

## SPANNING TRAILS AVOIDING AND CONTAINING GIVEN EDGES

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### Abstract

Let  $\kappa'(G)$  denote the edge connectivity of a graph  $G$ . For any disjoint subsets  $X, Y \subseteq E(G)$  with  $|Y| \leq \kappa'(G) - 1$ , a necessary and sufficient condition for  $G - Y$  to be a contractible configuration for  $G$  containing a spanning closed trail is obtained. We also characterize the structure of a graph  $G$  that has a spanning closed trail containing  $X$  and avoiding  $Y$  when  $|X| + |Y| \leq \kappa'(G)$ . These results are applied to show that if  $G$  is  $(s, t)$ -supereulerian (that is, for any disjoint subsets  $X, Y \subseteq E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ ,  $G$  has a spanning closed trail that contains  $X$  and avoids  $Y$ ) with  $\kappa'(G) = \delta(G) \geq 3$ , then for any permutation  $\alpha$  on the vertex set  $V(G)$ , the permutation graph  $\alpha(G)$  is  $(s, t)$ -supereulerian if and only if  $s + t \leq \kappa'(G)$ .

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\*Supported by the Planning Topics Key Project for the 13th Five-year Plan of Chongqing Education Sciences, grant numbers 2019-GX-118, and by General Project of Natural Science Foundation of Chongqing, China (No.CSTB2022NSCQ-MSX1370).

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**Keywords:**  $\alpha$ -permutation graph,  $(s, t)$ -supereulerian, edge connectivity, collapsible graph.

**2020 Mathematics Subject Classification:** 05C45, 05C40.

## 1. INTRODUCTION

Graphs considered are finite and loopless. We follow [5] for undefined terms and notations. A graph  $G$  is *nontrivial* if it contains at least one edge. As in [5], the connectivity, the edge connectivity and the minimum degree of a graph  $G$  are denoted by  $\kappa(G)$ ,  $\kappa'(G)$  and  $\delta(G)$ , respectively. For a subset  $X$  of  $V(G)$  or of  $E(G)$ , let  $G[X]$  denote the subgraph induced by  $X$ . For notational convenience, we often also use an edge subset  $X$  to denote the induced subgraph  $G[X]$ . When  $X \subseteq V(G)$ , we denote  $G - X = G[V(G) - X]$ ; when  $X \subseteq E(G)$ , we denote  $G - X$  to be a graph with the vertex set  $V(G)$  and the edge set  $E(G) - X$ . If  $X = \{x\}$ , we write  $G - x$  for  $G - \{x\}$  shortly.

Let  $O(G)$  denote the set of all odd degree vertices of a graph  $G$ . A graph  $G$  is *eulerian* if  $G$  is connected with  $O(G) = \emptyset$ . A graph is *supereulerian* if it has a spanning eulerian subgraph. Thus a graph  $G$  is supereulerian if and only if  $G$  has a spanning closed trail. The supereulerian problem was initiated by Boesch, Suffel and Tindell in [4], which seeks to characterize all supereulerian graphs. Pulleyblank [23] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been intensive studies on supereulerian graphs by many authors (see Catlin's survey [7], the supplements [12] and [17], among others).

The concept of  $(s, t)$ -supereulerian graphs was first raised in [20], as a model to generalize supereulerian graphs. Given two non-negative integers  $s$  and  $t$ , a graph  $G$  is  $(s, t)$ -supereulerian if for any disjoint subsets  $X, Y \subseteq E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ ,  $G - Y$  has a spanning closed trail that contains  $X$ . Clearly,  $G$  is supereulerian if and only if  $G$  is  $(0, 0)$ -supereulerian. Since every supereulerian graph must be 2-edge-connected, it follows that any  $(s, t)$ -supereulerian graph must be  $(t + 2)$ -edge-connected. Locally connected  $(s, t)$ -supereulerian graphs have been studied in [18] and [20], among others. In a recent paper [25], Xiong *et al.* showed that while determining if a graph  $G$  is  $(0, 0)$ -supereulerian is NP-complete, when  $t \geq 3$ , whether a graph  $G$  is  $(s, t)$ -supereulerian can be determined in polynomial time. This motivates our current research.

Throughout this paper, we let  $s, t$  be two non-negative integers. We are to investigate, for all values  $s$  and  $t$  with  $s + t \leq \kappa'(G)$ , the structural properties of an  $(s, t)$ -supereulerian graph  $G$  may have, and to apply our findings to study the  $(s, t)$ -supereulerianity of permutation graphs.

A useful tool to study  $(s, t)$ -supereulerian graphs is the elementary subdivision. An *elementary subdivision* of a graph  $G$  at an edge  $e = uv$  is an operation to obtain a new graph  $G(e)$  from  $G - e$  by adding a new vertex  $v_e$  and two new edges  $uv_e$  and  $v_e v$ . For a subset  $X \subseteq E(G)$ , we define  $G(X)$  to be the graph obtained from  $G$  by elementarily subdividing every edge of  $X$ . Thus,  $G$  has a spanning closed trail containing  $X$  if and only if  $G(X)$  is supereulerian.

Let  $2K_1$  be the edgeless graph on two vertices. For a subset  $Y \subseteq E(G)$ , the *contraction*  $G/Y$  is the graph obtained from  $G$  by identifying the two ends of each edge in  $Y$  and then by deleting the resulting loops. If  $H$  is a subgraph of  $G$ , we often use  $G/H$  for  $G/E(H)$ . If  $H$  is connected and  $v_H$  is the vertex in  $G/H$  onto which  $H$  is contracted, then  $H$  is the *preimage* of vertex  $v_H$ . In [6], Catlin introduced collapsible graphs as a powerful tool to study supereulerian graphs. A graph  $G$  is *collapsible* if for any  $R \subseteq V(G)$  with  $|R| \equiv 0 \pmod{2}$ ,  $G$  has a spanning connected subgraph  $S_R$  with  $O(S_R) = R$ . Let  $H_1, H_2, \dots, H_c$  be all maximal collapsible subgraphs of  $G$ . The *reduction* of  $G$ , denoted  $G'$ , is the graph  $G/(H_1 \cup H_2 \cup \dots \cup H_c)$ . A graph  $G$  is *reduced* if  $G' = G$ . Our main results in this paper are as follows.

**Theorem 1.1.** *Let  $G$  be a graph with  $\kappa'(G) \geq 4$  and let  $Y \subseteq E(G)$ . Each of the following holds.*

- (i) *When  $|Y| < \kappa'(G)$ ,  $G - Y$  is collapsible if and only if  $Y$  is not in a minimum edge-cut of  $G$  with  $|Y| = \kappa'(G) - 1$ .*
- (ii) *If  $|Y| \leq \kappa'(G)$  and  $G - Y$  is connected, then either  $G - Y$  is supereulerian, or the reduction of  $G - Y$  is a  $K_2$  or a  $K_{2,p}$ , where  $p$  is an odd integer.*

We observe that Theorem 1.1(i) and (ii) are generalizations of Theorem 1.5 and Theorem 1.6 in [14], respectively.

**Corollary 1.2** (Theorem 1.5 in [14]). *Let  $G$  be a graph with  $\kappa'(G) \geq 4$  and let  $Y \subset E(G)$  be an edge subset with  $|Y| \leq 3$ . Then  $G - Y$  is collapsible if and only if  $Y$  is not contained in a 4-edge-cut of  $G$  when  $|Y| = 3$ .*

It was mistakenly omitted “when  $|Y| = 3$ ” in the original statement of Corollary 1.2 (Theorem 1.5 in [14]) and in the end of argument. In fact, if  $G = K_5$  and  $Y$  consists of two adjacent edges in  $K_5$ , then  $G - Y$  is collapsible, which indicates that Corollary 1.2 is valid only for the case when  $|Y| = 3$ .

**Corollary 1.3** (Theorem 1.6 in [14]). *Let  $G$  be a graph with  $\kappa'(G) \geq 4$  and let  $Y \subset E(G)$  be an edge subset with  $|Y| \leq 4$ . Then  $G - Y$  is collapsible if and only if  $G - Y$  is not contractible to any member in  $\{2K_1, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$ .*

**Theorem 1.4.** *Let  $G$  be a graph with  $\kappa'(G) \geq 4$ . Each of the following holds.*

- (i) *If  $\kappa'(G) \geq s + t + 2$ , then  $G$  is  $(s, t)$ -supereulerian.*

- (ii) Suppose that  $\kappa'(G) \geq s+t+1$  and  $X, Y \subset E(G)$  are two disjoint subsets with  $|X| \leq s$  and  $|Y| \leq t$ . Then,  $G - Y$  has a spanning closed trail containing all edges in  $X$  if and only if  $Y$  is not in any minimum edge-cut of  $G$  with  $|Y| = \kappa'(G) - 1$ .
- (iii) Suppose that  $\kappa'(G) \geq s+t$ . Then  $G$  is not  $(s, t)$ -supereulerian if and only if for some disjoint edge subsets  $X, Y \subseteq E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ , one of the following holds.
  - (a)  $Y$  is in a  $(|Y| + 1)$ -edge-cut of  $G$ .
  - (b) The reduction of  $G - (X \cup Y)$  is a  $2K_1$ , if  $|X| = s$  is odd.
  - (c) The reduction of  $G - Y$  is a member in  $\{2K_1, K_2, K_{2,p} : p \text{ is odd}\}$ , if  $|Y| = \kappa'(G)$ .
  - (d) The reduction of  $(G - Y)(X)$  is a  $K_{2,3}$ , if  $|X \cup Y| = 4 = \kappa'(G)$  with  $1 \leq |X| \leq 2$ .

Let  $j(s, t)$  denote the smallest integer such that every graph  $G$  with  $\kappa'(G) \geq j(s, t)$  is  $(s, t)$ -supereulerian. The value of  $j(s, t)$  was determined in [25] as Theorem 1.2. The original statement missed the case of  $(s, t) = (4, 0)$ , so we corrected it as a corollary of Theorem 1.4 as follows.

**Corollary 1.5** (Theorem 1.2 in [25]).

(1)

$$j(s, t) = \begin{cases} \max\{4, t+2\}, & \text{if } 0 \leq s \leq 1, \text{ or } (s, t) \in \{(2, 0), (2, 1), (3, 0), (4, 0)\}; \\ 5, & \text{if } (s, t) \in \{(2, 2), (3, 1)\}; \\ s+t+\frac{1-(-1)^s}{2}, & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases}$$

The arguments to justify Corollary 1.5 also lead to the following corollary.

**Corollary 1.6.** Let  $G$  be a graph with  $\kappa'(G) < s+t \leq |E(G)|$ . Then,  $G$  is  $(s, t)$ -supereulerian if and only if  $G$  is eulerian and  $t = 0$ .

In this paper, we use  $S_n$  to denote the permutation group of degree  $n$ . Let  $G$  be a graph with vertices  $v_1, v_2, \dots, v_n$ , and let  $G_x$  and  $G_y$  be two copies of  $G$ , with vertex sets  $\{x_1, x_2, \dots, x_n\}$  and  $\{y_1, y_2, \dots, y_n\}$ , respectively, such that  $v_i \mapsto x_i$  and  $v_i \mapsto y_i$  are graph isomorphisms between  $G$  and  $G_x$ ,  $G$  and  $G_y$ , respectively. For each permutation  $\alpha$  in  $S_n$ , we follow [11, 24] to define the  $\alpha$ -permutation graph over  $G$  to be the graph  $\alpha(G)$  that consists of two vertex disjoint copies  $G_x$  and  $G_y$  of  $G$ , along with the edges  $x_i y_{\alpha(i)}$  for each  $1 \leq i \leq n$ . For example, the best known permutation graph is the Petersen graph. In recent years, with the introduction of computer network wiring problems, studies on permutation graphs derived from practical problems have attracted the attention of many graph theory researchers. Prior results on the connectivity, edge connectivity and minimum degree of permutation graphs can be found in [1, 2, 3, 10, 11, 16, 19, 21, 22], and among others.

**Theorem 1.7.** *Let  $G$  be an  $(s, t)$ -supereulerian graph on  $n$  vertices with  $\kappa'(G) \geq 3$ . If  $s + t \leq \kappa'(G) + 1$ , and  $\kappa'(G) \neq \delta(G)$  when  $s + t = \kappa'(G) + 1$ , then  $\alpha(G)$  is  $(s, t)$ -supereulerian for each  $\alpha \in S_n$ .*

**Theorem 1.8.** *Let  $G$  be an  $(s, t)$ -supereulerian graph on  $n$  vertices with  $\kappa'(G) = \delta(G) \geq 3$  and let  $\alpha \in S_n$ . Then,  $\alpha(G)$  is  $(s, t)$ -supereulerian if and only if  $s + t \leq \kappa'(G)$ .*

Needed mechanism will be presented and developed in Section 2, together with some auxiliary results. In Section 3, the main results will be proved. Some discussions on an application to permutation graphs and future work will be addressed in the last two sections.

## 2. PRELIMINARIES

Throughout this paper, for two integers  $m, n$  with  $m < n$ , we denote  $[m, n] = \{m, m+1, \dots, n\}$ . For two vertex subsets  $S, T$  of a graph  $G$ , let  $E_G[S, T] = \{xy : x \in S, y \in T\}$  and  $\partial_G(S) = E_G[S, V(G) - S]$ . We denote  $d_G(v) = |\partial_G(\{v\})|$  to be the *degree* of vertex  $v \in V(G)$ . For two subgraphs  $H_1, H_2$  of  $G$ , we write  $E_G[H_1, H_2]$  for  $E_G[V(H_1), V(H_2)]$ . For two graphs  $G_1, G_2$ , let  $G_1 \cup G_2$  be a graph with the vertex set  $V(G_1) \cup V(G_2)$  and the edge set  $E(G_1) \cup E(G_2)$ .

We will present some of the former results that are needed in our argument. The first summarizes certain properties of collapsible graphs and reduced graphs.

**Theorem 2.1.** *Let  $G$  be a connected graph and  $H$  be a collapsible subgraph of  $G$ . Each of the following holds.*

- (i) (Catlin, Lemma 3 in [6]) *Let  $J$  be a subgraph of  $G$ . If  $G$  is collapsible (respectively, supereulerian), then  $G/J$  is collapsible (respectively, supereulerian).*
- (ii) (Catlin, Theorem 8 in [6])  *$G$  is collapsible if and only if  $G/H$  is collapsible. In particular,  $G$  is collapsible if and only if the reduction of  $G$  is a  $K_1$ .*
- (iii) (Catlin, Theorem 8 in [6])  *$G$  is supereulerian if and only if  $G/H$  is supereulerian.*
- (iv) (Catlin *et al.*, Theorem 3 in [9]) *If each edge of  $G$  is in a cycle of length 2 or 3, then  $G$  is collapsible.*

The *spanning tree packing number* of  $G$ , denoted  $\tau(G)$ , is the maximum number of edge-disjoint spanning trees of  $G$ . Let  $F(G)$  be the minimum number of extra edges that must be added to  $G$  so that the resulting graph has two edge-disjoint spanning trees. Hence,  $\tau(G) \geq 2$  if and only if  $F(G) = 0$ .

**Theorem 2.2.** *Let  $G$  be a connected graph and  $G'$  be the reduction of  $G$ . Each of the following holds.*

- (i) (Catlin, Theorem 2 in [6]) If  $\kappa'(G) \geq 4$ , then  $F(G) = 0$ , and so  $G$  is collapsible.
- (ii) (Catlin, Theorem 7 in [6]) If  $F(G) \leq 1$ , then  $G' \in \{K_1, K_2\}$ .
- (iii) (Catlin *et al.*, Theorem 1.3 in [8]) If  $F(G) \leq 2$ , then  $G' \in \{K_1, K_2, K_{2,t} : t \geq 1\}$ .

**Theorem 2.3** (Piazza and Ringeisen, Theorem 4.2 in [22]). *Let  $G$  be a connected graph on  $n$  vertices with  $\kappa(G) = \delta(G)$ . Then,  $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = \delta(\alpha(G)) = \delta(G) + 1$  for each  $\alpha \in S_n$ .*

**Observation 2.4.** *Let  $G$  be a graph on  $n$  vertices with  $\kappa'(G) \geq 2$ . Then, for each  $\alpha \in S_n$ ,  $\kappa'(G) = \delta(G)$  if and only if  $\kappa'(\alpha(G)) = \kappa'(G) + 1$ .*

**Proof.** Suppose that  $\kappa'(G) = \delta(G)$ . By the definition of  $\alpha(G)$ ,  $\kappa'(\alpha(G)) \geq \kappa'(G) + 1$ . Since  $\kappa'(\alpha(G)) \leq \delta(\alpha(G)) = \delta(G) + 1 = \kappa'(G) + 1$ , we have the equality holds and then we are done.

Conversely, suppose that  $\kappa'(\alpha(G)) = \kappa'(G) + 1$ . Let  $W$  be a minimum edge-cut of  $\alpha(G)$  and let  $H_1, H_2$  be the two components of  $\alpha(G) - W$ . We may assume that  $|V(H_1)| \leq |V(H_2)|$ . Let  $G_1$  and  $G_2$  be the two copies of  $G$  in  $\alpha(G)$ , and let  $U_i = V(G_i) \cap V(H_1)$  and  $V_i = V(G_i) \cap V(H_2)$  for each  $i = 1, 2$ . Since  $G$  is connected,  $E_{\alpha(G)}[U_i, V_i] \neq \emptyset$  for some  $i = 1, 2$ . We may assume that  $E_{\alpha(G)}[U_1, V_1] \neq \emptyset$ . Since  $E_{\alpha(G)}[U_1, V_1]$  is also an edge-cut of  $G_1$ ,  $\kappa'(G) \leq |E_{\alpha(G)}[U_1, V_1]| < |E_{\alpha(G)}[H_1, H_2]| = \kappa'(\alpha(G)) = \kappa'(G) + 1$ , which indicates that  $|E_{\alpha(G)}[U_1, V_1]| = \kappa'(G)$  and  $|V(H_1)| = |U_1| = 1$  as  $\kappa'(G) \geq 2$ . Then,  $\delta(G) \leq |\partial_{G_1}(U_1)| = \kappa'(G)$  and so  $\delta(G) = \kappa'(G)$ . ■

### 3. PROOFS OF THE MAIN RESULTS

#### 3.1. Proofs of Theorems 1.1 and 1.4

Theorems 1.1 and 1.4 will be proved in this subsection. We start with two corollaries of the following theorem.

**Theorem 3.1** (Corollary 2.4 in [25]). *Let  $G$  be a graph, and  $\epsilon, k, \ell$  be integers with  $\epsilon \in \{0, 1\}$  and  $2 \leq k \leq \ell$ . The following are equivalent.*

- (i)  $\kappa'(G) \geq 2\ell + \epsilon$ .
- (ii) For any  $X \subseteq E(G)$  with  $|X| \leq 2\ell - k + \epsilon$ ,  $\tau(G - X) \geq k$ .

**Corollary 3.2.** *Let  $G$  be a graph with  $\kappa'(G) \geq 4$  and let  $\epsilon \in \{0, 1\}$ . If an edge subset  $X \subseteq E(G)$  satisfies  $|X| \leq \kappa'(G) - \epsilon$ , then  $F(G - X) \leq 2 - \epsilon$ .*

**Proof.** Let  $X_1 \subseteq X$  with  $|X_1| = \min\{|X|, 2 - \epsilon\}$ . Then  $|X - X_1| \leq \kappa'(G) - 2$ . As  $\kappa'(G) \geq 4$ , by Theorem 3.1,  $\tau(G - (X - X_1)) \geq 2$ . It implies that  $F(G - X) \leq |X_1| \leq 2 - \epsilon$ . ■

**Corollary 3.3.** *Let  $H_1, H_2$  be two subgraphs of a graph  $G$  with  $|E_G[H_1, H_2]| = \kappa'(G) \geq 4$ . Then,  $\tau(H_1) \geq 2$  and  $\tau(H_2) \geq 2$ . Consequently,  $H_1$  and  $H_2$  are both collapsible.*

**Proof.** Let  $Z \subset E_G[H_1, H_2]$  with  $|Z| = 2$  and  $Z' = E_G[H_1, H_2] - Z$ . Then  $|Z'| = \kappa'(G) - 2$ . By Theorem 3.1,  $\tau(G - Z') \geq 2$ . Since  $Z$  is the minimum edge-cut of  $G - Z'$  and  $|Z| = 2$ , it indicates that  $\tau(H_i) \geq 2$  for each  $i = 1, 2$ . Then, each  $H_i$  is collapsible by Theorem 2.2(i). ■

**Proof of Theorem 1.1.** Suppose that  $G$  is a graph with  $\kappa'(G) \geq 4$  and  $Y \subseteq E(G)$ .

(i) (*Necessity*) Suppose that  $|Y| < \kappa'(G)$  and  $G - Y$  is collapsible. This implies that  $\kappa'(G - Y) \geq 2$ . Then  $Y$  is not lying in any minimum edge-cut of  $G$  when  $|Y| = \kappa'(G) - 1$ .

(*Sufficiency*) Conversely, suppose that  $|Y| < \kappa'(G)$  and  $Y$  is not in any minimum edge-cut of  $G$  with  $|Y| = \kappa'(G) - 1$ . If  $|Y| \leq \kappa'(G) - 2$ , then, by Theorem 3.1,  $\tau(G - Y) \geq 2$ . It implies that  $G - Y$  is collapsible by Theorem 2.2(i). Now we consider that  $|Y| = \kappa'(G) - 1$ . Since there is no edge-cut of  $G$  of size  $\kappa'(G)$  that contains  $Y$ ,  $\kappa'(G - Y) \geq 2$ . As  $\kappa'(G) \geq 4$  and  $|Y| = \kappa'(G) - 1$ , by Corollary 3.2,  $F(G - Y) \leq 1$ . As  $\kappa'(G - Y) \geq 2$ , by Theorem 2.2(ii),  $G - Y$  is collapsible.

(ii) Suppose that  $G - Y$  is connected and  $|Y| \leq \kappa'(G)$ . By Corollary 3.2,  $F(G - Y) \leq 2$ . By Theorem 2.2(iii), either  $G - Y$  is collapsible and then  $G - Y$  is supereulerian; or the reduction of  $G - Y$  is a  $K_2$  or a  $K_{2,p}$ , for some integer  $p \geq 1$ . If  $p$  is even, then as  $K_{2,p}$  is eulerian, it follows by Theorem 2.1(iii) that  $G - Y$  is supereulerian. Hence if  $G - Y$  is not supereulerian, then  $p$  is odd. This completes the proof of Theorem 1.1. ■

To prove Theorem 1.4, we need some additional lemmas, as shown below.

**Lemma 3.4.** *Let  $X$  and  $Y$  be two disjoint edge subsets of  $G$ . If  $G - (X \cup Y)$  is collapsible, then  $G - Y$  has a spanning closed trail containing all edges in  $X$ .*

**Proof.** Let  $R = O(G[X])$ . By the definition of collapsible graphs,  $G - (X \cup Y)$  has a spanning connected subgraph  $L_R$  with  $O(L_R) = R$ . Define  $L = G[E(L_R) \cup X]$ . Then  $O(L) = \emptyset$  and  $V(L) = V(L_R) = V(G)$ . Hence  $L$  is a spanning closed trail of  $G$  with  $X \subseteq E(L)$ , and so the lemma is proved. ■

**Lemma 3.5.** *Let  $G$  be a graph with  $\kappa'(G) \geq 4$ . For every two disjoint edge subsets  $X, Y \subset E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ , each of the following holds.*

- (i) *If  $\kappa'(G) \geq s + t + 2$ , then  $G - (X \cup Y)$  is collapsible.*
- (ii) *If  $\kappa'(G) \geq s + t + 1$ , then either  $G - (X \cup Y)$  is collapsible, or the reduction of  $G - (X \cup Y)$  is a  $K_2$ .*

**Proof.** Assume that the edge subsets  $X$  and  $Y$  are given as stated in the hypotheses of the lemma.

(i) Since  $|X \cup Y| \leq s + t \leq \kappa'(G) - 2$ , it follows by Theorem 3.1, that  $\tau(G - (X \cup Y)) \geq 2$ , and so by Theorem 2.2(i),  $G - (X \cup Y)$  is collapsible.

(ii) By Lemma 3.5(i), it suffices to assume that  $|X \cup Y| = \kappa'(G) - 1$ . By Corollary 3.2,  $F(G - (X \cup Y)) \leq 1$ . By Theorem 2.2(ii), either  $G - (X \cup Y)$  is collapsible, or the reduction of  $G - (X \cup Y)$  is a  $K_2$ . This proves (ii). ■

Recall that  $G(X)$  is the graph obtained from  $G$  by elementarily subdividing every edge of  $X$ . When  $X = \{e_1, e_2, \dots, e_s\}$ , we write  $G(e_1, e_2, \dots, e_s)$  for  $G(\{e_1, e_2, \dots, e_s\})$  and denote  $V_X = \{v_e : e \in X\}$ .

**Lemma 3.6.** *Let  $G$  be a graph and let  $X, Y \subseteq E(G)$  be two disjoint subsets with  $1 \leq |X| \leq 2$  and  $4 \leq |X \cup Y| \leq \kappa'(G)$  satisfying*

- (i)  $G - (X \cup Y)$  is connected,
- (ii)  $G - Y$  is collapsible, and
- (iii) the reduction of  $(G - Y)(X)$  is a  $K_{2,p}$  ( $p \geq 2$ ).

*Then,  $\kappa'(G) = |X \cup Y| = 4$  and  $|X| + 1 \leq p \leq 4$ . Moreover,  $(G - Y)(X)$  has no nontrivial collapsible subgraph that contains  $v_e$  for each  $e \in X$ .*

**Proof.** Assume that  $X = \{e_1\}$  or  $\{e_1, e_2\}$ . Let  $w_1, w_2$  be the two vertices of degree  $p$ , and let  $v_1, v_2, \dots, v_p$  be the vertices of degree two in the reduction of  $(G - Y)(X)$ .

Let  $X' = \{e \in X : (G - Y)(X) \text{ has no nontrivial collapsible subgraph that contains } v_e\}$ . We claim that  $X = X'$ . If not, for each  $e_i \in X - X'$ , let  $L_i$  be the maximal nontrivial collapsible subgraph of  $(G - Y)(X)$  that contains  $v_{e_i}$ . Note that when  $|X - X'| = 2$ ,  $L_1$  and  $L_2$  may be the same. Let  $N_i$  be the graph obtained from  $L_i$  by contracting one incident edge of each  $v_{e_i} \in V(L_i)$ , that is,  $N_i \cong (G - Y)[V(L_i) - V_X]$  for each  $i$ . As  $G - Y$  is collapsible, we have  $(G - Y)(X) / (\bigcup_i L_i) = (G - Y) / (\bigcup_i N_i)$  is collapsible by Theorem 2.1(i). As  $L_i$  is collapsible, then, applying Theorem 2.1(ii),  $(G - Y)(X)$  is collapsible, contrary to the condition of (iii). Thus,  $(G - Y)(X)$  has no nontrivial collapsible subgraph that contains  $v_e$  for each  $e \in X$ .

Then, we may assume that for each  $1 \leq i \leq |X|$ ,  $v_i = v_{e_i}$ . Since  $G - (X \cup Y)$  is connected, we have  $p > |X|$  and denote  $J_i$  to be the preimage of  $v_i$  for each  $i > |X|$ . Let  $H_i$  be the preimage of  $w_i$  for each  $i \in \{1, 2\}$ , and let  $\mathcal{J} = \{H_1, H_2, J_{|X|+1}, \dots, J_p\}$  (see Figure 1). Since

$$\begin{aligned}
 (2) \quad 2(p - |X|) + 2p + 2|Y| &\geq \sum_{J \in \mathcal{J}} |\partial_G(J)| \geq (2 + p - |X|)\kappa'(G) \\
 &\geq (2 + p - |X|)|X \cup Y|,
 \end{aligned}$$



we have  $|X \cup Y| \leq 4$ . As  $|X \cup Y| \geq 4$ , the equalities hold in (2). It shows that for each  $J \in \mathcal{J}$ ,

$$(3) \quad |\partial_G(J)| = \kappa'(G) = |X \cup Y| = 4.$$

When  $|X| = 1$ , by (3), each  $\partial_G(J_i)$  contains at least two edges in  $Y$ . It follows that  $p \leq 4$ . Thus,  $2 \leq p \leq 4$ . When  $|X| = 2$ , by (3), each  $\partial_G(J_i)$  contains all edges in  $Y$ , which implies that  $p \leq 4$ . Thus,  $3 \leq p \leq 4$ . ■

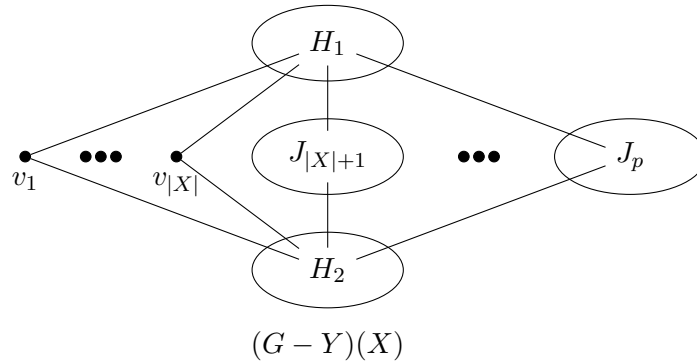


Figure 1. Illustration of the proof of Lemma 3.6.

**Proof of Theorem 1.4.** By Lemma 3.4 and Lemma 3.5(i), Theorem 1.4(i) holds. In the rest of the proofs, we let  $k = \kappa'(G)$ .

(ii) (*Necessity*) Suppose that  $G - Y$  has a spanning closed trail containing all edges in  $X$ . If  $Y$  is in a  $k$ -edge-cut of  $G$  with  $|Y| = k - 1$ , then  $\kappa'(G - Y) = 1$ , which contradicts with our assumption that  $G - Y$  has a spanning closed trail. Thus,  $Y$  is not in any  $k$ -edge-cut of  $G$  with  $|Y| = k - 1$ .

(*Sufficiency*) Suppose that  $Y$  is not in a  $k$ -edge-cut of  $G$  with  $|Y| = k - 1$ . If  $k \geq s + t + 2$ , then by Theorem 1.4(i), we are done. Now, we consider that  $|X| + |Y| = s + t = k - 1$ . It follows by Lemma 3.5(ii),  $G - (X \cup Y)$  is collapsible, or the reduction of  $G - (X \cup Y)$  is a  $K_2$ . If  $G - (X \cup Y)$  is collapsible, then, by Lemma 3.4,  $G - Y$  has a spanning closed trail containing  $X$ . Thus, we only need to consider the situation of the reduction of  $G - (X \cup Y)$  being a  $K_2$ . Let  $w_1 w_2$  be the only edge in the reduction of  $G - (X \cup Y)$ , and let  $H_1, H_2$  be the preimages of  $w_1, w_2$ , respectively. As  $|X| + |Y| = k - 1$ ,  $(X \cup Y) \subset E_G[H_1, H_2]$ . Since  $Y$  is not in any  $k$ -edge-cut of  $G$  with  $|Y| = k - 1$ , it shows that  $t \leq k - 2$  and  $X \neq \emptyset$ . Let  $X = \{e_1, e_2, \dots, e_s\}$  and  $L = (G - Y)(X)$ . Since every edge in  $L/(H_1 \cup H_2) = \{w_1 w_2\} \cup (\bigcup_{1 \leq i \leq s} \{w_1 v_{e_i}, w_2 v_{e_i}\})$  lies in a cycle of length 3, where  $v_{e_i}$  is the new vertex obtained by elementarily subdividing edge  $e_i \in X$ ,

by Theorem 2.1(iv),  $L/(H_1 \cup H_2)$  is collapsible, and so  $L$  is collapsible as well by Theorem 2.1(ii). Then  $L$  is supereulerian, which indicates that  $G - Y$  has a spanning closed trail containing all edges in  $X$ .

(iii) (*Sufficiency*) Suppose that for some disjoint edge subsets  $X, Y \subset E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ , one of Theorem 1.4(iii)(a)–(d) holds. Then,  $G - Y$  does not have a spanning closed trail containing all edges in  $X$ . This shows that  $G$  is not  $(s, t)$ -supereulerian.

(*Necessity*) Suppose that  $G$  is not  $(s, t)$ -supereulerian. Then, there exist two disjoint edge subsets  $X, Y \subset E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$  such that

(4)  $G - Y$  does not have a spanning closed trail containing all edges in  $X$ .

We aim to show that one of Theorem 1.4(iii)(a)–(d) holds. If  $s + t < k$ , then by Theorem 1.4(ii) and (4),  $Y$  is in a minimum edge-cut of  $G$  with  $|Y| = k - 1$ , which is Theorem 1.4(iii)(a). Now we consider that  $|X \cup Y| = s + t = k$ . Let  $X = \{e_1, e_2, \dots, e_s\}$  and distinguish among the following two cases.

*Case 1.*  $G - (X \cup Y)$  is disconnected. Let  $H_1$  and  $H_2$  be the two components of  $G - (X \cup Y)$  and so  $E_G[H_1, H_2] = X \cup Y$ . By Corollary 3.3, each  $H_i$  is collapsible. Then, the reduction of  $G - (X \cup Y)$  is a  $2K_1$ . Let  $w_1, w_2$  be the two vertices of the reduction of  $G - (X \cup Y)$ . If  $X \neq \emptyset$  and  $|X|$  is even, then  $\bigcup_{1 \leq i \leq s} \{w_1 v_{e_i}, w_2 v_{e_i}\}$  is eulerian. It follows by Theorem 2.1(ii) that  $(G - Y)(X)$  is supereulerian, which implies that  $G - Y$  has a spanning closed trail containing all edges in  $X$ , a contradiction to (4). Thus, either  $|X|$  is odd or  $|Y| = k$ , that is, either Theorem 1.4(iii)(b) or (c).

*Case 2.*  $G - (X \cup Y)$  is connected. As  $|X \cup Y| = \kappa'(G) \geq 4$ , by Corollary 3.2,  $F(G - (X \cup Y)) \leq 2$ . By Theorem 2.2(iii), Lemma 3.4, and (4), the reduction of  $G - (X \cup Y)$  is a member of  $\{K_2, K_{2,p} : p \geq 1\}$ .

*Subcase 2.1.* The reduction of  $G - (X \cup Y)$  is a  $K_2$ . Let  $w_1 w_2$  be the only edge of the reduction of  $G - (X \cup Y)$ . Denote  $H_i$  be the preimage of  $w_i$  for each  $i = 1, 2$ .

We claim that  $X \cap E_G[H_1, H_2] = \emptyset$ . If  $X \cap E_G[H_1, H_2] \neq \emptyset$ , let  $X \cap E_G[H_1, H_2] = \{e_1, e_2, \dots, e_{s'}\}$  where  $s - 1 \leq s' \leq s$ . Since every edge in  $L = \{w_1 w_2\} \cup (\bigcup_{1 \leq i \leq s'} \{w_1 v_{e_i}, w_2 v_{e_i}\})$  lies in a cycle of length 3, by Theorem 2.1(iv),  $L$  is collapsible. Since  $s + t = \kappa'(G) \leq |E_G[H_1, H_2]| \leq 1 + |X \cup Y| = 1 + s + t$ , either  $|E_G[H_1, H_2]| = \kappa'(G) + 1$ , or  $|E_G[H_1, H_2]| = \kappa'(G)$  and  $|(X \cup Y) \cap E(H_i)| = 1$  for exactly one  $i \in \{1, 2\}$ , say  $\{e\} = (X \cup Y) \cap E(H_1)$ . If  $|E_G[H_1, H_2]| = \kappa'(G) + 1$ , or  $|E_G[H_1, H_2]| = \kappa'(G)$  and  $e \in Y$ , then  $(G - Y)(X)/(H_1 \cup H_2) = L$  is collapsible, by whence Theorem 2.1(ii),  $(G - Y)(X)$  is collapsible, a contradiction to (4). If  $|E_G[H_1, H_2]| = \kappa'(G)$  and  $e \in X$ , then by Corollary 3.3,  $\tau(H_i) \geq 2$  for each  $i = 1, 2$ , and so  $F(H_1(e)) \leq 1$  and  $\kappa'(H_1(e)) \geq 2$ , which implies that  $H_1(e)$  is

collapsible by Theorem 2.2(ii). Since  $(G - Y)(X)/(H_1(e) \cup H_2) = L$  is collapsible, by Theorem 2.1(ii),  $(G - Y)(X)$  is collapsible, a contradiction to (4).

Then,  $X \cap E_G[H_1, H_2] = \emptyset$ . It shows that if the reduction of  $G - (X \cup Y)$  is a  $K_2$ , then it will be either Theorem 1.4(iii)(a) or (c).

*Subcase 2.2.* The reduction of  $G - (X \cup Y)$  is a  $K_{2,p}$  ( $p \geq 1$ ).

*Subcase 2.2.1.*  $|Y| = k$ . Then  $X = \emptyset$ . If  $p$  is even, then  $(G - (X \cup Y))' = (G - Y)' \cong K_{2,p}$  is eulerian. By Theorem 2.1(ii) that  $G - Y$  is supereulerian, contrary to (4). Thus in this case,  $p$  must be an odd integer, which implies Theorem 1.4(iii)(c).

*Subcase 2.2.2.*  $|Y| = k - 1$ . Then  $X = \{e_1\}$ . By Corollary 3.2,  $F(G - Y) \leq 1$ . It follows by Theorem 2.2(ii) that either  $G - Y$  is collapsible, or  $(G - Y)' \cong K_2$ . If  $(G - Y)' \cong K_2$ , then, since  $(G - (X \cup Y))' = (G - (\{e_1\} \cup Y))' \cong K_{2,p}$  ( $p \geq 1$ ), we have  $p = 1$  and  $\kappa'(G) \leq 2$ , which contradicts with the assumption of  $\kappa'(G) \geq 4$ .

Now, we assume that  $G - Y$  is collapsible. As  $F(G - Y) \leq 1$ , we have  $F((G - Y)(e_1)) \leq 2$ . Let  $G_1 = (G - Y)(e_1)$ . Since  $\kappa'(G - Y) \geq 2$ ,  $\kappa'(G_1) \geq 2$ . Then, by Theorem 2.2(iii) and (4),  $G'_1 \cong K_{2,q}$  ( $q \geq 2$ ). By Lemma 3.6 that  $|Y| = 3$ ,  $\kappa'(G) = 4$  and  $2 \leq q \leq 4$ . If  $q = 2$  or  $4$ , then  $G'_1$  is eulerian and so by Theorem 2.1(ii) that  $G_1 = G(e_1) - Y$  is supereulerian, which means that  $G - Y$  contains a spanning closed trail containing  $X = \{e_1\}$ , contrary to (4). Then,  $q = 3$ , and the reduction  $G'_1 = ((G - Y)(X))' \cong K_{2,3}$ , leading to Theorem 1.4(iii)(d).

*Subcase 2.2.3.*  $|Y| \leq k - 2$ . In this case, let  $X_1 = \{e_1, e_2\}$  and  $X_2 = X - X_1$ . As  $|X_2 \cup Y| = k - 2$ , by Theorem 3.1,  $\tau(G - (X_2 \cup Y)) \geq 2$ . Then, by Theorem 2.2(i),  $G - (X_2 \cup Y)$  is collapsible, and so  $\kappa'(G - (X_2 \cup Y)) \geq 2$ . Let  $G_2 = (G - (X_2 \cup Y))(e_1, e_2)$ . It follows that  $\kappa'(G_2) \geq 2$  and  $F(G_2) \leq 2$ . Then, by Theorem 2.2(iii),  $G'_2 \in \{K_1, K_{2,q} : q \geq 2\}$ .

If  $G'_2 \cong K_1$ , which means that  $G_2 = G(e_1, e_2) - (X_2 \cup Y)$  is collapsible, then by Lemma 3.4,  $G(e_1, e_2) - Y$  contains all edges in  $X_2$ . It follows that  $G - Y$  contains all edges in  $X$ , which contradicts to (4).

If  $G'_2 \cong K_{2,q}$  ( $q \geq 2$ ), then let  $w_1, w_2$  be the two vertices of degree  $q$ , and  $v_1, v_2, \dots, v_q$  be vertices of degree two in  $G'_2$ . Let  $H_i$  be the preimage of  $w_i$  for each  $i = 1, 2$ , and  $J_i$  be the preimage of  $v_i$  for each  $i \in [1, q]$ . By Lemma 3.6,  $|X_2 \cup Y| = 2$ ,  $\kappa'(G) = 4$  and  $3 \leq q \leq 4$ . Let  $v_1 = v_{e_1}$  and  $v_2 = v_{e_2}$ .

*Subcase 2.2.3.1.*  $q = 3$ . In this case, there is exactly one edge in  $X_2 \cup Y$  crossing  $H_i$  and  $J_3$  in  $G$  for each  $i$ . If  $|X_2| = 0$ , it is Theorem 1.4(iii)(d). If  $|X_2| = 1$ , then we may assume that  $e_3 \in E_G[J_3, H_1]$ . Let  $L_1$  be the reduction of  $G(X) - Y$ . Then  $L_1 = G'_2 \cup \{v_3v_{e_3}, w_1v_{e_3}\}$ . As  $L_1 - w_2v_3$  is eulerian,  $L_1$  is supereulerian, which implies that  $G - Y$  has a spanning closed trail containing  $X = \{e_1, e_2, e_3\}$ , contrary to (4). If  $|X_2| = 2$ , then  $Y = \emptyset$  and  $G(X)$  is collapsible,

which means that  $G$  has a spanning closed trail containing all edges in  $X$ , contrary to (4).

*Subcase 2.2.3.2.*  $q = 4$ . In this case,  $G'_2$  is eulerian and  $E_G[J_3, J_4] = X_2 \cup Y$ . When  $|X_2| = 0$ ,  $G'_2 = (G(X) - Y)'$  being eulerian implies that  $G(X) - Y$  is supereulerian, which contradicts to (4).

When  $|X_2| = 1$ ,  $X_2 = \{e_3\}$ . As  $G_2 = (G - Y)(e_1, e_2, e_3) - v_{e_3}$ , let  $L_2 = G'_2 \cup \{v_3v_{e_3}, v_4v_{e_3}\}$  (see Figure 2 for an illustration). Note that  $F(K_{3,3} - e) = 2$ , where  $e \in E(K_{3,3})$ . It follows by Theorem 2.2(iii) that  $L_2[w_1, w_2, v_{e_2}, v_3, v_4, v_{e_3}] \cong K_{3,3} - e$  is collapsible. As  $L_2/L_2[w_1, w_2, v_{e_2}, v_3, v_4, v_{e_3}]$  is a cycle of length 2 that is collapsible, by Theorem 2.1(ii),  $L_2$  is collapsible. This implies that  $G(X) - Y$  is supereulerian, which contradicts to (4).

When  $|X_2| = 2$ ,  $X_2 = \{e_3, e_4\}$ . Let  $L_3 = G'_2 \cup \{v_{e_3}v_3, v_{e_3}v_4, v_{e_4}v_3, v_{e_4}v_4\}$ . Since  $L_3$  is eulerian,  $G(X) - Y$  is supereulerian, which contradicts to (4). ■

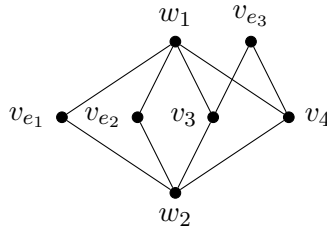


Figure 2. Illustration of the proof of Subcase 2.2.3.2 in Theorem 1.4.

### 3.2. Proofs of Corollaries 1.5 and 1.6

In the subsection, we shall prove Corollary 1.6 and provide a sketch of proof of Corollary 1.5 applying Theorem 1.4. Let us start with a necessary condition of  $(s, t)$ -supereulerian graphs.

**Proposition 3.7.** *If  $G$  is an  $(s, t)$ -supereulerian graph, then  $t \leq \kappa'(G) - 2$  and*

$$s \leq \begin{cases} |E(G)|, & \text{if } G \text{ is eulerian and } t = 0; \\ 2 \left\lfloor \frac{\kappa'(G) - t}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

**Proof.** Let  $k = \kappa'(G)$  and let  $W$  be a minimum edge-cut of  $G$ . Pick an edge subset  $Y \subseteq W$  with  $|Y| \leq t$ . Since  $G$  is  $(s, t)$ -supereulerian,  $G - Y$  has a spanning closed trail  $\Gamma$ . Since  $W$  is an edge-cut of  $G$ ,  $|E(\Gamma) \cap W| \geq 2$  and so  $|Y| \leq |W - E(\Gamma)| \leq k - 2$ . By arbitrary of  $Y$  with  $|Y| \leq t$ , we have  $t \leq k - 2$ .

If  $G$  is eulerian, then  $G$  has a spanning closed trail containing all edges in  $E(G)$ . This means that  $G$  is  $(|E(G)|, 0)$ -supereulerian. Now we assume that  $G$  is non-eulerian or  $t \geq 1$ .

We claim that  $s + t \leq k$ , and when  $s + t = k$ ,  $s \equiv 0 \pmod{2}$ . If not, then we pick an edge subset  $X' \subseteq W$  satisfying that  $|X'| \leq s$ ,  $|X'| \equiv 1 \pmod{2}$  and  $|X'|$  is maximized. Let  $Y' = W - X'$ . Then  $|Y'| \leq 1 \leq t$ . Since  $G$  is  $(s, t)$ -supereulerian,  $G - Y'$  has a spanning closed trail  $\Gamma'$  containing all edges in  $X'$ . Since  $W$  is an edge-cut of  $G$ ,  $X' = E(\Gamma') \cap W \neq \emptyset$  and  $|X'| = |E(\Gamma') \cap W| \equiv 0 \pmod{2}$ , which contradicts with that  $|X'| \equiv 1 \pmod{2}$ . Thus,  $s + t \leq k$ , and when  $s + t = k$ ,  $s \equiv 0 \pmod{2}$ . This follows that  $s \leq 2 \lfloor \frac{k-t}{2} \rfloor$ . ■

By Proposition 3.7, we verify Corollary 1.6 as follows.

**Proof of Corollary 1.6.** Suppose that  $G$  is eulerian and  $t = 0$ . Then for any non-negative integer  $s \leq |E(G)|$ ,  $G$  is  $(s, 0)$ -supereulerian.

Conversely, suppose that  $G$  is  $(s, t)$ -supereulerian, and that either  $G$  is non-eulerian or  $t > 0$ . By Proposition 3.7,  $s \leq 2 \lfloor \frac{\kappa'(G)-t}{2} \rfloor$  and  $t \leq \kappa'(G) - 2$ . This follows that  $s + t \leq \kappa'(G)$ , which contradicts with the assumption of  $\kappa'(G) < s + t$ . Thus, if  $G$  is  $(s, t)$ -supereulerian,  $G$  is eulerian and  $t = 0$ . ■

Recall that  $j(s, t)$  denotes the smallest integer such that every graph  $G$  with  $\kappa'(G) \geq j(s, t)$  is  $(s, t)$ -supereulerian.

**Theorem 3.8** (Examples 3.1(iii) and 3.2(iii) in [25]). *Each of the following holds.*

- (i)  $j(s, t) \geq 4$ ;
- (ii)  $j(2, 2) \geq 5$ .

By Proposition 3.7, Corollary 1.6 and Theorem 3.8(i), we obtain the following corollary (Proposition 1.1 in [25]) immediately. If a graph  $G$  is eulerian, then  $G$  is  $(s, 0)$ -supereulerian where  $s \leq |E(G)|$ . It was mistakenly omitted the condition that  $G$  is non-eulerian or  $t \geq 1$  in the original statement of Corollary 3.9 (Proposition 1.1 in [25]). So we corrected as follows.

**Corollary 3.9** (Proposition 1.1 in [25]). *Let  $G$  be an  $(s, t)$ -supereulerian graph. If  $G$  is non-eulerian or  $t \geq 1$ , then*

$$\kappa'(G) \geq \begin{cases} \max \{4, t + 2\}, & \text{if } s = 0; \\ \max \left\{ 4, s + t + \frac{1 - (-1)^s}{2} \right\}, & \text{if } s \geq 1. \end{cases}$$

Now, we can provide a schetch of proof of Corollay 1.5 as follows.

**Schetch of proof of Corollary 1.5.** Let  $m$  be the right hand side of (1). Let  $G$  be a graph with  $\kappa'(G) \geq m$ . If  $(s, t) = (4, 0)$ , or  $2 \leq s \equiv 0 \pmod{2}$  and  $s + t \geq 5$ , then  $s + t = m$ , and so  $G$  is  $(s, t)$ -supereulerian by Theorem 1.4(iii); otherwise, then  $s + t \leq m - 1$ , and so  $G$  is  $(s, t)$ -supereulerian by Theorem 1.4(ii) as  $t < m - 1$ . Thus, by the definition of  $j(s, t)$ ,  $j(s, t) \leq m$ .

Note that every eulerian graph with  $s$  edges is  $(s, 0)$ -supereulerian. It indicates that to show that  $j(s, t) \geq m$ , it suffices to prove that  $\kappa'(G_1) \geq m$  where  $G_1$  is  $(s, t)$ -supereulerian and  $G_1$  is non-eulerian when  $t = 0$ . Then, by Theorem 3.8(ii) and Corollary 3.9, we have  $\kappa'(G_1) \geq m$  (please see [25] for detailed proof in this part). ■

### 3.3. Proofs of Theorems 1.7 and 1.8

In this subsection, we shall verify Theorems 1.7 and 1.8.

**Proof of Theorem 1.7.** Suppose that  $G$  is  $(s, t)$ -supereulerian with  $\kappa'(G) \geq 3$ . Let  $X, Y \subset E(\alpha(G))$  be two disjoint edge subsets with  $|X| \leq s$  and  $|Y| \leq t$ . Let  $k = \kappa'(\alpha(G))$ .

If  $s + t \leq \kappa'(G)$ , then, as  $k \geq \kappa'(G) + 1 \geq 4$ ,  $s + t \leq k - 1$ . Since  $G$  is  $(s, t)$ -supereulerian, by Proposition 3.7,  $|Y| \leq t \leq \kappa'(G) - 2 \leq k - 3$ . Thus, by Theorem 1.4(ii),  $\alpha(G) - Y$  has a spanning closed trail containing all edges in  $X$ , which implies that  $\alpha(G)$  is  $(s, t)$ -supereulerian.

If  $s + t = \kappa'(G) + 1$  and  $\kappa'(G) \neq \delta(G)$ , then, as  $G$  is  $(s, t)$ -supereulerian, by Corollary 1.6,  $G$  is eulerian and  $t = 0$ . It shows that  $s = \kappa'(G) + 1$ . As  $3 \leq \kappa'(G) \neq \delta(G)$ , by Observation 2.4,  $\kappa'(\alpha(G)) \geq \kappa'(G) + 2 \geq 5$ . Since  $s \leq \kappa'(\alpha(G)) - 1$  and  $t = 0$ , by Theorem 1.4(ii),  $\alpha(G) - Y$  has a spanning closed trail containing all edges in  $X$ , which implies that  $\alpha(G)$  is  $(s, t)$ -supereulerian. ■

By Corollary 1.6 and Theorem 1.7, we have the following corollary directly.

**Corollary 3.10.** *Let  $G$  be an  $(s, t)$ -supereulerian graph on  $n$  vertices with  $\kappa'(G) \geq 3$ . If  $G$  is non-eulerian or  $t \geq 1$ , then  $\alpha(G)$  is  $(s, t)$ -supereulerian for each  $\alpha \in S_n$ .*

**Proof of Theorem 1.8.** Suppose that  $G$  is an  $(s, t)$ -supereulerian graph with  $\kappa'(G) = \delta(G) \geq 3$ . By Theorem 1.7, it suffices to show the necessity of Theorem 1.8. Suppose that  $\alpha(G)$  is  $(s, t)$ -supereulerian. We argue by contradiction and assume that  $s + t > \kappa'(G)$ . Since  $G$  is  $(s, t)$ -supereulerian, by Corollary 1.6,  $G$  is eulerian and  $t = 0$ . This indicates that  $\alpha(G)$  is non-eulerian by the definition of  $\alpha(G)$ . Since  $\alpha(G)$  is  $(s, t)$ -supereulerian and  $t = 0$ , by Proposition 3.7,  $\kappa'(G) < s \leq 2 \left\lfloor \frac{\kappa'(\alpha(G))}{2} \right\rfloor$ . As  $G$  is eulerian,  $\kappa'(G)$  is even. It follows that  $\kappa'(\alpha(G)) \geq \kappa'(G) + 2$ , which contradicts the assumption of  $\kappa'(G) = \delta(G)$  by Observation 2.4. ■

## 4. AN APPLICATION TO ITERATED PERMUTATION GRAPHS

Suppose that  $G$  is a wheel, or an  $n$ -dimensional hypercube  $Q_n$  ( $n \geq 3$ ), or a complete graph  $K_n$  ( $n \geq 4$ ), or a complete bipartite graph  $K_{m,n}$  ( $\min\{m, n\} \geq 3$ ). If  $G$  is  $(s, t)$ -supereulerian, then by Theorem 1.8, for each  $\alpha \in S_{|V(G)|}$ ,  $\alpha(G)$  is  $(s, t)$ -supereulerian if and only if  $s + t \leq \kappa'(G)$ .

Let  $G$  be a graph on  $n$  vertices and let  $A = (\alpha_0, \alpha_1, \alpha_2, \dots)$  be a permutation sequence where  $\alpha_i \in S_{2^{i_n}}$ . We define  $G^0(A) = G$ , and the  $i$ -th iterated permutation graph of  $G$  with respect to the sequence  $A$  is defined recursively as  $G^i(A) = \alpha_{i-1}(G^{i-1}(A))$ , for each positive integer  $i$ . If we do not emphasize the sequence  $A$ , we use  $G^i$  for  $G^i(A)$ . For example, the  $n$ -th iterated permutation graph of  $K_1$  is a hypercube variant, a twisted hypercube of dimension  $n$  (see Definition 1.1 in [26]). By the definition of iterated permutation graphs, as well as Theorem 2.3 and Observation 2.4, we obtain the following observation.

**Observation 4.1.** *Let  $G$  be a connected graph. For each integer  $m \geq 0$ , each of the following holds.*

- (i) *if  $\kappa'(G) = \delta(G)$ , then  $\kappa'(G^m) = \delta(G^m) = \delta(G) + m$ ;*
- (ii) *if  $\kappa(G) = \delta(G)$ , then  $\kappa(G^m) = \kappa'(G^m) = \delta(G^m) = \delta(G) + m$ .*

Given two non-negative integers  $s, t$ , a permutation sequence  $A$ , and a graph  $G$ . By Theorem 1.4(i), when  $\kappa'(G^m) \geq s + t + 2$ ,  $G^m$  is  $(s, t)$ -supereulerian. It follows by Theorem 1.7,  $G^{m+1}$  is also  $(s, t)$ -supereulerian. Therefore, there must exist a smallest integer  $m$  such that  $G^m$  is  $(s, t)$ -supereulerian. In Table 1, we list the edge connectivity  $\kappa'(G^m)$ , which are constructed by some special graphs.

 Table 1. Edge connectivity of  $\alpha(G)$  and  $G^m$  of some special graphs.

$G$	$\kappa(\alpha(G)) = \delta(\alpha(G))$	$\kappa'(G^m)$
Nontrivial tree	2	$m + 1$
$n$ -cycle $C_n$	3	$m + 2$
wheel $W_n$	4	$m + 3$
hypercube $Q_n$	$n + 1$	$n + m$
complete graph $K_n$	$n$	$n + m - 1$
complete bipartite graph $K_{n_1, n_2}$	$\min\{n_1, n_2\} + 1$	$\min\{n_1, n_2\} + m$

In general, for given integers  $s$  and  $t$ , it is an interesting question that how to find the smallest  $m$  such that  $G^m$  is  $(s, t)$ -supereulerian for a connected graph  $G$ . Let  $f(G)$  denote a graphical function and define  $\bar{f}(G)$  to be the maximum value of  $f(H)$  taken over all subgraphs  $H$  of  $G$ . As indicated in [15], for certain network reliability measures  $f$ , networks  $G$  with  $f(G) = \bar{f}(G)$  are important for network

survivability, and so the study of  $\bar{f}(G)$  is of interest. The following theorem gives some new and feasible ideas to find the smallest  $m$ .

**Theorem 4.2.** *Let  $G$  be a connected graph with  $n$  vertices. Each of the following holds.*

- (i) (Corollary 2.2 in [16])  $\kappa'(\alpha(G)) = \delta(\alpha(G))$ , if and only if  $2\kappa'(G) \geq \delta(G) + 1$  for any  $\alpha \in S_n$ .
- (ii) (Corollary 2.3 in [16]) If  $\kappa'(G) = \bar{\delta}(G)$ , then for any  $\alpha \in S_n$ ,  $\kappa'(\alpha(G)) = \bar{\delta}(\alpha(G))$ .
- (iii) (Theorem 2.5 in [16]) If  $\kappa'(G) = \bar{\kappa}'(G)$  and  $\delta(G) = \bar{\delta}(G)$ , then for any  $\alpha \in S_n$ , we have both  $\kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G))$  and  $\delta(\alpha(G)) = \bar{\delta}(\alpha(G))$ .

One can start with any graph  $G$  which satisfies Theorem 4.2 as the initial process to construct large survivable networks by repeatedly taking permutation graphs as  $G^m$ . Then for any given non-negative integer  $s$  and  $t$ , we can apply Theorem 1.4 and Theorem 4.2 to  $G^m$  to find the smallest values of  $m$  such that  $G^m$  is  $(s, t)$ -supereulerian.

## 5. CONCLUSION

In this paper, we characterized the  $(s, t)$ -supereulerianity of a graph  $G$  when  $s + t \leq \kappa'(G)$ . Using this result, for an  $(s, t)$ -supereulerian graph  $G$ , we also obtained a relationship between the  $(s, t)$ -supereulerianity of the permutation graph of  $G$  and the edge connectivity of  $G$ . It shows that if a graph  $G$  has the property that it has a spanning closed trail traversing some given edge set of size at most  $s$  and avoiding some given edge set of size at most  $t$ , then when the edge connectivity of  $G$  is large enough than  $s + t$ , the permutation graph of  $G$  which is a bigger structure will remain this property.

There are some questions that might be of interest for the future work.

1. For a given  $(s, t)$ -supereulerian graph  $G$  and given disjoint subsets  $X, Y \subset E(G)$  with  $|X| \leq s$  and  $|Y| \leq t$ , can we find a spanning closed trail  $T$  of  $G - Y$  such that  $X \subset E(T)$  and  $|E(T)|$  is minimized? If such spanning closed trail  $T$  visits every vertex exactly once, then  $G$  has a hamiltonian cycle containing all edges in  $X$  and avoiding all edges in  $Y$ . As we known, to determine if a graph has a hamiltonian cycle is a NP-complete problem (Theorem 3.4 in [13]). Thus, this generalized optimization question is interest of its own.
2. Pulleyblank [23] proved that determining whether a graph is  $(0, 0)$ -supereulerian, even when restricted to planar graphs, is NP-complete. Lately, it has been shown that when  $t \geq 3$ ,  $(s, t)$ -supereulerianity is polynomially determinable



in [25]. It is currently not known whether it is polynomially determinable or NP-complete when  $t = 1$  or  $2$ .

3. For interconnection network models, such as hypercubes and hypercube variants, for which values of  $s$  and  $t$ , could they be  $(s, t)$ -supereulerian?

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Received 21 October 2022

Revised 6 June 2023

Accepted 6 June 2023

Available online 19 July 2023