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# CRITICAL ASPECTS IN BROADCAST DOMINATION

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**Abstract**

A dominating broadcast labeling of a graph $G$ is a function $f : V(G) \to \{0, 1, 2, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leqslant e(v)$ for all $v \in V(G)$ and

$$\bigcup_{\substack{v \in V(G)\\ f(v)>0}} [\{u \in V(G) : d(u, v) \leqslant f(v)\}] = V(G),$$

where $e(v)$ is the eccentricity of $v$. The cost of $f$ is $\sum_{v \in V(G)} f(v)$. The minimum of costs over all the dominating broadcast labelings of $G$ is called the broadcast domination number $\gamma_b(G)$ of $G$. In this paper, we introduce the critical aspects in broadcast domination and study it with respect to edge deletion and edge addition. A graph $G$ is said to be $k$-$\gamma_b^+$-edge-critical ($k$-$\gamma_b^-$-edge-critical) if $\gamma_b(G - e) > \gamma_b(G)$, for every edge $e \in E(G)$ (if $\gamma_b(G + e) < \gamma_b(G)$, for every edge $e \notin E(G)$), where $\gamma_b(G) = k$. We give a necessary and sufficient condition for a graph to be $k$-$\gamma_b^+$-edge-critical. We characterize $k$-$\gamma_b^-$-edge-critical graphs for $k = 1, 2$, and give necessary conditions of the same for $k \geqslant 3$. Further, we define the broadcast bondage number and the broadcast reinforcement number of a graph, and give tight upper bounds for them.

**Keywords:** dominating broadcast labeling, broadcast domination number, critical graph, bondage number, reinforcement number.

**2020 Mathematics Subject Classification:** 05C69, 05C78.

## 1. Introduction

Theory of domination is a very rich area in graph theory. Many real-life problems have motivated the introduction of different types of dominations such as total domination, connected domination, distance-$k$ domination, etc. In 1968, Liu [15] has discussed about a concept of domination in a communication network, where cities are vertices and edge exists between two vertices if there is a communication link between the corresponding cities. The target is to select a set of minimum number of cities where broadcast stations can be placed in order to broadcast messages to all the cities in the network. However, it has been assumed that each broadcast station broadcasts only to its neighboring cities. Later, in 2001, Erwin [5] has introduced the concept of *broadcast domination* with the motivation of generalizing the domination concept in the communication network, proposed by Liu [15], where a broadcast station can broadcast further, depending on the capacity of the station.

The theory of broadcast domination has a wide applications in much more general scenario. In a communication network, let cities/sections of a region be the vertices and any possible communication links be the edges. The target is to find the positions of the facility so that the cost of the network gets minimized and the whole network/region enjoys the facility. Here, facility can be anything like broadcast towers, hospitals, shopping malls, police stations etc. Now, if there is any fault in some communication links or some new communication links are to be established, then there is a possibility of change in the cost. Analyzing fault tolerance of any network upon node or link failure has motivated the study of critical aspects in domination theory. Consequently, effect on domination number due to vertex removal, edge removal and edge inclusion has attracted a lot of research. In this paper, we introduce the critical aspects in broadcast domination and study it with respect to edge deletion and edge addition. All the graphs considered here are simple and undirected.

In a graph $G$, a set $S \subseteq V(G)$ is said to be a *dominating set* if every vertex in $G$ is either belongs to $S$ or adjacent to some vertex of $S$. The *domination number* $\gamma(G)$ is the minimum size of a dominating set in $G$. A *broadcast labeling* of a graph $G$ is a function $f : V(G) \to \{0, 1, 2, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leqslant e(v)$ for all $v \in V(G)$, where $e(v)$ is the eccentricity of $v$. The cost of $f$ is $\sum_{v \in V(G)} f(v)$ and is denoted by $\sigma(f)$. A vertex $v$ is a *broadcast vertex* if $f(v) \geqslant 1$, and the set of all broadcast vertices is denoted by $V_f^+(G)$ or simply $V_f^+$. A vertex $u \in V(G)$ is said to be *f-dominated*, if there exists a vertex $v \in V_f^+$ such that $d(u, v) \leqslant f(v)$. It is clear that a broadcast vertex $f$-dominates itself. For each vertex $v \in V_f^+$, the *closed f-neighborhood* $N_f[v]$ of $v$ is the set $\{u \in V(G) : d(u, v) \leqslant f(v)\}$. A vertex $u$ is said to be a *boundary vertex* of a broadcast vertex $v$ if $d(u, v) = f(v)$ and the set of boundary vertices of a broadcast vertex $v$ is denoted by $B_f(v)$. We say a broadcast vertex $v$ *uses an edge xy* if a shortest path from $v$ to a boundary vertex of $v$ contains $xy$. A broadcast labeling $f$ is a *dominating broadcast labeling* if $\bigcup_{v\in V_f^+}(N_f[v]) = V(G)$ and the minimum of costs over all the dominating broadcast labelings of $G$ is called the *broadcast domination number* $\gamma_b(G)$ of $G$. If the graph is disconnected, then the broadcast domination number of the graph is the sum of the broadcast domination numbers of its components. A dominating broadcast labeling $f$ is an *optimal dominating broadcast labeling* if $\sigma(f) = \gamma_b(G)$. A dominating broadcast labeling $f$ is said to be an *efficient dominating broadcast labeling* if every vertex is $f$-dominated by exactly one broadcast vertex. For convenience, we use DBL and ODBL in place of dominating broadcast labeling and optimal dominating broadcast labeling, respectively. If the codomain of a dominating broadcast labeling of a graph $G$ is restricted to $\{0, 1\}$, then the broadcast domination number of $G$ matches with the domination number of $G$. It is a fact that not all graphs have an efficient dominating set; whereas Dunbar *et al.* [4] have proved that every graph has an efficient optimal dominating broadcast labeling. Though, finding the domination number of a graph is a $NP$-hard problem, in 2006, Heggernes and Lokshtanov [11] have shown that finding the broadcast domination number of a graph is a polynomial time problem.

A vertex $v \in V(G)$ is called a *critical vertex* if $\gamma(G - v) \neq \gamma(G)$. Similarly, an edge $e \in E(G)$ ($e \in E(\overline{G})$) is referred as a *critical edge* if $\gamma(G - e) \neq \gamma(G)$ ($\gamma(G+e) \neq \gamma(G)$). Deletion of any edge (addition of any edge) does not decrease (increase) the domination number of a graph, and deletion of any edge (addition of any edge) increases (decreases) the domination number by at most 1. But in the case of vertex deletion, the change in the domination number cannot be predicted. In the light of this, vertices of $G$, and edges of $G$ and $\overline{G}$ are categorized as below.

$$\begin{aligned}
V^0(G) &= \{v \in V(G) : \gamma(G - v) = \gamma(G)\},\\
V^+(G) &= \{v \in V(G) : \gamma(G - v) > \gamma(G)\},\\
V^-(G) &= \{v \in V(G) : \gamma(G - v) < \gamma(G)\},\\
ER^0(G) &= \{e \in E(G) : \gamma(G - e) = \gamma(G)\},\\
ER^+(G) &= \{e \in E(G) : \gamma(G - e) > \gamma(G)\},\\
EA^0(G) &= \{e \in E(\overline{G}) : \gamma(G + e) = \gamma(G)\},\\
EA^-(G) &= \{e \in E(\overline{G}) : \gamma(G + e) < \gamma(G)\}.
\end{aligned}$$

Haynes *et al.* [10] have classified graphs in six following classes.

(a) CVR (Changing Vertex Removal): $\gamma(G - v) \neq \gamma(G)$ for all $v \in V(G)$,

(b) CER (Changing Edge Removal): $\gamma(G - e) > \gamma(G)$ for all $e \in E(G)$,

(c) CEA (Changing Edge Addition): $\gamma(G + e) < \gamma(G)$ for all $e \in E(\overline{G})$,

(d) UVR (Unchanging Vertex Removal): $\gamma(G - v) = \gamma(G)$ for all $v \in V(G)$,

(e) UER (Unchanging Edge Removal): $\gamma(G - e) = \gamma(G)$ for all $e \in E(G)$,

(f) UEA (Unchanging Edge Addition): $\gamma(G + e) = \gamma(G)$ for all $e \in E(\overline{G})$.

Due to our line of research in this paper, we focus on the literature of the classes CER and CEA only. If a graph $G \in$ CER, then $\gamma(G - e) = \gamma(G) + 1$ for all $e \in E(G)$. Similarly, if a graph $G \in$ CEA, then $\gamma(G + e) = \gamma(G) - 1$ for all $e \in E(\overline{G})$. Bauer *et al.* [1] and Walikar and Acharya [23] have studied the class CER in terms of $\gamma^+$-critical graphs, whereas Sumner and Blitch [20] have initiated the study of the class CEA in the name of $k$-$\gamma$-critical graphs, where the domination number of the graph is $k$. Further, Bauer *et al.* [1] have defined the concept of *bondage number* $b(G)$ of a graph $G$ as the minimum number of edges whose deletion increases the domination number. Similar to the bondage number, Kok and Mynhardt [14] have introduced the concept of *reinforcement number* $r(G)$ of a graph $G$ as the minimum number of edges to be added to $G$ that decreases the domination number.

In the framework of broadcast domination, we call a vertex $v \in V(G)$ $\gamma_b$-*critical vertex* if $\gamma_b(G - v) \neq \gamma_b(G)$, and an edge $e \in E(G)$ ($e \in E(\overline{G})$) $\gamma_b$-*critical edge* if $\gamma_b(G - e) \neq \gamma_b(G)$ ($\gamma_b(G + e) \neq \gamma_b(G)$). One can observe that the removal of an edge or addition of an edge cannot decrease or increase the broadcast domination number, respectively. Hence, analogous to those notions as in domination, we define the following.

$$\begin{aligned}
V_b^0(G) &= \{v \in V(G) : \gamma_b(G - v) = \gamma_b(G)\},\\
V_b^+(G) &= \{v \in V(G) : \gamma_b(G - v) > \gamma_b(G)\},\\
V_b^-(G) &= \{v \in V(G) : \gamma_b(G - v) < \gamma_b(G)\},\\
ER_b^0(G) &= \{e \in E(G) : \gamma_b(G - e) = \gamma_b(G)\},\\
ER_b^+(G) &= \{e \in E(G) : \gamma_b(G - e) > \gamma_b(G)\},\\
EA_b^0(G) &= \{e \in E(\overline{G}) : \gamma_b(G + e) = \gamma_b(G)\},\\
EA_b^-(G) &= \{e \in E(\overline{G}) : \gamma_b(G + e) < \gamma_b(G)\}.
\end{aligned}$$

Further, we classify graphs as below.

(a) $\mathrm{CVR}_b$ (Changing Vertex Removal): $\gamma_b(G - v) \neq \gamma_b(G)$ for all $v \in V(G)$,

(b) $\mathrm{CER}_b$ (Changing Edge Removal): $\gamma_b(G - e) > \gamma_b(G)$ for all $e \in E(G)$,

(c) $\mathrm{CEA}_b$ (Changing Edge Addition): $\gamma_b(G + e) < \gamma_b(G)$ for all $e \in E(\overline{G})$,

(d) $\mathrm{UVR}_b$ (Unchanging Vertex Removal): $\gamma_b(G - v) = \gamma_b(G)$ for all $v \in V(G)$,

(e) $\mathrm{UER}_b$ (Unchanging Edge Removal): $\gamma_b(G - e) = \gamma_b(G)$ for all $e \in E(G)$,

(f) $\mathrm{UEA}_b$ (Unchanging Edge Addition): $\gamma_b(G + e) = \gamma_b(G)$ for all $e \in E(\overline{G})$.

Any graph of the class $\mathrm{CER}_b$ is called as $\gamma_b^+$-edge-critical graph, and graph of the class $\mathrm{CEA}_b$ is called as $\gamma_b^-$-edge-critical graph. If $\gamma_b(G) = k$, then we call $\gamma_b^+$-edge-critical graph as $k$-$\gamma_b^+$-edge-critical graph and $\gamma_b^-$-edge-critical graph as $k$-$\gamma_b^-$-edge-critical graph. Moreover, we introduce the notions of *broadcast bondage number* and *broadcast reinforcement number* of a graph. The broadcast bondage number $b_b(G)$ of a graph $G$ of size at least 1 is defined as the minimum number of edges to be deleted from $G$ to increase $\gamma_b(G)$. The minimum number of edges to be added to a graph $G$ to decrease $\gamma_b(G)$ is called as the broadcast reinforcement number, denoted by $r_b(G)$.

The rest of the paper is organized in the following manner. A brief survey on dominating broadcast labeling and critical concepts in domination, with respect to edge deletion and edge addition, are given in Section 1.1. Our results are distributed to Section 2 and Section 3. Study on $\gamma_b^+$-edge-critical graphs comprises Section 2. We give a necessary and sufficient condition for a graph to be $\gamma_b^+$-edge-critical. Later in this section, we give results regarding the broadcast bondage number and determine the exact values for $K_n, K_{m,n}, P_n$ and $C_n$. We find a tight upper bound to $b_b(G)$ and give a relation between the broadcast bondage numbers of a graph and its spanning subgraph with the same broadcast domination number. We study $\gamma_b^-$-edge-critical graphs in Section 3. We characterize $k$-$\gamma_b^-$-edge-critical graphs for $k = 1, 2$, and prove that radius is 3 for 3-$\gamma_b^-$-edge-critical graphs. We show that some classes of trees are not $\gamma_b^-$-edge-critical and consequently we prove that the sets of connected $\gamma_b^-$-edge-critical graphs and connected $\gamma_b^+$-edge-critical graphs are exclusive. Further, we find few tight upper bounds for $r_b(G)$ and give few sufficient conditions for a graph to have $r_b(G) = 1$. We present concluding remarks and some open problems in Section 4.

### 1.1. A brief literature survey

Erwin [5] has given a bound of the broadcast domination number for any connected graph, as below.

**Theorem 1** [5]**.** *For a non-trivial connected graph $G$,* $\left\lceil \frac{\mathrm{diam}(G)+1}{3} \right\rceil \leqslant \gamma_b(G) \leqslant \min\{\gamma(G), \mathrm{rad}(G)\}$.

Dunbar *et al.* [4] have suggested three classes of graphs. If $\gamma_b(G) = \gamma(G)$, then $G$ is said to be 1-*cap* or Type-I graph. If $\gamma_b(G) = \mathrm{rad}(G)$, then $G$ is said to be *radial* or Type-II graph. Type-III graphs are those for which $\gamma_b(G) < \min\{\gamma(G), \mathrm{rad}(G)\}$. The relevant studies on them can be found in [3, 5, 13, 16, 19]. Erwin [5] has proved the following results.

**Theorem 2** [5]**.** *For a connected graph $G$, if* $\min\{\gamma(G), \mathrm{rad}(G)\} = k$*,* $1 \leqslant k \leqslant 3$*, then* $\gamma_b(G) = k$*.*

**Theorem 3** [5]**.** *Let $G$ be a connected graph. If $\gamma_b(G) = 2$, then*

(a) $\mathrm{rad}(G) = 2$ *or*

(b) $\gamma(G) = 2$ *and* $\mathrm{rad}(G) = 3$.

**Theorem 4** [5]**.** *Let $f$ be an optimal dominating broadcast labeling of a connected graph $G$. Then $V_f^+ = \{v\}$ if and only if $f(v) = \mathrm{rad}(G)$ and $v$ is a central vertex of $G$.*

**Theorem 5** [5]**.** *For path $P_n$ and cycle $C_n$, $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil = \gamma_b(C_n)$.*

Brešar and Špacapan [2] have proved that for any spanning subgraph $H$ of $G$, $\gamma_b(G) \leqslant \gamma_b(H)$. Herke [12] has proved the following result which gives a relation between the broadcast domination numbers of a connected graph and its spanning trees.

**Theorem 6** [12]**.** *If $G$ is a connected graph and $S(G)$ is the set of all spanning trees of $G$, then $\gamma_b(G) = \min\{\gamma_b(T) : T \in S(G)\}$.*

A DBL $f$ of $G$ is said to be a *radial labeling* if $f(v) = \mathrm{rad}(G)$ for some central vertex $v$, and $f(u) = 0$ if $u \neq v$. If a graph is radial, then one of its ODBLs is a radial labeling. A graph $G$ is said to be *uniquely radial graph* if radial labeling is the only ODBL of $G$. Mynhardt and Woodlinger [17] have characterized uniquely radial trees.

Bauer *et al.* [1] and Walikar and Acharya [23] have independently characterized $\gamma^+$-critical graphs. A *galaxy* is a forest whose each component is a star.

**Theorem 7** [1, 23]**.** *A graph is $\gamma^+$-critical if and only if it is a galaxy.*

Some bounds for the bondage number are given below. The maximum degree of a graph $G$ is denoted by $\Delta(G)$.

**Theorem 8.** (a) [1, 7] *For a non-trivial tree $T$, $b(T) \leqslant 2$.*

(b) [1, 7] *For any non-empty graph $G$, $b(G) \leqslant \delta'(G) - 1$, where $\delta'(G) = \min\{\deg(u) + deg(v) : uv \in E(G)\}$.*

(c) [7] *If $G$ is a connected graph of order at least 2, then $b(G) \leqslant |V(G)| - 1$.*

(d) [7] *If $G$ is a non-empty graph with $\gamma(G) \geqslant 2$, then $b(G) \leqslant (\gamma(G)-1)\Delta(G)+1$.*

(e) [7] *For any connected graph $G$ of order at least 2, $b(G) \leqslant |V(G)| - \gamma(G) + 1$.*

(f) [9] *For any graph $G$ having edge connectivity $\kappa$, $b(G) \leqslant \Delta(G) + \kappa - 1$.*

Teschner [22] has characterized trees with bondage number 1, and Hartnell and Rall [8] have characterized trees with bondage number 2. Fink *et al.* [7] have determined the exact values of the bondage numbers for $K_n$, $C_n$, $P_n$ and

$K_{n_1,n_2,n_3,\ldots,n_t}$. For more bounds on the bondage number, one may look into [18, 21, 24].

Sumner and Blitch [20] have characterized $k$-$\gamma$-critical graphs for $k = 1, 2$. They have shown that $K_n$ ($n \geqslant 2$) is the only 1-$\gamma$-critical graph. They have proved that a graph is 2-$\gamma$-critical graph if and only if the complement of that graph is a galaxy. Hence, by Theorem 7, we can say a graph is 2-$\gamma$-critical if and only if the complement of the graph is $\gamma^+$-critical. Further, they have given many necessary conditions for 3-$\gamma$-critical graphs.

**Theorem 9** [20]**.** (a) *Every* 3-$\gamma$-*critical graph contains a triangle.*

(b) *If $G$ is a connected* 3-$\gamma$-*critical graph of even order, then $G$ has a perfect matching.*

(c) *Let $G$ be a connected* 3-$\gamma$-*critical graph. Then, for $k \geqslant 1$, the number of vertices in $G$ of degree at most $k$ is less than or equal to* $3k$.

(d) *The diameter of* 3-$\gamma$-*critical graph is at most* 3.

Wojcicka [25] has proved that every connected 3-$\gamma$-critical graph of order at least 6 has a Hamiltonian path. Favaron *et al.* [6] have proved that the diameter of a $k$-$\gamma$-critical graph is at most $2k - 2$. They have further shown that there exists a $k$-$\gamma$-critical graph of diameter $\left\lfloor \frac{3k}{2} \right\rfloor - 1$ and diameter of 4-$\gamma$-critical graphs is at most 5.

Kok and Mynhardt [14] have determined $r(G)$ for path graphs, cycle graphs, Cartesian product $K_s \square K_r$, and union, join and corona of two graphs $G$ and $H$. If $A \subseteq V(G)$ and $u \in A$, then the *private neighbor set* of $u$ with respect to $A$, $pn[u, A]$, is defined as the set $\{v \in V(G) : N[v] \cap A = \{u\}\}$. Let $\epsilon(A) = \min\{|pn[u, A]| : u \in A\}$. The *private neighborhood number* of a graph $G$, $\epsilon(G)$, is defined as $\min\{\epsilon(D) : D$ is a dominating set of $G$ of minimum cardinality$\}$. Kok and Mynhardt [14] have given the following results.

**Theorem 10** [14]**.** (a) *For any graph $G$ of order $n$, $\gamma(G) \leqslant n - \Delta(G) - r(G) + 1$.*

(b) *Let $G$ be a graph with domination number at least* 2. *Then $r(G) = \eta(G)$, where $\eta(G) = \min\{|V(G) \setminus N[A]| : A \subseteq V(G), |A| = \gamma(G) - 1\}$.*

(c) *For any graph $G$ with $\gamma(G) \geqslant 2$, $r(G) \leqslant \epsilon(G)$. Moreover, equality holds if $r(G) = 1$.*

(d) *For any graph $G$ of order $n$, $r(G) \leqslant \epsilon(G) \leqslant \frac{n}{\gamma(G)}$. Moreover, for any integers $r, s, t$ with $2 \leqslant r \leqslant s$ and $t \geqslant 2$, there exists a connected graph $G$ with $r(G) = r, \epsilon(G) = s$ and $\gamma(G) = t$.*

## 2. $\gamma_b^+$-EDGE-CRITICAL GRAPHS

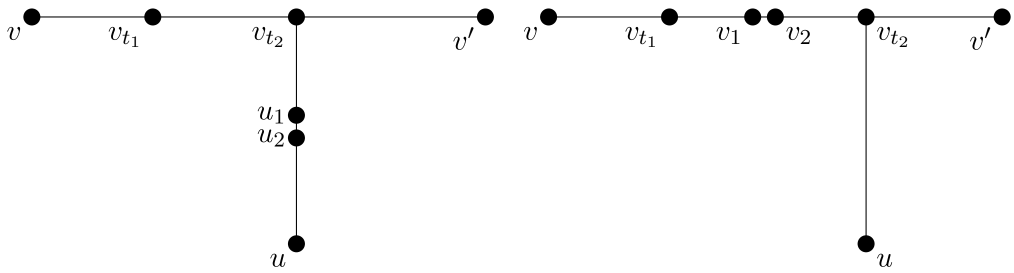
In this section, we give a necessary and sufficient condition for a graph to be $\gamma_b^+$-edge-critical. If $H$ is a subgraph of a connected graph $G$, then for any vertex

$u \in V(G)$, $d_G(u, H)$ or simply $d(u, H)$ is $\min\{d_G(u, x) : x \in V(H)\}$. Let $v$ be a broadcast vertex corresponding to a broadcast labeling $f$ of $G$. Then for any vertex $u \in V(G)$, the *cost of* $v$ *at* $u$ is defined as $f(v) - d(u, v)$ and is denoted by $c_v^f(u)$ or simply $c_v(u)$. Before going to our contribution, we state a result due to Erwin [5].

**Lemma 11** [5]**.** *Let $G$ be a connected graph and $f$ be an optimal dominating broadcast labeling of $G$. Then for every pair $u, v$ of distinct vertices with $f(u) \leqslant f(v)$, $f(u) \leqslant \left\lceil \frac{d(u,v)}{2} \right\rceil$.*

**Lemma 12.** *If $f$ is an optimal dominating broadcast labeling of a tree $T$ with $|V_f^+| \geqslant 2$, then there exists an edge $e$ such that $f$ is an optimal dominating broadcast labeling of $T - e$ also.*

***Proof.*** Let $v$ be a broadcast vertex such that $f(v) = \max\{f(x) : x \in V_f^+\}$. If $N_f[v] \cap N_f[x] = \emptyset$ for all broadcast vertices $x \in V_f^+ \setminus \{v\}$, then there exists an edge $xy$ such that $x \in B_f(v)$ and $y \in B_f(u)$ for some broadcast vertex $u$ $(\neq v)$, which proves our claim.




(a) Case $c_u(v_{t_2}) \leqslant c_v(v_{t_2})$. (b) Case $c_u(v_{t_2}) > c_v(v_{t_2})$.

Figure 1. Schematic diagram for the proof of Lemma 12.

Now, we consider that $N_f[v] \cap N_f[x] \neq \emptyset$ for some $x \in V_f^+ \setminus \{v\}$. Let $v'$ be a boundary vertex of $v$ and $P$ be the $v, v'$-path. Due to Lemma 11, $v \notin N_f[x]$ for all $x \in V_f^+ \setminus \{v\}$. Let $v_{t_1}$ be the first vertex on $P$, after $v$, which is $f$-dominated by some other broadcast vertex, say $u$. Let $v_{t_2}$ be the vertex on $v_{t_1}, v'$-path such that $d(u, v_{t_2}) = d(u, P)$. As $f$ is an ODBL of $T$, $c_v(u) < f(v)$.

If $c_u(v_{t_2}) \leqslant c_v(v_{t_2})$, then, as $f$ is an ODBL, there exists an edge $u_1u_2$ on $v_{t_2}, u$-path such that $c_u(u_1) \leqslant c_v(u_1)$ and $c_u(u_2) \geqslant c_v(u_2)$. One can look at Figure 1(a). Let $A \subseteq N_f[w_1]$ be the set of vertices $f$-dominated by some broadcast vertex $w_1(\neq u, v)$, using the edge $u_1u_2$, in $T$. First we consider, $c_{w_1}(u_2) > c_{w_1}(u_1)$. If we assume $c_{w_1}(u_1) > c_v(u_1)$, then $c_{w_1}(u_1) > c_u(u_1)$. So, $c_{w_1}(v_{t_1}) > 0$ which is a contradiction to the existence of the vertex $v_{t_1}$. Therefore, $c_{w_1}(u_1) \leqslant c_v(u_1)$ and hence $v$ $f$-dominates the vertices of $A$ in $T - u_1u_2$. Now, we consider

$c_{w_1}(u_2) < c_{w_1}(u_1)$. If $c_{w_1}(u_2) > c_u(u_2)$, then $c_{w_1}(v_{t_1}) > 0$, which is a contradiction. Therefore, $c_{w_1}(u_2) \leqslant c_u(u_2)$ and hence $u$ $f$-dominates the vertices of $A$ in $T - u_1u_2$. Hence, $f$ is an ODBL of $T - u_1u_2$.

Now, we consider the case when $c_u(v_{t_2}) > c_v(v_{t_2})$. Then, as $f$ is an ODBL, there exists an edge $v_1v_2$ on $v_{t_2}, v_{t_1}$-path such that $c_u(v_1) \leqslant c_v(v_1)$ and $c_u(v_2) \geqslant c_v(v_2)$. One can refer to Figure 1(b). Let $B \subseteq N_f[w_2]$ be the set of vertices $f$-dominated by some broadcast vertex $w_2$ $(\neq u, v)$, using the edge $v_1v_2$, in $T$. First we consider, $c_{w_2}(v_2) > c_{w_2}(v_1)$. If $c_{w_2}(v_1) > c_v(v_1)$, then $c_{w_2}(v_{t_1}) > 0$, which is a contradiction. Therefore, $c_{w_2}(v_1) \leqslant c_v(v_1)$ and hence $v$ $f$-dominates the vertices of $B$ in $T - v_1v_2$. Now, we consider $c_{w_2}(v_2) < c_{w_2}(v_1)$. If $c_{w_2}(v_2) > c_u(v_2)$, then again $c_{w_2}(v_{t_1}) > 0$, which is again a contradiction. Therefore $c_{w_2}(v_2) \leqslant c_u(v_2)$ and hence $u$ $f$-dominates the vertices of $B$ in $T - v_1v_2$. Hence, $f$ is an ODBL of $T - v_1v_2$. ■

**Theorem 13.** *A connected graph is $\gamma_b^+$-edge-critical if and only if it is a uniquely radial tree.*

***Proof.*** Let $G$ be a connected $\gamma_b^+$-edge-critical graph. Suppose that $G$ contains a cycle. We know that, by Theorem 6, $G$ has a spanning tree $T$ such that $\gamma_b(G) = \gamma_b(T)$. So, there exists an edge in $E(G) \setminus E(T)$ such that deletion of it does not alter $\gamma_b(G)$. Therefore, $G$ is not a $\gamma_b^+$-edge-critical graph. Hence, $G$ is a tree. If $G$ has an ODBL $f$ with $|V_f^+| \geqslant 2$, then, by Lemma 12, $G$ has an edge $e$ such that $\gamma_b(G) = \gamma_b(G - e)$. Therefore, $G$ is not $\gamma_b^+$-edge-critical and hence, $G$ is a uniquely radial tree.

For the converse, suppose $T$ is an uniquely radial tree and $e$ is any edge in $T$. Then, we have $\gamma_b(T) \leqslant \gamma_b(T - e)$. Since number of broadcast vertices in any ODBL of $T - e$ is at least 2, we have $\gamma_b(T) \neq \gamma_b(T - e)$. ■

**Remark 14.** The characterization given in Theorem 13 can be seen as a characterization of uniquely radial tree, which is different from the characterization provided by Mynhardt and Woodlinger [17].

Now, we define a subclass of spider graphs and prove that they are $\gamma_b^+$-edge-critical graphs.

**Theorem 15.** *If $G$ is a spider graph with $\operatorname{rad}(G) = k$ and having at least $k$ pendant vertices at distance $k$ from the maximum degree vertex, then $G$ is a $k$-$\gamma_b^+$-edge-critical graph.*

***Proof.*** By Theorem 1, we have $\gamma_b(G) \leqslant k$. Let $v$ be the central vertex and $\{u_1, u_2, u_3, \ldots, u_k, \ldots, u_m\}$ be the set of pendant vertices of $G$ which are at distance $k$ from $v$. Let $f$ be an ODBL of $G$. As $\sigma(f) \leqslant k$, a broadcast vertex that $f$-dominates $u_i$ must lie on $v, u_i$-path. If $v \notin V_f^+$, or $v \in V_f^+$ and $f(v) < k$, then

$u_i$s are $f$-dominated by distinct broadcast vertices. Since $\deg(v) \geqslant 3$ and $f$ is a DBL of $G$, we get $\sigma(f) > k$ which is a contradiction. Therefore, for any ODBL $f$ of $G$, $f(v) = k$ and $f(x) = 0$, when $x \neq v$. Therefore, $G$ is a uniquely radial tree and hence the result follows from Theorem 13. ∎

### 2.1. Broadcast bondage number

In the beginning of the section, we determine the exact values of the broadcast bondage numbers of $K_n, K_{m,n}, P_n$ and $C_n$. Later in the section, we give a tight upper bound for $b_b(G)$ involving the order and the size of $G$. For any subset $X \subseteq E(G)$ of a graph $G$, $G - X$ is the subgraph of $G$ obtained by deleting all the edges of $G$ in $X$.

**Theorem 16.** *For the complete graph $K_n$, $n \geqslant 2$, $b_b(G) = \lceil \frac{n}{2} \rceil$.*

***Proof.*** For any graph $G$ of order $n$, $\gamma_b(G) = 1$ if and only if $G$ has a vertex of degree $n-1$. For $K_n$, $\lceil \frac{n}{2} \rceil$ is the minimum number of edges to be deleted from $K_n$ to make the degree of each and every vertex less than $n-1$. Therefore, $b_b(G) = \lceil \frac{n}{2} \rceil$. ∎

**Theorem 17.** *For the complete bipartite graph $K_{m,n}$, $m \geqslant 1$ and $n \geqslant 1$ with $(m,n) \neq (2,2)$, $b_b(K_{m,n}) = \min\{m,n\}$.*

***Proof.*** For $m = 1$, $\gamma_b(K_{1,n}) = 1$ and deleting any edge of $K_{1,n}$ results to an isolated vertex and $K_{1,n-1}$. Hence, for any $n \geqslant 1$, $b_b(K_{1,n}) = 1$. For $m, n \geqslant 2$, we know $\gamma_b(K_{m,n}) = 2$. Without loss of generality, we assume $n \leqslant m$. Let $A$ and $B$ be the partite sets of order $m$ and $n$, respectively. Since deleting all the edges incident on a vertex of $A$ increases the broadcast domination number of $K_{m,n}$, we have $b_b(K_{m,n}) \leqslant n$. Now, deletion of less than $n$ number of edges from $K_{m,n}$ leaves a vertex of $A$ with eccentricity 2. Therefore, $b_b(K_{m,n}) = n$ or more generally $b_b(K_{m,n}) = \min\{m,n\}$. It easy to see that $b_b(K_{2,2}) = 3$. ∎

**Theorem 18.** *For any path $P_n$, $n \geqslant 2$*

$$b_b(P_n) = \begin{cases} 2 & \textit{if } n \equiv 1 \pmod 3, \\ 1 & \textit{otherwise.} \end{cases}$$

***Proof.*** For any path $P_n$, we have $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$. Let $P_n$: $v_1 v_2 v_3 \cdots v_n$.

(a) If $n = 3k$ for some natural number $k$, then $\gamma_b(P_{3k}) = k$. Now, $\gamma_b(P_{3k} - v_1v_2) = 1 + \lceil \frac{3k-1}{3} \rceil = k + 1 > \gamma_b(P_{3k})$.

(b) If $n = 3k+2$ for some natural number $k$, then $\gamma_b(P_{3k+2}) = k+1$. Now, $\gamma_b(P_{3k+2} - v_1v_2) = 1 + \lceil \frac{3k+2-1}{3} \rceil = k + 2 > \gamma_b(P_{3k+2})$.

(c) If $n = 3k+1$ for some positive natural number $k$, then $\gamma_b(P_{3k+1}) = k+1$. Since $\gamma_b(P_{3k+1} - \{v_1v_2, v_2v_3\}) = 2 + \lceil \frac{3k+1-2}{3} \rceil = k + 2 > \gamma_b(P_{3k+1})$, we have

$b_b(P_{3k+1}) \leqslant 2$. For any edge $e$ in $P_{3k+1}$, the graph $P_{3k+1} - e$ is a disjoint union of two shorter paths $P_{n_1}$ and $P_{n_2}$. Since $n = 3k+1$, either of the following cases are possible.

*Case* 1. $n_1 = 3k_1$ and $n_2 = 3k_2 + 1$ for some non-negative integers $k_1$ and $k_2$. Then,

$$\gamma_b(P_{n_1}) + \gamma_b(P_{n_2}) = \left\lceil \frac{n_1}{3} \right\rceil + \left\lceil \frac{n_2}{3} \right\rceil = k_1 + k_2 + 1 = \gamma_b(P_{3k+1}).$$

*Case* 2. $n_1 = 3k_1 + 2$ and $n_2 = 3k_2 + 2$ for some non-negative integers $k_1$ and $k_2$. Then,

$$\gamma_b(P_{n_1}) + \gamma_b(P_{n_2}) = \left\lceil \frac{n_1}{3} \right\rceil + \left\lceil \frac{n_2}{3} \right\rceil = k_1 + 1 + k_2 + 1 = k_1 + k_2 + 2 = \gamma_b(P_{3k+1}).$$

Hence, $b_b(P_{3k+1}) = 2$. ■

**Corollary 19.** *For any cycle $C_n$, $n \geqslant 3$,*

$$b_b(C_n) = \begin{cases} 3 & \textit{if } n \equiv 1 \pmod 3, \\ 2 & \textit{otherwise.} \end{cases}$$

***Proof.*** Since $\gamma_b(C_n) = \left\lceil \frac{n}{3} \right\rceil = \gamma_b(P_n)$ and deleting any one edge from a cycle gives a path of same order, the result follows from Theorem 18. ■

In a graph $G$, two leaves are said to be *twin leaves* if they are adjacent to the same vertex. For a pendant vertex, its adjacent vertex is said to be the *support vertex* of it. The next result gives the broadcast bondage number of a graph having twin leaves.

**Theorem 20.** *If a graph $G$ has twin leaves, then $b_b(G) = 1$.*

***Proof.*** Every graph has an ODBL in which none of the broadcast vertices is a leaf. Let $u_1$ and $u_2$ be a pair of twin leaves adjacent to a vertex $v$ in $G$. If, for any DBL $f$ of $G$, $u_1$ is $f$-dominated by some broadcast vertex $w$ ($\neq u_1, u_2$), then $u_2$ is also $f$-dominated by $w$. In any ODBL of $G - vu_1$, $u_1$ must be a broadcast vertex with cost 1. Let $f_1$ be any ODBL of $G - vu_1$ with $u_2 \notin V_{f_1}^{+}(G - vu_1)$. Then, $V_{f_1}^{+}(G - vu_1) \setminus \{u_1\}$ $f_1$-dominates the non-trivial component of $G - vu_1$. Therefore, $\gamma_b(G - vu_1) \geqslant \gamma_b(G) + 1$ and hence the proof. ■

Now, we give an upper bound of $b_b(G)$ in terms of the order and the size of $G$. The bound is sharp for some classes of paths and cycles.

**Lemma 21.** *For any tree $T$, $b_b(T) \leqslant 2$.*

***Proof.*** Let $u$ be an end point of a diametrical path of a tree $T$, with support vertex $v$. Let $w$ be the other vertex on the diametrical path, adjacent to $v$. In any ODBL of $T - (\{vu\}, \{vw\})$, $u$ and $v$ (or the other vertex if the component containing $v$ is $P_2$) must be a broadcast vertex with cost 1. If $f$ is an ODBL of $T - (\{vu\}, \{vw\})$, then $V_f^+(T - (\{vu\}, \{vw\})) \setminus \{u, v\}$ $f$-dominates the component which contains $w$. Therefore, we have a DBL $g$ of $T$ such that $g(x) = f(x)$, when $x \in V_f^+(T - (\{vu\}, \{vw\})) \setminus \{u, v\}$, $g(v) = 1$, and $g(x) = 0$ otherwise. Then, $\gamma_b(T - (\{vu\}, \{vw\})) \geqslant \gamma_b(T) + 1$ and hence $b_b(T) \leqslant 2$. ■

**Theorem 22.** *For any connected graph $G$ of order $n$ and size $m$, $b_b(G) \leqslant m - n + 3$.*

***Proof.*** For any connected graph $G$, $m - (n - 1)$ number of edges to be deleted from $G$ to get a spanning tree of $G$. By Theorem 6, there exists a spanning tree $T$ of $G$ such that $\gamma_b(G) = \gamma_b(T)$. Then by Lemma 21, $b_b(G) \leqslant b_b(T) + (m - n + 1) \leqslant m - n + 3$. ■

The upper bound in Theorem 22 is further improved for the following classes of graphs.

**Proposition 23.** *For any even integer $n$, let $K_n'$ be the graph obtained from $K_n$ by deleting a perfect matching. Then, $b_b(K_n') \leqslant n - 1$.*

***Proof.*** Since degree of every vertex of $K_n'$ is $n - 2$, $\gamma_b(K_n') = 2$. Let $v$ be any vertex of $K_n'$. Now, we delete all the edges incident to $v$ and an edge incident to the vertex which is not adjacent to $v$. Then, in the new graph, $v$ is an isolated vertex and the broadcast domination number of this new graph is 3. Hence, $b_b(K_n') \leqslant n - 1$. ■

**Proposition 24.** *For any odd integer $n$, let $K_n'$ be the graph obtained from $K_n$ by deleting a maximum matching. Then, $b_b(K_n') = 1$. Moreover, if $K_n''$ is the graph obtained from $K_n'$ by deleting an edge incident to the vertex of degree $n - 1$, then $b_b(K_n'') \leqslant n - 2$.*

***Proof.*** The graph $K_n'$ has exactly one vertex of degree $n - 1$ and hence $\gamma_b(K_n') = 1$. Therefore, $\gamma_b(K_n'') = 2$ and hence $b_b(K_n') = 1$.

The graph $K_n''$ has the broadcast domination number 2 and degrees of all the vertices are $n - 2$ except one which has degree $n - 3$. After deleting all the edges incident to the vertex of degree $n - 3$ and then deleting the edge between two maximum degree vertices, we have a non-trivial component with the broadcast domination number 2 and a trivial component. Therefore, $b_b(K_n'') \leqslant n - 2$. ■

The proposition below deals with a relation between the broadcast bondage numbers of a graph and a spanning subgraph of it.

**Proposition 25.** *Let $H$ be a spanning subgraph of $G$ with $\gamma_b(G) = \gamma_b(H)$. Then,*

$$b_b(G) - (|E(G)| - |E(H)|) \leqslant b_b(H) \leqslant b_b(G).$$

***Proof.*** Let $E_G$ be a set of minimum number of edges of $G$ whose deletion from $G$ increases $\gamma_b(G)$ and $E_H$ be a set of minimum number of edges of $H$ whose deletion from $H$ increases $\gamma_b(H)$. Since $\gamma_b(G) = \gamma_b(H)$, then deleting edges of $S \cup E_H$, from $G$, increase $\gamma_b(G)$, where $S = E(G) \setminus E(H)$. Therefore, $b_b(G) \leqslant (|E(G)| - |E(H)|) + |E_H|$ and hence we have our first inequality. For the second inequality, as $\gamma_b(H) = \gamma_b(G) < \gamma_b((G - S) - (E_G \setminus S)) = \gamma_b(H - (E_G \setminus S))$, $b_b(H) \leqslant |E_G \setminus S| \leqslant b_b(G)$. ■

## 3. $\gamma_b^-$-EDGE-CRITICAL GRAPHS

In this section, we first characterize $k$-$\gamma_b^-$-edge-critical graphs for $k = 1, 2$. Next, we give a necessary condition for 3-$\gamma_b^-$-edge-critical graphs in terms of the radius of the graph. Finally, we prove that some classes of trees are not $\gamma_b^-$-edge-critical graphs.

By convention, the complete graph $K_n$, $n \geqslant 1$, is the only 1-$\gamma_b^-$-edge-critical graph. The next theorem gives a characterization for 2-$\gamma_b^-$-edge-critical graphs.

**Theorem 26.** *A connected graph $G$ is 2-$\gamma_b^-$-edge-critical if and only if $\overline{G}$ is a galaxy.*

***Proof.*** Let $G$ be a graph such that $\overline{G}$ is a galaxy. Adding any edge in $G$ implies an edge gets deleted from $\overline{G}$ leaving an isolated vertex in $\overline{G}$. Hence, $G$ is 2-$\gamma_b^-$-edge-critical. Conversely, if $G$ is 2-$\gamma_b^-$-edge-critical then adding any edge leaves a vertex adjacent to all the other vertices in $G$. So, every edge in $\overline{G}$ must be incident to at least one pendant vertex in $\overline{G}$. Thus, each component of $\overline{G}$ must be a star graph. ■

As there is a relation between the theory of domination and broadcast domination, the study of critical aspects match to some extent. For $k \leqslant 2$, the characterization is the same for both the cases, but for $k \geqslant 3$, the theory gets more evolved and complicated in its own merit. From now onwards, unless we mention, all graphs are connected.

For any 1-cap graph, if it is $k$-$\gamma$-critical graph, then it is $k$-$\gamma_b^-$-edge-critical graph. But the converse is not true, at least for $k = 3$. It is to be noted that if a graph $G$ is 3-$\gamma_b^-$-edge-critical, then $\gamma_b(G + e) = 2$ for any edge $e \in \overline{G}$.

Now we construct a class of 1-cap graphs which are 3-$\gamma_b^-$-edge-critical but not 3-$\gamma$-critical. For an integer $p \geqslant 3$, we consider four sets of vertices $A = \{a\}$, $B = \{b_1, b_2, b_3, \ldots, b_p\}$, $C = \{c_1, c_2, c_3, \ldots, c_p\}$ and $D = \{d_1, d_2, d_3, \ldots, d_p\}$. The adjacencies among the vertices are as follows.

(i) $a \leftrightarrow b_i$ for all $i = 1, 2, 3, \ldots, p$.

(ii) $a \leftrightarrow c_i$ for all $i = 2, 3, 4, \ldots, p$.

(iii) $c_i \leftrightarrow d_i$ for all $i = 1, 2, 3, \ldots, p$.

(iv) Each vertex of $B$ is adjacent to $p-1$ number of vertices of $C$ and each vertex of $C$ is adjacent to $p-1$ number of vertices of $B$, such that $b_i \leftrightarrow c_i$ for all $i = 1, 2, 3, \ldots, p$ and $b_p \not\leftrightarrow c_1$.

(v) The vertices of $B, C \setminus \{c_1\}$ and $D \setminus \{d_1\}$, individually, form complete graphs.

We denote this class as $\mathcal{G}_p$ and a graph of $\mathcal{G}_4$ is shown in Figure 2. Let $G$ be any graph in $\mathcal{G}_p$. It is clear that $\gamma_b(G) = \gamma(G) = \text{rad}(G) = 3$. It is easy to observe that for any graph $G \in \mathcal{G}_p$, $\gamma(G + d_1d_2)$ is still 3. Thus $G$ is not a 3-$\gamma$-critical graph. Now, we prove that adding any edge to $G$ decreases the radius to 2.

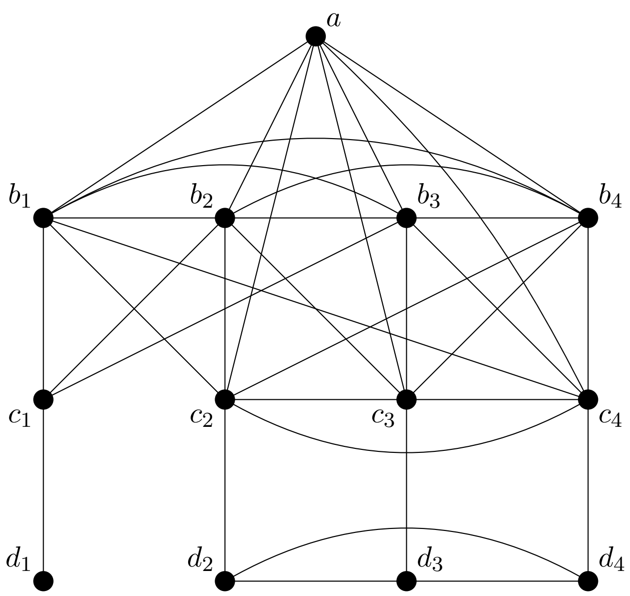



Figure 2. A graph in $\mathcal{G}_4$.

**Theorem 27.** *The graphs in the class $\mathcal{G}_p$, $p \geqslant 3$, is 3-$\gamma_b^-$-edge-critical.*

***Proof.*** The proof follows from Table 1 which gives a vertex of eccentricity 2 for every new edge added to any graph $G$ of $\mathcal{G}_p, p \geqslant 3$. ∎

| New edge | A vertex of eccentricity 2 | New edge | A vertex of eccentricity 2 |
|---|---|---|---|
| $ac_1$ | $a$ | $ad_1$ | $a$ |
| $ad_i$, $2 \leqslant i \leqslant p$ | $b_j$, such that $b_j \nleftrightarrow c_i$ in $G$ | $b_ic_j, 1 \leqslant i,j \leqslant p$ such that $c_j \nleftrightarrow b_i$ in $G$ | $b_i$ |
| $b_id_j$, $1 \leqslant i,j \leqslant p$, such that $b_i \leftrightarrow c_j$ in $G$ | $b_l$, such that $b_l \nleftrightarrow c_j$ in $G$ | $b_id_j$, $1 \leqslant i,j \leqslant p$, such that $b_i \nleftrightarrow c_j$ in $G$ | $b_i$ |
| $c_1c_i$, $2 \leqslant i \leqslant p$ | $c_i$ | $d_1d_i$, $2 \leqslant i \leqslant p$ | $c_i$ |
| $c_id_j$, $1 \leqslant i,j \leqslant p$ and $i \neq j$ | $b_l$, such that $b_l \nleftrightarrow c_j$ in $G$ | | |

Table 1. A vertex of eccentricity 2 corresponding to every newly added edge.

Not only for the graphs in the class $\mathcal{G}_p, p \geqslant 3$, the radius is 3, the radius equals 3 is necessary for a 3-$\gamma_b^-$-edge-critical graph.

**Theorem 28.** *If $G$ is a connected 3-$\gamma_b^-$-edge-critical graph, then* $\mathrm{rad}(G) = 3$.

***Proof.*** Let $G$ be a 3-$\gamma_b^-$-edge-critical graph. Since $\gamma_b(G) = 3$ and $\gamma_b(G) \leqslant \mathrm{rad}(G)$ (Theorem 1), the eccentricity of each vertex in $G$ is greater than 2. To show that $\mathrm{rad}(G) = 3$, it is enough to show that $G$ has a vertex of eccentricity 3. Let $f$ be an ODBL of $G$. If $|V_f^+| = 1$, then, by Theorem 4, we have $\mathrm{rad}(G) = 3$. Let $V_f^+ = \{v_1, v_2\}$ and, without loss of generality, let $f(v_1) = 1$ and $f(v_2) = 2$. If the vertices of $N[v_1]$ form a clique, then $e_G(u) = 3$, where $u$ is the neighbor of $v_2$ along a shortest $v_1, v_2$-path. If $N[v_1]$ does not form a clique, then we consider the graph $G' = G + x_1x_2$, where $x_1$ and $x_2$ are any two non-adjacent vertices of $N(v_1)$. Then $\gamma_b(G') = 2$ and, by Theorem 3, either $\gamma(G') = 2$ and $\mathrm{rad}(G') = 3$, or $\mathrm{rad}(G') = 2$.

(a) $\gamma(G') = 2$ and $\mathrm{rad}(G') = 3$. For any ODBL of $G'$, exactly one of $x_1$ and $x_2$ must be a broadcast vertex. Without loss of generality, let $\{x_1, v\}$ be a set of broadcast vertices, corresponding to an ODBL of $G'$. Then $d_{G'}(x_1, v) = 3$. Let $x$ be the neighbor of $x_1$ along a shortest $x_1, v$-path. There is at least one neighbor of $v$ at distance 3 and all the other neighbors of $v$ are at distance at most 3 from $x$. Since, in $G$, we have $x_1v_1x_2$ in place of $x_1x_2$, the distance (in $G$) between $x$ and any neighbor of $x_1$ is at most 3. Therefore $e_G(x) = 3$.

(b) $\mathrm{rad}(G') = 2$. Let $w$ be a central vertex of $G'$. Let $y$ be a farthest vertex from $w$ in $G$. Then $d_{G'}(w, y) = 2$ and the shortest $w, y$-path in $G'$ must contain the edge $x_1x_2$. Now, we get a shortest $w, y$-path in $G$ by replacing the edge $x_1x_2$ with the path $x_1v_1x_2$, in the $w, y$-path in $G'$. Therefore, $e_G(w) = 3$.

Let $V_f^+ = \{u_1, u_2, u_3\}$ with $f(u_i) = 1$ for $i = 1, 2, 3$. If the vertices of $N[u_i]$, for all $i = 1, 2, 3$, form a clique, then it is easy to observe that $\mathrm{rad}(G) = 3$. If not, without loss of generality, the vertices of $N[u_1]$ does not form a clique. Let $x_1'$ and $x_2'$ be any two non-adjacent vertices of $N(u_1)$. Then, by applying the same arguments as above on the graph $G + x_1'x_2'$, we show that $G$ has a vertex of eccentricity 3. ■

It is quite interesting to study whether trees are $\gamma_b^-$-edge-critical or not. Now, we categorize trees that are not $\gamma_b^-$-edge-critical graphs.

**Lemma 29.** *Let $T$ be a tree and $C$ be the cycle formed by adding an edge $e$ in $T$. Then $\gamma_b(C) \leqslant \gamma_b(T+e)$.*

***Proof.*** Let $f$ be an efficient ODBL of $T+e$ and $V$ be a smallest subset of $V_f^+(T+e)$ which $f$-dominates $C$. Now, we define a DBL $g$ of $C$ such that for any $x \in V(C)$,

$$g(x) = \begin{cases} f(v) & \text{if for some } v \in V,\ d_{T+e}(v,x) = d_{T+e}(v,C), \\ 0 & \text{otherwise.} \end{cases}$$

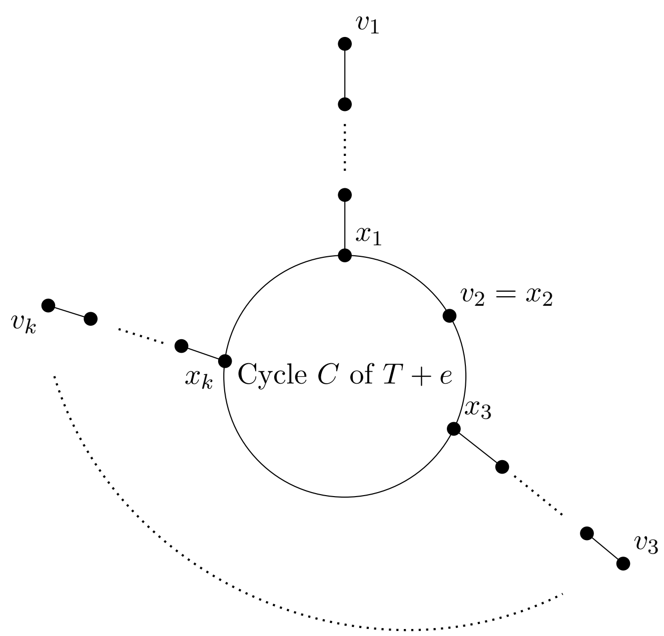



Figure 3. The smallest subset of $V_f^+(T+e)$ is $V = \{v_1, v_2, v_3, \ldots, v_k\}$ and the set $X(\subseteq V(C)) = \{x_1, x_2, x_3, \ldots, x_k\}$ such that $d_{T+e}(v_i, x_i) = d_{T+e}(v_i, C)$. The DBL $g$ of $C$ is $g(x_i) = f(v_i)$ for $i = 1, 2, 3, \ldots, k$ and $g(x) = 0$ for all $x \in V(C) \setminus X$.

An illustration of the function $g$ is given in Figure 3. Then, $\sigma(g) = \sum_{v \in V} f(v) \leqslant \gamma_b(T+e)$. Hence $\gamma_b(C) \leqslant \gamma_b(T+e)$. ■

**Proposition 30.** *Path graphs are not $\gamma_b^-$-edge-critical.*

***Proof.*** Adding an edge in between the end vertices of the path results in a cycle of same order. As the broadcast domination number of a path and a cycle, of same order, is equal, path is not $\gamma_b^-$-edge-critical graph. ■

**Theorem 31.** *Every connected 3-$\gamma_b^-$-edge-critical graph contains a cycle.*

***Proof.*** Let $G$ be a connected 3-$\gamma_b^-$-edge-critical graph. On contrary, suppose that $G$ does not contain a cycle. Due to Proposition 30, further we assume $G$ is not a path. By Theorem 28, we have $\mathrm{rad}(G) = 3$. Let $D$ be a diametrical path in $G$. Now, we add an edge $e$ between the end vertices of the path $D$. Thus, we get a cycle $C$ in $G+e$ of length 6 or 7. First, we consider the cycle $C$ is of length 6. Then, $\gamma_b(C) = 2$. Now, we prove that, $\gamma_b(G+e) \neq 2$. On the contrary, we assume $\gamma_b(G+e) = 2$. Then, every ODBL of $G+e$ has either 1 or 2 broadcast vertices and they must lie on $C$. Moreover, radial labeling cannot be an ODBL of $G+e$ as $\mathrm{rad}(G+e) = 3$. Then, $G+e$ has only 1-cap labeling. If the broadcast vertices of 1-cap labelings are not the pendant vertices of $D$, then the broadcast domination number of $G$ must be 2, a contradiction to the fact that $\gamma_b(G) = 3$. If one of the broadcast vertices is a non-pendant vertex of $D$, then we add an edge $e'$ in $G$, among a pair of neighbors of the non-pendant broadcast vertex, such that one end point of $e'$ lie on $D$ and other not lying on $D$. Then, it is easy to observe that $\gamma_b(G+e') = 3$, and hence $\gamma_b(G) = \gamma_b(G+e')$, which is again a contradiction to the fact that $G$ is a 3-$\gamma_b^-$-edge-critical graph. Therefore, if the length of the cycle $C$ is 6, then $\gamma_b(G+e) = 3$. As by Lemma 29, we have $\gamma_b(C) \leqslant \gamma_b(G+e)$, so if the cycle $C$ is of length 7, then $\gamma_b(G+e) = 3$. Hence, we get a contradiction that $G$ is not a 3-$\gamma_b^-$-edge-critical graph. ■

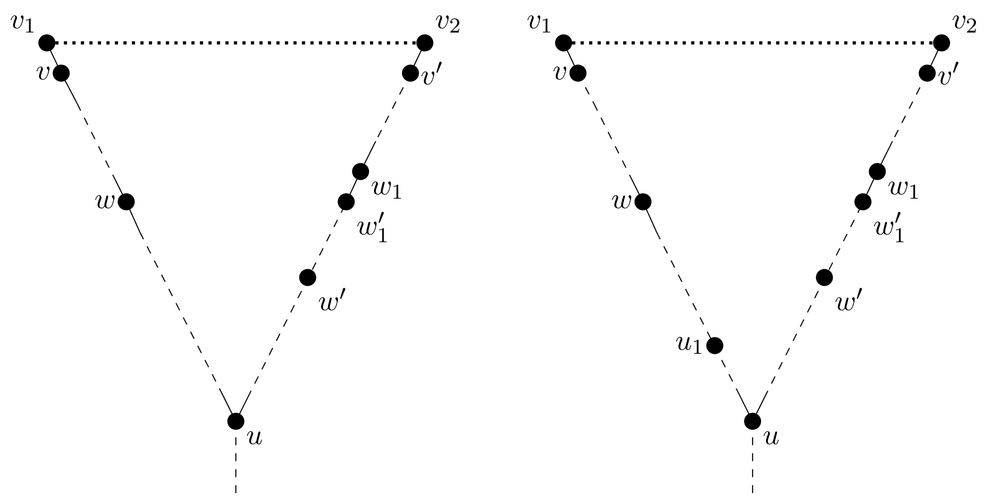
Now, we prove a subclass of trees and uniquely radial trees are not $\gamma_b^-$-edge-critical graphs. For a tree $T$, other than path, let $v_1$ and $v_2$ be two pendant vertices such that $v_1, v_2$-path has only one vertex of degree more than 2 and the degrees of all the other vertices are less than or equal to 2. Let $u$ be the vertex of degree greater than 2 in the $v_1, v_2$-path. Let $\mathcal{T}$ be the subclass of trees such that every tree in $\mathcal{T}$ has at least a triplet of vertices $(v_1, v_2, u)$, as mentioned above, such that $|d(u, v_1) - d(u, v_2)| \equiv 0 \pmod 3$.

**Theorem 32.** *Every tree in $\mathcal{T}$ is not $\gamma_b^-$-edge-critical.*

***Proof.*** Let $T$ be a tree in $\mathcal{T}$. Let $P_1$ and $P_2$ be the $u, v_1$-path and $u, v_2$-path in $T$, respectively. Let $v$ be the neighbor of $v_1$ on the path $P_1$ and $v'$ be the neighbor of $v_2$ on the path $P_2$. Let $f$ be an efficient ODBL of $T' = T + v_1v_2$. Now, we construct an ODBL $g$ of $T$ such that $\sigma(g) = \sigma(f)$, which in turn proves that $T$ is not $\gamma_b^-$-edge-critical. If none of the broadcast vertices of $f$ uses the edge $v_1v_2$, then $f = g$. Let $w$ be the broadcast vertex $f$-dominating both $v_1$ and $v_2$ using the edge $v_1v_2$.

*Case* 1. $d(u, v_1) = d(u, v_2)$.

*Subcase* 1.1. $u \notin N_f[w]$ or $c_w(u) = 0$. Without loss of generality, let $w$ be on $P_1$ and $w_1$ be the boundary vertex of $w$ on $P_2$ such that $w, w_1$-path contains the edge $v_1v_2$. Let $w_1'$ be the neighbor of $w_1$ on $P_2$ and let $w'$ be the broadcast vertex $f$-dominates $w_1'$. For reference, one may look at Figure 4(a).




(a) Reference to *Subcase* 1.1. (b) Reference to *Subcase* 1.2.

Figure 4. Schematic diagram for the proof of Theorem 32.

(i) $d(v_2, w_1)$ is even and $c_w(v_2) > 0$. It can be observed that, if we make $f(w) = 0$ and the cost of the neighbor of $w$ towards $u$ as $f(w) - 1$, and $f(w') = 0$ and the cost of the neighbor of $w'$ towards $v_2$ as $f(w') + 1$, then $f$ will still be an ODBL of $T'$. Let $w_{new}$ be a vertex on $w, u$-path, along $P_1$, such that $d(w, w_{new}) = \frac{d(v_2,w_1)}{2}$. Let $w'_{new}$ be the vertex on $w', w_1'$-path, along $P_2$, such that $d(w', w'_{new}) = \frac{d(v_2,w_1)}{2} - 1$. Now, we have the following ODBL $g$ of $T$.

$$g(x) = \begin{cases} f(w) - \frac{d(v_2,w_1)}{2} & \text{if } x = w_{new}, \\ f(w') + \frac{d(v_2,w_1)}{2} - 1 & \text{if } x = w'_{new}, \\ 1 & \text{if } x = v', \\ 0 & \text{if } x = w, w', \\ f(x) & \text{otherwise.} \end{cases}$$

(ii) $c_w(v_2) = 0$. If $f(w) \geqslant 3$, then we have an ODBL $g$ of $T$ as below.

$$g(x) = \begin{cases} f(w) - 2 & \text{if } x \text{ is the vertex on } w, u\text{-path, along } P_1, \text{ and } d(x, w) = 2, \\ 1 & \text{if } x = v, \\ f(w') + 1 & \text{if } x \text{ is the neighbor of } w' \text{ on } w', v_2\text{-path along } P_2, \\ 0 & \text{if } x = w, w', \\ f(x) & \text{otherwise.} \end{cases}$$

Now, we prove the existence of an ODBL $g$, only when $f(w) = 1$, as if $f(w) = 2$, then we can always get an ODBL $f'$ of $T'$ with $f'(v_1) = 1$, in the following manner. Let $\overline{w}$ be the next broadcast vertex, after $w$, in $w, u$-path along $P_1$.

$$f'(x) = \begin{cases} 1 & \text{if } x = v_1, \\ f(\overline{w}) + 1 & \text{if } x \text{ is the neighbor of } \overline{w} \text{ in } v_1, \overline{w}\text{-path, along } P_1, \\ 0 & \text{if } x = w, \overline{w}, \\ f(x) & \text{otherwise.} \end{cases}$$

Let $u'$ be the broadcast vertex $f$-dominating $u$. If $u'$ lies on $P_1$ and $u' \neq u$, then we get an ODBL $g$ of $T$ by shifting the costs of all the vertices of $P_1$ and $P_2$ one place counter-clockwise. Let $u'$ be the broadcast vertex $f$-dominating $u$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$. For $i = 1, 2$, let $P_i'$ be the path induced by the vertices of the path $P_i$, which are not $f$-dominated by $u'$. Let $P$ be the path induced by $V(P_1') \cup V(P_2')$ in $T'$. If $|V(P_i')| = 3k + 2$, for $i = 1, 2$, then we get an ODBL $g$ of $T$ by replacing the labeling of vertices of $P$ by ODBLs of paths $P_1'$ and $P_2'$. If $|V(P_i)| = 3k$, for $i = 1, 2$, then, as $f(w) = 1$, the cost of $P$, in $f$, is greater than $2k$.

Now, we consider $|V(P_i')| = 3k + 1$, for $i = 1, 2$. If $u' \neq u$, then we construct a DBL $g'$ of $T'$ as below.

$$g'(x) = \begin{cases} f(u') + 1 & \text{if } x \text{ is the neighbor of } u' \text{ in } u, u'\text{-path}, \\ 0 & \text{if } x = u', \\ f(x) & \text{otherwise.} \end{cases}$$

Let $P_i''$ be the path induced by the vertices which are not $g'$-dominated by $\overline{u'}$, of path $P_i'$, for $i = 1, 2$. Now, we have a DBL $g$ of $T$ as follows.

$$g(x) = \begin{cases} h(x) & \text{if } x \in V(P_1'') \text{ and } h \text{ is an ODBL of } P_1'', \\ h(x) & \text{if } x \in V(P_2'') \text{ and } h \text{ is an ODBL of } P_1'', \\ 0 & \text{if } x \in (V(P_1') \cup V(P_2')) \setminus (V(P_1'') \cup V(P_2'')), \\ g'(x) & \text{otherwise.} \end{cases} \tag{1}$$

As $|V(P_i'')| = |V(P_i')| - 2$, for $i = 1, 2$, $\sigma(g) = \sigma(g') - 1 = \sigma(f)$. Hence, $g$ is an ODBL of $T$. If $u' = u$, then we construct a DBL $g'$ of $T'$ as below.

$$g'(x) = \begin{cases} f(u') + 1 & \text{if } x = u', \\ f(x) & \text{otherwise.} \end{cases}$$

Let $P_i''$ denotes the path induced by the vertices which are not $g'$-dominated by $\overline{u'}$, of path $P_i'$, for $i = 1, 2$. Then, we have an ODBL $g$ of $T$ as described in Equation (1).

Now, let $u'$ lie on $P_2$ and $u' \neq u$. If $d(u', u) \geqslant 2$, we get an ODBL $g$ of $T$ by shifting the costs of all the vertices of $P_1$ and $P_2$ two places clockwise. If $d(u', u) = 1$, then by shifting the costs of all the vertices of $P_1$ and $P_2$ one place clockwise, $v_2$ becomes a broadcast vertex of cost 1 and $u = u'$. Then, applying similar arguments as above, we get an ODBL $g$ of $T$.

(iii) $d(v_2, w_1)$ is odd. If we make $f(w) = 0$ and the cost of the neighbor of $w$ towards $u$ as $f(w) - 1$, and $f(w') = 0$ and the cost of the neighbor of $w'$ towards $v_2$ as $f(w') + 1$, then $f$ will still be an ODBL of $T'$. Let $w_{new}$ be a vertex on $w, u$-path, along $P_1$, such that $d(w, w_{new}) = \frac{d(v_2, w_1)+1}{2}$. Let $w'_{new}$ be a vertex on $w', w_1$-path, along $P_2$, such that $d(w', w'_{new}) = \frac{d(v_2, w_1)+1}{2}$. Then, we have the ODBL $g$ of $T$ below.

$$g(x) = \begin{cases} f(w) - \frac{d(v_2, w_1)+1}{2} & \text{if } x = w_{new}, \\ f(w') + \frac{d(v_2, w_1)+1}{2} & \text{if } x = w'_{new}, \\ 0 & \text{if } x = w, w', \\ f(x) & \text{otherwise.} \end{cases}$$

*Subcase* 1.2. $c_w(u) > 0$. Since $c_w(u) < d(u, v_1)$, there exists a vertex $u_1$ on $P_1$, such that $d(u, u_1) = c_w(u)$. Let $\overline{w}$ be a central vertex of the $u_1, w_1$-path that contains the edge $v_1v_2$. For reference, one may look into Figure 4(b). Then, we have an ODBL $f'$ of $T'$ as follows.

$$f'(x) = \begin{cases} f(w) - c_w(u) & \begin{array}{l}\text{if } x \text{ is the vertex on } w, w_1\text{-path containing the edge} \\ v_1v_2 \text{ and } d(w, x) = c_w(u),\end{array} \\ c_w(u) & \text{if } x = u, \\ 0 & \text{if } x = w, \\ f(x) & \text{otherwise.} \end{cases}$$

Now, we apply the same arguments as in *Subcase* 1.1 to get an ODBL $g$ of $T$.

*Case* 2. $d(u, v_1) \neq d(u, v_2)$. Without loss of generality, we assume $d(u, v_1) > d(u, v_2)$. Let $w$ lies on $P_1$. If $u \notin N_f[w]$ or $c_w(u) = 0$, then the arguments of the proof are analogous to that of *Subcase* 1.1, except for the case $f(w) \leqslant 2, c_w(v_2) = 0$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$ for which all the possible orders of $P_1'$ and $P_2'$ are given in Table 2. Path of Type $A$ is not possible, as we can get a DBL of $T'$ of cost lesser than $\gamma_b(T')$. For paths of Type $B$ and Type $C$, we get an ODBL $g$ of $T$ as mentioned in *Subcase* 1.1. When $c_w(u) > 0$, then the proof is same as that of *Subcase* 1.2.

| Type | $\lvert V(P_1')\rvert$ | $\lvert V(P_2')\rvert$ |
|---|---|---|
| $A$ | $3k_1$ | $3k_2$ |
| $B$ | $3k_1+1$ | $3k_2+1$ |
| $C$ | $3k_1+2$ | $3k_2+2$ |

Table 2. Orders of paths $P_1'$ and $P_2'$.

If $w$ lie on $P_2$, then also the proof is similar to that of *Subcase* 1.1 and *Subcase* 1.2, except for the case $f(w) \leqslant 2, c_w(v_1) = 0$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$. Then, similarly, path of Type $A$ is not possible and for paths of Type $B$ and Type $C$, we get an ODBL $g$ of $T$. ■

**Corollary 33.** *If a graph $G$ has twin leaves, then $G$ is not $\gamma_b^-$-edge-critical graph.*

For any tree $T$, other than path, if for every triplet $(v_1, v_2, u)$, $|d(u, v_1) - d(u, v_2)| \not\equiv 0 (\mathrm{mod}\ 3)$, then we discuss $\gamma_b^-$-edge criticality of those trees in the remark below. The claims of the remark can be obtained by similar arguments as in the proof of Theorem 32. All the notations in the following remark are in accordance with Theorem 32.

**Remark 34.** 1. Let for an optimal dominating broadcast labeling $f$ of $T'$ and for a triplet $(v_1, v_2, u)$ of $T$, if $w$ lies on $P_1$, $u \notin N_f[w]$ or $c_w(u) = 0$, except $f(w) \leqslant 2, c_w(v_2) = 0$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$, or $c_w(u) > 0$, then $T$ has an optimal dominating broadcast labeling such that $\gamma_b(T) = \gamma_b(T')$. If, for every optimal dominating broadcast labeling of $T'$ and for every triplet of $T$ with $w$ lying on $P_1$, we have $f(w) \leqslant 2, c_w(v_2) = 0, u \notin N_f[w]$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$, then we have an optimal dominating broadcast labeling $g$ of $T$ such that $\sigma(g) = \sigma(f)$ if $(|V(P_1')|, |V(P_2')|) \in \{(3k_1, 3k_2+1), (3k_1+1, 3k_2), (3k_1+2, 3k_2)\}$. Also, $(|V(P_1')|, |V(P_2')|) \notin \{(3k_1, 3k_2+2), (3k_1+1, 3k_2+2)\}$, or else $T'$ have a dominating broadcast labeling of lesser cost than that of $f$.

2. Let for an optimal dominating broadcast labeling $f$ of $T'$ and for a triplet $(v_1, v_2, u)$ of $T$, if $w$ lies on $P_2$, $u \notin N_f[w]$ or $c_w(u) = 0$, except $f(w) \leqslant 2, c_w(v_1) = 0$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$, or $c_w(u) > 0$, then $T$ has an optimal dominating broadcast labeling such that $\gamma_b(T) = \gamma_b(T')$. If, for every optimal dominating broadcast labeling of $T'$ and for every triplet of $T$ with $w$ lying on $P_2$, we have $f(w) \leqslant 2, c_w(v_1) = 0, u \notin N_f[w]$ and $u' \notin (V(P_1) \cup V(P_2)) \setminus \{u\}$, then we

have an optimal dominating broadcast labeling $g$ of $T$ such that $\sigma(g) = \sigma(f)$ if $(|V(P_1')|, |V(P_2')|) \in \{(3k_1, 3k_2+1), (3k_1, 3k_2+2)\}$. Also, $(|V(P_1')|, |V(P_2')|) \notin \{(3k_1+1, 3k_2), (3k_1+2, 3k_2+1), (3k_1+2, 3k_2)\}$, or else $T'$ have a dominating broadcast labeling of lesser cost than that of $f$.

**Theorem 35.** *A uniquely radial tree is not $\gamma_b^-$-edge-critical graph.*

***Proof.*** Let $T$ be a uniquely radial tree. On the contrary, we assume that $T$ is a $\gamma_b^-$-edge-critical graph. Then, by Corollary 33, $T$ has no twin leaves. Let $x_1$ be one end point of a diametrical path of $T$ with support vertex $x_2$. Since $T$ has no twin leaves, $\deg(x_2) = 2$ and its other adjacent vertex be $x_3$. Let $T' = T + x_1x_3$. The graph $T'$ has three spanning trees, where $T' - x_1x_3$ (i.e., $T$) is isomorphic to $T' - x_2x_3$. Let us consider the spanning tree $T_1 = T' - x_1x_2$. Then, by Theorem 6, $\gamma_b(T_1) = \gamma_b(T') < \gamma_b(T)$.

Let $f$ be an ODBL of $T_1$. Clearly, $x_1$ and $x_2$ are not broadcast vertices of $f$. If $\gamma_b(T_1) \leqslant \operatorname{rad}(T) - 2$, then we have a broadcast labeling $f_1$ of $T$ such that

$$f_1(u) = \begin{cases} f(u) & \text{for all } u \in V(T_1) \setminus \{x_2\}, \\ 1 & u = x_2. \end{cases}$$

Then, $f_1$ is a DBL of $T$ with $\sigma(f_1) < \operatorname{rad}(T) = \gamma_b(T)$, which is a contradiction. Therefore, $\gamma_b(T_1) = \operatorname{rad}(T) - 1$. Moreover, the same broadcast labeling $f_1$ is an ODBL of $T$ with more than one broadcast vertex, which contradicts the fact that $T$ is uniquely radial. Therefore, $T$ is not $\gamma_b^-$-edge-critical. ■

Hence, we have the following result.

**Theorem 36.** *There is no intersection among the classes of connected $\gamma_b^-$-edge-critical graphs and connected $\gamma_b^+$-edge-critical graphs.*

***Proof.*** Theorem 13 implies that any connected $\gamma_b^+$-edge-critical graph must be a uniquely radial tree. Therefore, the proof follows from Theorem 35. ■

**Remark 37.** The disconnected graph $\overline{(K_n - e)}$, $n \geqslant 3$ is the only graph which is both $\gamma_b^+$-edge-critical and $\gamma_b^-$-edge-critical.

### 3.1. Broadcast reinforcement number

We dedicate this section to broadcast reinforcement number of a graph. We show some similar kind of identities as given in Theorem 10(a) and Theorem 10(b). Further, we give an improved upper bound for the class of graphs which are not uniquely radial and prove some sufficient conditions for a graph to have $r_b(G) = 1$.

As there is no scope of further reduction of the broadcast domination number of a graph $G$ with $\gamma_b(G) = 1$, the broadcast reinforcement number, for such graphs, is defined as 0. Now, we give a tight upper bound of $r_b(G)$, involving the order of the graph, $\gamma_b(G)$ and $\Delta(G)$.

**Theorem 38.** *For any graph $G$ of order $n$ and $\gamma_b(G) \geqslant 2$, $r_b(G) \leqslant n - \Delta(G) - \gamma_b(G) + 1$.*

***Proof.*** Let $v$ be a maximum degree vertex of $G$. Now, we obtain a graph $G'$, from $G$, by making $v$ adjacent to exactly $n - \Delta(G) - \gamma_b(G) + 1$ more number of vertices in $G$. It is easy to observe that, the broadcast labeling $f$ defined as $f(v) = 1$, $f(u) = 1$ for all $u \in V(G') \setminus N[v]$, and $f(x) = 0$ otherwise, is a DBL of $G'$. Since $\deg_{G'}(v) = n - \gamma_b(G) + 1$, we have $\gamma_b(G') < \gamma_b(G)$. Therefore, $r_b(G) \leqslant n - \Delta(G) - \gamma_b(G) + 1$. ■

**Remark 39.** Since adding $n - \Delta(G) - 2$ number of edges in a graph $G$ does not produce a vertex of degree $n - 1$, we have $r_b(G) = n - \Delta(G) - 1$ if $\gamma_b(G) = 2$.

Now, we give an improved upper bound for all graphs which are not uniquely radial graphs. The graph $P_9$ has the broadcast reinforcement number 2 and proves the sharpness of the bound given in the theorem below.

**Theorem 40.** *For any graph $G$ which is not uniquely radial and $\gamma_b(G) \geqslant 3$, $r_b(G) \leqslant 2$.*

***Proof.*** Since $G$ is not a uniquely radial graph, it has an ODBL with at least two broadcast vertices. We prove the upper bound in two cases.

*Case* 1. Let $f$ be an ODBL of $G$ with $|V_f^+| \geqslant 2$ and not all broadcast vertices have cost 1. Then, there exist two broadcast vertices $v_1$ and $v_2$ such that $f(v_1) \geqslant f(v_2)$ and $f(v_1) \geqslant 2$. Let $G' = G + v_1v_2$ and we define a broadcast $f_1$ of $G'$ such that

$$f_1(u) = \begin{cases} f(v_1) + 1 & \text{if } u = v_1 \text{ and } f(v_1) = f(v_2) \text{ or} \\ f(v_1) & \text{if } u = v_1 \text{ and } f(v_1) > f(v_2), \\ 0 & \text{if } u = v_2, \\ f(u) & \text{otherwise.} \end{cases}$$

Then, $f_1$ is a DBL of $G'$ and $\gamma_b(G') \leqslant \sigma(f_1) < \sigma(f) = \gamma_b(G)$. Hence, $r_b(G) = 1$.

*Case* 2. If all ODBLs, other than the radial labeling, of $G$ are 1-cap labelings, then for each vertex $v \in V_f^+$, $f(v) = 1$, where $f$ is a non-radial labeling of $G$. Let $v_1, v_2$ and $v_3$ be three broadcast vertices and $G'' = G + \{v_1v_2, v_1v_3\}$. Now, we define a broadcast $f_2$ such that

$$f_2(u) = \begin{cases} 2 & \text{if } u = v_1, \\ 0 & \text{if } u = v_2, v_3, \\ f(u) & \text{otherwise.} \end{cases}$$

Then, $f_2$ is a DBL of $G''$ and $\gamma_b(G'') \leqslant \sigma(f_1) < \sigma(f) = \gamma_b(G)$. Hence, $r_b(G) \leqslant 2$ in this case. ■

**Theorem 41.** *If $G$ is a Type* III *graph, then $r_b(G) = 1$.*

***Proof.*** Any Type III graph has an ODBL of type as mentioned in Case 1 of Theorem 40. Hence, the proof. ∎

We give a tight upper bound for $r_b(G)$ in terms of a parameter which is of similar kind as given in Theorem 10(b). Any graph with broadcast domination number 2 proves the tightness of the bound. For a graph $G$ with $\gamma_b(G) \geqslant 2$, we define

$$\eta_b(G) = \min\{|V(G)\backslash N_f[V_f^+]| : f \text{ is a broadcast labeling of } G \text{ and } \sigma(f) = \gamma_b(G)-1\},$$

where $N_f[V_f^+] = \bigcup_{v\in V_f^+} N_f[v]$.

**Theorem 42.** *For any graph $G$ with $\gamma_b(G) \geqslant 2$, $r_b(G) \leqslant \eta_b(G)$.*

***Proof.*** By the definition of $\eta_b(G)$, $G$ has a broadcast labeling $f$ with $\sigma(f) = \gamma_b(G)-1$ and $|V(G)\backslash N_f[V_f^+]| = \eta_b(G)$. Let $v$ be a broadcast vertex corresponding to the broadcast labeling $f$. Then, adding edges from $v$ to every vertex of $V(G) \setminus N_f[V_f^+]$ yields a graph $G'$ with $\gamma_b(G') \leqslant \sigma(f) < \gamma_b(G)$. Therefore, $r_b(G) \leqslant \eta_b(G)$. ∎

Now, we show that the non-emptiness of the set $V_b^-(G)$ is a necessary and sufficient condition for a graph $G$ with $\eta_b(G) = 1$.

**Proposition 43.** *For any graph $G$ with $\gamma_b(G) \geqslant 2$, $\eta_b(G) = 1$ if and only if $V_b^-(G) \neq \emptyset$.*

***Proof.*** Let $\eta_b(G) = 1$. Then, there exists a broadcast labeling $f$ of $G$ such that $\sigma(f) = \gamma_b(G) - 1$ and only one vertex $v \in V(G)$ is not $f$-dominated. Therefore, $f$ is a DBL of the graph $G - v$ and thus $\gamma_b(G - v) < \gamma_b(G)$. Hence, $v \in V_b^-(G)$.

Conversely, let $u \in V_b^-(G)$. Let $g$ be an ODBL of $G - u$. It is clear that $\gamma_b(G-u) = \gamma_b(G)-1$; or else we have a DBL of $G$ whose cost is less than $\gamma_b(G)$, which is not possible. Then, $g$ is a broadcast labeling of $G$, $\sigma(g) = \gamma_b(G) - 1$ and $V(G) \setminus N_g[V_g^+] = \{u\}$. Therefore, $\eta_b(G) = 1$. ∎

**Theorem 44.** *If $V_b^-(G) \neq \emptyset$ for any graph $G$ with $\gamma_b(G) \geqslant 2$, then $r_b(G) = 1$.*

***Proof.*** The proof follows from Proposition 43 and Theorem 42. ∎

**Theorem 45.** *If $b_b(T) = 2$ for any tree $T$, then $r_b(T) = 1$.*

***Proof.*** Let $T$ be a tree and $b_b(T) = 2$. Let $v$ be a leaf of $T$ with support vertex $u$. Since $b_b(T) = 2$, $\gamma_b(T) = \gamma_b(T - e)$ for any edge $e \in E(T)$. For the edge

$uv \in E(T)$, $T - uv$ has two components and we denote the component containing vertex $u$ as $T_u$. Now,

$$\gamma_b(T) = \gamma_b(T - uv) = 1 + \gamma_b(T_u).$$

Therefore,

$$\gamma_b(T) > \gamma_b(T_u). \tag{2}$$

Since $T_u$ is isomorphic to $T - v$, $\gamma_b(T - v) = \gamma_b(T_u)$. Then, Equation (2) implies $\gamma_b(T) > \gamma_b(T - v)$ and hence $v \in V_b^-(T)$. Since $V_b^-(T) \neq \emptyset$, by Theorem 44, we have $r_b(T) = 1$. ■

We conclude the section by determining the exact value of the broadcast reinforcement number of the hypercube $Q_n$. Brešar and Špacapan [2] have determined the broadcast domination number of hypercubes $Q_n$ as below.

**Theorem 46** [2]. *For hypercube $Q_n$,*

$$\gamma_b(Q_n) = \begin{cases} n & \text{for } n = 1 \text{ and } 2, \\ n-1 & \text{for } n \geqslant 3. \end{cases}$$

Moreover, for $n \geqslant 3$, they have provided an ODBL of $Q_n$, which assigns values $n - k - 1$ and $k$, for $1 \leqslant k \leqslant n - 2$, to two of its antipodal vertices $u$ and $v$, respectively, and zero to all the other vertices. The broadcast reinforcement number of $Q_n$ is given below.

**Proposition 47.** *For hypercube $Q_n$,*

$$r_b(Q_n) = \begin{cases} n-1 & \text{for } n = 1 \text{ and } 2, \\ 4 & \text{for } n = 3, \\ 1 & \text{for } n \geqslant 4. \end{cases}$$

***Proof.*** For $n = 1$ and 2, the broadcast reinforcement number of $Q_n$ is easy to observe and by Remark 39, we have $r_b(Q_3) = 4$. For $n \geqslant 4$, $Q_n$ is not a radial graph and it has an ODBL which satisfies the condition of Case 1 in Theorem 40. Therefore, $r_b(Q_n) = 1$ for $n \geqslant 4$. ■

## 4. CONCLUSION

This is the introductory paper on critical aspects in broadcast domination. We presented an overview of critical aspects of dominating broadcast labeling and studied edge-critical graphs with respect to both edge deletion and edge addition.

Here, we present some open problems which come out naturally in this article along the course of study.

1. We know deletion of an edge or addition of an edge increases or decreases the domination number of a graph by at most 1, which is not the case in broadcast domination. What is $\max\{\gamma_b(G-e)-\gamma_b(G) : e \in E(G)\}$ and $\max\{\gamma_b(G)-\gamma_b(G+e) : e \in E(\overline{G})\}$?

2. In Lemma 21, we proved that for any tree $T$, $b_b(T) \leqslant 2$. So, we propose to characterize the trees for which the broadcast bondage number is 1 or 2. One can look for different bounds for $b_b(G)$ and we believe that the bound given in Theorem 22 can be improved further for many classes of graphs.

3. One basic question is still remain unsolved, whether trees are $\gamma_b^-$-edge-critical graphs? We strongly believe the question has a negative answer. Moreover, study of $k$-$\gamma_b^-$-edge-critical graphs, for $k \geqslant 4$, is completely open.

4. In Theorem 40, we showed that $r_b(G) \leqslant 2$ when $G$ is not uniquely radial graph and $\gamma_b(G) \geqslant 3$. One can study $r_b(G)$ when $G$ is uniquely radial graph and $\gamma_b(G) \geqslant 3$. Moreover, determining the broadcast reinforcement number for different classes of graphs is also interesting.

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