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k-FAULT-TOLERANT GRAPHS FOR p DISJOINT COMPLETE GRAPHS OF ORDER c

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Abstract

Vertex-fault-tolerance was introduced by Hayes in 1976, and since then it has been systematically studied in different aspects. In this paper, we study graphs of order $cp + k$ that are k-vertex-fault-tolerant for p disjoint complete graphs of order c, i.e., graphs in which removing any k vertices leaves a graph that has p disjoint complete graphs of order c as a subgraph. In this paper, we analyze some properties of such graphs for any value of k. The main contribution is to describe such graphs that have the smallest possible number of edges for $k = 1$, $p \geq 1$, and $c \geq 3$.

Keywords: fault-tolerance, interconnection network, factor, algorithm, k-critical graph.

2020 Mathematics Subject Classification: 05C35, 05C70, 05C85, 68M10, 68M15.

1. Introduction

Throughout the paper, we deal with simple undirected graphs. For standard terms and notations in graph theory, the reader is referred to the books of Diestel [6] and Brandstädt *et al.* [3].

Given a graph H and a positive integer k, a graph G is called *vertex k-fault-tolerant with respect to H*, denoted by k-FT(H), if $G - S$ contains a subgraph isomorphic to H for every $S \subset V(G)$ with $|S| \leq k$.

Vertex-fault-tolerance was introduced by Hayes [10] in 1976 as a graph theoretic model of computer or communication networks working correctly in the presence of faults. The main motivation for the problem of constructing k-fault-tolerant graphs is in finding fault-tolerant architectures. A graph H represents the desired interconnection network and a k-FT(H) graph G allows one to emulate the graph H even in the presence of k vertex (processor) faults.

The problem has been systematically studied with different quality measures of k-fault-tolerant graphs. Hayes [10] and Ajtai *et al.* [1] considered k-FT(H) graphs with $|V(H)| + k$ vertices and the number of edges as small as possible. A different quality measure of k-FT(H) graphs was introduced by Ueno *et al.* [16], and independently by Dudek *et al.*, [7], where the authors were interested in k-FT(H) graphs having as few edges as possible, disregarding the number of vertices (see also [14, 20]). Yet another setup was studied by Alon and Chung [2], Ueno and Yamada [15], and Zhang [18]. They allowed $O(k)$ spare vertices in k-FT(H) graphs and focused on minimizing the maximum degree (giving priority to the scalability of a network). Other results on k-fault-tolerance can be found, for example, in [5, 9, 12, 21].

In this paper, we study the variant introduced by Hayes, i.e., given a graph H, we analyze k-FT(H) graphs of order $|V(H)|+k$ and minimum size. Hayes [10] characterized k-FT(H) graphs of order $|V(H)|+k$ in the case where H is a path, a cycle, or a tree of a special type [10]. Some results related to constructing a k-fault-tolerant supergraph for an arbitrary graph H have also been published (e.g., see [4, 8]). We focus on k-FT(pK_c) graphs, where pK_c is the union of p vertex disjoint complete graphs of order c, for $k, p \geq 1$, $c \geq 3$. Our main contribution is to describe minimum k-FT(pK_c) graphs of order $pc + k$ for $k = 1$ and any values of p and c (Theorem 13).

1.1. Motivations

Even though cliques are among the most popular concepts used to model cohesive clusters in different graph-based applications, such as social, biological, and communication networks, the k-FT(pK_c) graphs have not received much attention yet. But they could be applied in the design of topologies for computing systems. E.g., companies that process large volumes of data in a regular/repetitive manner might use such designs to optimize their data centers. There are computational loads that require a given degree of parallelism to achieve the expected response timings. On the other hand, throughput requirements impose minimal limits on the number of such clusters to be operational at a given time. If processing a batch of data requires the collaboration of c machines, and up to k machines may fail at the same time, a k-FT(pK_c) design becomes highly relevant.

Consider the case of k-FT(K_c), i.e., where we need to preserve only one clique of size c. This case is related to the minimum vertex blocker clique problem, in which one searches for a subset of vertices of minimum cardinality to be removed from a graph G so that the maximum (weighted) clique in the remaining graph is of size (weight) at most a given integer $c \geq 1$. This problem was studied in [13], where an exact algorithm based on row generation was proposed. Note that in the case where the weights of all vertices in G are 1, the minimum vertex blocker corresponds to enforcing the clique number to be at most c. So, if G is a k-FT(K_{c+1}) graph, then the minimum vertex blocker (for K_c) in G has at least $k+1$ vertices. But, in general, the vertex blocker clique problem is concerned with the existence of just one clique of size c, whereas k-FT(K_c) considers the existence of a cover of the whole vertex set with disjoint cliques of size c. It might be interesting to extend the study of blockers to this case as well.

The paper is organized as follows. In Section 2, we provide some definitions and present basic properties of k-FT(pK_c) graphs, analyze their connectivity and separators of size k, and present an upper bound on the size of minimum k-FT(pK_c) graphs of order $pc+k$. In Section 3, we prove that this upper bound is tight when $k = 1$ and fully characterize minimum 1-FT(pK_c) graphs of order $pc+1$. Finally, we present some concluding comments in Section 4.

2. k-FT(pK_c) graphs

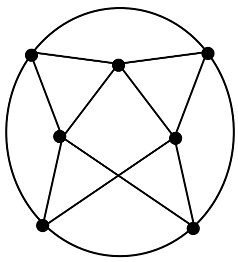
Given a graph $G = (V, E)$ and a vertex v, $v \in V$, we use $N_G(v)$ to denote the *neighborhood* of v in G, i.e., the set of *neighbors* of v, i.e., vertices y, $y \in V$, such that $\{v, y\} \in E$. Given a set of vertices U, $U \subset V$, $N_G(U)$ denotes the set of vertices in $V \setminus U$ that have a neighbor in U. We use $N_G[v]$ to denote the *closed neighborhood* of v, i.e., $N_G[v] = N_G(v) \cup \{v\}$. Likewise, we have $N_G[U] = N_G(U) \cup U$. When it does not lead to confusion, we omit the subscript, writing just $N(v), N[v], N(U)$ and $N[U]$. We use K_c to denote a complete graph on c vertices. The vertex set of a complete graph is called a *clique*.

2.1. Basic properties

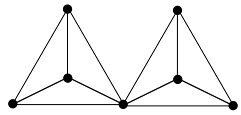
Let us start with the main definition.

Definition. Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, and $c \geq 2$. Let $G = (V, E)$ be a graph. We say that G is k-FT(pK_c) if $G - S$ contains the union of p disjoint complete graphs K_c as a subgraph, for any S, $S \subset V$ and $|S| \leq k$. We say that G is *minimal* k-FT(pK_c) if $|V| = pc + k$ and no proper subgraph of G is k-FT(pK_c). We say that G is *minimum k-FT(pK_c)* if $|V| = pc + k$ and there is no k-FT(pK_c) graph G' with $|V(G')| = |V|$ and $|E(G')| < |E|$.

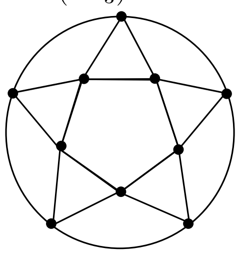
Note that the property of being k-FT(pK_c) is monotone, i.e., it is preserved when adding an edge to a graph. So any k-FT(pK_c) graph can be transformed into a minimal k-FT(pK_c) graph by successively removing edges, one by one in arbitrary order until it is not possible to remove another edge without losing the property. However, a k-FT(pK_c) minimal graph is not necessarily k-FT(pK_c) minimum.



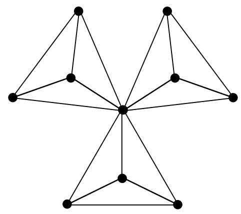
(a) A 1-FT$(2K_3)$ minimal graph.



(b) A 1-FT$(2K_3)$ minimum graph.



(c) A 1-FT$(3K_3)$ minimal graph.



(d) A 1-FT$(3K_3)$ minimum graph.

Figure 1. Examples of k-FT(pK_c) graphs.

An example of 1-FT$(2K_3)$ minimal graph that is not minimum can be found in Figure 1(a). Indeed, it has 13 edges, whereas the graph in Figure 1(b) has 12 edges. It is easy to check that both graphs are 1-FT$(2K_3)$ minimal, but only the graph in Figure 1(b) is minimum. Note that the construction in Figure 1(a) can be generalized to larger values of k, p, and c. Figure 1(c) gives another simple example.

It is easy to check that, in order to prove that G is k-FT(pK_c), it is enough to verify that $G-S$ contains the union of p disjoint complete graphs K_c as a subgraph, for any S, $S \subset V$ with $|S| = k$ (equality instead of weak inequality). For the sake of simplicity, some of the following proofs use this observation without stating it explicitly.

For $c = 2$ and $G = (V, E)$ such that $|V| = 2p + k$, the concept of k-FT(pK_c) graphs has been widely studied under the name *k-factor critical* graphs. This idea was first introduced and studied for $k = 2$ by Lovász [11] under the term of *bicritical graph.* For $k > 2$ it was introduced by Yu in 1993 [17], and independently by Favaron in 1996 [9]. In [19], Zhang *et al.* showed that if $G = (V, E)$ is minimum

k-FT(pK_2), then $|E| = (k+1)|V|/2$.

In what follows, we focus on the cases with $c \geq 3$. Let us start with a few simple lemmas that give some basic properties of k-FT(pK_c) graphs.

Lemma 1. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, and $c \geq 3$. Let $G = (V, E)$ be a k-FT(pK_c) graph with $|V| = pc+k$. Then every vertex x, $x \in V$:*

1. *belongs to a subgraph isomorphic to K_c in G,*
2. *is of degree at least $c + k - 1$ in G,*
3. *belongs to a subgraph isomorphic to K_c in G', $G' = G - S$, for any S with $S \subset V \setminus \{x\}$ and $|S| = k$.*

Proof. Let x be any vertex of G. Choose any S with $S \subset V \setminus \{v\}$ and $|S| = k$, such that $|S \cap N_G(v)|$ is maximum over all such subsets. Notice that S exists, as $|V| = pc + k$ and p, c, k are positive integers. Since G is k-FT(pK_c), $G' = G - S$ contains p disjoint subgraphs isomorphic to K_c. Moreover, $|G'| = pc$, thus x belongs to one of them, which proves item 1. Moreover, it implies that $d_{G'}(x) \geq c - 1$. By the choice of S, it is easy to see that $S \subset N_G(x)$. Thus $d_G(x) \geq c + k - 1$, which proves item 2. Clearly, a similar reasoning applies for any choice of S, with $S \subset V \setminus \{v\}$ and $|S| = k$, which proves item 3. ■

In the following, we will use interchangeably the expressions that a 0-FT(pK_c) graph G' contains p disjoint complete graphs K_c and that its vertex set $V(G')$ contains p disjoint cliques of size c.

Lemma 2. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, and $c \geq 3$. Let $G = (V, E)$ be a k-FT(pK_c) graph with $|V| = pc+k$. Then every vertex x, $x \in V$, with $d(x) = c + k - 1$ belongs to a subgraph of G isomorphic to K_{c+k}. Moreover, if W is a separator of size k in G and A is a component of $G - W$ of order c, then $W \cup V(A)$ is a clique.*

Proof. Let x be any vertex of G with $d_G(x) = c + k - 1$. For any choice of S with $|S| = k$ and $S \subset N_G(x)$, by Lemma 1, x belongs to a copy of K_c in G', $G' = G - S$. Since $c \geq 3$, $N_{G'}(x)$ is a clique of size at least 2. Therefore, $N_G[x]$ is also a clique.

For the second part of the lemma, note that $V(A)$ contains a vertex of degree $c + k - 1$. The conclusion follows. ■

2.2. Connectivity and separators

Now let us proceed with some observations on the connectivity of k-FT(pK_c) graphs.

Lemma 3. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, and $c \geq 3$. Let $G = (V, E)$ be a k-$\mathrm{FT}(pK_c)$ graph with $|V| = pc + k$. Then G is $(c+k-1)$-edge-connected.*

Proof. Suppose that G is not $(c + k - 1)$-edge-connected. It implies that there exists a separating set of at most $c+k-2$ edges F, $F \subset E$. Let A be a component of $G - F$. Choose any S, with $S \subset V$ and $|S| = k$, such that $|V(A)| - b \not\equiv 0 \pmod{c}$ and $|\widehat{F}| \geq k$, where $b = |S \cap V(A)|$ and $\widehat{F} = \{f \in F \mid f \cap S \neq \emptyset\}$. It is easy to check that such S exists and $|F'| < c - 1$, where $F' = F \setminus \widehat{F}$. Let $A' = A - S$ and $G' = G - S$. It is easy to check that, since $|F'| < c - 1$, G' does not contain a copy of K_c intersecting both $V(A')$ and $V \setminus S \setminus V(A')$. On the other hand, since G is k-$\mathrm{FT}(pK_c)$ and $|V| = pc + k$, G' contains p disjoint copies of K_c, where every vertex of G' belongs to one of them. But the order of A' is not a multiple of c, a contradiction. ∎

Lemma 4. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, and $c \geq 3$. Let $G = (V, E)$ be a k-$\mathrm{FT}(pK_c)$graph with $|V| = pc + k$. Then G is k-connected.*

Proof. Suppose that G is not k-connected. So there is a separator W in G with $|W| < k$. Moreover, we can choose S, with $S \subset V$, $|S| = k$, and $W \subsetneq S$, in such a way that the order of one of the components of G', $G' = G - S$, is not a multiple of c. On the other hand, since G is k-$\mathrm{FT}(pK_c)$, G' contains p disjoint copies of K_c, a contradiction. ∎

Let us analyze some properties of k-$\mathrm{FT}(pK_c)$ graphs $G = (V, E)$ with $|V| = pc + k$ that contain a separator of size k.

Lemma 5. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, $c \geq 3$, and $k < c$. Let $G = (V, E)$ be a k-$\mathrm{FT}(pK_c)$ graph with $|V| = pc + k$. Let W, with $W \subset V$ and $|W| = k$, be a separator in G. Let $\{A_i\}_{i=1}^{z}$ be the components of G', $G' = G - W$, and let $p_i = |V(A_i)|/c$ for every i, $1 \leq i \leq z$. Then $\sum_{i=1}^{z} p_i = p$ and G_i, $G_i = G[V(A_i) \cup W]$, is k-$\mathrm{FT}(pK_c)$ for every i, $1 \leq i \leq z$. Moreover, if W is a clique and G is minimum k-$\mathrm{FT}(pK_c)$, then G_i is minimum k-$\mathrm{FT}(pK_c)$ for every i, $1 \leq i \leq z$.*

Proof. Since G is k-$\mathrm{FT}(pK_c)$ and $|W| = k$, G' contains p disjoint cliques of size c. Since W is a separator in G, then each of these cliques is included in $V(A_i)$ for some i, $1 \leq i \leq z$. Moreover, since $\sum_{i=1}^{z} |V(A_i)| = pc$, each $V(A_i)$, $1 \leq i \leq z$, is the union of exactly p_i of these cliques and $|V(A_i)| = p_i c$. So p_i is an integer for every i, $1 \leq i \leq z$, and $\sum_{i=1}^{z} p_i = p$.

Take any G_i, $1 \leq i \leq z$. In order to prove that G_i is k-$\mathrm{FT}(p_i K_c)$, we need to show that $G_i - S$ also contains p_i disjoint cliques of size c for any other choice of a set of vertices S, $S \subset V(G_i)$, of size k.

Choose any S with $S \subset V(G_i)$ and $|S| = k$. Let $G_i'' = G_i - S$. Note that $|G_i''| = p_i c$. Let us show that G_i'' contains p_i disjoint cliques of size c. Let $G'' = G - S$. Since G is k-FT(pK_c), G'' contains p disjoint cliques of size c. Let us use $\mathcal{C}$ to denote this set of cliques. Moreover, let $\mathcal{C}_i$ be the set of elements of $\mathcal{C}$ that intersect $V(A_i)$. Since W is a separator in G, the cliques in $\mathcal{C}_i$ are subsets of $V(G_i)$. By the choice of $\mathcal{C}$, they also are subsets of $V(G_i'')$. Since $|S| = k$ and $k < c$, there is $|V(A_i) \setminus S| > (p_i - 1)c$. Thus $|\mathcal{C}_i| = p_i$. This concludes the proof that G_i is k-FT$(p_i K_c)$.

For the final part of the lemma, assume that W is a clique and G is minimum k-FT(pK_c). Choose any G_i, like above. So G_i is k-FT$(p_i K_c)$. Towards a contradiction, suppose it is not minimum k-FT$(p_i K_c)$. Consider a graph $\widehat{G_i}$ that is minimum k-FT$(p_i K_c)$. Since $k < c$, by Lemma 1, $\widehat{G_i}$ contains a subgraph isomorphic to K_k, let $\widehat{W}$ denote its vertex set. Let $\widehat{G}$ be the graph obtained from G by removing A_i and adding $\widehat{G_i}$, identifying the vertices of W with the ones of $\widehat{W}$. It is easy to check that $\widehat{G}$ is k-FT(pK_c) and $|E(\widehat{G})| < |E|$, a contrary to the choice of G. ∎

Let us show that, in a graph G that is k-FT(pK_c), replacing the closed neighborhood of a vertex of degree $c + k - 1$ with a copy of K_k, gives a graph that is k-FT$(p'K_c)$ with $p' = p - 1$.

Lemma 6. *Let k, p, and c be integers with $k = 1$, $p \geq 1$, and $c \geq 3$. Let $G = (V, E)$ be a k-FT(pK_c) graph with $|V| = pc + k$. Let x, $x \in V$, be any vertex with $d_G(x) = c + k - 1$. Then the graph G' obtained from G by replacing $N_G[x]$ with a copy of K_k, and making every vertex of this K_k adjacent to every vertex in $N_G(N_G[x])$ (neighborhood of the closed neighborhood of x), is k-FT$(p'K_c)$, with $p' = p - 1$.*

Proof. We need to show that, for any S', $S' \subset V(G')$ and $|S'| = k$, the vertices of $G' - S'$ can be partitioned into $p - 1$ disjoint cliques of size c. Take any such S'. Note that $S' \cap N_G[x] = \emptyset$ (since $S' \subset V(G')$).

Note that, by Lemma 2, $N_G[x]$ is isomorphic to K_{c+k}. Let $\widehat{K}$ be the subgraph isomorphic to K_k added in G'.

Given S' to be removed from G', let us construct the corresponding S to be removed from G. If $S' \cap V(\widehat{K}) = \emptyset$, then just let $S = S'$. Otherwise, let $i = |S' \cap V(\widehat{K})|$. By construction, $i \leq k$. Let us construct S from S' by replacing the vertices in $S' \cap V(\widehat{K})$ by i arbitrary vertices in $N(x)$. It is possible, since $i \leq k$ and $k \leq c + k - 1$.

Since G is k-FT(pK_c), the vertices of $G - S$ can be partitioned into p disjoint cliques of size c. Let us use $\mathcal{C}$ to denote this set of cliques. Let us use C_x to denote the clique in $\mathcal{C}$ that contains x. Note that $C_x \cap V(G') = \emptyset$ and there are exactly $k - i$ other vertices in $N_G(x) \setminus C_v \setminus S$ that are covered by other cliques in $\mathcal{C}$. So there exists a bijection b between $N_G(x) \setminus C_x \setminus S$ and $V(\widehat{K}) \setminus S'$.

Let us construct $\mathcal{C}'$ by removing C_x from it, and adapting the other cliques by replacing the vertices in $N_G(x) \setminus C_x \setminus S$ by the corresponding vertices in $V(\widehat{K}) \setminus S'$, based on the bijection b. Since every vertex in $\widehat{K}$ is adjacent to every vertex in $N_G(N[x])$, the sets thus obtained are cliques indeed. So $\mathcal{C}'$ is a partition of $V(G') \setminus S'$ as needed. ■

Note that, in the statement of Lemma 6, if $N_G[x]$ is a separator in G, then the copy of K_k it is replaced by in G' is a separator of size k that is a clique, so Lemma 5 applies.

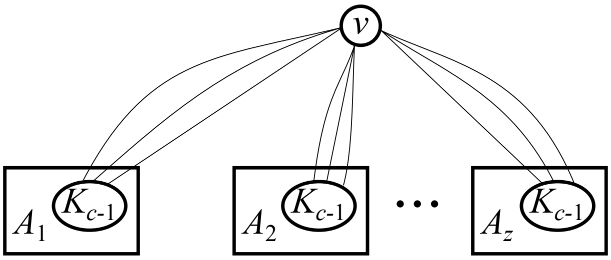


Figure 2. A separator $W = \{v\}$ and z connected components.

Lemma 7. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, $c \geq 3$, and $k < c$. Let $G = (V, E)$ be a k-FT(pK_c) graph with $|V| = pc + k$. Let W, with $W \subset V$ and $|W| = k$, be a separator in G. Let $\{A_i\}_{i=1}^{z}$ be the components of G', $G' = G - W$. Then, for every vertex x in W and every A_i, $1 \leq i \leq z$, there exists a $V_{i,x}$, $V_{i,x} \subset V(A_i)$, with $|V_{i,x}| = c - 1$, such that $\{x\} \cup V_{i,x}$ is a clique.*

Proof. Choose any A_i, $1 \leq i \leq z$. Let $G_i = G[V(A_i) \cup W]$. By Lemma 5, G_i is k-FT$(p_i K_c)$ for some positive integer p_i. Choose any x in W and any x_i in $V(A_i)$. Let $W' = W \cup \{x_i\} \setminus \{x\}$. Since G_i is k-FT$(p_i K_c)$, $V(G_i) \setminus W'$ can be partitioned into cliques of size c. Since $V(G_i) \setminus W' = V(A_i) \cup \{x\} \setminus \{x_i\}$, there exists a clique $\{x\} \cup V_{i,x}$ of size c as needed. ■

In Figure 2, we can observe an example of the situation described in Lemma 7 for $k = 1$.

Note that, in particular, Lemma 7, means that in a k-FT(pK_c) graph G of order $pc + k$, with $k \geq 0$, $p \geq 1$, $c \geq 3$, and $k < c$, given a separator W of size k, all components of $G - W$ are *full*, i.e., the neighborhood of the vertex set of each of them equals W.

2.3. Upper bound on the size of minimum k-FT(pK_c) graphs

Finally, let us give a simple upper bound on the size of a minimum k-FT(pK_c) graph.

Lemma 8. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, $c \geq 3$. Let $G = (V, E)$ be a minimum k-FT(pK_c) graph. Then* $|E| \leq \left(\binom{c}{2} + ck\right) p + \binom{k}{2}$.

Proof. Let H be the graph obtained by taking p disjoint copies of K_c, one copy of K_k, and making each vertex of each copy of K_c adjacent to each vertex of K_k. It is easy to check that H is k-FT(pK_c) and $|E(H)| = \left(\binom{c}{2} + ck\right) p + \binom{k}{2}$. The conclusion follows. ∎

It can be easily checked that the graphs constructed in the proof of Lemma 8 are minimal k-FT(pK_c). Our main result, Theorem 13 presented in Section 3, shows that these graphs are also minimum k-FT(pK_c) when $k = 1$. We conjecture that they are minimum when $k > 1$ too.

The construction from the proof of Lemma 8 can be easily generalized using the notions of tree-decomposition and chordal graph. Let us start by recalling the definition of the former.

Definition (Section 12.3 in [6])**.** Let G be a graph, T a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the nodes t of T. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following three conditions:

1. $V(G) = \bigcup_{t \in V(T)} V_t$;
2. for every edge $e \in E(G)$ there exists $t \in V(T)$ such that $e \subseteq V_t$;
3. $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the path between t_1 and t_3 in T.

T is the *decomposition tree* of $(T, \mathcal{V})$, and the elements of $\mathcal{V}$ are the *parts* of $(T, \mathcal{V})$.

Given two nodes t_1, t_2 adjacent in a decomposition tree T, $V_{t_1} \cap V_{t_2}$ is the *adhesion set* of V_{t_1} and V_{t_2}. The *adhesion* of a tree-decomposition is the maximum size of its adhesion sets.

With the notion of tree-decomposition, we obtain the following characterization of chordal graphs.

Proposition 9 (Proposition 12.3.6 in [6])**.** *G is chordal if and only if G has a tree-decomposition in which every part is a clique.*

It is easy to check that, without loss of generality, the tree-decomposition of G in Proposition 9 can be restricted to have the set of parts equal to the set of maximal cliques of G. In this case, the adhesion sets of the tree-decomposition are the minimal separators of G (which are cliques).

Now we can present a generalization of the construction from the proof of Lemma 8. Based on Proposition 9, it is easy to check the validity of the following lemma.

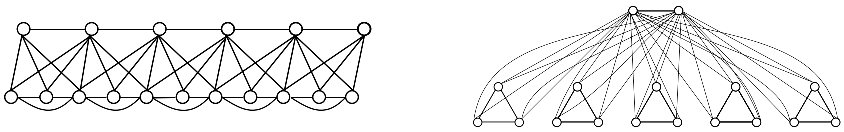


Figure 3. Examples of 2-FT$(5K_3)$ graphs.

Lemma 10. *Let k, p, and c be integers with $k \geq 0$, $p \geq 1$, $c \geq 3$. Let $G = (V, E)$ be a chordal graph of order $pc + k$ in which all minimal separators are of size k and all maximal cliques are of size $k + c$. Then G is a k-FT(pK_c) graph with $|E| = \left(\binom{c}{2} + ck\right) p + \binom{k}{2}$.*

In Figure 3 we can see examples of graphs described in Lemma 10.

3. Minimum 1-FT(pK_c) graphs

In this section we focus on k-FT(pK_c) graphs G with $|V(G)| = pc + k$, for $k = 1$, $p \geq 1$, and $c \geq 3$.

Recall that *a block* is a maximal connected subgraph without a cutvertex. So, every block is a maximal 2-connected subgraph, a bridge, or an isolated vertex. Conversely, every such subgraph is a block. Different blocks of a graph G overlap on at most one vertex, which is then a cutvertex of G. Every edge of G lies in a unique block, and G is the union of its blocks. Let A be the set of cutvertices of G, and $\mathcal{B}$ the set of its blocks. We then have a natural bipartite graph on $A \cup \mathcal{B}$ formed by the edges $\{a, B\}$ with $a \in B$, and the following lemma holds.

Lemma 11 (Lemma 3.1.4 in [6])**.** *The block graph of a connected graph is a tree.*

Lemma 12. *Let p, and c be integers with $p \geq 1$ and $c \geq 3$. Let $G = (V, E)$ be a 1-FT(pK_c) graph with $|V| = pc + 1$. If every block of G is isomorphic to K_{c+1}, then G is a chordal graph in which all minimal separators are of size 1 and all maximal cliques are of size $c + 1$.*

Proof. By Lemma 3, G is c-edge-connected. Since $c \geq 3$, G has no isolated vertices nor bridges, so all blocks of G are maximal 2-connected subgraphs of G.

Consider the block graph of G. By Lemma 11, it is a tree. It is easy to check that this tree gives a tree-decomposition of G in which every part is a clique. So, by Proposition 9, it is a chordal graph. ■

Theorem 13. *Let p and c be integers with $p \geq 1$ and $c \geq 3$. Let $G = (V, E)$ be a 1-FT(pK_c) graph with $|V| = pc + 1$. If $|E| \leq \binom{c+1}{2}p$, then every block of G is isomorphic to K_{c+1}.*

Proof. Reasoning towards a contradiction, suppose that G is a counterexample of the smallest possible order, i.e., for all $\widehat{p}$, $\widehat{p} < p$, 1-FT($\widehat{p}K_c$) graphs $\widehat{G} = (\widehat{V}, \widehat{E})$ with $|\widehat{V}| = \widehat{p}c + 1$ and $|\widehat{E}| \leq \binom{c+1}{2}\widehat{p}$ satisfy the conclusion of the theorem. Notice that $p > 1$, since K_{c+1} is the only 1-FT(K_c) graph satisfying the hypotheses, it also satisfies the conclusions.

By Lemma 3, G has neither bridges nor isolated vertices. So all blocks of G are maximal 2-connected subgraphs of G.

Let us show that G is 2-connected. Towards a contradiction, suppose it is not. Consider the block graph $BG(G)$ of G. Take a block B of G that is a leaf in $BG(G)$, and let u be the cutvertex adjacent to B in $BT(G)$. Thus $u \in V(B)$.

Let $G[R]$ be the subgraph induced by $R = V \setminus V(B) \cup \{u\}$. By Lemma 5, B and $G[R]$ are 1-FT($p_B K_c$) and 1-FT($p_R K_c$), respectively, for some positive integers p_B and p_R such that $p_B + p_R = p$. There is $|E(B)| \leq \binom{c+1}{2}p_B$ or $|E(G[R])| \leq \binom{c+1}{2}p_R$, since otherwise we would have $|E(G)| = |E(B)| + |E(G[R])| > \binom{c+1}{2}p_B + \binom{c+1}{2}p_R = \binom{c+1}{2}p$, a contradiction.

Suppose that $|E(B)| \leq \binom{c+1}{2}p_B$. Then B is isomorphic to K_{c+1}, since B is 2-connected, G is the smallest counterexample, and B is a leaf in $BG(G)$. So we have $|E(B)| = \binom{c+1}{2}p_B$, which implies that $|E(G[R])| \leq \binom{c+1}{2}p_R$. By similar reasoning, $G[R]$ also satisfies the conclusion of the theorem. Therefore G satisfies the conclusion itself, a contradiction.

So there is $|E(B)| > \binom{c+1}{2}p_B$ and $|E(G[R])| < \binom{c+1}{2}p_R$. Again, since G is the smallest counterexample, $G[R]$ satisfies the conclusion of the theorem. By Lemma 12, we get that $|E(G[R])| = \binom{c+1}{2}p_R$, and so $|E| > \binom{c+1}{2}p$, a contradiction. So G is 2-connected.

Let us show that G contains a vertex v of degree c. By Lemma 1, there is $d(x) \geq c$ for every vertex $x \in V$. Towards a contradiction, suppose there is $d(x) > c$ for every vertex $x \in V$. So we have $|E| \geq (pc+1)(c+1)/2 = \binom{c+1}{2}p + \frac{c+1}{2}$, a contrary to $|E| \leq \binom{c+1}{2}p$.

Let $G' = (V', E') = G/N[v]$ be the graph obtained from G by contracting the closed neighborhood of v in G. Let v' be the vertex obtained from contracting $N[v]$. By Lemma 6, $G' = (V', E')$ is 1-FT($p'K_c$) with $p' = p - 1$.

The contraction of $N[v]$ eliminated the $\binom{c+1}{2}$ edges of $G[N[v]]$ and there exists a natural injection from the newly added edges incident to v', to the removed edges incident to one vertex in $N[v]$ and another in $V \setminus N[v]$. So there is $|E(G')| \leq \binom{c+1}{2}p'$. Moreover, G' satisfies the conclusion of the theorem, since G is the smallest counterexample.

If v' is a cutvertex, since G is 2-connected, then v' is the only cutvertex in G' (any other cutvertex of G' would also be a cutvertex in G, a contradiction.). Otherwise, G' has no cutvertices.

Suppose that v' is not a cutvertex. So G' is 2-connected and, not being a counterexample, G' is isomorphic to K_{c+1}. So $p' = 1$ and G' has $\binom{c+1}{2}$ edges.

Moreover, G has $2\binom{c+1}{2}$ edges. Indeed, the contraction diminished the number of edges by at least $\binom{c+1}{2}$ and, satisfying the hypotheses of the theorem, G cannot have more than $2\binom{c+1}{2}$ edges.

So G is composed of a subgraph isomorphic to K_{c+1} corresponding to $N[v]$, denoted by $\widehat{K_{c+1}}$, a subgraph isomorphic to K_c corresponding to the other vertices, denoted by $\widehat{K_c}$, and a matching of size c between $\widehat{K_{c+1}}$ and $\widehat{K_c}$. Indeed, there must be an edge between each vertex of $\widehat{K_c}$ and a vertex of $\widehat{K_{c+1}}$ to get a K_{c+1} after the contraction, and the number of such edges must be c to give the total count of $2\binom{c+1}{2}$ in G.

Now let x be any vertex in $\widehat{K_c}$. There is $d_G(x) = c$ and, by Lemma 2, x belongs to a subgraph of G isomorphic to K_{c+1}, a contrary to the above observation on the structure of G.

Now suppose that v' is the unique cutvertex in G'. So G', not being a counterexample, is isomorphic to p' copies of K_{c+1} sharing the vertex v'. So G' has exactly $\binom{c+1}{2}p'$ edges, and the contraction diminished the number of edges by exactly $\binom{c+1}{2}$. On the other hand, $N[v]$ is a separator in G, and every component of $G - N[v]$ is a K_c. Let $\widehat{K_c}$ be one of these components. By an argument similar to that of the case when v' is not a cutvertex, $G[V(\widehat{K_c}) \cup V(N_G[v])]$ is composed of a K_{c+1}, a K_c, and a matching between them, and G is not 1-FT(pK_c). A contradiction. ∎

4. Conclusions

In this paper, we have provided an upper bound on the number of edges in minimum k-FT(pK_c) graphs for $k \geq 1$, $p \geq 1$ and $c \geq 3$ (Lemma 10). We have shown that this bound is tight and given a complete characterization of minimum k-FT(pK_c) for $k = 1$ (Theorem 1). We conjecture that this bound is also tight for $k > 1$, $p \geq 1$, $c \geq 3$, and $k < c$. In particular, minimum k-FT(pK_c) graphs correspond to the subclass of chordal graphs described in Lemma 10.

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