Discussiones Mathematicae
Graph Theory 44 (2024) 1429–1447
https://doi.org/10.7151/dmgt.2505

SPANNING TRAILS AVOIDING AND CONTAINING GIVEN EDGES

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Abstract

Let $\kappa'(G)$ denote the edge connectivity of a graph G. For any disjoint subsets $X, Y \subseteq E(G)$ with $|Y| \leq \kappa'(G) - 1$, a necessary and sufficient condition for $G - Y$ to be a contractible configuration for G containing a spanning closed trail is obtained. We also characterize the structure of a graph G that has a spanning closed trail containing X and avoiding Y when $|X| + |Y| \leq \kappa'(G)$. These results are applied to show that if G is (s,t)-supereulerian (that is, for any disjoint subsets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y) with $\kappa'(G) = \delta(G) \geq 3$, then for any permutation α on the vertex set $V(G)$, the permutation graph $\alpha(G)$ is (s,t)-supereulerian if and only if $s+t \leq \kappa'(G)$.

*Supported by the Planning Topics Key Project for the 13th Five-year Plan of Chongqing Education Sciences, grant numbers 2019-GX-118, and by General Project of Natural Science Foundation of Chongqing, China (No.CSTB2022NSCQ-MSX1370).

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Keywords: α-permutation graph, (s,t)-supereulerian, edge connectivity, collapsible graph.

2020 Mathematics Subject Classification: 05C45, 05C40.

1. Introduction

Graphs considered are finite and loopless. We follow [5] for undefined terms and notations. A graph G is *nontrivial* if it contains at least one edge. As in [5], the connectivity, the edge connectivity and the minimum degree of a graph G are denoted by $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$, respectively. For a subset X of $V(G)$ or of $E(G)$, let $G[X]$ denote the subgraph induced by X. For notational convenience, we often also use an edge subset X to denote the induced subgraph $G[X]$. When $X \subseteq V(G)$, we denote $G-X = G[V(G)-X]$; when $X \subseteq E(G)$, we denote $G-X$ to be a graph with the vertex set $V(G)$ and the edge set $E(G)-X$. If $X = \{x\}$, we write $G-x$ for $G-\{x\}$ shortly.

Let $O(G)$ denote the set of all odd degree vertices of a graph G. A graph G is *eulerian* if G is connected with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. Thus a graph G is supereulerian if and only if G has a spanning closed trail. The supereulerian problem was initiated by Boesch, Suffel and Tindell in [4], which seeks to characterize all supereulerian graphs. Pulleyblank [23] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, there have been intensive studies on supereulerian graphs by many authors (see Catlin's survey [7], the supplements [12] and [17], among others).

The concept of (s,t)-supereulerian graphs was first raised in [20], as a model to generalize supereulerian graphs. Given two non-negative integers s and t, a graph G is (s,t)-*supereulerian* if for any disjoint subsets $X, Y \subseteq E(G)$ with $|X| \le s$ and $|Y| \le t$, $G-Y$ has a spanning closed trail that contains X. Clearly, G is supereulerian if and only if G is $(0,0)$-supereulerian. Since every supereulerian graph must be 2-edge-connected, it follows that any (s,t)-supereulerian graph must be $(t+2)$-edge-connected. Locally connected (s,t)-supereulerian graphs have been studied in [18] and [20], among others. In a recent paper [25], Xiong *et al.* showed that while determining if a graph G is $(0,0)$-supereulerian is NP-complete, when $t \ge 3$, whether a graph G is (s,t)-supereulerian can be determined in polynomial time. This motivates our current research.

Throughout this paper, we let s,t be two non-negative integers. We are to investigate, for all values s and t with $s+t \le \kappa'(G)$, the structural properties of an (s,t)-supereulerian graph G may have, and to apply our findings to study the (s,t)-supereulerianicity of permutation graphs.

A useful tool to study (s,t)-supereulerian graphs is the elementary subdivision. An *elementary subdivision* of a graph G at an edge $e = uv$ is an operation to obtain a new graph $G(e)$ from $G - e$ by adding a new vertex v_e and two new edges uv_e and $v_e v$. For a subset $X \subseteq E(G)$, we define $G(X)$ to be the graph obtained from G by elementarily subdividing every edge of X. Thus, G has a spanning closed trail containing X if and only if $G(X)$ is supereulerian.

Let $2K_1$ be the edgeless graph on two vertices. For a subset $Y \subseteq E(G)$, the *contraction* G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. If H is connected and v_H is the vertex in G/H onto which H is contracted, then H is the *preimage* of vertex v_H. In [6], Catlin introduced collapsible graphs as a powerful tool to study supereulerian graphs. A graph G is *collapsible* if for any $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph S_R with $O(S_R) = R$. Let $H_1, H_2, \ldots, H_c$ be all maximal collapsible subgraphs of G. The *reduction* of G, denoted G', is the graph $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$. A graph G is *reduced* if $G' = G$. Our main results in this paper are as follows.

Theorem 1.1. *Let G be a graph with $\kappa'(G) \geq 4$ and let $Y \subseteq E(G)$. Each of the following holds.*

(i) *When $|Y| < \kappa'(G)$, $G-Y$ is collapsible if and only if Y is not in a minimum edge-cut of G with $|Y| = \kappa'(G) - 1$.*

(ii) *If $|Y| \leq \kappa'(G)$ and $G - Y$ is connected, then either $G - Y$ is supereulerian, or the reduction of $G - Y$ is a K_2 or a $K_{2,p}$, where p is an odd integer.*

We observe that Theorem 1.1(i) and (ii) are generalizations of Theorem 1.5 and Theorem 1.6 in [14], respectively.

Corollary 1.2 (Theorem 1.5 in [14])**.** *Let G be a graph with $\kappa'(G) \geq 4$ and let $Y \subset E(G)$ be an edge subset with $|Y| \leq 3$. Then $G - Y$ is collapsible if and only if Y is not contained in a 4-edge-cut of G when $|Y| = 3$.*

It was mistakingly omitted "when $|Y| = 3$" in the original statement of Corollary 1.2 (Theorem 1.5 in [14]) and in the end of argument. In fact, if $G = K_5$ and Y consists of two adjacent edges in K_5, then $G - Y$ is collapsible, which indicates that Corollary 1.2 is valid only for the case when $|Y| = 3$.

Corollary 1.3 (Theorem 1.6 in [14])**.** *Let G be a graph with $\kappa'(G) \geq 4$ and let $Y \subset E(G)$ be an edge subset with $|Y| \leq 4$. Then $G - Y$ is collapsible if and only if $G - Y$ is not contractible to any member in $\{2K_1, K_2, K_{2,2}, K_{2,3}, K_{2,4}\}$.*

Theorem 1.4. *Let G be a graph with $\kappa'(G) \geq 4$. Each of the following holds.*

(i) *If $\kappa'(G) \geq s + t + 2$, then G is (s,t)-supereulerian.*

(ii) *Suppose that $\kappa'(G) \geq s+t+1$ and $X, Y \subset E(G)$ are two disjoint subsets with $|X| \leq s$ and $|Y| \leq t$. Then, $G - Y$ has a spanning closed trail containing all edges in X if and only if Y is not in any minimum edge-cut of G with $|Y| = \kappa'(G) - 1$.*

(iii) *Suppose that $\kappa'(G) \geq s+t$. Then G is not (s,t)-supereulerian if and only if for some disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, one of the following holds.*

(a) *Y is in a $(|Y|+1)$-edge-cut of G.*

(b) *The reduction of $G - (X \cup Y)$ is a $2K_1$, if $|X| = s$ is odd.*

(c) *The reduction of $G - Y$ is a member in $\{2K_1, K_2, K_{2,p} : p \text{ is odd}\}$, if $|Y| = \kappa'(G)$.*

(d) *The reduction of $(G - Y)(X)$ is a $K_{2,3}$, if $|X \cup Y| = 4 = \kappa'(G)$ with $1 \leq |X| \leq 2$.*

Let $j(s,t)$ denote the smallest integer such that every graph G with $\kappa'(G) \geq j(s,t)$ is (s,t)-supereulerian. The value of $j(s,t)$ was determined in [25] as Theorem 1.2. The original statement missed the case of $(s,t) = (4,0)$, so we corrected it as a corollary of Theorem 1.4 as follows.

Corollary 1.5 (Theorem 1.2 in [25])**.**
(1)
$$j(s,t) = \begin{cases} \max\{4, t+2\}, & \text{if } 0 \leq s \leq 1, \text{ or } (s,t) \in \{(2,0),(2,1),(3,0),(4,0)\}; \\ 5, & \text{if } (s,t) \in \{(2,2),(3,1)\}; \\ s+t+\frac{1-(-1)^s}{2}, & \text{if } s \geq 2 \text{ and } s+t \geq 5. \end{cases}$$

The arguments to justify Corollary 1.5 also lead to the following corollary.

Corollary 1.6. *Let G be a graph with $\kappa'(G) < s+t \leq |E(G)|$. Then, G is (s,t)-supereulerian if and only if G is eulerian and $t = 0$.*

In this paper, we use S_n to denote the permutation group of degree n. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$, and let G_x and G_y be two copies of G, with vertex sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$, respectively, such that $v_i \longmapsto x_i$ and $v_i \longmapsto y_i$ are graph isomorphisms between G and G_x, G and G_y, respectively. For each permutation α in S_n, we follow [11, 24] to define the *α-permutation graph* over G to be the graph $\alpha(G)$ that consists of two vertex disjoint copies G_x and G_y of G, along with the edges $x_i y_{\alpha(i)}$ for each $1 \leq i \leq n$. For example, the best known permutation graph is the Petersen graph. In recent years, with the introduction of computer network wiring problems, studies on permutation graphs derived from practical problems have attracted the attention of many graph theory researchers. Prior results on the connectivity, edge connectivity and minimum degree of permutation graphs can be found in [1, 2, 3, 10, 11, 16, 19, 21, 22], and among others.

Theorem 1.7. *Let G be an (s,t)-supereulerian graph on n vertices with $\kappa'(G) \geq 3$. If $s+t \leq \kappa'(G)+1$, and $\kappa'(G) \neq \delta(G)$ when $s+t = \kappa'(G)+1$, then $\alpha(G)$ is (s,t)-supereulerian for each $\alpha \in S_n$.*

Theorem 1.8. *Let G be an (s,t)-supereulerian graph on n vertices with $\kappa'(G) = \delta(G) \geq 3$ and let $\alpha \in S_n$. Then, $\alpha(G)$ is (s,t)-supereulerian if and only if $s+t \leq \kappa'(G)$.*

Needed mechanism will be presented and developed in Section 2, together with some auxiliary results. In Section 3, the main results will be proved. Some discussions on an application to permutation graphs and future work will be addressed in the last two sections.

2. Preliminaries

Throughout this paper, for two integers m, n with $m < n$, we denote $[m,n] = \{m, m+1, \ldots, n\}$. For two vertex subsets S, T of a graph G, let $E_G[S,T] = \{xy : x \in S, y \in T\}$ and $\partial_G(S) = E_G[S, V(G)-S]$. We denote $d_G(v) = |\partial_G(\{v\})|$ to be the *degree* of vertex $v \in V(G)$. For two subgraphs H_1, H_2 of G, we write $E_G[H_1,H_2]$ for $E_G[V(H_1), V(H_2)]$. For two graphs G_1, G_2, let $G_1 \cup G_2$ be a graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

We will present some of the former results that are needed in our argument. The first summarizes certain properties of collapsible graphs and reduced graphs.

Theorem 2.1. *Let G be a connected graph and H be a collapsible subgraph of G. Each of the following holds.*

(i) (Catlin, Lemma 3 in [6]) *Let J be a subgraph of G. If G is collapsible (respectively, supereulerian), then G/J is collapsible (respectively, supereulerian).*

(ii) (Catlin, Theorem 8 in [6]) *G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction of G is a K_1.*

(iii) (Catlin, Theorem 8 in [6]) *G is supereulerian if and only if G/H is supereulerian.*

(iv) (Catlin *et al.*, Theorem 3 in [9]) *If each edge of G is in a cycle of length 2 or 3, then G is collapsible.*

The *spanning tree packing number* of G, denoted $\tau(G)$, is the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of extra edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. Hence, $\tau(G) \geq 2$ if and only if $F(G) = 0$.

Theorem 2.2. *Let G be a connected graph and G' be the reduction of G. Each of the following holds.*

(i) (Catlin, Theorem 2 in [6]) *If $\kappa'(G) \geq 4$, then $F(G) = 0$, and so G is collapsible.*

(ii) (Catlin, Theorem 7 in [6]) *If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$.*

(iii) (Catlin *et al.*, Theorem 1.3 in [8]) *If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t} : t \geq 1\}$.*

Theorem 2.3 (Piazza and Ringeisen, Theorem 4.2 in [22])**.** *Let G be a connected graph on n vertices with $\kappa(G) = \delta(G)$. Then, $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = \delta(\alpha(G)) = \delta(G) + 1$ for each $\alpha \in S_n$.*

Observation 2.4. *Let G be a graph on n vertices with $\kappa'(G) \geq 2$. Then, for each $\alpha \in S_n$, $\kappa'(G) = \delta(G)$ if and only if $\kappa'(\alpha(G)) = \kappa'(G) + 1$.*

Proof. Suppose that $\kappa'(G) = \delta(G)$. By the definition of $\alpha(G)$, $\kappa'(\alpha(G)) \geq \kappa'(G) + 1$. Since $\kappa'(\alpha(G)) \leq \delta(\alpha(G)) = \delta(G) + 1 = \kappa'(G) + 1$, we have the equality holds and then we are done.

Conversely, suppose that $\kappa'(\alpha(G)) = \kappa'(G) + 1$. Let W be a minimum edge-cut of $\alpha(G)$ and let H_1, H_2 be the two components of $\alpha(G) - W$. We may assume that $|V(H_1)| \leq |V(H_2)|$. Let G_1 and G_2 be the two copies of G in $\alpha(G)$, and let $U_i = V(G_i) \cap V(H_1)$ and $V_i = V(G_i) \cap V(H_2)$ for each $i = 1, 2$. Since G is connected, $E_{\alpha(G)}[U_i, V_i] \neq \emptyset$ for some $i = 1, 2$. We may assume that $E_{\alpha(G)}[U_1, V_1] \neq \emptyset$. Since $E_{\alpha(G)}[U_1, V_1]$ is also an edge-cut of G_1, $\kappa'(G) \leq |E_{\alpha(G)}[U_1, V_1]| < |E_{\alpha(G)}[H_1, H_2]| = \kappa'(\alpha(G)) = \kappa'(G) + 1$, which indicates that $|E_{\alpha(G)}[U_1, V_1]| = \kappa'(G)$ and $|V(H_1)| = |U_1| = 1$ as $\kappa'(G) \geq 2$. Then, $\delta(G) \leq |\partial_{G_1}(U_1)| = \kappa'(G)$ and so $\delta(G) = \kappa'(G)$. ■

3. Proofs of the Main Results

3.1. Proofs of Theorems 1.1 and 1.4

Theorems 1.1 and 1.4 will be proved in this subsection. We start with two corollaries of the following theorem.

Theorem 3.1 (Corollary 2.4 in [25])**.** *Let G be a graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0, 1\}$ and $2 \leq k \leq \ell$. The following are equivalent.*

(i) $\kappa'(G) \geq 2\ell + \epsilon$.

(ii) *For any $X \subseteq E(G)$ with $|X| \leq 2\ell - k + \epsilon$, $\tau(G - X) \geq k$.*

Corollary 3.2. *Let G be a graph with $\kappa'(G) \geq 4$ and let $\epsilon \in \{0, 1\}$. If an edge subset $X \subseteq E(G)$ satisfies $|X| \leq \kappa'(G) - \epsilon$, then $F(G - X) \leq 2 - \epsilon$.*

Proof. Let $X_1 \subseteq X$ with $|X_1| = \min\{|X|, 2 - \epsilon\}$. Then $|X - X_1| \leq \kappa'(G) - 2$. As $\kappa'(G) \geq 4$, by Theorem 3.1, $\tau(G - (X - X_1)) \geq 2$. It implies that $F(G - X) \leq |X_1| \leq 2 - \epsilon$. ■

Corollary 3.3. *Let H_1, H_2 be two subgraphs of a graph G with $|E_G[H_1, H_2]| = \kappa'(G) \geq 4$. Then, $\tau(H_1) \geq 2$ and $\tau(H_2) \geq 2$. Consequently, H_1 and H_2 are both collapsible.*

Proof. Let $Z \subset E_G[H_1, H_2]$ with $|Z| = 2$ and $Z' = E_G[H_1, H_2] - Z$. Then $|Z'| = \kappa'(G) - 2$. By Theorem 3.1, $\tau(G - Z') \geq 2$. Since Z is the minimum edge-cut of $G - Z'$ and $|Z| = 2$, it indicates that $\tau(H_i) \geq 2$ for each $i = 1, 2$. Then, each H_i is collapsible by Theorem 2.2(i). ∎

***Proof of Theorem* 1.1.** Suppose that G is a graph with $\kappa'(G) \geq 4$ and $Y \subseteq E(G)$.

(i) (*Necessity*) Suppose that $|Y| < \kappa'(G)$ and $G - Y$ is collapsible. This implies that $\kappa'(G - Y) \geq 2$. Then Y is not lying in any minimum edge-cut of G when $|Y| = \kappa'(G) - 1$.

(*Sufficiency*) Conversely, suppose that $|Y| < \kappa'(G)$ and Y is not in any minimum edge-cut of G with $|Y| = \kappa'(G) - 1$. If $|Y| \leq \kappa'(G) - 2$, then, by Theorem 3.1, $\tau(G - Y) \geq 2$. It implies that $G - Y$ is collapsible by Theorem 2.2(i). Now we consider that $|Y| = \kappa'(G) - 1$. Since there is no edge-cut of G of size $\kappa'(G)$ that contains Y, $\kappa'(G - Y) \geq 2$. As $\kappa'(G) \geq 4$ and $|Y| = \kappa'(G) - 1$, by Corollary 3.2, $F(G - Y) \leq 1$. As $\kappa'(G - Y) \geq 2$, by Theorem 2.2(ii), $G - Y$ is collapsible.

(ii) Suppose that $G - Y$ is connected and $|Y| \leq \kappa'(G)$. By Corollary 3.2, $F(G - Y) \leq 2$. By Theorem 2.2(iii), either $G - Y$ is collapsible and then $G - Y$ is supereulerian; or the reduction of $G - Y$ is a K_2 or a $K_{2,p}$, for some integer $p \geq 1$. If p is even, then as $K_{2,p}$ is eulerian, it follows by Theorem 2.1(iii) that $G - Y$ is supereulerian. Hence if $G - Y$ is not supereulerian, then p is odd. This completes the proof of Theorem 1.1. ∎

To prove Theorem 1.4, we need some additional lemmas, as shown below.

Lemma 3.4. *Let X and Y be two disjoint edge subsets of G. If $G - (X \cup Y)$ is collapsible, then $G - Y$ has a spanning closed trail containing all edges in X.*

Proof. Let $R = O(G[X])$. By the definition of collapsible graphs, $G - (X \cup Y)$ has a spanning connected subgraph L_R with $O(L_R) = R$. Define $L = G[E(L_R) \cup X]$. Then $O(L) = \emptyset$ and $V(L) = V(L_R) = V(G)$. Hence L is a spanning closed trail of G with $X \subseteq E(L)$, and so the lemma is proved. ∎

Lemma 3.5. *Let G be a graph with $\kappa'(G) \geq 4$. For every two disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, each of the following holds.*

(i) *If $\kappa'(G) \geq s + t + 2$, then $G - (X \cup Y)$ is collapsible.*

(ii) *If $\kappa'(G) \geq s + t + 1$, then either $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2.*

Proof. Assume that the edge subsets X and Y are given as stated in the hypotheses of the lemma.

(i) Since $|X \cup Y| \leq s + t \leq \kappa'(G) - 2$, it follows by Theorem 3.1, that $\tau(G - (X \cup Y)) \geq 2$, and so by Theorem 2.2(i), $G - (X \cup Y)$ is collapsible.

(ii) By Lemma 3.5(i), it suffices to assume that $|X \cup Y| = \kappa'(G) - 1$. By Corollary 3.2, $F(G - (X \cup Y)) \leq 1$. By Theorem 2.2(ii), either $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2. This proves (ii). ∎

Recall that $G(X)$ is the graph obtained from G by elementarily subdividing every edge of X. When $X = \{e_1, e_2, \ldots, e_s\}$, we write $G(e_1, e_2, \ldots, e_s)$ for $G(\{e_1, e_2, \ldots, e_s\})$ and denote $V_X = \{v_e : e \in X\}$.

Lemma 3.6. *Let G be a graph and let $X, Y \subseteq E(G)$ be two disjoint subsets with $1 \leq |X| \leq 2$ and $4 \leq |X \cup Y| \leq \kappa'(G)$ satisfying*

(i) *$G - (X \cup Y)$ is connected,*

(ii) *$G - Y$ is collapsible, and*

(iii) *the reduction of $(G - Y)(X)$ is a $K_{2,p}$ ($p \geq 2$).*

Then, $\kappa'(G) = |X \cup Y| = 4$ and $|X| + 1 \leq p \leq 4$. Moreover, $(G - Y)(X)$ has no nontrivial collapsible subgraph that contains v_e for each $e \in X$.

Proof. Assume that $X = \{e_1\}$ or $\{e_1, e_2\}$. Let w_1, w_2 be the two vertices of degree p, and let $v_1, v_2, \ldots, v_p$ be the vertices of degree two in the reduction of $(G - Y)(X)$.

Let $X' = \{e \in X : (G - Y)(X)$ has no nontrivial collapsible subgraph that contains $v_e\}$. We claim that $X = X'$. If not, for each $e_i \in X - X'$, let L_i be the maximal nontrivial collapsible subgraph of $(G - Y)(X)$ that contains v_{e_i}. Note that when $|X - X'| = 2$, L_1 and L_2 may be the same. Let N_i be the graph obtained from L_i by contracting one incident edge of each $v_{e_i} \in V(L_i)$, that is, $N_i \cong (G - Y)[V(L_i) - V_X]$ for each i. As $G - Y$ is collapsible, we have $(G - Y)(X)/(\bigcup_i L_i) = (G - Y)/(\bigcup_i N_i)$ is collapsible by Theorem 2.1(i), As L_i is collapsible, then, applying Theorem 2.1(ii), $(G - Y)(X)$ is collapsible, contrary to the condition of (iii). Thus, $(G - Y)(X)$ has no nontrivial collapsible subgraph that contains v_e for each $e \in X$.

Then, we may assume that for each $1 \leq i \leq |X|$, $v_i = v_{e_i}$. Since $G - (X \cup Y)$ is connected, we have $p > |X|$ and denote J_i to be the preimage of v_i for each $i > |X|$. Let H_i be the preimage of w_i for each $i \in \{1, 2\}$, and let $\mathcal{J} = \{H_1, H_2, J_{|X|+1}, \ldots, J_p\}$ (see Figure 1). Since

$$\begin{aligned} 2(p - |X|) + 2p + 2|Y| &\geq \sum_{J \in \mathcal{J}} |\partial_G(J)| \geq (2 + p - |X|)\kappa'(G) \\ &\geq (2 + p - |X|)|X \cup Y|, \end{aligned} \tag{2}$$

we have $|X \cup Y| \leq 4$. As $|X \cup Y| \geq 4$, the equalities hold in (2). It shows that for each $J \in \mathcal{J}$,

$$|\partial_G(J)| = \kappa'(G) = |X \cup Y| = 4. \tag{3}$$

When $|X| = 1$, by (3), each $\partial_G(J_i)$ contains at least two edges in Y. It follows that $p \leq 4$. Thus, $2 \leq p \leq 4$. When $|X| = 2$, by (3), each $\partial_G(J_i)$ contains all edges in Y, which implies that $p \leq 4$. Thus, $3 \leq p \leq 4$. ■

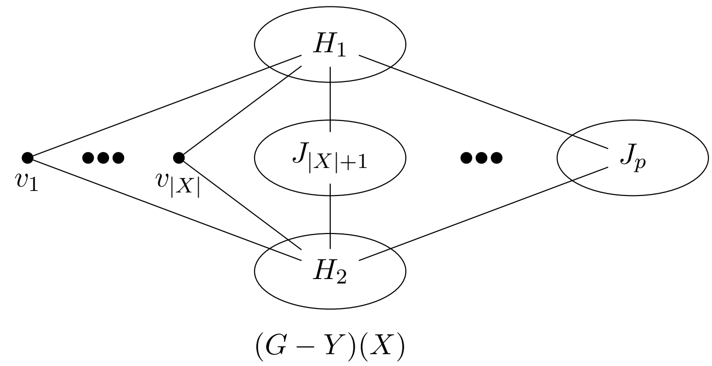

Figure 1. Illustration of the proof of Lemma 3.6.

***Proof of Theorem* 1.4.** By Lemma 3.4 and Lemma 3.5(i), Theorem 1.4(i) holds. In the rest of the proofs, we let $k = \kappa'(G)$.

(ii) (*Necessity*) Suppose that $G - Y$ has a spanning closed trail containing all edges in X. If Y is in a k-edge-cut of G with $|Y| = k-1$, then $\kappa'(G-Y) = 1$, which contradicts with our assumption that $G - Y$ has a spanning closed trail. Thus, Y is not in any k-edge-cut of G with $|Y| = k-1$.

(*Sufficiency*) Suppose that Y is not in a k-edge-cut of G with $|Y| = k-1$. If $k \geq s+t+2$, then by Theorem 1.4(i), we are done. Now, we consider that $|X| + |Y| = s+t = k-1$. It follows by Lemma 3.5(ii), $G - (X \cup Y)$ is collapsible, or the reduction of $G - (X \cup Y)$ is a K_2. If $G - (X \cup Y)$ is collapsible, then, by Lemma 3.4, $G - Y$ has a spanning closed trail containing X. Thus, we only need to consider the situation of the reduction of $G - (X \cup Y)$ being a K_2. Let w_1w_2 be the only edge in the reduction of $G - (X \cup Y)$, and let H_1, H_2 be the preimages of w_1, w_2, respectively. As $|X| + |Y| = k-1$, $(X \cup Y) \subset E_G[H_1, H_2]$. Since Y is not in any k-edge-cut of G with $|Y| = k-1$, it shows that $t \leq k-2$ and $X \neq \emptyset$. Let $X = \{e_1, e_2, \ldots, e_s\}$ and $L = (G-Y)(X)$. Since every edge in $L/(H_1 \cup H_2) = \{w_1w_2\} \cup (\bigcup_{1 \leq i \leq s}\{w_1v_{e_i}, w_2v_{e_i}\})$ lies in a cycle of length 3, where v_{e_i} is the new vertex obtained by elementarily subdividing edge $e_i \in X$, by Theorem 2.1(iv), $L/(H_1 \cup H_2)$ is collapsible, and so L is collapsible as well by Theorem 2.1(ii). Then L is supereulerian, which indicates that $G - Y$ has a spanning closed trail containing all edges in X.

(iii) (*Sufficiency*) Suppose that for some disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, one of Theorem 1.4(iii)(a)–(d) holds. Then, $G - Y$ does not have a spanning closed trail containing all edges in X. This shows that G is not (s,t)-supereulerian.

(*Necessity*) Suppose that G is not (s,t)-supereulerian. Then, there exist two disjoint edge subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$ such that

(4) $\quad G - Y$ does not have a spanning closed trail containing all edges in X.

We aim to show that one of Theorem 1.4(iii)(a)-(d) holds. If $s + t < k$, then by Theorem 1.4(ii) and (4), Y is in a minimum edge-cut of G with $|Y| = k - 1$, which is Theorem 1.4(iii)(a). Now we consider that $|X \cup Y| = s + t = k$. Let $X = \{e_1, e_2, \ldots, e_s\}$ and distinguish among the following two cases.

Case 1. $G-(X \cup Y)$ is disconnected. Let H_1 and H_2 be the two components of $G - (X \cup Y)$ and so $E_G[H_1, H_2] = X \cup Y$. By Corollary 3.3, each H_i is collapsible. Then, the reduction of $G - (X \cup Y)$ is a $2K_1$. Let w_1, w_2 be the two vertices of the reduction of $G - (X \cup Y)$. If $X \neq \emptyset$ and $|X|$ is even, then $\bigcup_{1 \leq i \leq s}\{w_1v_{e_i}, w_2v_{e_i}\}$ is eulerian. It follows by Theorem 2.1(ii) that $(G - Y)(X)$ is supereulerian, which implies that $G - Y$ has a spanning closed trail containing all edges in X, a contradiction to (4). Thus, either $|X|$ is odd or $|Y| = k$, that is, either Theorem 1.4(iii)(b) or (c).

Case 2. $G-(X \cup Y)$ is connected. As $|X \cup Y| = \kappa'(G) \geq 4$, by Corollary 3.2, $F(G - (X \cup Y)) \leq 2$. By Theorem 2.2(iii), Lemma 3.4, and (4), the reduction of $G - (X \cup Y)$ is a member of $\{K_2, K_{2,p} : p \geq 1\}$.

Subcase 2.1. The reduction of $G - (X \cup Y)$ is a K_2. Let w_1w_2 be the only edge of the reduction of $G - (X \cup Y)$. Denote H_i be the preimage of w_i for each $i = 1, 2$.

We claim that $X \cap E_G[H_1, H_2] = \emptyset$. If $X \cap E_G[H_1, H_2] \neq \emptyset$, let $X \cap E_G[H_1, H_2] = \{e_1, e_2, \ldots, e_{s'}\}$ where $s - 1 \leq s' \leq s$. Since every edge in $L = \{w_1w_2\} \cup (\bigcup_{1 \leq i \leq s'}\{w_1v_{e_i}, w_2v_{e_i}\})$ lies in a cycle of length 3, by Theorem 2.1(iv), L is collapsible. Since $s+t = \kappa'(G) \leq |E_G[H_1, H_2]| \leq 1+|X \cup Y| = 1+s+t$, either $|E_G[H_1, H_2]| = \kappa'(G)+1$, or $|E_G[H_1, H_2]| = \kappa'(G)$ and $|(X \cup Y) \cap E(H_i)| = 1$ for exactly one $i \in \{1, 2\}$, say $\{e\} = (X \cup Y) \cap E(H_1)$. If $|E_G[H_1, H_2]| = \kappa'(G)+1$, or $|E_G[H_1, H_2]| = \kappa'(G)$ and $e \in Y$, then $(G - Y)(X)/(H_1 \cup H_2) = L$ is collapsible, by whence Theorem 2.1(ii), $(G - Y)(X)$ is collapsible, a contradiction to (4). If $|E_G[H_1, H_2]| = \kappa'(G)$ and $e \in X$, then by Corollary 3.3, $\tau(H_i) \geq 2$ for each $i = 1, 2$, and so $F(H_1(e)) \leq 1$ and $\kappa'(H_1(e)) \geq 2$, which implies that $H_1(e)$ is collapsible by Theorem 2.2(ii). Since $(G-Y)(X)/(H_1(e)\cup H_2) = L$ is collapsible, by Theorem 2.1(ii), $(G-Y)(X)$ is collapsible, a contradiction to (4).

Then, $X \cap E_G[H_1, H_2] = \emptyset$. It shows that if the reduction of $G-(X\cup Y)$ is a K_2, then it will be either Theorem 1.4(iii)(a) or (c).

Subcase 2.2. The reduction of $G-(X\cup Y)$ is a $K_{2,p}$ $(p \geq 1)$.

Subcase 2.2.1. $|Y| = k$. Then $X = \emptyset$. If p is even, then $(G-(X\cup Y))' = (G-Y)' \cong K_{2,p}$ is eulerian. By Theorem 2.1(ii) that $G-Y$ is supereulerian, contrary to (4). Thus in this case, p must be an odd integer, which implies Theorem 1.4(iii)(c).

Subcase 2.2.2. $|Y| = k-1$. Then $X = \{e_1\}$. By Corollary 3.2, $F(G-Y) \leq 1$. It follows by Theorem 2.2(ii) that either $G-Y$ is collapsible, or $(G-Y)' \cong K_2$. If $(G-Y)' \cong K_2$, then, since $(G-(X\cup Y))' = (G-(\{e_1\}\cup Y))' \cong K_{2,p}$ $(p \geq 1)$, we have $p = 1$ and $\kappa'(G) \leq 2$, which contradicts with the assumption of $\kappa'(G) \geq 4$.

Now, we assume that $G-Y$ is collapsible. As $F(G-Y) \leq 1$, we have $F((G-Y)(e_1)) \leq 2$. Let $G_1 = (G-Y)(e_1)$. Since $\kappa'(G-Y) \geq 2$, $\kappa'(G_1) \geq 2$. Then, by Theorem 2.2(iii) and (4), $G_1' \cong K_{2,q}$ $(q \geq 2)$. By Lemma 3.6 that $|Y| = 3$, $\kappa'(G) = 4$ and $2 \leq q \leq 4$. If $q = 2$ or 4, then G_1' is eulerian and so by Theorem 2.1(ii) that $G_1 = G(e_1) - Y$ is supereulerian, which means that $G-Y$ contains a spanning closed trail containing $X = \{e_1\}$, contrary to (4). Then, $q = 3$, and the reduction $G_1' = ((G-Y)(X))' \cong K_{2,3}$, leading to Theorem 1.4(iii)(d).

Subcase 2.2.3. $|Y| \leq k-2$. In this case, let $X_1 = \{e_1, e_2\}$ and $X_2 = X - X_1$. As $|X_2 \cup Y| = k-2$, by Theorem 3.1, $\tau(G-(X_2\cup Y)) \geq 2$. Then, by Theorem 2.2(i), $G-(X_2\cup Y)$ is collapsible, and so $\kappa'(G-(X_2\cup Y)) \geq 2$. Let $G_2 = (G-(X_2\cup Y))(e_1, e_2)$. It follows that $\kappa'(G_2) \geq 2$ and $F(G_2) \leq 2$. Then, by Theorem 2.2(iii), $G_2' \in \{K_1, K_{2,q} : q \geq 2\}$.

If $G_2' \cong K_1$, which means that $G_2 = G(e_1, e_2) - (X_2 \cup Y)$ is collapsible, then by Lemma 3.4, $G(e_1, e_2) - Y$ contains all edges in X_2. It follows that $G-Y$ contains all edges in X, which contradicts to (4).

If $G_2' \cong K_{2,q}$ $(q \geq 2)$, then let w_1, w_2 be the two vertices of degree q, and $v_1, v_2, \ldots, v_q$ be vertices of degree two in G_2'. Let H_i be the preimage of w_i for each $i = 1, 2$, and J_i be the preimage of v_i for each $i \in [1, q]$. By Lemma 3.6, $|X_2 \cup Y| = 2$, $\kappa'(G) = 4$ and $3 \leq q \leq 4$. Let $v_1 = v_{e_1}$ and $v_2 = v_{e_2}$.

Subcase 2.2.3.1. $q = 3$. In this case, there is exactly one edge in $X_2 \cup Y$ crossing H_i and J_3 in G for each i. If $|X_2| = 0$, it is Theorem 1.4(iii)(d). If $|X_2| = 1$, then we may assume that $e_3 \in E_G[J_3, H_1]$. Let L_1 be the reduction of $G(X) - Y$. Then $L_1 = G_2' \cup \{v_3 v_{e_3}, w_1 v_{e_3}\}$. As $L_1 - w_2 v_3$ is eulerian, L_1 is supereulerian, which implies that $G-Y$ has a spanning closed trail containing $X = \{e_1, e_2, e_3\}$, contrary to (4). If $|X_2| = 2$, then $Y = \emptyset$ and $G(X)$ is collapsible, which means that G has a spanning closed trail containing all edges in X, contrary to (4).

Subcase 2.2.3.2. $q = 4$. In this case, G_2' is eulerian and $E_G[J_3, J_4] = X_2 \cup Y$. When $|X_2| = 0$, $G_2' = (G(X) - Y)'$ being eulerian implies that $G(X) - Y$ is supereulerian, which contradicts to (4).

When $|X_2| = 1$, $X_2 = \{e_3\}$. As $G_2 = (G - Y)(e_1, e_2, e_3) - v_{e_3}$, let $L_2 = G_2' \cup \{v_3 v_{e_3}, v_4 v_{e_3}\}$ (see Figure 2 for an illustration). Note that $F(K_{3,3} - e) = 2$, where $e \in E(K_{3,3})$. It follows by Theorem 2.2(iii) that $L_2[w_1, w_2, v_{e_2}, v_3, v_4, v_{e_3}] \cong K_{3,3} - e$ is collapsible. As $L_2/L_2[w_1, w_2, v_{e_2}, v_3, v_4, v_{e_3}]$ is a cycle of length 2 that is collapsible, by Theorem 2.1(ii), L_2 is collapsible. This implies that $G(X) - Y$ is supereulerian, which contradicts to (4).

When $|X_2| = 2$, $X_2 = \{e_3, e_4\}$. Let $L_3 = G_2' \cup \{v_{e_3}v_3, v_{e_3}v_4, v_{e_4}v_3, v_{e_4}v_4\}$. Since L_3 is eulerian, $G(X) - Y$ is supereulerian, which contradicts to (4). ■

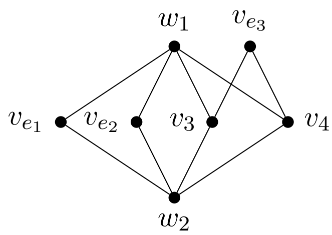

Figure 2. Illustration of the proof of Subcase 2.2.3.2 in Theorem 1.4.

3.2. Proofs of Corollaries 1.5 and 1.6

In the subsection, we shall prove Corollary 1.6 and provide a schetch of proof of Corollary 1.5 applying Theorem 1.4. Let us start with a necessary condition of (s, t)-supereulerian graphs.

Proposition 3.7. *If G is an (s, t)-supereulerian graph, then $t \le \kappa'(G) - 2$ and*

$$s \le \begin{cases} |E(G)|, & \text{if } G \text{ is eulerian and } t = 0; \\ 2\left\lfloor \dfrac{\kappa'(G) - t}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

Proof. Let $k = \kappa'(G)$ and let W be a minimum edge-cut of G. Pick an edge subset $Y \subseteq W$ with $|Y| \le t$. Since G is (s, t)-supereulerian, $G - Y$ has a spanning closed trail Γ. Since W is an edge-cut of G, $|E(\Gamma) \cap W| \ge 2$ and so $|Y| \le |W - E(\Gamma)| \le k - 2$. By arbitrary of Y with $|Y| \le t$, we have $t \le k - 2$.

If G is eulerian, then G has a spanning closed trail containing all edges in $E(G)$. This means that G is $(|E(G)|, 0)$-supereulerian. Now we assume that G is non-eulerian or $t \geq 1$.

We claim that $s + t \leq k$, and when $s + t = k$, $s \equiv 0 \pmod 2$. If not, then we pick an edge subset $X' \subseteq W$ satisfying that $|X'| \leq s$, $|X'| \equiv 1 \pmod 2$ and $|X'|$ is maximized. Let $Y' = W - X'$. Then $|Y'| \leq 1 \leq t$. Since G is (s,t)-supereulerian, $G - Y'$ has a spanning closed trail Γ' containing all edges in X'. Since W is an edge-cut of G, $X' = E(\Gamma') \cap W \neq \emptyset$ and $|X'| = |E(\Gamma') \cap W| \equiv 0 \pmod 2$, which contradicts with that $|X'| \equiv 1 \pmod 2$. Thus, $s + t \leq k$, and when $s + t = k$, $s \equiv 0 \pmod 2$. This follows that $s \leq 2\left\lfloor \frac{k-t}{2} \right\rfloor$. ■

By Proposition 3.7, we verify Corollary 1.6 as follows.

***Proof of Corollary* 1.6.** Suppose that G is eulerian and $t = 0$. Then for any non-negative integer $s \leq |E(G)|$, G is $(s, 0)$-supereulerian.

Conversely, suppose that G is (s,t)-supereulerian, and that either G is non-eulerian or $t > 0$. By Proposition 3.7, $s \leq 2\left\lfloor \frac{\kappa'(G)-t}{2} \right\rfloor$ and $t \leq \kappa'(G) - 2$. This follows that $s+t \leq \kappa'(G)$, which contradicts with the assumption of $\kappa'(G) < s+t$. Thus, if G is (s,t)-supereulerian, G is eulerian and $t = 0$. ■

Recall that $j(s,t)$ denotes the smallest integer such that every graph G with $\kappa'(G) \geq j(s,t)$ is (s,t)-supereulerian.

Theorem 3.8 (Examples 3.1(iii) and 3.2(iii) in [25])**.** *Each of the following holds.*

(i) $j(s,t) \geq 4$;

(ii) $j(2,2) \geq 5$.

By Proposition 3.7, Corollary 1.6 and Theorem 3.8(i), we obtain the following corollary (Proposition 1.1 in [25]) immediately. If a graph G is eulerian, then G is $(s,0)$-supereulerian where $s \leq |E(G)|$. It was mistakingly omitted the condition that G is non-eulerian or $t \geq 1$ in the original statement of Corollary 3.9 (Proposition 1.1 in [25]). So we corrected as follows.

Corollary 3.9 (Proposition 1.1 in [25])**.** *Let G be an (s,t)-supereulerian graph. If G is non-eulerian or $t \geq 1$, then*

$$\kappa'(G) \geq \begin{cases} \max\{4, t+2\}, & \text{if } s = 0; \\ \max\left\{4, s+t+\dfrac{1-(-1)^s}{2}\right\}, & \text{if } s \geq 1. \end{cases}$$

Now, we can provide a schetch of proof of Corollay 1.5 as follows.

***Schetch of proof of Corollary* 1.5.** Let m be the right hand side of (1). Let G be a graph with $\kappa'(G) \geq m$. If $(s,t) = (4,0)$, or $2 \leq s \equiv 0 \pmod 2$ and $s+t \geq 5$, then $s+t = m$, and so G is (s,t)-supereulerian by Theorem 1.4(iii); otherwise, then $s+t \leq m-1$, and so G is (s,t)-supereulerian by Theorem 1.4(ii) as $t < m-1$. Thus, by the definition of $j(s,t)$, $j(s,t) \leq m$.

Note that every eulerian graph with s edges is $(s,0)$-supereulerian. It indicates that to show that $j(s,t) \geq m$, it suffices to prove that $\kappa'(G_1) \geq m$ where G_1 is (s,t)-supereulerian and G_1 is non-eulerian when $t = 0$. Then, by Theorem 3.8(ii) and Corollary 3.9, we have $\kappa'(G_1) \geq m$ (please see [25] for detailed proof in this part). ■

3.3. Proofs of Theorems 1.7 and 1.8

In this subsection, we shall verify Theorems 1.7 and 1.8.

***Proof of Theorem* 1.7.** Suppose that G is (s,t)-supereulerian with $\kappa'(G) \geq 3$. Let $X, Y \subset E(\alpha(G))$ be two disjoint edge subsets with $|X| \leq s$ and $|Y| \leq t$. Let $k = \kappa'(\alpha(G))$.

If $s+t \leq \kappa'(G)$, then, as $k \geq \kappa'(G)+1 \geq 4$, $s+t \leq k-1$. Since G is (s,t)-supereulerian, by Proposition 3.7, $|Y| \leq t \leq \kappa'(G)-2 \leq k-3$. Thus, by Theorem 1.4(ii), $\alpha(G)-Y$ has a spanning closed trail containing all edges in X, which implies that $\alpha(G)$ is (s,t)-supereulerian.

If $s+t = \kappa'(G)+1$ and $\kappa'(G) \neq \delta(G)$, then, as G is (s,t)-supereulerian, by Corollary 1.6, G is eulerian and $t = 0$. It shows that $s = \kappa'(G)+1$. As $3 \leq \kappa'(G) \neq \delta(G)$, by Observation 2.4, $\kappa'(\alpha(G)) \geq \kappa'(G)+2 \geq 5$. Since $s \leq \kappa'(\alpha(G))-1$ and $t = 0$, by Theorem 1.4(ii), $\alpha(G)-Y$ has a spanning closed trail containing all edges in X, which implies that $\alpha(G)$ is (s,t)-supereulerian. ■

By Corollary 1.6 and Theorem 1.7, we have the following corollary directly.

Corollary 3.10. *Let G be an (s,t)-supereulerian graph on n vertices with $\kappa'(G) \geq 3$. If G is non-eulerian or $t \geq 1$, then $\alpha(G)$ is (s,t)-supereulerian for each $\alpha \in S_n$.*

***Proof of Theorem* 1.8.** Suppose that G is an (s,t)-supereulerian graph with $\kappa'(G) = \delta(G) \geq 3$. By Theorem 1.7, it suffices to show the necessity of Theorem 1.8. Suppose that $\alpha(G)$ is (s,t)-supereulerian. We argue by contradiction and assume that $s+t > \kappa'(G)$. Since G is (s,t)-supereulerian, by Corollary 1.6, G is eulerian and $t = 0$. This indicates that $\alpha(G)$ is non-eulerian by the definition of $\alpha(G)$. Since $\alpha(G)$ is (s,t)-supereulerian and $t = 0$, by Proposition 3.7, $\kappa'(G) < s \leq 2\left\lfloor \frac{\kappa'(\alpha(G))}{2} \right\rfloor$. As G is eulerian, $\kappa'(G)$ is even. It follows that $\kappa'(\alpha(G)) \geq \kappa'(G)+2$, which contradicts the assumption of $\kappa'(G) = \delta(G)$ by Observation 2.4. ■

4. An Application to Iterated Permutation Graphs

Suppose that G is a wheel, or an n-dimensional hypercube Q_n ($n \geq 3$), or a complete graph K_n ($n \geq 4$), or a complete bipartite graph $K_{m,n}$ ($\min\{m,n\} \geq 3$). If G is (s,t)-supereulerian, then by Theorem 1.8, for each $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is (s,t)-supereulerian if and only if $s+t \leq \kappa'(G)$.

Let G be a graph on n vertices and let $\mathrm{A} = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ be a permutation sequence where $\alpha_i \in S_{2^i n}$. We define $G^0(\mathrm{A}) = G$, and the *i-th iterated permutation graph of G with respect to the sequence* A is defined recursively as $G^i(\mathrm{A}) = \alpha_{i-1}(G^{i-1}(\mathrm{A}))$, for each positive integer i. If we do not emphasize the sequence A, we use G^i for $G^i(\mathrm{A})$. For example, the n-th iterated permutation graph of K_1 is a hypercube variant, a twisted hypercube of dimension n (see Definition 1.1 in [26]). By the definition of iterated permutation graphs, as well as Theorem 2.3 and Observation 2.4, we obtain the following observation.

Observation 4.1. *Let G be a connected graph. For each integer $m \geq 0$, each of the following holds.*

(i) *if $\kappa'(G) = \delta(G)$, then $\kappa'(G^m) = \delta(G^m) = \delta(G) + m$;*

(ii) *if $\kappa(G) = \delta(G)$, then $\kappa(G^m) = \kappa'(G^m) = \delta(G^m) = \delta(G) + m$.*

Given two non-negative integers s, t, a permutation sequence A, and a graph G. By Theorem 1.4(i), when $\kappa'(G^m) \geq s+t+2$, G^m is (s,t)-supereulerian. It follows by Theorem 1.7, G^{m+1} is also (s,t)-supereulerian. Therefore, there must exist a smallest integer m such that G^m is (s,t)-supereulerian. In Table 1, we list the edge connectivity $\kappa'(G^m)$, which are constructed by some special graphs.

Table 1. Edge connectivity of $\alpha(G)$ and G^m of some special graphs.

G	$\kappa(\alpha(G)) = \delta(\alpha(G))$	$\kappa'(G^m)$
Nontrivial tree	2	$m+1$
n-cycle C_n	3	$m+2$
wheel W_n	4	$m+3$
hypercube Q_n	$n+1$	$n+m$
complete graph K_n	n	$n+m-1$
complete bipartite graph K_{n_1,n_2}	$\min\{n_1,n_2\}+1$	$\min\{n_1,n_2\}+m$

In general, for given integers s and t, it is an interesting question that how to find the smallest m such that G^m is (s,t)-supereulerian for a connected graph G. Let $f(G)$ denote a graphical function and define $\overline{f}(G)$ to be the maximum value of $f(H)$ taken over all subgraphs H of G. As indicated in [15], for certain network reliability measures f, networks G with $f(G) = \overline{f}(G)$ are important for network survivability, and so the study of $\overline{f}(G)$ is of interest. The following theorem gives some new and feasible ideas to find the smallest m.

Theorem 4.2. *Let G be a connected graph with n vertices. Each of the following holds.*

(i) (Corollary 2.2 in [16]) $\kappa'(\alpha(G) = \delta(\alpha(G))$, *if and only if* $2\kappa'(G) \geq \delta(G) + 1$ *for any* $\alpha \in S_n$.

(ii) (Corollary 2.3 in [16]) *If* $\kappa'(G) = \overline{\delta}(G)$, *then for any* $\alpha \in S_n$, $\kappa'(\alpha(G)) = \overline{\delta}(\alpha(G))$.

(iii) (Theorem 2.5 in [16]) *If* $\kappa'(G) = \overline{\kappa}'(G)$ *and* $\delta(G) = \overline{\delta}(G)$, *then for any* $\alpha \in S_n$, *we have both* $\kappa'(\alpha(G)) = \overline{\kappa}'(\alpha(G))$ *and* $\delta(\alpha(G)) = \overline{\delta}(\alpha(G))$.

One can start with any graph G which satisfies Theorem 4.2 as the initial process to construct large survivable networks by repeatedly taking permutation graphs as G^m. Then for any given non-negative integer s and t, we can apply Theorem 1.4 and Theorem 4.2 to G^m to find the smallest values of m such that G^m is (s,t)-supereulerian.

5. Conclusion

In this paper, we characterized the (s,t)-supereulerianicity of a graph G when $s + t \leq \kappa'(G)$. Using this result, for an (s,t)-supereulerian graph G, we also obtained a relationship between the (s,t)-supereulerianicity of the permutation graph of G and the edge connectivity of G. It shows that if a graph G has the property that it has a spanning closed trail traversing some given edge set of size at most s and avoiding some given edge set of size at most t, then when the edge connectivity of G is large enough than $s + t$, the permutation graph of G which is a bigger structure will remain this property.

There are some questions that might be of interest for the future work.

1. For a given (s,t)-supereuleian graph G and given disjoint subsets $X, Y \subset E(G)$ with $|X| \leq s$ and $|Y| \leq t$, can we find a spanning closed trail T of $G - Y$ such that $X \subset E(T)$ and $|E(T)|$ is minimized? If such spanning closed trail T visits every vertex exactly once, then G has a hamiltonian cycle containing all edges in X and avoiding all edges in Y. As we known, to determine if a graph has a hamiltonian cycle is a NP-complete problem (Theorem 3.4 in [13]). Thus, this generalized optimization question is interest of its own.

2. Pulleyblank [23] proved that determining whether a graph is $(0,0)$-supereulerian, even when restricted to planar graphs, is NP-complete. Lately, it has been shown that when $t \geq 3$, (s,t)-supereulerianicity is polynomially determinable

in [25]. It is currently not known whether it is polynomially determinable or NP-complete when $t = 1$ or 2.

3. For interconnection network models, such as hypercubes and hypercube variants, for which values of s and t, could they be (s, t)-supereulerian?

References

[1] C. Balbuena, X. Marcote and P. García-Vázquez, *On restricted connectivities of permutation graphs*, Networks **45** (2005) 113–118.
https://doi.org/10.1002/net.20056

[2] C. Balbuena, P. García-Vázquez and X. Marcote, *Reliability of interconnection networks modelled by a product of graphs*, Networks **48** (2006) 114–120.
https://doi.org/10.1002/net.20124

[3] J.C. Bermond, C. Delorme and G. Farhi, *Large graphs with given degree and diameter* II, J. Combin. Theory Ser. B **36** (1984) 32–84.
https://doi.org/10.1016/0095-8956(84)90012-1

[4] F.T. Boesch, C. Suffel and R. Tindell, *The spanning subgraphs of eulerian graphs*, J. Graph Theory **1** (1977) 79–84.
https://doi.org/10.1002/jgt.3190010115

[5] A.J. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, 1976).

[6] P.A. Catlin, *A reduction method to find spanning Eulerian subgraphs*, J. Graph Theory **12** (1988) 29–44.
https://doi.org/10.1002/jgt.3190120105

[7] P.A. Catlin, *Supereulerian graphs: A survey*, J. Graph Theory **16** (1992) 177–196.
https://doi.org/10.1002/jgt.3190160209

[8] P.A. Catlin, Z.-Y. Han and H.-J. Lai, *Graphs without spanning closed trails*, Discrete Math. **160** (1996) 81–91.
https://doi.org/10.1016/S0012-365X(95)00149-Q

[9] P.A. Catlin, T. Iqbalunnisa, T.N. Janakiraman and N. Srinivasan, *Hamilton cycles and closed trails in iterated line graphs*, J. Graph Theory **14** (1990) 347–364.
https://doi.org/10.1002/jgt.3190140308

[10] G. Chartrand and J. Frechen, *On the chromatic number of permutation graphs*, in: Proof Techniques in Graph Theory, Proceedings of the Second Ann Arbor Graph Theory Conference, F. Harary (Ed(s)) (Academic Press, New York, London, 1969) 21–24.

[11] G. Chartrand and F. Harary, *Planar permutation graphs*, Ann. Inst. Henri Poincaré Probab. Stat. **3** (1967) 433–438.

[12] Z.H. Chen and H.-J. Lai, *Reduction techniques for supereulerian graphs and related topics—a survey*, in: Combinatorics and Graph Theory'95, T.-H. Gu (Ed(s)) (World Sci. Pub. River Edge N.J., 1995) 53–69.

[13] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (W.H. Freeman and Company, 1979).

[14] R. Gu, H.-J. Lai, Y. Liang, Z. Miao and M. Zhang, *Collapsible subgraphs of a* 4-*edge-connected graph*, Discrete Appl. Math. **260** (2019) 272–277.
https://doi.org/10.1016/j.dam.2019.01.033

[15] A.M. Hobbs, *Network survivability*, in: Appl. Discrete Math., J.G. Michaels and K.H. Rosen (Ed(s)) (McGraw-Hill, 1991) 332–353.

[16] H.-J. Lai, *Large survivable nets and the generalized prisms*, Discrete Appl. Math. **61** (1995) 181–185.
https://doi.org/10.1016/0166-218X(94)00131-V

[17] H.-J. Lai, Y. Shao and H. Yan, *An update on supereulerian graphs*, WSEAS Transactions on Mathematics **12** (2013) 926–940.

[18] L. Lei, X. Li and B. Wang, *On* (s,t)*-supereulerian locally connected graphs*, in: International Conference on Computational Science, Y. Shi, G.D. Albada, J. Dongarra and P.M.A. Sloot (Ed(s)) (Springer, 2007) 384–388.

[19] L. Lei and X. Li, *A note on the connectivity of generalized prisms*, J. Southwest China Normal University (Natural Science Edition) **33** (2008) 1–3.

[20] L. Lei, X. Li, B. Wang and H.-J. Lai, *On* (s,t)*-supereulerian graphs in locally highly connected graphs*, Discrete Math. **310** (2010) 929–934.
https://doi.org/10.1016/j.disc.2009.08.012

[21] X. Li, L. Lei and H.-J. Lai, *The connectivity of generalized graph products*, Inform. Process. Lett. **136** (2018) 37–40.
https://doi.org/10.1016/j.ipl.2018.03.014

[22] B.L. Piazza and R.D. Ringeisen, *Connectivity of generalized prisms over* G, Discrete Appl. Math. **30** (1991) 229–233.
https://doi.org/10.1016/0166-218X(91)90047-Z

[23] W.R. Pulleyblank, *A note on graphs spanned by Eulerian graphs*, J. Graph Theory **3** (1979) 309–310.
https://doi.org/10.1002/jgt.3190030316

[24] R.D. Ringeisen, *On cycle permutation graphs*, Discrete Math. **51** (1984) 265–275.
https://doi.org/10.1016/0012-365X(84)90007-4

[25] W. Xiong, S. Song and H.-J. Lai, *Polynomially determine if a graph is* $(s,3)$*-supereulerian*, Discrete Math. **344(12)** (2021) 112601.
https://doi.org/10.1016/j.disc.2021.112601

[26] X. Zhu, *A hypercube variant with small diameter*, J. Graph Theory **85** (2017) 651–660.
https://doi.org/10.1002/jgt.22096

Received 21 October 2022
Revised 6 June 2023
Accepted 6 June 2023
Available online 19 July 2023