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SEMITOTAL FORCING IN CLAW-FREE CUBIC GRAPHS

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Abstract

For an isolate-free graph $G = (V(G), E(G))$, a set $S \subseteq V(G)$ is called a semitotal forcing set of G if it is a forcing set (or a zero forcing set) of G and every vertex in S is within distance 2 of another vertex of S. The semitotal forcing number $F_{t2}(G)$ is the minimum cardinality of a semitotal forcing set in G. In this paper, we prove that if $G \neq K_4$ is a connected claw-free cubic graph of order n, then $F_{t2}(G) \leq \frac{3}{8}n + 1$. The graphs achieving equality in this bound are characterized, an infinite set of graphs.

Keywords: semitotal forcing number, claw-free, cubic.

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1. Introduction

In this paper, by a graph we always mean a simple finite undirected graph; if we admit multiple edges, we always talk about a multigraph.

The concept of a (zero) forcing set, along with the related (zero) forcing number, of a simple graph was introduced in [2] to study the maximum nullity/minimum rank of the family of symmetric matrices associated with the graph. Independently, this parameter was introduced by Burgarth *et al.* [4] in conjunction with control of quantum systems; in this context it is known as the graph infection number. In addition, the (zero) forcing number was considered in connection with logic circuits [5] and dynamical systems [22].

For any two-coloring of vertex set V of a graph G, say black and white for two used colors, define a following *color-change rule*: a white vertex v is converted to black if it is the only white neighbor of some black vertex u. In this case, we say *u forces v*, write $u \to v$ and refer to u as a *forcing vertex*. Let S be a subset of V. Define a two-coloring of G as coloring S black, the others white. The *derived set* $D(S)$ of S is the set of black vertices obtained by iteratively applying the color-change rule until no more changes are possible. Moreover, applying the color-change rule iteratively results in *forcing chains* $v_1 \to v_2 \to \cdots \to v_k$, where v_i forces v_{i+1} for $1 \leq i \leq k-1$. If $D(S) = V$, then we say S is a *forcing set* (*also called a zero forcing set*) of G. The procedure of coloring a graph using the color-change rule applied for S is called a *forcing process* with respect to S. A *minimum forcing set* of G is a forcing set of G of minimum cardinality, and the *forcing number*, denoted by $F(G)$, is the cardinality of a minimum forcing set. In addition, if S is a forcing set of G and $G[S]$ contains no isolated vertex, then S is a *total forcing set* of G. The *total forcing number* of G is the cardinality of a minimum total forcing set in G, denoted by $F_t(G)$.

In this paper, we focus on the semitotal forcing, which was first introduced by Chen in [7]. A set S of vertices in G is a *semitotal forcing set* of G if it is a forcing set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal forcing number*, denoted by $F_{t2}(G)$, is the cardinality of a minimum semitotal forcing set of G.

Since every total forcing set is also a semitotal forcing set, and since every semitotal forcing set is a forcing set, we have the following chain of inequalities [7]. For every isolate-free graph G, $F(G) \leq F_{t2}(G) \leq F_t(G)$. We remark that the gap between the semitotal forcing number with forcing number and total forcing number for graphs can be arbitrary large, such as a graph $G \in \mathcal{N}_{cubic}$. Forcing and its variants are heavily studied in graph theory and we refer the reader to [1–3, 6, 8–15, 17–21, 23, 24].

Forcing and total forcing of connected claw-free cubic graphs have been studied in [8, 9, 11]. Let $G \neq K_4$ be a connected claw-free cubic graph of order n. Davila and Henning [8] showed that $F_t(G) \leq \frac{1}{2}n$, with equality if and only if $G \in \mathcal{N}_{cubic}$ or G is the prism $C_3 \square K_2$; and then these two authors [11] showed that $F(G) \leq \frac{1}{3}n + 1$ unless $G = N_2$. Chen [7] showed that $F_{t2}(G) \leq \frac{1}{2}n$, with equality if and only if G is the diamond-necklace N_2 or the prism $C_3 \square K_2$. In this paper, we improve this upper bound on the semitotal forcing number of G: $F_{t2}(G) \leq \frac{3}{8}n + 1$, and the graphs achieving equality in this bound are characterized i.e., Theorem 13.

2. Preliminaries

In this section, we give some basic definitions and list or prove some lemmas and theorems as preliminaries, which will be used in the proof of our main results.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number of its vertices, denote $n = |V(G)|$, and its *size* is the number of its edges, denote $m = |E(G)|$. If $uv \in E(G)$, then we say u, v are *adjacent*, u is a *neighbor* of v and vice versa. Let $N_G(v)$ be the set of neighbours of a vertex v in a graph G, and let $d_G(v) = |N_G(v)|$ be the *degree* of a vertex v in a graph G. A graph is *isolate-free* if it does not contain an isolated vertex; that is, a vertex of degree 0. A graph is *cubic* if every vertex has degree three. The *distance* between u and v is the length of a shortest (u, v)-path in G, denoted by $d_G(u, v)$. If the graph G is clear from the context, we write V, E, $N(v)$, $d(G)$ and $d(u, v)$ shortened. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, 2, \ldots, k\}$.

A graph $H = (V(H), E(H))$ is called a *subgraph* of $G = (V(G), E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset X of $V(G)$, the *induced subgraph* by X, denoted by $G[X]$, is the graph with vertex set X, in which two vertices are adjacent if and only if they are adjacent in G. We denote by $G - X$ the induced subgraph $G[V \setminus X]$, if $X = \{x\}$, write $G - x$ for short. A graph is *H-free* if it does not contain H as an induced subgraph.

We denote a path, a cycle and a complete graph on n vertices by P_n, C_n and K_n, respectively. A complete bipartite graph with parts of sizes a and b is denoted by $K_{a,b}$. A complete graph K_3 is called a *triangle* and a complete bipartite graph $K_{1,3}$ is called a *claw*. The complete graph K_4 minus one edge is called a *diamond*.

Two vertices u and v in a nontrivial connected graph G are *twins* if u and v have the same neighbors in $V(G) \setminus \{u, v\}$.

Observation 1. *If u and v are twins of a connected graph G, then every forcing set of G contains at least one vertex of $\{u, v\}$.*

Lemma 2. *Let G be a connected cubic graph and let T be an induced triangle of G satisfying that there exists a minimum semitotal forcing set S containing only one vertex of T, say u. Then there exists a forcing process with respect to S such that u does not force any other vertex of T.*

Proof. Suppose $V(T) = \{u, v, w\}$. Note that $u \in S$ and $v, w \notin S$. Without loss of generality, we may assume that v becomes black before w. Then v can be forced by its neighbor different from u and w, and further $v \to w$. Thus, u does not force v and w. ■

Lemma 3. *Let G be a connected cubic graph containing an induced diamond D, where $V(D) = \{a, b, c, d\}$ and ab is the missing edge in D. Then there exists a minimum semitotal forcing set S such that $c \in S$ and $d \notin S$. In addition, c is not a forcing vertex if $|S \cap \{a, b, c, d\}| \leq 2$; c forces d if $|S \cap \{a, b, c, d\}| = 3$.*

Proof. Since c and d are twins of G, a minimum semitotal forcing set S contains at least one vertex of $\{c, d\}$. Without loss of generality, $c \in S$. Note that S contains at most three vertices of D, otherwise $S \setminus \{d\}$ is a semitotal forcing set smaller than S, a contradiction. If $|S \cap \{a, b, c, d\}| = 3$, then $S \cap \{a, b, c, d\} = \{a, b, c\}$ is satisfied. Hence we assume that $|S \cap \{a, b, c, d\}| \in \{1, 2\}$. If $d \notin S$, there is nothing to prove. Now consider $d \in S$. If $a \in S$, then $(S \setminus \{d\}) \cup \{b\}$ is also a minimum semitotal forcing set satisfying the statement of the lemma. Thus, we may assume $a \notin S$ and similarly $b \notin S$. Without loss of generality, assume that a becomes black before b. Hence, a must be forced by its neighbor different from c and d, and $(S \setminus \{d\}) \cup \{b\}$ satisfies the statement of the lemma.

Let S be a minimum semitotal forcing set of G such that $c \in S$ and $d \notin S$. If $|S \cap \{a, b, c, d\}| = 1$, then $a, b, d \notin S$. By Lemma 2, c is not a forcing vertex. If $|S \cap \{a, b, c, d\}| = 2$, without loss of generality, assume that $a \in S$ and $b \notin S$. By Lemma 2, c does not force b and d. Since $a \in S$, c is not a forcing vertex. If $|S \cap \{a, b, c, d\}| = 3$, we let $c \to d$ in the forcing process. ■

The following property of connected claw-free cubic graphs is established in [16].

Lemma 4 [16]**.** *If $G \neq K_4$ is a connected claw-free cubic graph of order n, then the vertex set $V(G)$ can be uniquely partitioned into sets each of which induces a triangle or a diamond in G.*

Following the notation introduced in [16], we refer to such a partition as a *triangle-diamond partition* of G, abbreviated Δ*-D-partition*. Further we call every triangle and diamond induced by a set in our Δ-D-partition a unit of the partition. A unit that is a triangle we call a *triangle-unit* and a unit that is a diamond we call a *diamond-unit*. (Note that a triangle-unit is a triangle that does not belong to a diamond.) We say that two units in the Δ-D-partition are *adjacent* if there is an edge joining a vertex in one unit to a vertex in the other unit.

In what follows, we give the definition of a diamond-necklace, diamond-bracelet, diamond-chain, double triangle-chain, and triangle-necklace, coined by Henning and Löwenstein [16], Davila and Henning [11], respectively.

Definition [16]**.** For $k \geq 2$ an integer, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies $D_1, D_2, \ldots, D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i. Let N_k be obtained from the disjoint union of these k diamonds by adding the edges $\{a_i b_{i+1} \mid i \in [k]\}$ with addition taken modulo k (and so $a_k b_{k+1} = a_k b_1$). We call N_k a *diamond-necklace* with k diamonds. Let $\mathcal{N}_{cubic} = \{N_k \mid k \geq 2\}$.

A diamond-necklace, N_6, with six diamonds is shown in Figure 1. By Lemma 4, we note that if G is a connected claw-free cubic graph with no triangle-units,

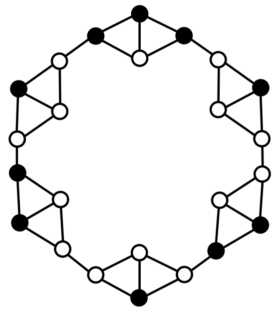


Figure 1. A diamond-necklace N_6, where the black vertices form a minimum semitotal forcing set of N_6.

then $G \in \mathcal{N}_{cubic}$. The semitotal forcing number of a diamond-necklace was determined by Chen in [7].

Theorem 5 [7]**.** *Let $G = N_k \in \mathcal{N}_{cubic}$ have order $n = 4k$. Then $F_{t2}(G) = \left\lceil \frac{3k+1}{2} \right\rceil = \left\lfloor \frac{3n}{8} \right\rfloor + 1$.*

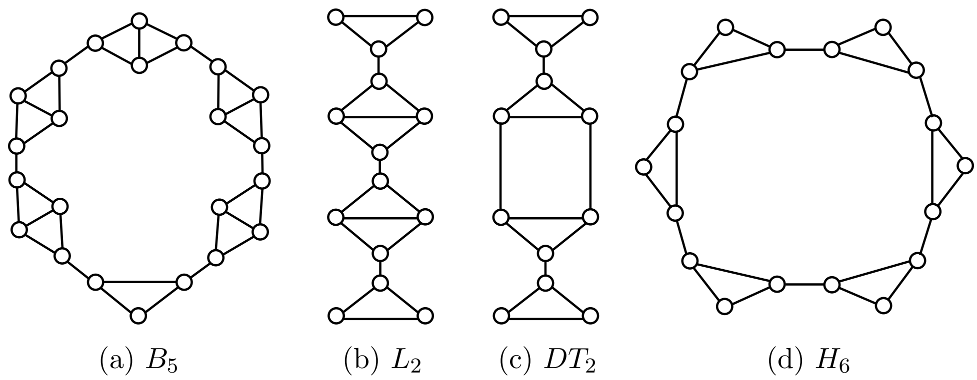


Figure 2. A diamond-bracelet B_5, a diamond-chain L_2, a double triangle-chain DT_2 and a triangle-necklace H_6.

Definition [16]**.** For $k \geq 1$ an integer, a *diamond-bracelet* B_k (see Figure 2(a) for example) with k diamonds is defined as follows. Let B_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, \ldots, D_{k+1}$ by removing the diamond D_{k+1} and adding a triangle T_1 with $V(T_1) = \{x_1, y_1, z_1\}$, and adding the edges $y_1 b_1$ and $z_1 a_k$.

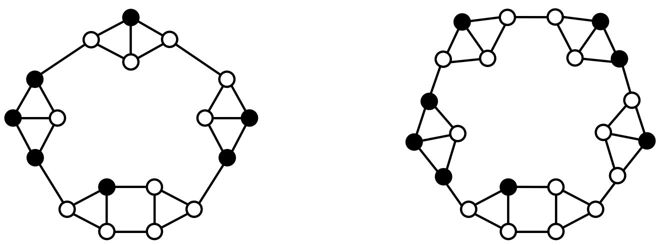
Definition [16]**.** For $k \geq 1$ an integer, a *diamond-chain* L_k (see Figure 2(b) for example) with k diamonds is defined as follows. Let L_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, \ldots, D_{k+1}$ by removing the diamond D_{k+1} and adding two disjoint triangles T_1 and T_2 and adding an edge joining b_1 to a vertex of T_1 and adding an edge joining a_k to a vertex of T_2.

Definition [16]**.** A *double triangle-chain* DT_2 (see Figure 2(c) for example) is obtained from a diamond-chain L_2 with two diamonds by replacing the two diamonds with two triangles and adding two edges joining these two triangles.

Definition [11]**.** For $k \geq 2$ an integer, let H_k (see Figure 2(d) for example) be the graph constructed as follows. Take k disjoint copies $T_1, T_2, \ldots, T_k$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$. Let H_k be obtained from the disjoint union of these k triangles by adding the edges $\{x_i y_{i+1} \,|\, i \in [k]\}$ with addition taken modulo k (and so $x_k y_{k+1} = x_k y_1$). We call H_k a *triangle-necklace* with k triangles.

We next define a diamond-armlet.

Definition. For $k \geq 1$ an integer, a *diamond-armlet* A_k with k diamonds is defined as follows. Let A_k be obtained from a diamond-necklace N_{k+1} with $k+1$ diamonds $D_1, D_2, \ldots, D_{k+1}$ by removing the diamond D_{k+1} and adding two triangles T_1 and T_2 with $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2]$, and adding the edges $x_1 b_1$, $x_2 a_k$, $y_1 y_2$ and $z_1 z_2$. Let $\mathcal{A}_{cubic} = \{A_k | k \geq 1\}$.



(a) A_3 (b) A_4

Figure 3. Two diamond-armlets A_3 and A_4, where the black vertices form a minimum semitotal forcing set of their respective graphs.

Two diamond-armlets, A_3 and A_4, are shown in Figure 3(a) and Figure 3(b), respectively. The semitotal forcing number of a diamond-armlet is determined as follows.

Theorem 6. *Let $G = A_k \in \mathcal{A}_{cubic}$ have order $n = 4k+6$. Then $F_{t2}(G) = \frac{3n}{8} + \frac{1}{4}$ when k is odd; $F_{t2}(G) = \frac{3n}{8} - \frac{1}{4}$ when k is even.*

Proof. Let $G = A_k \in \mathcal{A}_{cubic}$ have order $n = 4k + 6$. For convenience, we write $A = \{a_i \,|\, 1 \leq i \leq k\}$, $B = \{b_i \,|\, 1 \leq i \leq k\}$, $C = \{c_i \,|\, 1 \leq i \leq k\}$, and $D = \{d_i \,|\, 1 \leq i \leq k\}$.

We first show that $F_{t2}(G) \leq \lceil \frac{3k}{2} \rceil + 2$. Let $S = C \cup \{a_i \,|\, 1 \leq i \leq k$ and i is odd$\} \cup \{b_1, y_1\}$. Then the set S is a semitotal forcing set of G. Thus, $F_{t2}(G) \leq |S| = k + \lceil \frac{k}{2} \rceil + 2 = \lceil \frac{3k}{2} \rceil + 2$.

We next show that $F_{t2}(G) \geq \lceil \frac{3k}{2} \rceil + 2$. Let S' be an arbitrary semitotal forcing set of G. By Observation 1, S' contains at least one vertex of $\{c_i, d_i\}$ for every $i \in [k]$. Renaming vertices if necessary, we may assume that $c_i \in S'$. Moreover, at least one vertex of $\{y_1, y_2, z_1, z_2\}$ belongs to S', otherwise, S' is not a forcing set, a contradiction. Without loss of generality, assume that $y_1 \in S'$. Let S'' be the set of vertices in S' that belong to $A \cup B \cup D \cup \{x_1, x_2, y_2, z_1, z_2\}$. Now we consider two cases. In the case where k is even. Since every vertex in S' is within distance 2 of another vertex of S', $|S''| \geq \lceil \frac{k}{2} \rceil + 1$. In the case where k is odd. Similarly, in order to guarantee that S' is semitotal, we have $|S''| \geq \lceil \frac{k}{2} \rceil$. And then every vertex in $G[S']$ has degree at most 1 if $|S''| = \lceil \frac{k}{2} \rceil$. Since G is cubic, the first forcing vertex in the forcing process has degree 2 in $G[S']$, a contradiction. Thus, $|S''| \geq \lceil \frac{k}{2} \rceil + 1$. In both cases, $|S'| = |C| + 1 + |S''| \geq k + 1 + \lceil \frac{k}{2} \rceil + 1 = \lceil \frac{3k}{2} \rceil + 2$. This implies that $F_{t2}(G) \geq \lceil \frac{3k}{2} \rceil + 2$.

To conclude, when k is odd, $F_{t2}(G) = \lceil \frac{3k}{2} \rceil + 2 = \frac{3n}{8} + \frac{1}{4}$; when k is even, $F_{t2}(G) = \lceil \frac{3k}{2} \rceil + 2 = \frac{3n}{8} - \frac{1}{4}$. ∎

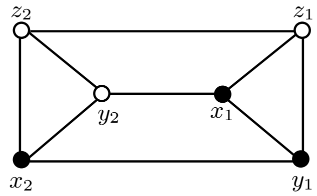
3. CONNECTED CLAW-FREE CUBIC GRAPHS

In this section, we establish an upper bound on the semitotal forcing number of a connected claw-free cubic graph $G \neq K_4$ in terms of its order.

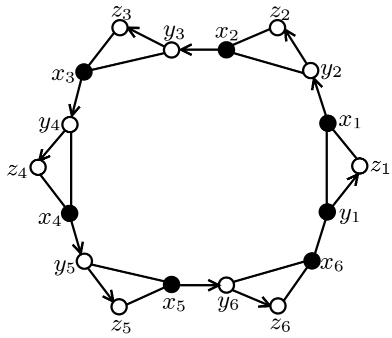
First, we give the following theorem for the special case where G is diamond-free.

Theorem 7. *If $G \neq K_4$ is a connected claw-free and diamond-free cubic graph of order n, then $F_{t2}(G) \leq \frac{1}{3}n + 1$.*

Proof. Since $G \neq K_4$ is a connected claw-free cubic graph, there is a unique Δ-D-partition of G by Lemma 4. Further every unit in G is a triangle-unit because G is diamond-free. Now we define the contraction multigraph of G, denoted G', to be the multigraph whose vertices correspond to the triangle-units in G and where two vertices in G' are joined by the number of edges joining the corresponding triangle-units in G. We note that the order of G' is precisely the number of triangle-units in G. Every vertex in G' has degree 3, and so G' is a cubic multigraph. Let $C : v_1 v_2 \cdots v_k v_1$ be a shortest cycle in G' where we allow 2-cycles, and so $k \geq 2$. For $i \in [k]$, let T_i be the triangle-unit in G associated with the vertex v_i, where $V(T_i) = \{x_i, y_i, z_i\}$ and where $x_i y_{i+1}$ is an edge with addition taken modulo k (and so $x_k y_{k+1}$ is the edge $x_k y_1$). Thus, G contains a subgraph as a triangle-necklace, H_k, with k triangles. We note that if $k = 2$, then either G is the prism $C_3 \Box K_2$ or G contains a H_2.



(a) $C_3 \Box K_2$



(b) H_6

Figure 4. Initialization for the proof of Theorem 7.

We construct a semitotal forcing set S of G such that $|S| \le \frac{1}{3}n + 1$. For this purpose, we set $S = \{y_1, x_1, x_2, \dots, x_k\}$ and color all vertices in S black and the others white, initially. At any step we let B be the set of vertices that have already been colored black, and W the set of all vertices that remain white. If $G = C_3 \Box K_2$, then $S = \{y_1, x_1, x_k\}$ is a minimum semitotal forcing set of G and $F_{t2}(G) = 3 = \frac{1}{3}n + 1$. (The graph $C_3 \Box K_2$ is shown in Figure 4(a).) Suppose that $G \ne C_3 \Box K_2$. Then the set $\{z_1, \dots, z_k\}$ is independent. Starting with this initial set S, we play the sequence of vertices $y_1, x_1, y_2, x_2, \dots, y_k$ in turn which force the vertices $z_1, y_2, z_2, y_3, \dots, z_k$, respectively, to be colored black. The initialization of the initial set S in the case of $k = 6$ is shown in Figure 4(b). Now $B = V(H_k)$ and $W \ne \emptyset$. Next, we extend S by Rule 1 and Rule 2, iteratively apply the color-change rule to the current set S, and in each step, we color all vertices in one triangle-unit or two triangle-units black.

Rule 1. There exists a triangle-unit T such that $V(T) \subseteq W$ and T has at least two vertices that are adjacent to vertices of B, then all vertices in this triangle-unit to become black.

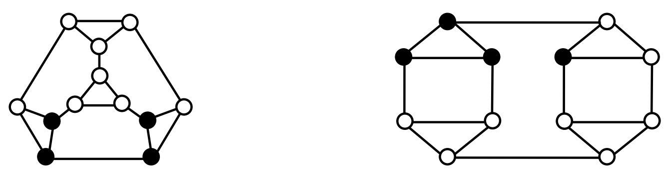
Rule 2. There exists a triangle-unit T such that $V(T) = \{x, y, z\} \subseteq W$ and T has only one vertex that is adjacent to a vertex of B. Renaming vertices if necessary, assume that x has a neighbor $v \in B$. Then T is adjacent to a triangle-unit T', where $V(T') = \{x', y', z'\} \subseteq W$. Renaming vertices if necessary, assume that y is adjacent to x'. Now we add $\{y, y'\}$ to S. Thus, $v \to x \to z$, $y \to x' \to z'$, and all vertices in $V(T) \cup V(T')$ become black. Note that $d(y, y') = 2$.

Since G is connected, Rule 1 and Rule 2 continue until all vertices become black. And in each step we add at most one vertex of a triangle-unit to S. Thus, S is a forcing set of G with $|S| \leq \frac{1}{3}n + 1$. It is easy to see that S is a semitotal forcing set of G. ■

Note that the only three connected claw-free cubic graphs that have order $n < 10$ are K_4, $C_3 \square K_2$ and N_2. If $G \in \mathcal{N}_{cubic}$, the semitotal forcing number of G was determined by Theorem 5. Next we add the restriction that G is not a diamond-necklace and $n \geq 10$.

Theorem 8. *If $G \notin \mathcal{N}_{cubic}$ is a connected claw-free cubic graph of order $n \geq 10$, then $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$.*

Proof. Since G is cubic, we note that n is even. We proceed by induction on the order $n \geq 10$. If $n = 10$, then $G = A_1$ and $F_{t2}(A_1) = 4 = \frac{3}{8}n + \frac{1}{4}$. If $n = 12$, since $G \neq N_3$, G can only be $G_{12.1}$ and $G_{12.2}$, shown in Figure 5, respectively, and $F_{t2}(G) = 4 < \frac{3}{8}n + \frac{1}{4}$. This establishes the base cases. Next, let $n \geq 14$ and assume that for every connected claw-free cubic graph $G' \notin \mathcal{N}_{cubic}$ of order $10 \leq n' < n$, $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let G be a connected claw-free cubic graph of order n. By Lemma 4, there is a unique Δ-D-partition of G. If every unit in the Δ-D-partition of G is a triangle-unit, i.e., G is diamond-free, we get $n \geq 18$, and then $F_{t2}(G) \leq \frac{n}{3} + 1 \leq \frac{3}{8}n + \frac{1}{4}$ by Theorem 7. Thus, we may assume that G has at least one diamond-unit. Since $G \notin \mathcal{N}_{cubic}$, G must contain a diamond-chain or a diamond-bracelet.



(a) $G_{12.1}$ (b) $G_{12.2}$

Figure 5. Two connected claw-free cubic graphs of order 12, where the black vertices form a minimum semitotal forcing set of their respective graphs.

We proceed further with the following series of claims.

Claim 9. *If G contains a diamond-chain L_2, then $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$.*

Proof. Suppose that G contains a diamond-chain L_2. Using our earlier notation, let D_1, D_2 be two consecutive diamonds in L_2, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where a_ib_i is the missing edge in D_i for $i \in [2]$ and a_1b_2 is an edge. Let w_1 be the neighbor of b_1 not in D_1, and let w_2 be the neighbor of a_2 not in D_2. By the definition of a diamond-chain L_2, we note that $w_1 \neq w_2$ and w_1w_2 is not an edge; the triangles containing w_1 and w_2 are vertex disjoint. Let G' be the graph obtained from G by removing all vertices in $V(D_1) \cup V(D_2)$ and adding the edge w_1w_2. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n - 8 \geq 6$.

If $G' = C_3 \Box K_2$, then $G = A_2$ and $F_{t2}(A_2) = 5 = \frac{3}{8}n - \frac{1}{4}$ is satisfied. If $G' \in \mathcal{N}_{cubic}$, then $G \in \mathcal{N}_{cubic}$, a contradiction. Now assume that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G'. If $|S' \cap \{w_1, w_2\}| = 0$, without loss of generality, we may suppose that w_1 becomes black before w_2. Then, $S = S' \cup \{c_1, c_2, a_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until w_1 becomes black. In this time, two neighbors (different from b_1) of w_1 are both black. Then $w_1 \to b_1 \to d_1$, and $a_1 \to b_2 \to d_2 \to a_2 \to w_2$. Finally, all remaining white vertices in G will become black using the same forcing chains as in G'. The newly added vertices guarantee that S is semitotal. Now consider $|S' \cap \{w_1, w_2\}| \in \{1, 2\}$. Without loss of generality, assume that $w_1 \in S'$. We claim that $S = S' \cup \{c_1, c_2, a_2\}$ is a semitotal forcing set of G. If w_1 is not a forcing vertex during the forcing process with respect to S' in G', then the same forcing chains in G' with respect to S' can be used in G with respect to S. Thus, all vertices in G' still become black. Finally, $a_2 \to d_2 \to b_2 \to a_1 \to d_1 \to b_1$. It follows that all vertices in G become black. If w_1 is a forcing vertex during the forcing process with respect to S' in G', let $N_{G'}(w_1) = \{w_2, u, v\}$ and divide into two cases. In the case of $w_1 \to w_2$, it will be replaced by two forcing chains in G: $w_1 \to b_1 \to d_1 \to a_1 \to b_2 \to d_2$, $a_2 \to w_2$. In the case of $w_1 \to u$ (or $w_1 \to v$), note that in this time step, w_2 is already black. So the forcing chains in G' with respect to S' remain valid in G until w_2 becomes black. Further, $a_2 \to d_2 \to b_2 \to a_1 \to d_1 \to b_1$. Finally, all remaining white vertices in G will become black using the same forcing chains as in G'. In either case, S is a semitotal forcing set of G. Therefore, we have $F_{t2}(G) \leq F_{t2}(G') + 3 \leq \frac{3}{8}n' + \frac{1}{4} + 3 = \frac{3}{8}(n-8) + \frac{1}{4} + 3 = \frac{3}{8}n + \frac{1}{4}$. □

We note that if G contains a diamond-chain L_k with $k \geq 2$, then G contains a diamond-chain L_2. Hence by Claim 9, we may assume that G does not contain a diamond-chain L_k with $k \geq 2$. Moreover, if G contains a diamond-bracelet B_k, then $k \leq 2$, otherwise, G contains a diamond-chain L_2, a contradiction. Therefore, next we will discuss when G contains a diamond-bracelet B_k, where $k \in [2]$, or every diamond-unit is adjacent to two distinct triangle-units to complete our proof. Before this, we give the following claim.

Claim 10. *If G contains a double triangle-chain DT_2, then $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$.*

Proof. Suppose that G contains a double triangle-chain DT_2, where T_1 and T_2 are the two triangle-units in DT_2 joined by two edges. Let $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2]$, where $y_1y_2, z_1z_2 \in E(G)$. Let w_i be the neighbor of x_i not in T_i for $i \in [2]$. By the definition of a double triangle-chain, we note that the triangles containing w_1 and w_2 are vertex disjoint. In particular, w_1 and w_2 are not adjacent. Let G' be the graph obtained from G by removing all vertices in $V(T_1) \cup V(T_2)$ and adding the edge w_1w_2. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n - 6 \geq 8$.

If $G' = N_k$ with $k \geq 2$, then $G = A_k \in \mathcal{A}_{cubic}$ and $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$ by Theorem 6. Suppose that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G'. If $|S' \cap \{w_1, w_2\}| = 0$, without loss of generality, we may suppose that w_1 becomes black before w_2. Then, the set $S = S' \cup \{x_1, y_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until w_1 becomes black. Further, $x_1 \to z_1 \to z_2$ and $y_1 \to y_2 \to x_2 \to w_2$. Then all remaining white vertices in G will become black using the same forcing chains as in G'. Now consider $|S' \cap \{w_1, w_2\}| \in \{1, 2\}$. Without loss of generality, assume that $w_1 \in S'$. If w_1 is not a forcing vertex during the forcing process with respect to S' in G', then $S = S' \cup \{x_2, y_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G and $w_1 \to x_1 \to z_1 \to z_2 \to y_2$. The newly added vertices guarantee that S is semitotal. If w_1 is a forcing vertex during the forcing process with respect to S' in G', then we divide into two cases. In the case of $w_1 \to w_2$, $S = S' \cup \{x_2, y_1\}$ is also a semitotal forcing set of G. Applying the color-change rule to S in G, $w_1 \to w_2$ is replaced by two forcing chains: $w_1 \to x_1 \to z_1 \to z_2 \to y_2$, $x_2 \to w_2$. The newly added vertices guarantee that S is semitotal. In the case of $w_1 \to u$, then $v \in S'$ by Lemma 2, where $\{u, v\} = N_{G'}(w_1) \setminus \{w_2\}$. Note that in this time step, w_2 is already black. One can easily verify that $S = S' \cup \{x_2, y_2\}$ is a semitotal forcing set of G. Therefore, we have $F_{t2}(G) \leq F_{t2}(G') + 2 \leq \frac{3}{8}n' + \frac{1}{4} + 2 = \frac{3}{8}(n - 6) + \frac{1}{4} + 2 < \frac{3}{8}n + \frac{1}{4}$. □

By Claim 10, we may assume that G contains no double triangle-chain DT_2.

Claim 11. *If G contains a diamond-bracelet B_k with $k \in [2]$, then $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$.*

Proof. Suppose that G contains a diamond-bracelet B_k with $k \in [2]$. Using our earlier notation, let D_i be a diamond of B_k for $i \in [k]$, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and a_ib_i is the missing edge in D_i. Let T_1 be the only triangle in B_k that $V(T_1) = \{x_1, y_1, z_1\}$ and y_1b_1 and z_1a_k are edges in B_k. Let U be the unit

(a) B_1 (b) B_2

Figure 6. The diamond-bracelets B_1 and B_2.

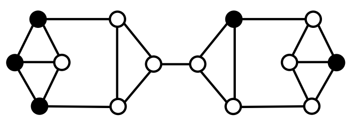
that is adjacent to T_1 and does not belong to B_k. Now we divide into two cases to discuss according to U.

Case 1. U is a triangle-unit.

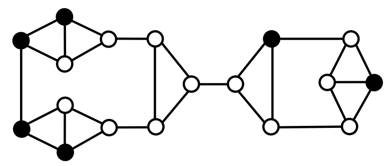
Suppose that U is a triangle-unit, say T, where $V(T) = \{x, y, z\}$ and $xx_1 \in E(G)$. Let u and v denote the neighbors of y and z, respectively, not in T. If $uv \in E(G)$, then u and v must be in the same triangle-unit, say T'. Clearly, G has a double triangle-chain DT_2 containing T and T', a contradiction. Hence, $uv \notin E(G)$.

Let G' be the graph obtained from G by removing all vertices in $V(B_k) \cup V(T)$ and adding the edge uv. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n - 6 - 4k \geq 4$ for $k \in [2]$. If $G' = K_4$, then G is the graph $G_{14.1}$ or $G_{18.1}$, shown in Figure 7(a) and Figure 7(b), respectively. If $G' = C_3 \Box K_2$, then G is the graph $G_{16.1}$ or $G_{20.1}$, shown in Figure 7(c) and Figure 7(d), respectively. If $G' = N_2$, then G is the graph $G_{18.1}$ or $G_{22.1}$, shown in Figure 7(b) and Figure 7(e), respectively. In all cases, $F_{t2}(G) < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_k$ with $k \geq 3$, then G contains a diamond-chain L_2, a contradiction. Now we assume that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G'.

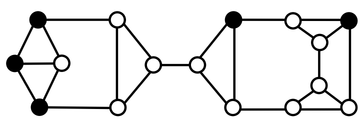
Suppose first $k = 1$ and $n' = n - 10$. If $|S' \cap \{u, v\}| = 0$, without loss of generality, we may suppose that u becomes black before v. Then, $S = S' \cup \{x, y_1, c_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until u becomes black. In this time, two neighbors (different from y) of u are black. Then $u \to y \to z \to v$ and $x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. The newly added vertices guarantee that S is semitotal. Now consider $|S' \cap \{u, v\}| \in \{1, 2\}$. Without loss of generality, assume that $u \in S'$. If u is not a forcing vertex during the forcing process with respect to S' in G', then $S = S' \cup \{z, y_1, c_1\}$ is a semitotal forcing set of G. This is because if we



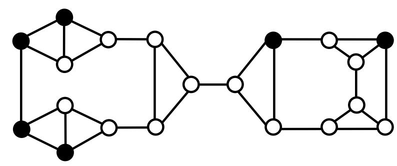
(a) $G_{14.1}$



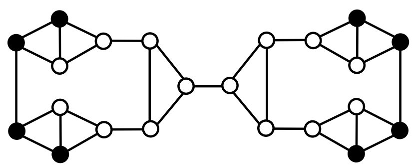
(b) $G_{18.1}$



(c) $G_{16.1}$



(d) $G_{20.1}$



(e) $G_{22.1}$

Figure 7. Five connected claw-free cubic graphs in the proof of Case 1 in Claim 11, where the black vertices form a semitotal forcing set of their respective graphs.

apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G. Further, $u \to y \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$ and all vertices of G will become black. If u is a forcing vertex during the forcing process with respect to S' in G', then we consider two cases. In the case of $u \to v$, $S = S' \cup \{z, y_1, c_1\}$ is also a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, $u \to v$ is replaced by two forcing chains: $u \to y \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$ and then $z \to v$. In the case of $u \to u_1$, then $u_2 \in S'$ by Lemma 2, where $\{u_1, u_2\} = N_{G'}(u) \setminus \{v\}$. Note that in this time step, v is already black. So, $S = (S' \setminus \{u\}) \cup \{y, z, y_1, c_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until v becomes black. Then $z \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$ and further $y \to u \to u_1$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. It is not hard to see that S is semitotal. Therefore, we have $F_{t2}(G) \leq F_{t2}(G') + 3 \leq \frac{3}{8}n' + \frac{1}{4} + 3 = \frac{3}{8}(n-10) + \frac{1}{4} + 3 < \frac{3}{8}n + \frac{1}{4}$.

Suppose $k = 2$ and $n' = n - 14$. If $|S' \cap \{u, v\}| = 0$, without loss of generality, we may suppose that u becomes black before v. Then, $S = S' \cup \{a_1, b_2, c_1, c_2\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until u becomes black. In this time, two neighbors (different from y) of u are both black. Then $u \to y$, $b_2 \to d_2 \to a_2 \to z_1$, and $a_1 \to d_1 \to b_1 \to y_1 \to x_1 \to x \to z \to v$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Now consider $|S' \cap \{u, v\}| \in \{1, 2\}$. Without loss of generality, assume that $u \in S'$. One can easily verify that $S' \cup \{a_1, b_2, c_1, c_2, z\}$ is a semitotal forcing set of G. Thus, we have $F_{t2}(G) \leq F_{t2}(G') + 5 \leq \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n-14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}$.

Case 2. U is a diamond-unit.

Suppose that U is a diamond-unit, say D, where $V(D) = \{a, b, c, d\}$ and ab is the missing edge in the diamond D and $x_1a \in E(G)$. Let U' be the unit that is adjacent to U and does not belong to B_k. Since U is a diamond-unit, U' is not a diamond-unit, otherwise G contains a diamond-chain L_2, a contradiction. Thus U' is a triangle-unit, say T', where $V(T') = \{x', y', z'\}$ and $bx' \in E(G)$. Let u' and v' denote the neighbors of y' and z', respectively, not in T'. If $u'v' \in E(G)$, then G contains a double triangle-chain DT_2, a contradiction. Hence, $u'v' \notin E(G)$.

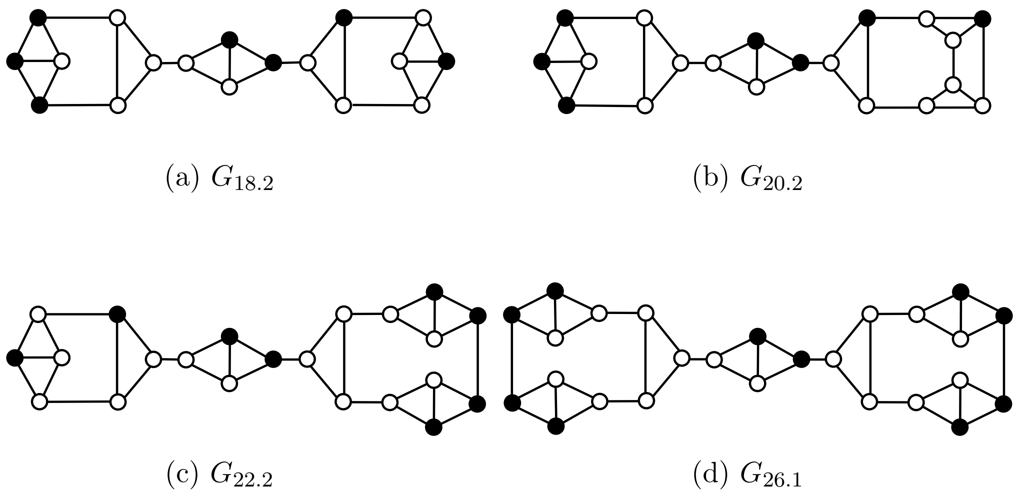


Figure 8. Four connected claw-free cubic graphs in the proof of Case 2 in Claim 11, where the black vertices form a semitotal forcing set of their respective graphs.

Suppose first that $k = 1$ and let G' be the graph obtained from G by removing all vertices in $V(B_1) \cup V(D) \cup V(T')$ and adding the edge $u'v'$. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n - 14 \geq 4$. If $G' = K_4$, then $G = G_{18.2}$ shown in Figure 8(a), and $F_{t2}(G) \leq 7 = \frac{3}{8}n + \frac{1}{4}$. If $G' = C_3 \Box K_2$, then $G = G_{20.2}$ shown in Figure 8(b), and $F_{t2}(G) \leq 7 < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_2$, then $G = G_{22.2}$ shown in Figure 8(c), and $F_{t2}(G) \leq 8 < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_k$ with $k \geq 3$, then G contains a diamond-chain L_2, a contradiction. Now assume that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G'. Applying the color-change rule to S' in G', without loss of generality, we may assume that u' becomes black no later than v'. We claim that $S = S' \cup \{c_1, y_1, c, x', z'\}$ is a semitotal forcing set of G. If $|S' \cap \{u', v'\}| = 2$, it is clear that S is a semitotal forcing set of G. If $|S' \cap \{u', v'\}| = 0$, then the forcing chains in G' with respect to S' remain valid in G until u' becomes black. In this time, two neighbors (different from y') of u' are both black. Further, $u' \to y'$ and then $x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$, $z' \to v'$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. If $|S' \cap \{u', v'\}| = 1$, recall our assumption that u' becomes black no later than v'. Thus, we have $u' \in S'$ and $v' \notin S'$. Next we divide into two cases. In the case of $u' \to v'$ in G', we replace with three forcing chains when we apply the color-change rule to S in G: $u' \to y'$, $x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$, $z' \to v'$. In the case of u' does not force v' in G', the same forcing chains in G' with respect to S' remain valid in G until v' becomes black. Then, $z' \to y'$ and $x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Thus, S is a forcing set of G. It is not hard to see that S is semitotal. Therefore, $F_{t2}(G) \leq F_{t2}(G') + 5 \leq \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n - 14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}$.

Suppose $k = 2$ and let G' be the graph obtained from G by removing all vertices in $V(B_2) \cup V(D)$, subdividing the edge $y'z'$ and denoting the resulting new vertex by w', and adding the edges $x'w'$. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n - 14 \geq 8$. If $G' = N_2$, then $G = G_{22.2}$ shown in Figure 8(c), and $F_{t2}(G) \leq 8 < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_3$, then $G = G_{26.1}$ shown in Figure 8(d), and $F_{t2}(G) \leq 10 = \frac{3}{8}n + \frac{1}{4}$. If $G' = N_k$ with $k \geq 4$, then G contains a diamond-chain L_2, a contradiction. Now assume that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G' with $x' \in S'$ and $w' \notin S'$ as in Lemma 3. Then $S = S' \cup \{a_1, b_2, c_1, c_2, c\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, first, $b_2 \to d_2 \to a_2 \to z_1$. Further, $a_1 \to d_1 \to b_1 \to y_1 \to x_1 \to a \to d \to b$. The forcing chains in G' with respect to S' remain valid in G except in the following cases. If $x' \to w'$ occurs in G', then $\{y', z'\} \subset S'$ by Lemma 3. Then $x' \to w'$ will be omitted in G and all remaining forcing chains in G' with respect to S'

will remain valid in G. Thus, S is a semitotal forcing set of G. If $y' \to w'$ occurs in G', then $z' \in S'$ or $w' \to z'$. In this case, either $y' \to w'$ will be omitted in G or $y' \to w' \to z'$ will be replaced by $y' \to z'$ in G. Finally, all remaining forcing chains in G' with respect to S' will remain valid in G and S is a semitotal forcing set. Lastly, if $z' \to w'$, a similar argument to that used above shows that S is a semitotal forcing set. Thus, $F_{t2}(G) \le F_{t2}(G') + 5 \le \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n-14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}$. □

By Claim 9 and Claim 11, we may assume that every diamond-unit is adjacent to two distinct triangle-units. Now G must contain a diamond-chain L_1. Using our earlier notation, let D_1 be the diamond in L_1, where $V(D_1) = \{a_1, b_1, c_1, d_1\}$ and where a_1b_1 is the missing edge in D_1. Let T_1 and T_2 be the two triangles in L_1, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2]$ and x_1b_1, x_2a_1 are edges. We note that T_1 and T_2 are two distinct triangle-units of G. If y_1 or z_1 is adjacent to y_2 or z_2, then G contains an induced subgraph isomorphic to H shown in Figure 9 by $n \ge 14$.

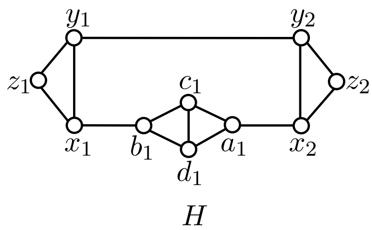


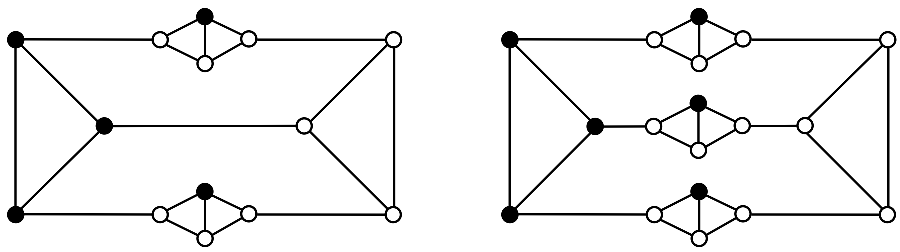
Figure 9. An induced subgraph H of G.

Claim 12. *If G contains an induced subgraph H shown in Figure* 9, *then* $F_{t2}(G) < \frac{3}{8}n + \frac{1}{4}$.

Proof. Let G' be the graph obtained from G by removing the vertices in $V(D) \cup \{x_2, y_2\}$ and adding the edges z_2x_1 and z_2y_1. Since G is a connected claw-free cubic graph, so also is the graph G'. We note that the subgraph of G' induced by $\{x_1, y_1, z_1, z_2\}$ is a diamond-unit where z_1z_2 is the missing edge in this unit. Let G' have order n'. Then $n' = n - 6 \ge 8$.

If $G' = N_2$, then $G = G_{14.2}$ shown in Figure 10(a), and $F_{t2}(G) \le 5 < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_k$ with $k \ge 3$, then G contains a diamond-chain L_2, a contradiction. Now assume that $G' \notin \mathcal{N}_{cubic}$ and $n' \ge 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \le \frac{3}{8}n' + \frac{1}{4}$. Let S' be a minimum semitotal forcing set of G' with $x_1 \in S'$ and $y_1 \notin S'$ as in Lemma 3. If $|S' \cap \{z_1, z_2\}| = 2$, it is clear that $(S' \backslash \{x_1\}) \cup \{a_1, b_1, c_1\}$ is a semitotal forcing set of G. Now consider $|S' \cap \{z_1, z_2\}| \in \{0, 1\}$. We claim that $S = S' \cup \{c_1, x_2\}$ is a semitotal forcing set of G. If $|S' \cap \{z_1, z_2\}| = 0$, without loss of generality, assume that z_1 becomes black before

z_2. Then, the same forcing chains in G' with respect to S' remain valid in G until z_1 becomes black. Further, $z_1 \to y_1 \to y_2 \to z_2$ and $x_1 \to b_1 \to d_1 \to a_1$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. If $|S' \cap \{z_1, z_2\}| = 1$, without loss of generality, assume that $z_1 \in S'$ and $z_2 \notin S'$. Then x_1 is not a forcing vertex in G' by Lemma 3. Next we consider two cases. In the case of $z_1 \to y_1 \to z_2$ in G', we replace with two forcing chains when we apply the color-change rule to S in G: $z_1 \to y_1 \to y_2 \to z_2$, $x_1 \to b_1 \to d_1 \to a_1$. In the case of z_2 is forced by its neighbor different from x_1 and y_1, then the same forcing chains in G' with respect to S' remain valid in G until z_2 becomes black. Further, $z_2 \to y_2 \to y_1$ and $x_1 \to b_1 \to d_1 \to a_1$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Thus, S is a semitotal forcing set of G. Therefore, $F_{t2}(G) \le F_{t2}(G') + 2 \le \frac{3}{8}n' + \frac{1}{4} + 2 = \frac{3}{8}(n-6) + \frac{1}{4} + 2 < \frac{3}{8}n + \frac{1}{4}$. □



(a) $G_{14.2}$ (b) $G_{18.3}$

Figure 10. Two connected claw-free cubic graphs $G_{14.2}$ and $G_{18.3}$, where the black vertices form a semitotal forcing set of their respective graphs.

By Claim 12, we assume that G contains no induced subgraph H. Now, G contains a diamond-chain L_1 with two distinct triangle-units T_1 and T_2, and neither y_1 nor z_1 is adjacent to y_2 or z_2. Let u and v be the neighbors of y_1 and z_1, respectively, not in T_1. If $uv \in E(G)$, then G contains a double triangle-chain DT_2, a contradiction. Hence, $uv \notin E(G)$.

Let G' be the graph obtained from G by removing all vertices in $V(T_1) \cup V(D_1)$, subdividing the edge y_2z_2 and denoting the resulting new vertex by w, and adding the edges x_2w and uv. Since G is a claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then $n' = n-6$. Since G is connected, either G' is connected of order $n' \ge 10$ (note that $\{u, v\} \cap \{y_2, z_2\} = \emptyset$), or G' has two components C_1 and C_2, where C_1 contains the vertex u and $n_1 = |V(C_1)| \ge 4$, C_2 contains the vertex x_2 and $n_2 = |V(C_2)| \ge 8$, $n' = n_1 + n_2$.

In the case where G' is connected. If $G' = N_3$, then $G = G_{18.3}$ shown in Figure 10(b), and $F_{t2}(G) \le 6 < \frac{3}{8}n + \frac{1}{4}$. If $G' = N_k$ with $k \ge 4$, then G contains a L_2, a contradiction. Hence, suppose that $G' \notin \mathcal{N}_{cubic}$ and $n' \geq 10$. Applying the inductive hypothesis to G', $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$.

In the case where G' is disconnected. Recall that $n_1 \geq 4$ and $n_2 \geq 8$. If $C_2 = N_k$ with $k \geq 2$, then G may contain a B_1, or a B_2, or a L_2, a contradiction. Hence, suppose that $C_2 \notin \mathcal{N}_{cubic}$ and $n_2 \geq 10$. Applying the inductive hypothesis to C_2, $F_{t2}(C_2) \leq \frac{3}{8}n_2 + \frac{1}{4}$. If $C_1 = K_4$, then G contains a B_1, a contradiction. If $C_1 = N_k$ with $k \geq 2$, then G may contain a B_2 or a L_2, a contradiction. If $C_1 = C_3 \Box K_2$, then C_2 is the graph obtained from G by removing all vertices in $V(C_1) \cup V(T_1) \cup V(D_1)$, subdividing the edge y_2z_2 and denoting the resulting new vertex by w, and adding the edges x_2w and uv. Now $n_2 = n - 12$. Let S_2 be a minimum semitotal forcing set of C_2 with $x_2 \in S_2$ and $w \notin S_2$ as in Lemma 3. We can easily get that $S_2 \cup \{a_1, c_1, y_1, u_1\}$ is a semitotal forcing set of G, where u_1 is a neighbor of u in G different from y_1. Thus, $F_{t2}(G) \leq F_{t2}(C_2) + 4 \leq \frac{3}{8}n_2 + \frac{1}{4} + 4 = \frac{3}{8}(n-12) + \frac{1}{4} + 4 < \frac{3}{8}n + \frac{1}{4}$. Hence, suppose that $C_1 \notin \mathcal{N}_{cubic}$ and $n_1 \geq 10$. Applying the inductive hypothesis to C_1, $F_{t2}(C_1) \leq \frac{3}{8}n_1 + \frac{1}{4}$. Therefore, $F_{t2}(G') = F_{t2}(C_1) + F_{t2}(C_2) \leq \frac{3}{8}n_1 + \frac{1}{4} + \frac{3}{8}n_2 + \frac{1}{4} = \frac{3}{8}n' + \frac{1}{2}$.

Now, we have $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{2}$ whether G' is connected or not. Let S' be a minimum semitotal forcing set of G' with $x_2 \in S'$ and $w \notin S'$ as in Lemma 3. Let $G'' = G' - w + y_2z_2$. Then S' is also a semitotal forcing set of G''. If $|S' \cap \{u, v\}| = 0$, without loss of generality, assume that u becomes black before v. Then, $S = S' \cup \{a_1, c_1\}$ is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, then the same forcing chains in G'' with respect to S' remain valid in G until u becomes black. Further, $u \to y_1$ and $a_1 \to d_1 \to b_1 \to x_1 \to z_1 \to v$. Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G''. If $|S' \cap \{u, v\}| = \{1, 2\}$, without loss of generality, assume $u \in S'$. If u is not a forcing vertex during the forcing process with respect to S' in G'', then $S' \cup \{z_1, c_1\}$ is a semitotal forcing set of G. The same forcing chains in G'' with respect to S' remain valid in G. Finally, $u \to y_1 \to x_1 \to b_1 \to d_1$. It follows that all vertices in G become black. Now assume that u is a forcing vertex during the forcing process with respect to S' in G''. If $u \to v$, then $S' \cup \{z_1, c_1\}$ is also a semitotal forcing set of G. This is because $u \to v$ in G'' will be replaced by two forcing chains in G: $u \to y_1 \to x_1 \to b_1 \to d_1 \to a_1$, $z_1 \to v$. If $u \to u_1$, then $u_2 \in S'$ by Lemma 2, where $\{u_1, u_2\} = N_G(u) \setminus \{y_1\}$. And in this time step, v is black. One can easily verify that $S = (S' \setminus \{u\}) \cup \{y_1, z_1, c_1\}$ is a semitotal forcing set of G. Therefore, $F_{t2}(G) \leq |S'| + 2 = F_{t2}(G') + 2 \leq \frac{3}{8}n' + \frac{1}{2} + 2 = \frac{3}{8}(n-6) + \frac{1}{2} + 2 = \frac{3}{8}n + \frac{1}{4}$. This completes the proof. ■

Combining $F_{t2}(C_3 \Box K_2) = 3 < \frac{3}{8}n + 1$ with Theorem 5 and Theorem 8, we get the following theorem immediately.

Theorem 13. *If $G \neq K_4$ is a connected claw-free cubic graph of order n, then $F_{t2}(G) \leq \frac{3}{8}n + 1$, with equality if and only if $G = N_k$ and k is even.*

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