Discussiones Mathematicae
Graph Theory 44 (2024) 1449–1469
https://doi.org/10.7151/dmgt.2502

ON THE EDGE-SUM DISTINGUISHING GAME

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Abstract

The *Edge-Sum Distinguishing game* (ESD game) is a graph labeling game proposed by Tuza in 2017. In such a game, the players, traditionally called Alice and Bob, alternately assign an unused label $f(v) \in \{1, \ldots, s\}$ to an unlabeled vertex v of a graph G, and the induced edge label $\phi(uv)$ of an edge $uv \in E(G)$ is given by $\phi(uv) = f(u) + f(v)$. Alice's goal is to end up with an injective vertex labeling of all vertices of G that induces distinct edge labels, and Bob's goal is to prevent this. Tuza also posed the following questions about the ESD game: given a simple graph G, for which values of s can Alice win the ESD game? And if Alice wins the ESD game with the set of labels $\{1, \ldots, s\}$, can she also win with $\{1, \ldots, s+1\}$? In this work, we partially answer these questions by presenting bounds on the number of consecutive non-negative integer labels necessary for Alice to win the ESD game on general and classical families of graphs.

Keywords: graph labeling, labeling game, maker-breaker game, edge-sum distinguishing game, combinatorial game.

2020 Mathematics Subject Classification: 05C57, 05C78.

1. Introduction

In this paper, all graphs $G = (V(G), E(G))$ are finite, undirected, simple such that $n = |V(G)|$ and $m = |E(G)|$.

In 1963, Sedláček [12] introduced the notion of a *magic labeling* of a connected graph G as a labeling of the edges of G with real numbers such that: (1) distinct edges are assigned distinct labels, and (2) the sum of the values assigned to all edges incident to a given vertex is the same for all vertices $v \in V(G)$. Since then, many graph labelings based on sums of integer labels have been proposed [5]. For example, in 1970, Kotzig and Rosa [9] introduced the notion of a *magic valuation*, which is a labeling $f\colon V(G) \to \{1, \dots, n\}$ of a graph G such that $S = \{f(u) + f(v)\colon uv \in E(G)\}$ consists of m consecutive integers. This labeling was later rediscovered by Enomoto *et al.* [3] and renamed as *super edge-magic labeling*.

In 1980, Graham and Sloane [6] defined the *harmonious labeling* as an injective function $f\colon V(G) \to \mathbb{Z}_m$ in which each edge $uv \in E(G)$ is labeled with $\phi(uv) = (f(u) + f(v)) \bmod m$, so that the resulting edge labels are distinct, and conjectured that every tree is harmonious. Liu and Zhang [10] proved that every graph is a subgraph of a harmonious graph.

In 1990, Hartsfield and Ringel [8] introduced *antimagic labelings* motivated by magic labelings. A graph with m edges is called *antimagic* if its edges can be labeled with distinct labels from $\{1, \dots, m\}$, such that the sums of the labels of the edges incident to each vertex are distinct. They conjectured that every graph except K_2 has an antimagic labeling.

For the reader interested in more examples of labelings constructed from sums of integer labels or in results on (anti)magic labelings, harmonious labelings and super edge-magic labelings, we suggest Gallian's dynamic survey [5].

In 2017, Tuza [14] introduced the *Edge-Sum Distinguishing labeling* (ESD labeling), defined as follows: given a graph G and a set of consecutive integer labels $\mathcal{L} = \{1, 2, \dots, s\}$, an ESD labeling of G is an injective labeling $f\colon V(G) \to \mathcal{L}$ such that, when we assign the edge label $\phi(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$, the (induced) edge labeling ϕ is injective. We note that the set of all possible edge labels induced by the vertex labeling f is represented by $\mathcal{L}_E = \{3, 4, \dots, 2s - 1\}$. Figure 1 exhibits a graph with an ESD labeling.

The ESD labeling was later investigated by Bok and Jedličková [1], who determined the minimum positive integer s for which many classical families of graphs admit an ESD labeling $f\colon V(G) \to \{1, \dots, s\}$.

Graph labelings are usually investigated from the perspective of determining whether a given graph has a required labeling or not [5]. An alternative perspective is to analyze graph labeling problems from the point of view of combinatorial games [2, 7, 13]. In fact, Tuza introduced the ESD labeling in connection to the

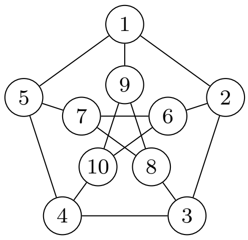


Figure 1. Petersen graph with an edge-sum distinguishing labeling.

study of a combinatorial game related to graph labelings with sums. In his seminal article, Tuza [14] surveyed the area of graph labeling games and presented two graph labeling games with sums [2, 7] based on magic labelings. Tuza [14] also proposed new variants of graph labeling games such as the *Graceful game*, studied by Frickes et al. [4], the *Edge-Difference Distinguishing game*, later investigated by Oliveira *et al.* [11], and the *Edge-Sum Distinguishing game*.

The *Edge-Sum Distinguishing game* (ESD game) is a type of *maker-breaker* game, where the players have opposite goals. In this game, Alice and Bob alternately assign a previously unused label $f(v) \in \mathcal{L} = \{1, \ldots, s\}$ to an unlabeled vertex v of a given graph G. If both ends of an edge $vw \in E(G)$ are already labeled, then the (induced) label $\phi(vw)$ of the edge vw is defined as $\phi(vw) = f(v) + f(w)$. A move is *legal* if after it all edge labels are distinct. Only legal moves are allowed in this game. Alice (the *maker*) wins if the graph G is fully ESD labeled, and Bob (the *breaker*) wins if he can prevent this (that is, Bob wins if, at some point, no more legal moves are allowed and the graph is not fully ESD labeled).

Tuza [14] posed the following questions about the ESD game.

Question 1. *Given a graph G and a set of consecutive non-negative integer labels $\mathcal{L} = \{1, \ldots, s\}$, for which values of s can Alice win the* ESD *game*?

Question 2. *If Alice can win the* ESD *game on a graph G with the set of labels $\mathcal{L} = \{1, \ldots, s\}$, can she also win with $\mathcal{L} = \{1, \ldots, s+1\}$*?

In this work, we investigate winning strategies for Alice and Bob on the ESD *g*ame for classical families of graphs, such as stars, paths, cycles, wheels and complete graphs. Furthermore, we partially answer Tuza's questions presenting bounds for the number of consecutive non-negative integer labels necessary for Alice to win the ESD game on a graph G.

2. Main Results

We begin this section by presenting definitions used throughout this paper. Let $G = (V(G), E(G))$ be a graph such that $n = |V(G)|$ and $m = |E(G)|$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$; in such a case, edge $e = uv$ and vertices u and v are called *incident*, and vertices u and v are also called *neighbors*. The *neighborhood* of $u \in V(G)$ is the set $N(u) = \{v \colon uv \in E(G)\}$. The *degree* $d(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. The *maximum degree* of G is the number $\Delta(G) = \max\{d(v) \colon v \in V(G)\}$. The *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path connecting u and v in G.

The *edge-sum distinguishing labeling number* $\sigma(G)$ of a graph G is the least positive integer s for which G has an ESD labeling $f \colon V(G) \to \{1, \dots, s\}$.

The *edge-sum distinguishing game number* $\sigma_g(G)$ of a graph G is the least positive integer s such that Alice has a winning strategy for the ESD game on G using the set of labels $\{1, \dots, s\}$. Our first results present bounds for the edge-sum distinguishing game number and its relation with the edge-sum distinguishing labeling number.

Lemma 3. *For every graph G, $\sigma_g(G) \geq \sigma(G)$.*

Proof. If Alice has a winning strategy to obtain an ESD labeling of G with labels from the set $\{1, \dots, \sigma_g(G)\}$, then $\sigma_g(G) \geq \sigma(G)$. ■

Lemma 4. *Let s be a positive integer. If a graph G with m edges has an* ESD *labeling $f \colon V(G) \to \{1, \dots, s\}$, then $m \leq 2s - 3$.*

Proof. Let G be a graph with an ESD labeling $f \colon V(G) \to \{1, \dots, s\}$. By the definition of ESD labeling, $\mathcal{L}_E = \{3, 4, \dots, 2s - 1\}$. Since $|\mathcal{L}_E| = 2s - 3$ and all edges of G receive distinct induced labels, we obtain that $m \leq 2s - 3$. ■

Corollary 5. *If a graph G with m edges has an* ESD *labeling $f \colon V(G) \to \{1, \dots, s\}$, then $s \geq \frac{m+3}{2}$.*

Corollary 5 establishes a lower bound for s that guarantees a necessary number of labels to each edge of G, and the next result states a necessary number of labels to each vertex of G in an ESD labeling.

Lemma 6. *For every graph G with n vertices and m edges, $\sigma_g(G) \geq \sigma(G) \geq \max\left\{n, \frac{m+3}{2}\right\}$.*

Proof. Since an ESD labeling is an injective mapping from the set of vertices on the set of integers from 1 to s, if the lower bound given by Corollary 5 is less than n, that is $m < 2n - 3$, then s is the maximum between $\left\{n, \frac{m+3}{2}\right\}$. ■

Bok and Jedličková [1] proved that if G is a graph with maximum degree Δ, then $\sigma_g(G) \leq (\Delta^2 + 1)n + \Delta\binom{n-1}{2}$. In the next theorem, we improve their result, by presenting a better upper bound for the parameter $\sigma_g(G)$, for an arbitrary graph G.

Theorem 7. *If G is a graph on n vertices and m edges, then*

$$\sigma_g(G) \leq n + \max\{d(u)(m - d(u)) \colon u \in V(G)\}.$$

Proof. Let G be a graph on n vertices and m edges, and let $\mathcal{L} = \{1, \ldots, s\}$ be a set of consecutive integer labels such that $s \geq n + \max\{d(u)(m - d(u)) \colon u \in V(G)\}$. Alice (or Bob) starts playing the ESD game on G, and our objective is to show a winning strategy for Alice.

For each vertex $w \in V(G)$, let $L(w)$ be the set of available labels for w. At the beginning of the game, $L(w) = \mathcal{L}$ for all $w \in V(G)$. At each round of the game, a player (Alice or Bob) chooses an unlabeled vertex and assigns to it an available label α such that $1 \leq \alpha \leq s$. Right after a player's move, the sets of available labels of the remaining unlabeled vertices are updated to maintain the property that these sets only contain available labels for the respective vertices. Thus, when it is Alice's turn, she always chooses an unlabeled vertex w and assign to w any label in the set of available labels $L(w)$, if $L(w) \neq \emptyset$. At the end of this proof, we show that $L(w) \neq \emptyset$ for every unlabeled vertex $w \in V(G)$ at any point of the game. Next, we describe how the sets of available labels of unlabeled vertices are updated right after each player's move.

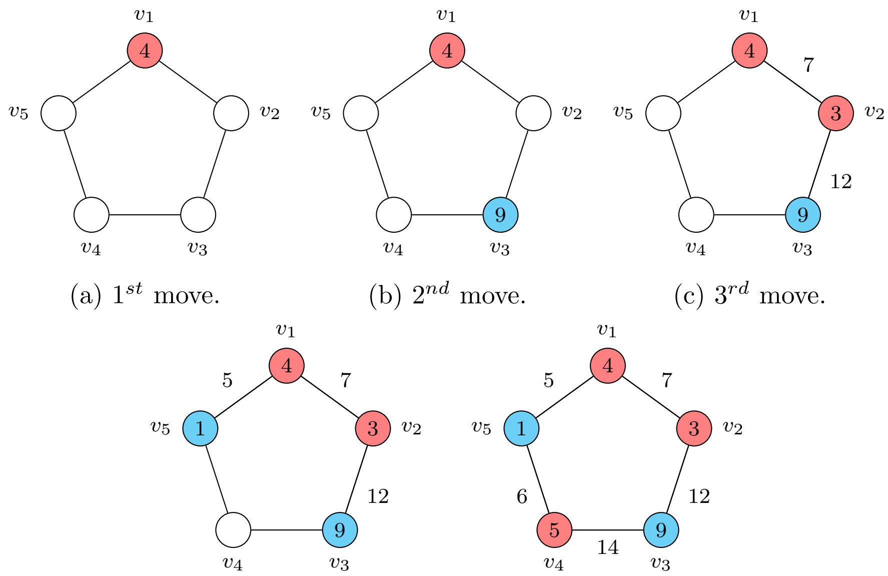
At the j-th move, a player (Alice or Bob) chooses an unlabeled vertex $v_j \in V(G)$ and assigns an available label $f(v_j)$ to v_j. Right after the j-th move, the set of available labels $L(u)$ of each remaining unlabeled vertex $u \in V(G)$ is updated. Only unused vertex labels and vertex labels that cannot generate repeated edge labels in future iterations can remain in each set. The sets of available labels are updated according to the following two steps.

1. For every unlabeled vertex $u \in V(G)$, remove $f(v_j)$ from $L(u)$. Note that, since an ESD labeling is injective, the label $f(v_j)$ cannot be assigned to more than one vertex.
2. For every unlabeled vertex $u \in V(G)$ and for every labeled vertex $u' \in N(u)$, delete from $L(u)$ every label ℓ such that $\ell + f(u') = \phi(e)$, for every edge $e \in E(G)$ that has both endpoints labeled.

Figure 2 illustrates the strategy described above.

Next, we determine the maximum number of labels that are deleted, throughout the game, from each set of available labels. First, note that exactly one label is deleted from each unlabeled vertex at each execution of Step 1. Thus, exactly $n - 1$ labels are deleted at Step 1 after the first $n - 1$ moves.

Now, we count how many labels are deleted after the first $n - 1$ executions of Step 2. Let u be the last unlabeled vertex. Right after $n - 1$ moves, all the neighbors of u are labeled and there are exactly $m - d(u)$ edges in G with induced labels ℓ'. For each neighbor $w \in N(u)$, each induced edge label ℓ' can preclude at most one vertex label from being assigned to vertex u, namely, the label $\ell' - f(w)$. Thus, each neighbor of u contributes for the deletion of at most $(m - d(u))$ labels from $L(u)$. Since u has $d(u)$ neighbors, the number of labels deleted from $L(u)$ after the first $n - 1$ executions of Step 2 is at most $d(u)(m - d(u))$. Therefore, the maximum number of labels that can be excluded on Step 2 is $\max\{d(u)(m - d(u)) \colon u \in V(G)\}$.



(a) 1^{st} move. (b) 2^{nd} move. (c) 3^{rd} move.

(d) 4^{th} move. (e) 5^{th} move.

Figure 2. A sequence of moves of the ESD game illustrating the strategy described in the proof of Theorem 7. A player chooses an unlabeled vertex v_i of the cycle C_5. At the beginning of the game, $L(v_i) = \mathcal{L} = \{1, \ldots, 11\}$, for every $v_i \in V(G)$. After each iteration, the set of available labels is updated for each vertex not labeled. In the last move we guarantee that there exists an available label that can be assigned for the last vertex v_i in the set $L(v_i)$.

From the previous analysis, we conclude that at most $(n-1)+\max\{d(u)(m-d(u)) \colon u \in V(G)\}$ labels are deleted from each set of available labels. Since $|\mathcal{L}|$ is greater than this value, we conclude that there is always an available label at each set $L(u)$ to be assigned to an unlabeled vertex u, and the result follows. ∎

We observe that the bound given by Theorem 7 is tight in the sense that graphs K_2 and $2K_2$ are the smallest graphs that attain the equality. The next result follows from the proof of Theorem 7 and partially answers Question 2 posed by Tuza.

Corollary 8. *If Alice wins the* ESD *game on G with the set of labels $\mathcal{L} = \{1, \ldots, s\}$, then she also wins with $\mathcal{L}' = \{1, \ldots, s+1\}$ for any integer $s \geq n + \max\{d(u)(m - d(u)) \colon u \in V(G)\}$.*

In this work, we have also implemented computational algorithms to analyze the game on small graphs. We applied backtracking technique which means that, at each player turn, we grow the tree of partial solutions, branching on the set of all possible moves. We prune the branching process for a partial solution as soon as we can decide which player wins the game at that position. For the general case, a partial solution is a winning position for a player X if X can reach a winning position on its next move. A position where all vertices have a label is a winning position for Alice. A position where there exist unlabelled vertices and the player of the turn could not make a legal move is a winning position for Bob. The computational results obtained are described throughout the next section.

2.1. ESD game on families of graphs

In this section, we present our analysis of the ESD game for some classical families of graphs. We define $\mathcal{L} = \{1, \ldots, s\}$ as the set of available vertex labels for an ESD game's match.

2.1.1. Stars

A *star* graph is a tree on n nodes isomorphic to the complete bipartite graph $K_{1,n-1}$, where one node has degree $n-1$ (central vertex), and the other $n-1$ nodes have degree 1.

Theorem 9. *If $K_{1,n-1}$ is a star, $n \geq 2$, then $\sigma(K_{1,n-1}) = \sigma_g(K_{1,n-1}) = n$.*

Proof. Independently of the label $\ell \in \{1, \ldots, n\}$ that is assigned to the central vertex of $K_{1,n-1}$, the labels in the set $\{1, \ldots, n\} \backslash \{\ell\}$ are all assigned to the remaining vertices of $K_{1,n-1}$ and induce distinct edge labels. ■

2.1.2. Paths

A *path* P_n is a graph with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} \colon 1 \leq i \leq n-1\}$. An ESD labelling $f \colon V(P_n) \to \{1, \ldots, n\}$ is easily obtained by assigning $f(v_i) = \lceil \frac{i}{2} \rceil$ if i is odd; or $f(v_i) = \lceil \frac{n}{2} \rceil + \frac{i}{2}$ otherwise. Therefore, $\sigma(P_n) = n$. By Theorem 7, $\sigma_g(P_n) \leq 2n-2$ for $2 \leq n \leq 4$, and $\sigma_g(P_n) \leq 3n-6$ for $n \geq 5$. The next result, establishes the winner of the ESD game on small paths.

Proposition 10. *If P_n is a path with $n \leq 11$ vertices, then*

$$\sigma_g(P_n) = \begin{cases} n & \textit{if } n \leq 3; \\ n+1 & \textit{if } 4 \leq n \leq 8; \\ n+2 & \textit{if } 9 \leq n \leq 11. \end{cases}$$

Moreover, if $\mathcal{L} = \{1, \ldots, n\}$ is a set of labels then, for P_4, the winner is the player who does not start the game and, for P_5, the winner is the player who starts the game.

Proof. Let $\mathcal{L} = \{1, \ldots, n\}$ be a set of labels and P_n be a path on n vertices. For $2 \leq n \leq 3$, the result follows from Theorem 9 since in these cases P_n is a star. Consider $n = 4$. First, suppose that Alice starts the game by assigning label $\ell \in \mathcal{L}$ to an arbitrary vertex $v \in V(P_4)$. On the second move, Bob assigns label $5 - \ell$ to a neighbor of v with a lower degree. Bob wins with this move since there will always be an edge with the repeated edge label 5. When Bob is the first player, he starts the game by assigning label $\ell \in \{1, 2\}$ to an arbitrary vertex $v \in V(G)$ (the case $\ell \in \{3, 4\}$ is complementary, that is, when a label α is played here, the label $n + 1 - \alpha$ is used on the same round of the complementary game). On the second move, Alice assigns label $5 - \ell$ to a vertex u at distance two apart from v. Alice wins since this partial labeling can always be extended to an ESD labeling of P_4. Hence, the winner on P_4 is the player who does not start the game.

Now, consider $n = 5$ and suppose that Bob starts the game by assigning label 5 to $v_3 \in V(P_5)$. On the second move, Alice assigns label $\ell \in \mathcal{L} \backslash \{5\}$ to another vertex u of P_5. Then, Bob assigns the label $5 - \ell$ to a neighbor of u on the third move, thus generating the edge label 5. Bob wins since the next moves generate a repeated edge label 5. Next, consider that Alice starts the game. She starts by assigning label 3 to vertex $v_2 \in V(P_5)$. Now, Bob chooses an arbitrary vertex from $\{v_1, v_3, v_4, v_5\}$. There are four cases to consider.

Case 1. Bob chooses v_1 on the second move: (i) if Bob assigns 1 to v_1, then Alice assigns 2 to v_3; (ii) if Bob assigns 2 to v_1, then Alice assigns 5 to v_4; (iii) if Bob assigns 4 to v_1, then Alice assigns 1 to v_4; and (iv) if Bob assigns 5 to v_1, then Alice assigns 1 to v_3. It can be checked by inspection that these partial labelings extend to an ESD labeling of P_5, independently of the next two moves.

Case 2. Bob chooses v_3 on the second move: (i) if Bob assigns 1 to v_3, then Alice assigns 4 to v_5; (ii) if Bob assigns 2 to v_3, then Alice assigns 1 to v_1; (iii) if Bob assigns 4 to v_3, then Alice assigns 5 to v_1; and (iv) if Bob assigns 5 to v_3, then Alice assigns 1 to v_1. It can be checked by inspection that these partial labelings extend to an ESD labeling of P_5, independently of the next two moves.

Case 3. Bob chooses v_4 on the second move: (i) if Bob assigns 1 to v_4, then Alice assigns 2 to v_5; (ii) if Bob assigns 2 to v_4, then Alice assigns 5 to v_1; (iii) if Bob assigns 4 to v_4, then Alice assigns 1 to v_1; and (iv) if Bob assigns 5 to v_4, then Alice assigns 4 to v_5. It can be checked by inspection that these partial labelings extend to an ESD labeling of P_5, independently of the next two moves.

Case 4. Bob chooses v_5 on the second move: (i) if Bob assigns 1 to v_5, then Alice assigns 4 to v_3; (ii) if Bob assigns 2 to v_5, then Alice assigns 1 to v_4; (iii) if Bob assigns 4 to v_5, then Alice assigns 1 to v_1; and (iv) if Bob assigns 5 to v_5, then Alice assigns 2 to v_3. It can be checked by inspection that these partial labelings extend to an ESD labeling of P_5, independently of the next two moves.

Therefore, Alice wins on P_5 when she starts.

Finally, we computationally determine the value of $\sigma_g(P_n)$ for $n \leq 11$ as follows: $\sigma_g(P_n) = n$ if $n \leq 3$, $\sigma_g(P_n) = n+1$ if $4 \leq n \leq 8$, and $\sigma_g(P_n) = n+2$ if $9 \leq n \leq 11$. ∎

2.1.3. mP_2 graphs

A mP_2 is a graph with vertex set $V(mP_2) = \{v_1, \ldots, v_{2m}\}$ and edge set $E(mP_2) = \{v_i v_{i+1} \colon i \text{ is odd and } 1 \leq i \leq 2m-1\}$. An ESD labelling $f \colon V(mP_2) \to \{1, \ldots, 2m\}$ is obtained by assigning $f(v_i) = i$ for every $v_i \in V(mP_2)$. Therefore, $\sigma(mP_2) = 2m$.

Theorem 11. *Let $\mathcal{L} = \{1, \ldots, 2m\}$ be a set of labels. If Alice starts the* ESD *game on mP_2, then Bob wins the game.*

Proof. The strategy for Bob to win is: if Alice assigns label ℓ to a vertex v, then Bob must play on vertex w, adjacent to v, assigning label $\ell+1$ if ℓ is odd or $\ell-1$ if ℓ is even. Bob must repeat this procedure until an unlabeled induced $2P_2$ is left. At this moment, two pair of labels $(a, a+1)$ and $(b, b+1)$, where a, b are odd, remain unused. And now, if Alice assigns $\ell' \in \{a, a+1, b, b+1\}$ to a vertex v on the remaining $2P_2$, then Bob assigns $a+b+1-\ell'$ to the vertex adjacent to v generating an edge with sum $a+b+1$. Therefore, for the last two moves of the game, the players have a pair of labels with sum $a+b+1$ and, consequently, it will not be possible to complete an ESD labeling, which means that Bob wins the game. ∎

Theorem 12. *Let $\mathcal{L} = \{1, \ldots, s\}$ be a set of labels. If Bob starts the* ESD *game on mP_2, then Alice wins the game for any value of $s \geq 2m$.*

Proof. First, consider that s is even. The strategy for Alice to win is: if Bob assigns label ℓ to a vertex v, then Alice must play on vertex w, adjacent to v, assigning label $\ell+1$ if ℓ is odd or $\ell-1$ if ℓ is even. Alice must repeat this procedure until all the vertices are labeled. It is easy to see that the sum of the extreme points of the edges are all odd and pairwise distinct. Hence, Alice wins.

Now, consider that s is odd. Alice should repeat the strategy of the previous case, playing pairs of the form $(\ell, \ell+1)$ where ℓ is odd, until Bob assign the label s to some vertex v. Alice replies this move assigning to the vertex w, adjacent to v, the label x where x is the greatest unused odd label of $\mathcal{L}$. This move is legal because $s+x$ is even and all the previous labeled edges have odd label. Clearly, the label $x+1$ is even, unused and, after the last move, it is the greatest unused label of $\mathcal{L}$ and the only vertex without an unused pair of the form $(\ell, \ell+1)$, where ℓ is odd. Alice continues to play following the same strategy as before for every move of Bob where he chooses a label less than $x+1$. If Bob assigns the label $x+1$ to some vertex v, Alice replies this move assigning to vertex w, adjacent to v, the label $y+1$ where $y+1$ is the greatest unused even label of $\mathcal{L}$. The edge label $y+1+x+1$ is even and we need to compare it with $s+x$ to verify if this is a legal move. Since $x+1 < s$ and $y+1 < x$, we have that $y+1+x+1 < s+x$, which implies that the move is legal. After this, we have y as the greatest (and the only one) vertex without an unused pair of the form $(\ell, \ell+1)$, where ℓ is odd. If Alice repeats the previous strategies, using the same analysis, we conclude that she and Bob will always have legal moves to play. Consequently, Alice wins the game. ■

2.1.4. Cycles

A *cycle* C_n, with $n \geq 3$ vertices, is a graph with vertex set $V(C_n) = \{v_1, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} \colon 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. From Lemma 6 and Theorem 7, it follows that $n \leq \sigma(C_n) \leq \sigma_g(C_n) \leq 3n-4$ for $n \geq 3$. In the next result, we show the proofs of the Proposition 3.5 posed by Tuza [14] on cycles C_3, C_4, and C_5, when Alice starts the game. We also establish the winner of the ESD game played with labels from $\{1, \ldots, n\}$ when Bob starts the game.

Proposition 13. *If C_n is a cycle with $n \leq 10$ vertices, then*

$$\sigma_g(C_n) = \begin{cases} n & \text{if } n = 3; \\ n+1 & \text{if } 4 \leq n \leq 5; \\ n+2 & \text{if } 6 \leq n \leq 9; \\ n+3 & \text{if } n = 10. \end{cases}$$

Moreover, if $\mathcal{L} = \{1, \ldots, n\}$ is a set of labels, Bob wins the game on C_5, independently of who starts the game; and, for C_4, the winner is the player who does not start the game.

Proof. Let $\mathcal{L} = \{1, \ldots, n\}$ be a set of labels and C_n a cycle on n vertices. Alice wins the game on C_3 because, independently of who starts the game, its vertices are assigned a cyclic permutation of $\{1, 2, 3\}$, thus generating edge labels $\mathcal{L}_E = \{3, 4, 5\}$. So, consider $n = 4$. First, suppose that Bob starts the game by assigning an arbitrary label $\ell \in \mathcal{L}$ to $v_1 \in V(C_4)$. On the second move, Alice assigns label $5 - \ell$ to v_3 and wins the game since an ESD labeling can always be obtained independently of the next players' moves. Now, suppose that Alice starts the game by assigning an arbitrary label $\ell \in \mathcal{L}$ to $v_1 \in V(C_4)$. On the second move, Bob assigns label $5-\ell$ to v_2, thus generating edge label 5. Bob wins the game since, regardless of Alice's next choice, there will always be a repeated edge label 5.

Next, consider $n = 5$. First, suppose, without loss of generality, that Alice starts the game by assigning an arbitrary label $\ell \in \mathcal{L}$ to vertex $v_1 \in V(C_5)$.

Case 1. $\ell = 1$ (the case $\ell = 5$ is complementary). On the second move, Bob assigns the label $6 - \ell$ to $v_2 \in V(C_5)$. In the third move, if Alice assigns a label $\alpha \in \{2, 4\}$ to a vertex $w \in \{v_3, v_4, v_5\}$, then Bob wins the game by assigning the label 3 to a vertex $v \in \{v_3, v_5\} \backslash \{w\}$ since a repeated edge label 6 would be generated in the next move. For the case where Alice assigns label 3 to a vertex $w \in \{v_3, v_5\}$ on the third move, she loses by the same reasoning of the previous case. So, consider that Alice assigns label 3 to v_4 on the third move. Thus, Bob assigns label 2 to v_3 and wins the game since a repeated edge label 5 would be generated in the next move.

Case 2. $\ell = 2$ (the case $\ell = 4$ is complementary). On the second move, Bob assigns the label 5 to $v_3 \in V(C_5)$. On the third move: (i) if Alice assigns label 1 (respectively 3) to v_2, then Bob assigns label 3 (respectively 1) to v_4; (ii) if Alice assigns label 4 to v_2, then Bob assigns label 3 to v_5; (iii) if Alice assigns label 1 to v_5, then Bob assigns label 3 to v_2; (iv) if Alice assigns label 3 to v_5, then Bob assigns label 4 to v_2; (v) if Alice assigns label 4 to v_5, then Bob assigns label 3 to v_2; (vi) if Alice assigns label 1 to v_4, then Bob assigns label 3 to v_2; (vii) if Alice assigns label 3 to v_4, then Bob assigns label 1 to v_2; and (viii) if Alice assigns label 4 to v_4, then Bob assigns label 3 to v_2. It can be checked by inspection that all these partial labelings cannot be extended to an ESD labeling of C_5.

Case 3. $\ell = 3$. On the second move, Bob assigns label 1 to v_3. On the third move: (i) if Alice assigns label 2 to v_2, then Bob assigns label 5 to v_5; (ii) if Alice assigns label 4 to v_2, then Bob assigns label 5 to v_4; (iii) if Alice assigns label 5 to v_2, then Bob assigns label 4 to v_4; (iv) if Alice assigns label 2 to v_5, then Bob assigns label 5 to v_2; (v) if Alice assigns label 4 to v_5, then Bob assigns label 5 to v_2; (vi) if Alice assigns label 5 to v_5, then Bob assigns label 2 to v_2; (vii) if Alice assigns label 2 to v_4, then Bob assigns label 5 to v_2; (viii) if Alice assigns label 4 to v_4, then Bob assigns label 5 to v_2; and (ix) if Alice assigns label 5 to v_4, then Bob assigns label 4 to v_2. It can be checked by inspection that all these nine partial labelings cannot be extended to an ESD labeling of C_5.

Therefore, Bob wins the game on C_5 when Alice starts.

Now, suppose that Bob starts the game by assigning label 2 to vertex v_1. Then, there are only two choices for Alice, either choosing a neighbor or a non-neighbor of v_1 for her second move. First, consider that Alice chooses a neighbor of v_1, say v_2. Thus: (i) if Alice assigns label 1 to v_2, then Bob assigns label 5 to v_5; (ii) if Alice assigns label 3 or 4 to v_2, then Bob assigns label 1 to v_4; and (iii) if Alice assigns label 5 to v_2, then Bob assigns label 3 to v_4. It can be checked by inspection that all these partial labelings cannot be extended to an ESD labeling of C_5. Next, consider that Alice chooses a non-neighbor of v_1, say v_3. Thus: (i) if Alice assigns label 1 to v_3, then Bob assigns label 3 to v_2; (ii) if Alice assigns label 3 to v_3, then Bob assigns label 4 to v_2; (iii) if Alice assigns label 4 to v_3, then Bob assigns label 5 to v_2; and (iv) if Alice assigns label 5 to v_3, then Bob assigns label 3 to v_2. It can be checked by inspection that all these partial labelings cannot be extended to an ESD labeling of C_5. Therefore, Bob also wins the game on C_5 when he starts.

Finally, we computationally verify the value of $\sigma_g(C_n)$ for $n \leq 10$ as follows:

$$\sigma_g(C_n) = \begin{cases} n+1 & \text{if } 4 \leq n \leq 5; \\ n+2 & \text{if } 6 \leq n \leq 9; \\ n+3 & \text{if } n = 10. \end{cases}$$

■

2.1.5. Wheels

A *wheel* W_{n-1} is a graph with n vertices and $m = 2n-2$ edges, $n \geq 4$, formed by connecting a single central vertex v_0 to all vertices $v_1, \ldots, v_{n-1}$ of a cycle C_{n-1}. Figure 3 exhibits ESD labelings for some wheels.

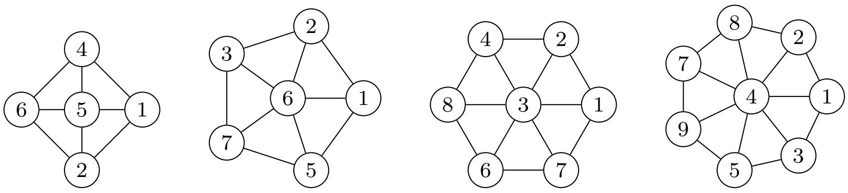


Figure 3. Wheels W_{n-1}, $5 \leq n \leq 8$, with ESD labelings.

The next result follows from Lemma 6 and Theorem 7.

Corollary 14. *If W_{n-1} is a wheel graph with $n \geq 4$, then $n+1 \leq \sigma_g(W_{n-1}) \leq (n-1)^2 + n$.*

For $n \geq 5$, the upper bound presented in Corollary 14 can be improved: adjusting the strategy presented in the proof of Theorem 7 in order to $v_0 \in V(W_n)$ be chosen on the first or second move.

Theorem 15. *If W_{n-1} is a wheel graph with $n \geq 5$, then $\sigma_g(W_{n-1}) \leq 6n-17$.*

Proof. Let $G = W_{n-1}$ be a wheel graph with $n \geq 5$. Also, let $\mathcal{L} = \{1, \ldots, 6n-17\}$ be a set of consecutive integer labels. Alice (or Bob) starts the ESD game on G and our objective is to show a winning strategy for Alice.

For each vertex $w \in V(G)$, define the set of available labels $\mathcal{L}(w)$ such that, at the beginning of the game, $\mathcal{L}(w) = \mathcal{L}$ for all $w \in V(G)$. At each round of the game, a player (Alice or Bob) chooses an unlabeled vertex and assigns to it an available label α such that $1 \leq \alpha \leq 6n-17$. Right after a player's move, the sets of available labels of the remaining unlabeled vertices are updated to maintain the property that these sets only contain available labels for the respective vertices. At the end of this proof, we show that $\mathcal{L}(w) \neq \emptyset$ for every unlabeled vertex $w \in V(G)$ at any round of the game.

If Alice starts the game, then she chooses $v_0 \in V(G)$, the central vertex of the wheel, and assigns to it any label in the set $\mathcal{L}(v_0)$. Otherwise, if Bob starts the game, and he does not chooses v_0 on his first move, then Alice chooses v_0 on the next move and assigns any label in the set $\mathcal{L}(v_0)$ to it. Thus, it is guaranteed that the central vertex is always labeled on the first or second round of the game.

Next, it is described how the sets of available labels of unlabeled vertices are updated right after each player's move. At the i-th move, $1 \leq i \leq n$, a player (Alice or Bob) chooses an unlabeled vertex $v_j \in V(G)$ and assigns v_j an available label $f(v_j) \in \mathcal{L}(v_j)$. Right after the i-th move, the set of available labels $\mathcal{L}(u)$ of each remaining unlabeled vertex $u \in V(G)$ is updated. Only unused vertex labels and vertex labels that cannot generate repeated edge labels in future iterations can remain in each set $\mathcal{L}(u)$. The sets of available labels are updated according to the following two steps.

(1) For every unlabeled vertex $u \in V(G)$, remove $f(v_j)$ from $\mathcal{L}(u)$.

(2) For every unlabeled vertex $u \in V(G)$ and for every labeled vertex $u' \in N(u)$, delete from $\mathcal{L}(u)$ every label ℓ such that $\ell + f(u') = \phi(e)$, for every edge $e \in E(G)$ that has both endpoints labeled.

Next, we determine the maximum number of labels that are deleted from each set of available labels, after the first $n-1$ moves. Note that exactly one label is deleted from each unlabeled vertex at each execution of Step (1). Thus, exactly $n-1$ labels are deleted at Step (1) after the first $n-1$ moves.

Now, we count how many labels are deleted after the first $n-1$ executions of Step (2). Let v_k be the last unlabeled vertex, $1 \leq k \leq n$. Note that $d(v_k) = 3$ since the central vertex of G was labeled at the beginning of the game. Right after the first $n-1$ moves, all the three neighbors of v_k are labeled and there are exactly $m-3 = 2n-5$ edges in G with induced labels $\phi(e)$. For each neighbor $w \in N(v_k)$, each induced edge label $\phi(e)$ can preclude at most one vertex label from being assigned to v_k, namely, the label $\phi(e) - f(w)$. However, the labeled edges incident with the neighbors of v_k can be disregarded from this counting since the label of their other endpoint was already deleted from $\mathcal{L}(v_k)$ at Step (1). Thus, each neighbor of v_k with degree 3 contributes to the deletion of at most $m-5 = 2n-7$ labels from $\mathcal{L}(v_k)$. Now, let us analyze the neighbor $v_0 \in N(v_k)$. Let $w \in \{v_1, \ldots, v_n\} \backslash \{v_k\}$. Note that there is only one way to generate an edge label $f(v_0) + f(w)$ having $f(v_0)$ fixed as one of the operands of the sum. Therefore, the labels of the $n-2$ labeled edges incident with v_0 cannot preclude vertex labels from being assigned to v_k anymore, since the problematic labels were already deleted from $\mathcal{L}(v_k)$ at Step (1). So, when considering the neighbor v_0, there are at most $m-3-(n-2) = (2n-2-3)-(n-2) = n-3$ edge labels that can preclude a vertex label from being assigned to v_k.

Hence, the total number of labels that is deleted from $\mathcal{L}(v_k)$ after the first $n-1$ moves is at most $(n-1)+(2n-7)+(2n-7)+(n-3) = 6n-18$. Since $|\mathcal{L}| > 6n-18$, there is always an available label at each set $\mathcal{L}(u)$ to be assigned to an unlabeled vertex u, and the result follows. ■

Proposition 16 presents basic properties of ESD labelings of wheels W_{n-1} using labels of the set $\{1,\ldots,n+1\}$. Proposition 16 is used in the proof of Theorem 18.

Proposition 16. *Let W_{n-1} be a wheel with n vertices, $n \geq 5$. If W_{n-1} has an* ESD *labeling $f\colon V(W_{n-1}) \to \{1,\ldots,n+1\}$, then we have the following.*

(a) $\mathcal{L}_E = \{3,\ldots,2n+1\}$ *and* $|\mathcal{L}_E| = |E(W_{n-1})|+1$.

(b) *If the edge labels $3,4,5$ are all induced on W_{n-1}, then either there exists a vertex with label 1 that is adjacent with three vertices with labels 2, 3 and 4; or W_{n-1} contains an induced cycle C_3 whose vertices have labels 1, 2 and 3. Furthermore, in the first case, with exception of the vertices with labels 2 and 3, one of these vertices is the central vertex of W_{n-1}. In the second case, one of these vertices is the central vertex of W_{n-1}.*

(c) *If the edge labels $2n-1, 2n, 2n+1$ are all induced on W_{n-1}, then either there exists a vertex with label $n+1$ that is adjacent with three vertices with labels n, $n-1$ and $n-2$; or W_{n-1} contains an induced cycle C_3 whose vertices have labels $n+1$, n and $n-1$. Furthermore, in the first case, with exception of the vertices with labels $n-1$ and n, one of these vertices is the central vertex of W_{n-1}. In the second case, one of these vertices is the central vertex of W_{n-1}.*

The following result follows immediately from Proposition 16.

Corollary 17. *Let W_{n-1} be a wheel with n vertices, $n \geq 5$. If W_{n-1} has an* ESD *labeling $f\colon V(W_{n-1}) \to \{1,\ldots,n+1\}$, then the central vertex v_0 of W_{n-1} has label $f(v_0) \in \{1,2,3,4,n-2,n-1,n,n+1\}$.*

Theorem 18. *Let $\mathcal{L} = \{1,\ldots,n+1\}$ be a set of labels. Bob wins the* ESD *game on W_{n-1}, $n \geq 7$, if he starts.*

Proof. Let W_{n-1} be as stated in the hypothesis. Bob and Alice play the ESD game on W_{n-1} using vertex labels from the set $\mathcal{L} = \{1,\ldots,n+1\}$. Also, note that $\mathcal{L}_E = \{3,\ldots,2n+1\}$. Bob starts the ESD game. On his first move, he assigns label 3 to the central vertex v_0, that has degree $n-1$. From now on, Bob's strategy consists on precluding the edge label 3 from being generated. Bob wins if the edge label 3 is not induced because, by item (a) of Proposition 16, all the three edge labels $2n-1, 2n, 2n+1$ must be induced on this case and, by item (c) of Proposition 16, one of the vertex labels $n-2$, $n-1$, n, $n+1$ must be assigned to the central vertex of W_n. This is a contradiction since 3 is already assigned to the central vertex and $3 < n-2$. Therefore, since Alice wants to win the game, she avoids using labels 1 and 2 in the first rounds of the game (at least as long as there are many pairs of unlabeled vertices two apart from each other). Hence, on the second round, Alice assigns a label $\ell \in \mathcal{L}\backslash\{1, 2, 3\}$ to a vertex $v_i \in V(W_{n-1})$. If $\ell = 4$, then Bob wins the game by assigning label 1 to v_{i+1}, thus generating the edge label 5 (note that the vertex label 2 cannot be assigned to a neighbor of v_{i+1} in order to induce the edge label 3 since it would generate the repeated edge label 5 with the central vertex). Now, suppose that $\ell \neq 4$, then, on the third move, Bob assigns label 4 to vertex v_{i+3}, thus generating edge label 7. On the fourth move, Alice assigns a label ℓ' to a vertex v. From the previous reasoning, we know that $\ell' \notin \{1, 2\}$. Now, on the fifth move, Bob assigns label 1 to an unlabelled neighbor of v_{i+3}, thus generating the edge labels 4 and 5. This precludes the vertex label 2 from being assigned to a vertex of W_{n-1} and, consequently, also precludes the edge label 3 from being generated. Therefore, in both cases, when $\ell = 4$ and $\ell \neq 4$, Bob wins the game. ∎

Theorem 19. *Let* $\mathcal{L} = \{1, \ldots, n+1\}$ *be a set of labels. For* $n \geq 9$*, Bob wins the* ESD *game on* W_{n-1} *when Alice starts.*

Proof. Let W_{n-1} be as stated in the hypothesis. Also, note that $\mathcal{L}_E = \{3, \ldots, 2n+1\}$. Bob and Alice play the ESD game on W_{n-1} using vertex labels from the set $\mathcal{L} = \{1, \ldots, n+1\}$. Alice starts the ESD game. If she chooses any vertex other than the central vertex v_0 on her first move, say v_i with $1 \leq i \leq n-1$, then Bob assigns a label $\ell \in \{n-3, n-4\}\backslash\{f(v_i)\}$ to v_0. Bob wins the game since, by Corollary 17, there is no ESD labeling f of W_{n-1} with $f(v_0) \in \{n-3, n-4\}$. Therefore, from now on, we consider that Alice starts the ESD game choosing the central vertex v_0 and that she assigns to it a label $\ell \in \{1, 2, 3, 4, n-2, n-1, n, n+1\}$. We split the proof into four cases.

Case 1. $f(v_0) \in \{1, n+1\}$. We first suppose $f(v_0) = 1$. By Proposition 16, $|\mathcal{L}_E| = |E(W_{n-1})| + 1$. Thus, Bob's winning strategy consists on preventing two edge labels from the set $\{2n+1, 2n, 2n-1, 2n-2\}$ from being induced.

In the second move, Bob assigns label $n+1$ to v_1. Note that the edge label $2n+1$ is induced only by the vertex labels n and $n+1$; and the edge label $2n$ is induced only by the vertex labels $n-1$ and $n+1$. Therefore, if none of the vertex labels n and $n-1$ is assigned to a neighbor of v_1, then Alice loses the game since the edge labels $2n+1$ and $2n$ are not induced. This implies that, until the fifth move, Alice has to assign a label $\ell \in \{n, n-1\}$ to a neighbor of v_1, say v_{n-1}. In the fourth move, Bob assigns an available label $\alpha \in \{2, 3\}$ to vertex v_2, thus preventing either $2n+1$ or $2n$ from being induced. In the fifth move, Alice chooses a new vertex and assigns a label β to it.

At this point of the game, the edge label $2n-1$ can only be induced by the vertex labels $n-1$ and n; and the edge label $2n-2$ can only be induced by the vertex labels $n-2$ and n. In fact, at most one of these labels can already have been induced. In order to win, Bob must prevent either $n-1$ or $n-2$ from being assigned to a neighbour of a vertex with label n. Just after the fifth move, we know, from the above discussion, that $f(v_{n-1}) \in \{n, n-1\}$. Then, on the sixth move, Bob assigns an available label from the set $\{n, n-1, n-2\}$ to a vertex not adjacent to the previously labeled vertices (with exception of v_0), thus winning the game. The analysis of the case $f(v_0) = n+1$ is complementary to the analyzes of the case $f(v_0) = 1$ (with edge label ℓ taken as $2n+4-\ell$ and with vertex label α taken as $n+2-\alpha$).

Case 2. $f(v_0) = 2$ (respectively $f(v_0) = n$). According to item (b) of Proposition 16, the labels 1 and 3 (respectively $n-1$ and $n+1$) must be assigned to adjacent vertices in order to induce the edge label 4 (respectively $2n$). So the strategy of Bob is to force these labels to be assigned to nonadjacent vertices. Thus, on the second move, Bob assigns label 4 (respectively $n-2$) to v_1, inducing the edge label 6 (respectively $2n-2$). Independently of Alice's next move, Bob assigns one of the labels 1 or 3 (respectively $n-1$ or $n+1$) to a vertex nonadjacent to the last vertex chosen by Alice. Therefore, Bob wins the game.

Case 3. $f(v_0) = 3$ (respectively $f(v_0) = n-1$). According to item (b) of Proposition 16, the labels 1 and 2 (respectively $n+1$ and n) must be assigned to adjacent vertices in order to induce the edge label 3 (respectively $2n+1$). On the second move, Bob assigns label 5 (respectively $n-3$) to v_1, inducing the edge label 8 (respectively $2n-4$). Independently of Alice's next move, Bob assigns one of the labels 1 or 2 (respectively $n+1$ or n) to a vertex nonadjacent to the last vertex chosen by Alice, thus winning the game.

Case 4. $f(v_0) = 4$ (respectively $f(v_0) = n-2$). According to item (b) of Proposition 16, the labels 2, 1 and 3 (respectively n, $n+1$ and n) must be assigned to consecutive vertices v_i, v_{i+1}, v_{i+2}, respectively, in order to induce the edge labels 3 and 4 (respectively $2n+1$ and $2n$). On the second move, Bob assigns label 1 (respectively $n+1$) to v_1, inducing the edge label 5 (respectively $2n-1$). Independently of Alice's next move, Bob assigns one of the labels 2 or 3 (respectively n or $n-1$) to a vertex nonadjacent to v_1. Therefore, Bob wins the game. ∎

Our algorithm determines the value of $\sigma_g(W_{n-1})$, for $4 \leq n \leq 10$. Also, from Theorems 18 and 19 we summarize in the following result.

Corollary 20. *If W_{n-1} is a wheel graph with $n \geq 4$, then $\sigma_g(W_{n-1}) > n+1$. Moreover, if W_{n-1} is a wheel with $n \leq 10$ vertices, then*

$$\sigma_g(W_{n-1}) = \begin{cases} 2n-2 & \textit{if } 4 \leq n \leq 5; \\ 2n-1 & \textit{if } 6 \leq n \leq 10. \end{cases}$$

2.1.6. Complete Graphs

A *complete graph* K_n is a graph on n vertices where any two of its vertices are adjacent. The next result follows from Lemma 6 and Theorem 7.

Corollary 21. *If K_n is a complete graph with $n \geq 2$ vertices, then*

$$\left\lceil \frac{n^2-n+6}{4} \right\rceil \leq \sigma(K_n) \leq \sigma_g(K_n) \leq \left\lfloor \frac{n^3-4n^2+7n-2}{2} \right\rfloor.$$

In the next result, we show that complete graphs with at least six vertices do not have an ESD labeling with labels from the set $\left\{1, \ldots, \left\lceil \frac{n^2-n+6}{4} \right\rceil \right\}$.

Theorem 22. *If K_n is a complete graph with $n \geq 6$ vertices, then $\sigma(K_n) > \left\lceil \frac{n^2-n+6}{4} \right\rceil$.*

Proof. Let $G = K_n$ be as stated in the hypothesis. Since G is a complete graph, we know that $|E(G)| = \frac{n(n-1)}{2}$. Suppose, by contradiction, that $\sigma(G) \leq \left\lceil \frac{n^2-n+6}{4} \right\rceil$. Thus, G has an ESD labeling $f \colon V(G) \to \left\{1, \ldots, \left\lceil \frac{n^2-n+6}{4} \right\rceil \right\}$. When $n \equiv 0, 1 \pmod 4$, $\left\lceil \frac{n^2-n+6}{4} \right\rceil = \frac{n^2-n+6+2}{4}$. On the other hand, when $n \equiv 2, 3 \pmod 4$, $\left\lceil \frac{n^2-n+6}{4} \right\rceil = \frac{n^2-n+6}{4}$. Note that the smallest edge label that can be induced is 3 and the largest edge label that can be induced is $2\left\lceil \frac{n^2-n+6}{4} \right\rceil - 1$. Thus, the possible induced edge labels belong to the set $\mathcal{L}_E = \left\{3, \ldots, 2\left\lceil \frac{n^2-n+6}{4} \right\rceil - 1\right\}$. Next, we calculate the cardinality of the set $\mathcal{L}_E$, depending on the value of n modulo 4.

Case 1. $n \equiv 0, 1 \pmod 4$. Thus,

$$\begin{aligned} |\mathcal{L}_E| &= \left(2\left\lceil \frac{n^2-n+6}{4} \right\rceil - 1\right) - 3 + 1 = \left(2\left(\frac{n^2-n+6+2}{4}\right) - 1\right) - 2 \\ &= \left(\frac{n^2-n+6+2}{2} - 1\right) - 2 = \left(\frac{n^2-n+6+2-2}{2}\right) - 2 \\ &= \frac{n^2-n}{2} + 3 - 2 = \frac{n(n-1)}{2} + 1 = |E(G)| + 1. \end{aligned}$$

Case 2. $n \equiv 2, 3 \pmod 4$. Thus,

$$|\mathcal{L}_E| = \left(2\left\lceil \frac{n^2-n+6}{4} \right\rceil - 1\right) - 3 + 1 = \left(\frac{2n^2-2n+12-4}{4}\right) - 2$$
$$= \left(\frac{2n^2-2n}{4}\right) = \left(\frac{n^2-n}{2}\right) = \frac{n(n-1)}{2} = |E(G)|.$$

By Cases 1 and 2, we conclude that $|\mathcal{L}_E| = |E(G)| + 1$ if $n \equiv 0, 1 \pmod 4$; or $|\mathcal{L}_E| = |E(G)|$ otherwise. First, consider G with $n \equiv 2, 3 \pmod 4$. In this case, $|\mathcal{L}_E| = |E(G)|$. This implies that all edge labels in the set $\mathcal{L}_E$ are induced on the edges of G, in particular, the smallest edge labels $3, 4$ and the two largest edge labels $\frac{2n^2-2n+8}{4}$ and $\frac{2n^2-2n+8}{4} - 1$. Note that: (a) the edge label 3 can only be induced by vertex labels 1 and 2; (b) the edge label 4 can only be induced by vertex labels 1 and 3; (c) the edge label $\frac{2n^2-2n+8}{4}$ can only be induced by vertex labels $\frac{n^2-n+6}{4}$ and $\frac{n^2-n+6}{4} - 1$; and (d) the edge label $\frac{2n^2-2n+8}{4} - 1$ can only be induced by vertex labels $\frac{n^2-n+6}{4}$ and $\frac{n^2-n+6}{4} - 2$. From the previous observations, we obtain that the six vertex labels $1, 2, 3, \frac{n^2-n+6}{4}, \frac{n^2-n+6}{4} - 1, \frac{n^2-n+6}{4} - 2$ are assigned to vertices of G. This contradicts the fact that f is an ESD labeling since the pairs of vertex labels $\left(1, \frac{n^2-n+6}{4}\right)$ and $\left(2, \frac{n^2-n+6}{4} - 1\right)$ induce repeated edge labels. Note that the pairs of vertex labels $\left(2, \frac{n^2-n+6}{4}\right)$ and $\left(3, \frac{n^2-n+6}{4} - 1\right)$ also induce repeated edge labels.

Now, consider G with $n \equiv 0, 1 \pmod 4$. In this case, $|\mathcal{L}_E| = |E(G)| + 1$. This implies that exactly one edge label in the set $\mathcal{L}_E$ is not induced on the edges of G. From the previous case, we know that if the four edge labels $3, 4, \frac{2n^2-2n+8}{4} - 1, \frac{2n^2-2n+8}{4}$ are all induced on the edges of G, then repeated edge labels are generated. This implies that exactly one of these four edge labels is not induced on the edges of G. We split the remaining of this proof into 4 cases depending of the missing edge label.

Case 1. The edge label 3 is not induced on G. In this case, the vertex labels 1, 3, $\frac{n^2-n+6}{4}$, $\frac{n^2-n+6}{4} - 1$, $\frac{n^2-n+6}{4} - 2$ are assigned to vertices of G. This contradicts the fact that f is an ESD labeling since the pairs of vertex labels $\left(1, \frac{n^2-n+6}{4}\right)$ and $\left(3, \frac{n^2-n+6}{4} - 2\right)$ induce repeated edge labels.

Case 2. The edge label 4 is not induced on G. In this case, the vertex labels 1, 2, $\frac{n^2-n+6}{4}$, $\frac{n^2-n+6}{4} - 1$, $\frac{n^2-n+6}{4} - 2$ are assigned to vertices of G. This contradicts the fact that f is an ESD labeling since the pairs of vertex labels $\left(1, \frac{n^2-n+6}{4}\right)$ and $\left(2, \frac{n^2-n+6}{4} - 1\right)$ induce repeated edge labels.

Case 3. The edge label $\frac{2n^2-2n+8}{4}$ is not induced on G. In this case, the vertex labels 1, 2, 3, $\frac{n^2-n+6}{4}$, $\frac{n^2-n+6}{4} - 2$ are assigned to vertices of G. This contradicts the fact that f is an ESD labeling since the pairs of vertex labels $\left(1, \frac{n^2-n+6}{4}\right)$ and $\left(3, \frac{n^2-n+6}{4} - 2\right)$ induce repeated edge labels.

Case 4. The edge label $\frac{2n^2-2n+8}{4} - 1$ is not induced on G. In this case, the vertex labels 1, 2, 3, $\frac{n^2-n+6}{4}$, $\frac{n^2-n+6}{4} - 1$ are assigned to vertices of G. This contradicts the fact that f is an ESD labeling since the pairs of vertex labels $\big(1, \frac{n^2-n+6}{4}\big)$ and $\big(2, \frac{n^2-n+6}{4} - 1\big)$ induce repeated edge labels.

Independently of the value of n modulo 4, a contradiction is reached. Therefore, we conclude that our initial assumption that $\sigma(G) \leq \left\lceil \frac{n^2-n+6}{4} \right\rceil$ is false. The correct affirmation is that $\sigma(G) > \left\lceil \frac{n^2-n+6}{4} \right\rceil$. ■

Our computational experiments shows that, for $4 \leq n \leq 5$, $\sigma_g(K_n) > \left\lceil \frac{n^2-n+6}{4} \right\rceil$. Also, from Theorem 22, we summarize in the following result.

Corollary 23. *If K_n is a complete graph on $n \geq 4$ vertices, then $\sigma_g(K_n) > \left\lceil \frac{n^2-n+6}{4} \right\rceil$.*

3. Concluding Remarks

In this work, we considered a combinatorial game defined by Tuza called Edge-Sum Distinguishing game (ESD game). Tuza proposed two questions about the game: given a simple graph G, for which values of s can Alice win the ESD game? And if Alice wins the ESD game with the set of labels $\{1, \ldots, s\}$, can she also win with $\{1, \ldots, s+1\}$?

For the first question of Tuza, on Theorem 7, we present a tight upper bound for the number of labels necessary for Alice to win the game on general graphs. We also show winning strategies for some classical graph classes such as stars, mP_2 and wheels.

We developed computational experiments to check the result of the game for some small graphs. We applied the backtracking technique, growing the tree of partial solutions by branching on the set of all possible moves and pruning the branching process as soon as we can decide which player wins the game.

We observe that, in this algorithm, at each turn of the game, a player needs to choose one vertex in the set of remaining vertices and a label in the set of remaining labels. Thus, at the k-th turn at the end of the game, the player has $(n-k+1) \cdot (s-k+1)$ possible choices. Since $s \geq n$, we have $\Omega(n! \times n!)$ possible configurations for the hole game. For this reason, we just considered graphs with at most 11 vertices to compute with our backtracking algorithm. For example, for paths with $n = 12$, we spent four days running the program and the answers obtained show that: Bob always wins with $n+1$, Alice always wins with $n+3$, but for $n+2$, the program did not complete the verification.

In our tests, we did not find a counterexample in which Alice wins the game with s labels but does not win with $s+1$ labels, and this means that the second question of Tuza remains an open problem.

Finally, given a graph G, we asked what is the minimum s such that Alice wins the ESD game independently of which player started the game, i.e., the $\sigma_g(G)$. Our algorithm computed the value of $\sigma_g(G)$ of G in some graph classes and these results are summarized in Table 1.

Graph G	$\sigma_g\,(G)$
$K_{1,n-1}$, $n \geq 2$	n
P_n, $n \leq 3$	n
P_n, $4 \leq n \leq 8$	$n+1$
P_n, $9 \leq n \leq 11$	$n+2$
C_n, $n = 3$	n
C_n, $4 \leq n \leq 5$	$n+1$
C_n, $6 \leq n \leq 9$	$n+2$
C_n, $n = 10$	$n+3$
W_{n-1}, $4 \leq n \leq 5$	$2n-2$
W_{n-1}, $6 \leq n \leq 10$	$2n-1$

Table 1. Graphs and their respective edge-sum distinguishing game number.

Acknowledgements

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and CAPES-PrInt project number 88881.310248/2018-01, CNPq and FAPERJ.

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Received 25 November 2022
Revised 7 June 2023
Accepted 11 June 2023
Available online 8 July 2023