction. It suffices to check $e(G)$ for the case $s_1 = \ell = 2$. Let $P' = y_1 y_2 \cdots y_7$. Recall that $s_1 = \max_{v \in Y\backslash\{y^*\}} d_{P'}(v)$. It follows from Lemma 17(1) and Lemma 17(2) that $d_{P'}(y_1) \leq 2$, $d_{P'}(y_7) \leq 2$, $d_{P'}(y_2)+d_{P'}(y_5) \leq 7$ and $d_{P'}(y_3)+d_{P'}(y_6) \leq 7$, yielding that $e(P) \leq \sum_{i=1}^{7} d_{P'}(y_i)/2+2\times 7+1 = 27$. Note that $N_{P'}(v) \subseteq \{y_3, y_5\}$ for any $v \in Y\backslash\{y^*\}$ and G' is $P_3 \cup P_5$-free. This implies that $d_{P'}(v_1) = d_{P'}(v_2) = 0$ for any $v_1, v_2 \in Y\backslash\{y^*\}$ with $v_1 v_2 \in E(G)$. Therefore, $e(G) \leq 27 + 2(n-9)$. It follows that $e(G) < \phi(n, 2\ell+7, 3)$ for $n < 21$ and $e(G) < \psi(n, 2\ell+4)+1$ for $n \geq 21$, a contradiction.

This completes the proof of Theorem 8. ■

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APPENDIX A. THE VALUES OF $f_\ell(s, s_1)$ FOR $15 \leq \ell \leq 19$

ℓ	s \ s_1	0	1	2	3	4	5	6	7	8	9
19	3	152	158.5	166	\	\	\	\	\	\	\
	4	142.5	148	154.5	162	\	\	\	\	\	\
	5	133	137.5	143	149.5	157	\	\	\	\	\
	6	123.5	127	131.5	137	143.5	\	\	\	\	\
	7	114	116.5	120	124.5	130	136.5	144	\	\	\
	8	104.5	106	108.5	112	116.5	122	128.5	136	\	\
	9	95	95.5	97	99.5	103	107.5	113	119.5	127	\
18	3	135	141	148	\	\	\	\	\	\	\
	4	126	131	137	144	\	\	\	\	\	\
	5	117	121	126	132	139	\	\	\	\	\
	6	108	111	115	120	126	133	\	\	\	\
	7	99	101	104	108	113	119	126	\	\	\
	8	90	91	93	96	100	105	111	118	\	\
	9	81	81	82	84	87	91	96	102	109	117
17	3	119	124.5	131	\	\	\	\	\	\	\
	4	110.5	115	120.5	127	\	\	\	\	\	\
	5	102	105.5	110	115.5	122	\	\	\	\	\
	6	93.5	96	99.5	104	109.5	116	\	\	\	\
	7	85	86.5	89	92.5	97	102.5	109	\	\	\
	8	76.5	77	78.5	81	84.5	89	94.5	101	\	\
16	3	104	109	115	\	\	\	\	\	\	\
	4	96	100	105	111	\	\	\	\	\	\
	5	88	91	95	100	106	\	\	\	\	\
	6	80	82	85	89	94	100	107	\	\	\
	7	72	73	75	78	82	87	93	\	\	\
	8	64	64	65	67	70	74	79	85	92	\
15	3	90	94.5	100	\	\	\	\	\	\	\
	4	82.5	86	90.5	96	\	\	\	\	\	\
	5	75	77.5	81	85.5	91	\	\	\	\	\
	6	67.5	69	71.5	75	79.5	85	\	\	\	\
	7	60	60.5	62	64.5	68	72.5	78	\	\	\

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