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## SEMITOTAL FORCING IN CLAW-FREE CUBIC GRAPHS

YI-PING LIANG, JIE CHEN AND SHOU-JUN XU

School of Mathematics and Statistics Gansu Center for Applied Mathematics, Lanzhou University Lanzhou, Gansu 730000, China

> e-mail: liangyp18@lzu.edu.cn chenjie21@lzu.edu.cn shjxu@lzu.edu.cn

### Abstract

For an isolate-free graph G = (V(G), E(G)), a set  $S \subseteq V(G)$  is called a semitotal forcing set of G if it is a forcing set (or a zero forcing set) of G and every vertex in S is within distance 2 of another vertex of S. The semitotal forcing number  $F_{t2}(G)$  is the minimum cardinality of a semitotal forcing set in G. In this paper, we prove that if  $G \neq K_4$  is a connected claw-free cubic graph of order n, then  $F_{t2}(G) \leq \frac{3}{8}n + 1$ . The graphs achieving equality in this bound are characterized, an infinite set of graphs.

Keywords: semitotal forcing number, claw-free, cubic.

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## 1. INTRODUCTION

In this paper, by a graph we always mean a simple finite undirected graph; if we admit multiple edges, we always talk about a multigraph.

The concept of a (zero) forcing set, along with the related (zero) forcing number, of a simple graph was introduced in [2] to study the maximum nullity/minimum rank of the family of symmetric matrices associated with the graph. Independently, this parameter was introduced by Burgarth *et al.* [4] in conjunction with control of quantum systems; in this context it is known as the graph infection number. In addition, the (zero) forcing number was considered in connection with logic circuits [5] and dynamical systems [22].

For any two-coloring of vertex set V of a graph G, say black and white for two used colors, define a following *color-change rule*: a white vertex v is converted to black if it is the only white neighbor of some black vertex u. In this case, we say u forces v, write  $u \to v$  and refer to u as a forcing vertex. Let S be a subset of V. Define a two-coloring of G as coloring S black, the others white. The derived set D(S) of S is the set of black vertices obtained by iteratively applying the color-change rule until no more changes are possible. Moreover, applying the color-change rule iteratively results in forcing chains  $v_1 \to v_2 \to \cdots \to v_k$ , where  $v_i$  forces  $v_{i+1}$  for  $1 \leq i \leq k-1$ . If D(S) = V, then we say S is a forcing set (also called a zero forcing set) of G. The procedure of coloring a graph using the color-change rule applied for S is called a forcing process with respect to S. A minimum forcing set of G is a forcing set of G of minimum cardinality, and the forcing number, denoted by F(G), is the cardinality of a minimum forcing set. In addition, if S is a forcing set of G and G[S] contains no isolated vertex, then S is a total forcing set of G. The total forcing number of G is the cardinality of a minimum total forcing set in G, denoted by  $F_t(G)$ .

In this paper, we focus on the semitotal forcing, which was first introduced by Chen in [7]. A set S of vertices in G is a semitotal forcing set of G if it is a forcing set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal forcing number, denoted by  $F_{t2}(G)$ , is the cardinality of a minimum semitotal forcing set of G.

Since every total forcing set is also a semitotal forcing set, and since every semitotal forcing set is a forcing set, we have the following chain of inequalities [7]. For every isolate-free graph G,  $F(G) \leq F_{t2}(G) \leq F_t(G)$ . We remark that the gap between the semitotal forcing number with forcing number and total forcing number for graphs can be arbitrary large, such as a graph  $G \in \mathcal{N}_{cubic}$ . Forcing and its variants are heavily studied in graph theory and we refer the reader to [1–3, 6, 8–15, 17–21, 23, 24].

Forcing and total forcing of connected claw-free cubic graphs have been studied in [8, 9, 11]. Let  $G \neq K_4$  be a connected claw-free cubic graph of order n. Davila and Henning [8] showed that  $F_t(G) \leq \frac{1}{2}n$ , with equality if and only if  $G \in \mathcal{N}_{cubic}$  or G is the prism  $C_3 \Box K_2$ ; and then these two authors [11] showed that  $F(G) \leq \frac{1}{3}n + 1$  unless  $G = N_2$ . Chen [7] showed that  $F_{t2}(G) \leq \frac{1}{2}n$ , with equality if and only if G is the diamond-necklace  $N_2$  or the prism  $C_3 \Box K_2$ . In this paper, we improve this upper bound on the semitotal forcing number of G:  $F_{t2}(G) \leq \frac{3}{8}n + 1$ , and the graphs achieving equality in this bound are characterized i.e., Theorem 13.

#### 2. Preliminaries

In this section, we give some basic definitions and list or prove some lemmas and theorems as preliminaries, which will be used in the proof of our main results. Let G = (V(G), E(G)) be a graph with vertex set V(G) and edge set E(G). The order of G is the number of its vertices, denote n = |V(G)|, and its size is the number of its edges, denote m = |E(G)|. If  $uv \in E(G)$ , then we say u, v are adjacent, u is a neighbor of v and vice versa. Let  $N_G(v)$  be the set of neighbours of a vertex v in a graph G, and let  $d_G(v) = |N_G(v)|$  be the degree of a vertex v in a graph G. A graph is isolate-free if it does not contain an isolated vertex; that is, a vertex of degree 0. A graph is cubic if every vertex has degree three. The distance between u and v is the length of a shortest (u, v)-path in G, denoted by  $d_G(u, v)$ . If the graph G is clear from the context, we write V, E, N(v), d(G) and d(u, v) shortened. For  $k \geq 1$  an integer, we use the standard notation  $[k] = \{1, 2, \ldots, k\}$ .

A graph H = (V(H), E(H)) is called a *subgraph* of G = (V(G), E(G)) if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . For a subset X of V(G), the *induced subgraph* by X, denoted by G[X], is the graph with vertex set X, in which two vertices are adjacent if and only if they are adjacent in G. We denote by G - X the induced subgraph  $G[V \setminus X]$ , if  $X = \{x\}$ , write G - x for short. A graph is H-free if it does not contain H as an induced subgraph.

We denote a path, a cycle and a complete graph on n vertices by  $P_n$ ,  $C_n$  and  $K_n$ , respectively. A complete bipartite graph with parts of sizes a and b is denoted by  $K_{a,b}$ . A complete graph  $K_3$  is called a *triangle* and a complete bipartite graph  $K_{1,3}$  is called a *claw*. The complete graph  $K_4$  minus one edge is called a *diamond*.

Two vertices u and v in a nontrivial connected graph G are *twins* if u and v have the same neighbors in  $V(G) \setminus \{u, v\}$ .

**Observation 1.** If u and v are twins of a connected graph G, then every forcing set of G contains at least one vertex of  $\{u, v\}$ .

**Lemma 2.** Let G be a connected cubic graph and let T be an induced triangle of G satisfying that there exists a minimum semitotal forcing set S containing only one vertex of T, say u. Then there exists a forcing process with respect to S such that u does not force any other vertex of T.

**Proof.** Suppose  $V(T) = \{u, v, w\}$ . Note that  $u \in S$  and  $v, w \notin S$ . Without loss of generality, we may assume that v becomes black before w. Then v can be forced by its neighbor different from u and w, and further  $v \to w$ . Thus, u does not force v and w.

**Lemma 3.** Let G be a connected cubic graph containing an induced diamond D, where  $V(D) = \{a, b, c, d\}$  and ab is the missing edge in D. Then there exists a minimum semitotal forcing set S such that  $c \in S$  and  $d \notin S$ . In addition, c is not a forcing vertex if  $|S \cap \{a, b, c, d\}| \le 2$ ; c forces d if  $|S \cap \{a, b, c, d\}| = 3$ . **Proof.** Since c and d are twins of G, a minimum semitotal forcing set S contains at least one vertex of  $\{c, d\}$ . Without loss of generality,  $c \in S$ . Note that S contains at most three vertices of D, otherwise  $S \setminus \{d\}$  is a semitotal forcing set smaller than S, a contradiction. If  $|S \cap \{a, b, c, d\}| = 3$ , then  $S \cap \{a, b, c, d\} =$  $\{a, b, c\}$  is satisfied. Hence we assume that  $|S \cap \{a, b, c, d\}| \in \{1, 2\}$ . If  $d \notin S$ , there is nothing to prove. Now consider  $d \in S$ . If  $a \in S$ , then  $(S \setminus \{d\}) \cup \{b\}$ is also a minimum semitotal forcing set satisfying the statement of the lemma. Thus, we may assume  $a \notin S$  and similarly  $b \notin S$ . Without loss of generality, assume that a becomes black before b. Hence, a must be forced by its neighbor different from c and d, and  $(S \setminus \{d\}) \cup \{b\}$  satisfies the statement of the lemma.

Let S be a minimum semitotal forcing set of G such that  $c \in S$  and  $d \notin S$ . If  $|S \cap \{a, b, c, d\}| = 1$ , then  $a, b, d \notin S$ . By Lemma 2, c is not a forcing vertex. If  $|S \cap \{a, b, c, d\}| = 2$ , without loss of generality, assume that  $a \in S$  and  $b \notin S$ . By Lemma 2, c does not force b and d. Since  $a \in S$ , c is not a forcing vertex. If  $|S \cap \{a, b, c, d\}| = 3$ , we let  $c \to d$  in the forcing process.

The following property of connected claw-free cubic graphs is established in [16].

**Lemma 4** [16]. If  $G \neq K_4$  is a connected claw-free cubic graph of order n, then the vertex set V(G) can be uniquely partitioned into sets each of which induces a triangle or a diamond in G.

Following the notation introduced in [16], we refer to such a partition as a triangle-diamond partition of G, abbreviated  $\Delta$ -D-partition. Further we call every triangle and diamond induced by a set in our  $\Delta$ -D-partition a unit of the partition. A unit that is a triangle we call a triangle-unit and a unit that is a diamond we call a diamond-unit. (Note that a triangle-unit is a triangle that does not belong to a diamond.) We say that two units in the  $\Delta$ -D-partition are adjacent if there is an edge joining a vertex in one unit to a vertex in the other unit.

In what follows, we give the definition of a diamond-necklace, diamondbracelet, diamond-chain, double triangle-chain, and triangle-necklace, coined by Henning and Löwenstein [16], Davila and Henning [11], respectively.

**Definition** [16]. For  $k \geq 2$  an integer, let  $N_k$  be the connected cubic graph constructed as follows. Take k disjoint copies  $D_1, D_2, \ldots, D_k$  of a diamond, where  $V(D_i) = \{a_i, b_i, c_i, d_i\}$  and where  $a_i b_i$  is the missing edge in  $D_i$ . Let  $N_k$  be obtained from the disjoint union of these k diamonds by adding the edges  $\{a_i b_{i+1} | i \in [k]\}$  with addition taken modulo k (and so  $a_k b_{k+1} = a_k b_1$ ). We call  $N_k$  a diamond-necklace with k diamonds. Let  $\mathcal{N}_{cubic} = \{N_k | k \geq 2\}$ .

A diamond-necklace,  $N_6$ , with six diamonds is shown in Figure 1. By Lemma 4, we note that if G is a connected claw-free cubic graph with no triangle-units,



Figure 1. A diamond-necklace  $N_6$ , where the black vertices form a minimum semitotal forcing set of  $N_6$ .

then  $G \in \mathcal{N}_{cubic}$ . The semitotal forcing number of a diamond-necklace was determined by Chen in [7].

**Theorem 5** [7]. Let  $G = N_k \in \mathcal{N}_{cubic}$  have order n = 4k. Then  $F_{t2}(G) = \left\lfloor \frac{3k+1}{2} \right\rceil = \left\lfloor \frac{3n}{8} \right\rfloor + 1$ .



Figure 2. A diamond-bracelet  $B_5$ , a diamond-chain  $L_2$ , a double triangle-chain  $DT_2$  and a triangle-necklace  $H_6$ .

**Definition** [16]. For  $k \ge 1$  an integer, a *diamond-bracelet*  $B_k$  (see Figure 2(a) for example) with k diamonds is defined as follows. Let  $B_k$  be obtained from a diamond-necklace  $N_{k+1}$  with k + 1 diamonds  $D_1, D_2, \ldots, D_{k+1}$  by removing the diamond  $D_{k+1}$  and adding a triangle  $T_1$  with  $V(T_1) = \{x_1, y_1, z_1\}$ , and adding the edges  $y_1b_1$  and  $z_1a_k$ .

**Definition** [16]. For  $k \ge 1$  an integer, a diamond-chain  $L_k$  (see Figure 2(b) for example) with k diamonds is defined as follows. Let  $L_k$  be obtained from a diamond-necklace  $N_{k+1}$  with k + 1 diamonds  $D_1, D_2, \ldots, D_{k+1}$  by removing the diamond  $D_{k+1}$  and adding two disjoint triangles  $T_1$  and  $T_2$  and adding an edge joining  $b_1$  to a vertex of  $T_1$  and adding an edge joining  $a_k$  to a vertex of  $T_2$ .

**Definition** [16]. A double triangle-chain  $DT_2$  (see Figure 2(c) for example) is obtained from a diamond-chain  $L_2$  with two diamonds by replacing the two diamonds with two triangles and adding two edges joining these two triangles.

**Definition** [11]. For  $k \ge 2$  an integer, let  $H_k$  (see Figure 2(d) for example) be the graph constructed as follows. Take k disjoint copies  $T_1, T_2, \ldots, T_k$  of a triangle, where  $V(T_i) = \{x_i, y_i, z_i\}$ . Let  $H_k$  be obtained from the disjoint union of these k triangles by adding the edges  $\{x_iy_{i+1} | i \in [k]\}$  with addition taken modulo k (and so  $x_ky_{k+1} = x_ky_1$ ). We call  $H_k$  a triangle-necklace with k triangles.

We next define a diamond-armlet.

**Definition.** For  $k \geq 1$  an integer, a diamond-armlet  $A_k$  with k diamonds is defined as follows. Let  $A_k$  be obtained from a diamond-necklace  $N_{k+1}$  with k + 1 diamonds  $D_1, D_2, \ldots, D_{k+1}$  by removing the diamond  $D_{k+1}$  and adding two triangles  $T_1$  and  $T_2$  with  $V(T_i) = \{x_i, y_i, z_i\}$  for  $i \in [2]$ , and adding the edges  $x_1b_1, x_2a_k, y_1y_2$  and  $z_1z_2$ . Let  $\mathcal{A}_{cubic} = \{A_k | k \geq 1\}$ .



Figure 3. Two diamond-armlets  $A_3$  and  $A_4$ , where the black vertices form a minimum semitotal forcing set of their respective graphs.

Two diamond-armlets,  $A_3$  and  $A_4$ , are shown in Figure 3(a) and Figure 3(b), respectively. The semitotal forcing number of a diamond-armlet is determined as follows.

**Theorem 6.** Let  $G = A_k \in \mathcal{A}_{cubic}$  have order n = 4k + 6. Then  $F_{t2}(G) = \frac{3n}{8} + \frac{1}{4}$  when k is odd;  $F_{t2}(G) = \frac{3n}{8} - \frac{1}{4}$  when k is even.

**Proof.** Let  $G = A_k \in \mathcal{A}_{cubic}$  have order n = 4k + 6. For convenience, we write  $A = \{a_i | 1 \le i \le k\}, B = \{b_i | 1 \le i \le k\}, C = \{c_i | 1 \le i \le k\}$ , and  $D = \{d_i | 1 \le i \le k\}$ .

We first show that  $F_{t2}(G) \leq \left\lceil \frac{3k}{2} \right\rceil + 2$ . Let  $S = C \cup \{a_i \mid 1 \leq i \leq k \text{ and } i \text{ is } \text{odd}\} \cup \{b_1, y_1\}$ . Then the set S is a semitotal forcing set of G. Thus,  $F_{t2}(G) \leq |S| = k + \left\lceil \frac{k}{2} \right\rceil + 2 = \left\lceil \frac{3k}{2} \right\rceil + 2$ .

We next show that  $F_{t2}(G) \ge \left\lceil \frac{3k}{2} \right\rceil + 2$ . Let S' be an arbitrary semitotal forcing set of G. By Observation 1, S' contains at least one vertex of  $\{c_i, d_i\}$  for every  $i \in [k]$ . Renaming vertices if necessary, we may assume that  $c_i \in S'$ . Moreover, at least one vertex of  $\{y_1, y_2, z_1, z_2\}$  belongs to S', otherwise, S' is not a forcing set, a contradiction. Without loss of generality, assume that  $y_1 \in S'$ . Let S'' be the set of vertices in S' that belong to  $A \cup B \cup D \cup \{x_1, x_2, y_2, z_1, z_2\}$ . Now we consider two cases. In the case where k is even. Since every vertex in S' is within distance 2 of another vertex of S',  $|S''| \ge \left\lceil \frac{k}{2} \right\rceil + 1$ . In the case where k is odd. Similarly, in order to guarantee that S' is semitotal, we have  $|S''| \ge \left\lceil \frac{k}{2} \right\rceil$ . And then every vertex in G[S'] has degree at most 1 if  $|S''| = \left\lceil \frac{k}{2} \right\rceil$ . Since G is cubic, the first forcing vertex in the forcing process has degree 2 in G[S'], a contradiction. Thus,  $|S''| \ge \left\lceil \frac{k}{2} \right\rceil + 1$ . In both cases,  $|S'| = |C| + 1 + |S''| \ge k + 1 + \left\lceil \frac{k}{2} \right\rceil + 1 = \left\lceil \frac{3k}{2} \right\rceil + 2$ .

To conclude, when k is odd,  $F_{t2}(G) = \left\lceil \frac{3k}{2} \right\rceil + 2 = \frac{3n}{8} + \frac{1}{4}$ ; when k is even,  $F_{t2}(G) = \left\lceil \frac{3k}{2} \right\rceil + 2 = \frac{3n}{8} - \frac{1}{4}$ .

## 3. Connected Claw-Free Cubic Graphs

In this section, we establish an upper bound on the semitotal forcing number of a connected claw-free cubic graph  $G \neq K_4$  in terms of its order.

First, we give the following theorem for the special case where G is diamond-free.

**Theorem 7.** If  $G \neq K_4$  is a connected claw-free and diamond-free cubic graph of order n, then  $F_{t2}(G) \leq \frac{1}{3}n + 1$ .

**Proof.** Since  $G \neq K_4$  is a connected claw-free cubic graph, there is a unique  $\Delta$ -D-partition of G by Lemma 4. Further every unit in G is a triangle-unit because G is diamond-free. Now we define the contraction multigraph of G, denoted G', to be the multigraph whose vertices correspond to the triangle-units in G and where two vertices in G' are joined by the number of edges joining the corresponding triangle-units in G. We note that the order of G' is precisely the number of triangle-units in G. Every vertex in G' has degree 3, and so G' is a cubic multigraph. Let  $C: v_1v_2 \cdots v_kv_1$  be a shortest cycle in G' where we allow 2-cycles, and so  $k \geq 2$ . For  $i \in [k]$ , let  $T_i$  be the triangle-unit in G associated

with the vertex  $v_i$ , where  $V(T_i) = \{x_i, y_i, z_i\}$  and where  $x_i y_{i+1}$  is an edge with addition taken modulo k (and so  $x_k y_{k+1}$  is the edge  $x_k y_1$ ). Thus, G contains a subgraph as a triangle-necklace,  $H_k$ , with k triangles. We note that if k = 2, then either G is the prism  $C_3 \Box K_2$  or G contains a  $H_2$ .



Figure 4. Initialization for the proof of Theorem 7.

We construct a semitotal forcing set S of G such that  $|S| \leq \frac{1}{3}n + 1$ . For this purpose, we set  $S = \{y_1, x_1, x_2, \ldots, x_k\}$  and color all vertices in S black and the others white, initially. At any step we let B be the set of vertices that have already been colored black, and W the set of all vertices that remain white. If  $G = C_3 \Box K_2$ , then  $S = \{y_1, x_1, x_k\}$  is a minimum semitotal forcing set of G and  $F_{t2}(G) = 3 = \frac{1}{3}n + 1$ . (The graph  $C_3 \Box K_2$  is shown in Figure 4(a).) Suppose that  $G \neq C_3 \Box K_2$ . Then the set  $\{z_1, \ldots, z_k\}$  is independent. Starting with this initial set S, we play the sequence of vertices  $y_1, x_1, y_2, x_2, \ldots, y_k$  in turn which force the vertices  $z_1, y_2, z_2, y_3, \ldots, z_k$ , respectively, to be colored black. The initialization of the initial set S in the case of k = 6 is shown in Figure 4(b). Now  $B = V(H_k)$ and  $W \neq \emptyset$ . Next, we extend S by Rule 1 and Rule 2, iteratively apply the color-change rule to the current set S, and in each step, we color all vertices in one triangle-unit or two triangle-units black.

**Rule 1.** There exists a triangle-unit T such that  $V(T) \subseteq W$  and T has at least two vertices that are adjacent to vertices of B, then all vertices in this triangle-unit to become black.

**Rule 2.** There exists a triangle-unit T such that  $V(T) = \{x, y, z\} \subseteq W$  and T has only one vertex that is adjacent to a vertex of B. Renaming vertices if necessary, assume that x has a neighbor  $v \in B$ . Then T is adjacent to a triangle-unit T', where  $V(T') = \{x', y', z'\} \subseteq W$ . Renaming vertices if necessary, assume that y is adjacent to x'. Now we add  $\{y, y'\}$  to S. Thus,  $v \to x \to z, y \to x' \to z'$ , and all vertices in  $V(T) \cup V(T')$  become black. Note that d(y, y') = 2.

Since G is connected, Rule 1 and Rule 2 continue until all vertices become black. And in each step we add at most one vertex of a triangle-unit to S. Thus, S is a forcing set of G with  $|S| \leq \frac{1}{3}n + 1$ . It is easy to see that S is a semitotal forcing set of G.

Note that the only three connected claw-free cubic graphs that have order n < 10 are  $K_4$ ,  $C_3 \Box K_2$  and  $N_2$ . If  $G \in \mathcal{N}_{cubic}$ , the semitotal forcing number of G was determined by Theorem 5. Next we add the restriction that G is not a diamond-necklace and  $n \ge 10$ .

**Theorem 8.** If  $G \notin \mathcal{N}_{cubic}$  is a connected claw-free cubic graph of order  $n \ge 10$ , then  $F_{t2}(G) \le \frac{3}{8}n + \frac{1}{4}$ .

**Proof.** Since G is cubic, we note that n is even. We proceed by induction on the order  $n \ge 10$ . If n = 10, then  $G = A_1$  and  $F_{t2}(A_1) = 4 = \frac{3}{8}n + \frac{1}{4}$ . If n = 12, since  $G \ne N_3$ , G can only be  $G_{12,1}$  and  $G_{12,2}$ , shown in Figure 5, respectively, and  $F_{t2}(G) = 4 < \frac{3}{8}n + \frac{1}{4}$ . This establishes the base cases. Next, let  $n \ge 14$ and assume that for every connected claw-free cubic graph  $G' \notin \mathcal{N}_{cubic}$  of order  $10 \le n' < n, F_{t2}(G') \le \frac{3}{8}n' + \frac{1}{4}$ . Let G be a connected claw-free cubic graph of order n. By Lemma 4, there is a unique  $\Delta$ -D-partition of G. If every unit in the  $\Delta$ -D-partition of G is a triangle-unit, i.e., G is diamond-free, we get  $n \ge 18$ , and then  $F_{t2}(G) \le \frac{n}{3} + 1 \le \frac{3}{8}n + \frac{1}{4}$  by Theorem 7. Thus, we may assume that G has at least one diamond-unit. Since  $G \notin \mathcal{N}_{cubic}$ , G must contain a diamond-chain or a diamond-bracelet.



Figure 5. Two connected claw-free cubic graphs of order 12, where the black vertices form a minimum semitotal forcing set of their respective graphs.

We proceed further with the following series of claims.

**Claim 9.** If G contains a diamond-chain  $L_2$ , then  $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$ .

**Proof.** Suppose that G contains a diamond-chain  $L_2$ . Using our earlier notation, let  $D_1, D_2$  be two consecutive diamonds in  $L_2$ , where  $V(D_i) = \{a_i, b_i, c_i, d_i\}$  and where  $a_i b_i$  is the missing edge in  $D_i$  for  $i \in [2]$  and  $a_1 b_2$  is an edge. Let  $w_1$  be the neighbor of  $b_1$  not in  $D_1$ , and let  $w_2$  be the neighbor of  $a_2$  not in  $D_2$ . By the definition of a diamond-chain  $L_2$ , we note that  $w_1 \neq w_2$  and  $w_1w_2$  is not an edge; the triangles containing  $w_1$  and  $w_2$  are vertex disjoint. Let G' be the graph obtained from G by removing all vertices in  $V(D_1) \cup V(D_2)$  and adding the edge  $w_1w_2$ . Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then  $n' = n - 8 \ge 6$ .

If  $G' = C_3 \Box K_2$ , then  $G = A_2$  and  $F_{t2}(A_2) = 5 = \frac{3}{8}n - \frac{1}{4}$  is satisfied. If  $G' \in \mathcal{N}_{cubic}$ , then  $G \in \mathcal{N}_{cubic}$ , a contradiction. Now assume that  $G' \notin \mathcal{N}_{cubic}$ and  $n' \geq 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G'. If  $|S' \cap \{w_1, w_2\}| = 0$ , without loss of generality, we may suppose that  $w_1$  becomes black before  $w_2$ . Then,  $S = S' \cup \{c_1, c_2, a_1\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until  $w_1$  becomes black. In this time, two neighbors (different from  $b_1$ ) of  $w_1$  are both black. Then  $w_1 \rightarrow b_1 \rightarrow d_1$ , and  $a_1 \rightarrow b_2 \rightarrow d_2 \rightarrow d_2$  $a_2 \rightarrow w_2$ . Finally, all remaining white vertices in G will become black using the same forcing chains as in G'. The newly added vertices guarantee that Sis semitotal. Now consider  $|S' \cap \{w_1, w_2\}| \in \{1, 2\}$ . Without loss of generality, assume that  $w_1 \in S'$ . We claim that  $S = S' \cup \{c_1, c_2, a_2\}$  is a semitotal forcing set of G. If  $w_1$  is not a forcing vertex during the forcing process with respect to S' in G', then the same forcing chains in G' with respect to S' can be used in G with respect to S. Thus, all vertices in G' still become black. Finally,  $a_2 \to d_2 \to b_2 \to a_1 \to d_1 \to b_1$ . It follows that all vertices in G become black. If  $w_1$  is a forcing vertex during the forcing process with respect to S' in G', let  $N_{G'}(w_1) = \{w_2, u, v\}$  and divide into two cases. In the case of  $w_1 \to w_2$ , it will be replaced by two forcing chains in  $G: w_1 \to b_1 \to d_1 \to a_1 \to b_2 \to d_2$ ,  $a_2 \to w_2$ . In the case of  $w_1 \to u$  (or  $w_1 \to v$ ), note that in this time step,  $w_2$  is already black. So the forcing chains in G' with respect to S' remain valid in Guntil  $w_2$  becomes black. Further,  $a_2 \rightarrow d_2 \rightarrow b_2 \rightarrow a_1 \rightarrow d_1 \rightarrow b_1$ . Finally, all remaining white vertices in G will become black using the same forcing chains as in G'. In either case, S is a semitotal forcing set of G. Therefore, we have  $F_{t2}(G) \le F_{t2}(G') + 3 \le \frac{3}{8}n' + \frac{1}{4} + 3 = \frac{3}{8}(n-8) + \frac{1}{4} + 3 = \frac{3}{8}n + \frac{1}{4}.$  $\square$ 

We note that if G contains a diamond-chain  $L_k$  with  $k \ge 2$ , then G contains a diamond-chain  $L_2$ . Hence by Claim 9, we may assume that G does not contain a diamond-chain  $L_k$  with  $k \ge 2$ . Moreover, if G contains a diamond-bracelet  $B_k$ , then  $k \le 2$ , otherwise, G contains a diamond-chain  $L_2$ , a contradiction. Therefore, next we will discuss when G contains a diamond-bracelet  $B_k$ , where  $k \in [2]$ , or every diamond-unit is adjacent to two distinct triangle-units to complete our proof. Before this, we give the following claim.

Claim 10. If G contains a double triangle-chain  $DT_2$ , then  $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$ .

**Proof.** Suppose that G contains a double triangle-chain  $DT_2$ , where  $T_1$  and  $T_2$  are the two triangle-units in  $DT_2$  joined by two edges. Let  $V(T_i) = \{x_i, y_i, z_i\}$  for  $i \in [2]$ , where  $y_1y_2, z_1z_2 \in E(G)$ . Let  $w_i$  be the neighbor of  $x_i$  not in  $T_i$  for  $i \in [2]$ . By the definition of a double triangle-chain, we note that the triangles containing  $w_1$  and  $w_2$  are vertex disjoint. In particular,  $w_1$  and  $w_2$  are not adjacent. Let G' be the graph obtained from G by removing all vertices in  $V(T_1) \cup V(T_2)$  and adding the edge  $w_1w_2$ . Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then  $n' = n - 6 \geq 8$ .

If  $G' = N_k$  with  $k \ge 2$ , then  $G = A_k \in \mathcal{A}_{cubic}$  and  $F_{t2}(G) \le \frac{3}{8}n + \frac{1}{4}$  by Theorem 6. Suppose that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \ge 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G'. If  $|S' \cap \{w_1, w_2\}| = 0$ , without loss of generality, we may suppose that  $w_1$  becomes black before  $w_2$ . Then, the set  $S = S' \cup \{x_1, y_1\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until  $w_1$  becomes black. Further,  $x_1 \to z_1 \to z_2$  and  $y_1 \to y_2 \to x_2 \to w_2$ . Then all remaining white vertices in G will become black using the same forcing chains as in G'. Now consider  $|S' \cap \{w_1, w_2\}| \in \{1, 2\}$ . Without loss of generality, assume that  $w_1 \in S'$ . If  $w_1$  is not a forcing vertex during the forcing process with respect to S' in G', then  $S = S' \cup \{x_2, y_1\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G and  $w_1 \rightarrow x_1 \rightarrow z_1 \rightarrow z_2 \rightarrow y_2$ . The newly added vertices guarantee that S is semitotal. If  $w_1$  is a forcing vertex during the forcing process with respect to S' in G', then we divide into two cases. In the case of  $w_1 \to w_2$ ,  $S = S' \cup \{x_2, y_1\}$  is also a semitotal forcing set of G. Applying the color-change rule to S in  $G, w_1 \to w_2$  is replaced by two forcing chains:  $w_1 \to x_1 \to z_1 \to z_2 \to y_2, x_2 \to w_2$ . The newly added vertices guarantee that S is semitotal. In the case of  $w_1 \to u$ , then  $v \in S'$  by Lemma 2, where  $\{u, v\} = N_{G'}(w_1) \setminus \{w_2\}$ . Note that in this time step,  $w_2$  is already black. One can easily verify that  $S = S' \cup \{x_2, y_2\}$  is a semitotal forcing set of G. Therefore, we have  $F_{t2}(G) \leq F_{t2}(G') + 2 \leq \frac{3}{8}n' + \frac{1}{4} + 2 = \frac{3}{8}(n-6) + \frac{1}{4} + 2 < \frac{3}{8}n + \frac{1}{4}$ .  $\square$ 

By Claim 10, we may assume that G contains no double triangle-chain  $DT_2$ .

**Claim 11.** If G contains a diamond-bracelet  $B_k$  with  $k \in [2]$ , then  $F_{t2}(G) \leq \frac{3}{8}n + \frac{1}{4}$ .

**Proof.** Suppose that G contains a diamond-bracelet  $B_k$  with  $k \in [2]$ . Using our earlier notation, let  $D_i$  be a diamond of  $B_k$  for  $i \in [k]$ , where  $V(D_i) = \{a_i, b_i, c_i, d_i\}$  and  $a_i b_i$  is the missing edge in  $D_i$ . Let  $T_1$  be the only triangle in  $B_k$  that  $V(T_1) = \{x_1, y_1, z_1\}$  and  $y_1 b_1$  and  $z_1 a_k$  are edges in  $B_k$ . Let U be the unit



Figure 6. The diamond-bracelets  $B_1$  and  $B_2$ .

that is adjacent to  $T_1$  and does not belong to  $B_k$ . Now we divide into two cases to discuss according to U.

Case 1. U is a triangle-unit.

Suppose that U is a triangle-unit, say T, where  $V(T) = \{x, y, z\}$  and  $xx_1 \in E(G)$ . Let u and v denote the neighbors of y and z, respectively, not in T. If  $uv \in E(G)$ , then u and v must be in the same triangle-unit, say T'. Clearly, G has a double triangle-chain  $DT_2$  containing T and T', a contradiction. Hence,  $uv \notin E(G)$ .

Let G' be the graph obtained from G by removing all vertices in  $V(B_k) \cup V(T)$ and adding the edge uv. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then  $n' = n - 6 - 4k \ge 4$  for  $k \in [2]$ . If  $G' = K_4$ , then G is the graph  $G_{14.1}$  or  $G_{18.1}$ , shown in Figure 7(a) and Figure 7(b), respectively. If  $G' = C_3 \Box K_2$ , then G is the graph  $G_{16.1}$  or  $G_{20.1}$ , shown in Figure 7(c) and Figure 7(d), respectively. If  $G' = N_2$ , then G is the graph  $G_{18.1}$  or  $G_{22.1}$ , shown in Figure 7(b) and Figure 7(e), respectively. In all cases,  $F_{t2}(G) < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_k$  with  $k \ge 3$ , then G contains a diamond-chain  $L_2$ , a contradiction. Now we assume that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \ge 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \le \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G'.

Suppose first k = 1 and n' = n - 10. If  $|S' \cap \{u, v\}| = 0$ , without loss of generality, we may suppose that u becomes black before v. Then,  $S = S' \cup \{x, y_1, c_1\}$  is a semitotal forcing set of G. This is because if we apply the colorchange rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until u becomes black. In this time, two neighbors (different from y) of u are black. Then  $u \to y \to z \to v$  and  $x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. The newly added vertices guarantee that S is semitotal. Now consider  $|S' \cap \{u, v\}| \in \{1, 2\}$ . Without loss of generality, assume that  $u \in S'$ . If u is not a forcing vertex during the forcing process with respect to S' in G', then  $S = S' \cup \{z, y_1, c_1\}$  is a semitotal forcing set of G. This is because if we



Figure 7. Five connected claw-free cubic graphs in the proof of Case 1 in Claim 11, where the black vertices form a semitotal forcing set of their respective graphs.

apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G. Further,  $u \to y \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$  and all vertices of G will become black. If u is a forcing vertex during the forcing process with respect to S' in G', then we consider two cases. In the case of  $u \to v$ ,  $S = S' \cup \{z, y_1, c_1\}$  is also a semitotal forcing set of G. This is because if we apply the color-change rule to S in G,  $u \to v$  is replaced by two forcing chains:  $u \to y \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$  and then  $z \to v$ . In the case of  $u \to u_1$ , then  $u_2 \in S'$  by Lemma 2, where  $\{u_1, u_2\} = N_{G'}(u) \setminus \{v\}$ . Note that in this time step, v is already black. So,  $S = (S' \setminus \{u\}) \cup \{y, z, y_1, c_1\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until v becomes black. Then  $z \to x \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$  and further  $y \to u \to u_1$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. It is not hard to see that S is semitotal. Therefore, we have  $F_{t2}(G) \leq F_{t2}(G') + 3 \leq \frac{3}{8}n' + \frac{1}{4} + 3 = \frac{3}{8}(n-10) + \frac{1}{4} + 3 < \frac{3}{8}n + \frac{1}{4}$ . Suppose k = 2 and n' = n - 14. If  $|S' \cap \{u, v\}| = 0$ , without loss of generality,

Suppose k = 2 and n' = n - 14. If  $|S' \cap \{u, v\}| = 0$ , without loss of generality, we may suppose that u becomes black before v. Then,  $S = S' \cup \{a_1, b_2, c_1, c_2\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, the same forcing chains in G' with respect to S' remain valid in G until ubecomes black. In this time, two neighbors (different from y) of u are both black. Then  $u \to y$ ,  $b_2 \to d_2 \to a_2 \to z_1$ , and  $a_1 \to d_1 \to b_1 \to y_1 \to x_1 \to x \to z \to v$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Now consider  $|S' \cap \{u, v\}| \in \{1, 2\}$ . Without loss of generality, assume that  $u \in S'$ . One can easily verify that  $S' \cup \{a_1, b_2, c_1, c_2, z\}$  is a semitotal forcing set of G. Thus, we have  $F_{t2}(G) \leq F_{t2}(G') + 5 \leq \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n-14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}$ .

Case 2. U is a diamond-unit.

Suppose that U is a diamond-unit, say D, where  $V(D) = \{a, b, c, d\}$  and ab is the missing edge in the diamond D and  $x_1a \in E(G)$ . Let U' be the unit that is adjacent to U and does not belong to  $B_k$ . Since U is a diamond-unit, U' is not a diamond-unit, otherwise G contains a diamond-chain  $L_2$ , a contradiction. Thus U' is a triangle-unit, say T', where  $V(T') = \{x', y', z'\}$  and  $bx' \in E(G)$ . Let u' and v' denote the neighbors of y' and z', respectively, not in T'. If  $u'v' \in E(G)$ , then G contains a double triangle-chain  $DT_2$ , a contradiction. Hence,  $u'v' \notin E(G)$ .



Figure 8. Four connected claw-free cubic graphs in the proof of Case 2 in Claim 11, where the black vertices form a semitotal forcing set of their respective graphs.

Suppose first that k = 1 and let G' be the graph obtained from G by removing

all vertices in  $V(B_1) \cup V(D) \cup V(T')$  and adding the edge u'v'. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then  $n' = n - 14 \ge 4$ . If  $G' = K_4$ , then  $G = G_{18.2}$  shown in Figure 8(a), and  $F_{t2}(G) \leq 7 = \frac{3}{8}n + \frac{1}{4}$ . If  $G' = C_3 \Box K_2$ , then  $G = G_{20.2}$  shown in Figure 8(b), and  $F_{t2}(G) \leq 7 < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_2$ , then  $G = G_{22,2}$  shown in Figure 8(c), and  $F_{t2}(G) \leq 8 < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_k$  with  $k \geq 3$ , then G contains a diamond-chain  $L_2$ , a contradiction. Now assume that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \geq 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G'. Applying the color-change rule to S' in G', without loss of generality, we may assume that u' becomes black no later than v'. We claim that  $S = S' \cup \{c_1, y_1, c, x', z'\}$  is a semitotal forcing set of G. If  $|S' \cap \{u', v'\}| = 2$ , it is clear that S is a semitotal forcing set of G. If  $|S' \cap \{u', v'\}| = 0$ , then the forcing chains in G' with respect to S' remain valid in G until u' becomes black. In this time, two neighbors (different from y') of u' are both black. Further,  $u' \to y'$  and then  $x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1, z' \to v'$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. If  $|S' \cap \{u', v'\}| = 1$ , recall our assumption that u' becomes black no later than v'. Thus, we have  $u' \in S'$  and  $v' \notin S'$ . Next we divide into two cases. In the case of  $u' \to v'$  in G', we replace with three forcing chains when we apply the colorchange rule to S in G:  $u' \to y', x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1,$  $z' \to v'$ . In the case of u' does not force v' in G', the same forcing chains in G' with respect to S' remain valid in G until v' becomes black. Then,  $z' \to y'$  and  $x' \to b \to d \to a \to x_1 \to z_1 \to a_1 \to d_1 \to b_1$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Thus, S is a forcing set of G. It is not hard to see that S is semitotal. Therefore,  $F_{t2}(G) \leq F_{t2}(G') + 5 \leq \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n-14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}.$ 

Suppose k = 2 and let G' be the graph obtained from G by removing all vertices in  $V(B_2) \cup V(D)$ , subdividing the edge y'z' and denoting the resulting new vertex by w', and adding the edges x'w'. Since G is a connected claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then  $n' = n - 14 \ge 8$ . If  $G' = N_2$ , then  $G = G_{22.2}$  shown in Figure 8(c), and  $F_{t2}(G) \le 8 < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_3$ , then  $G = G_{26.1}$  shown in Figure 8(d), and  $F_{t2}(G) \le 10 = \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_k$  with  $k \ge 4$ , then G contains a diamond-chain  $L_2$ , a contradiction. Now assume that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \ge 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \le \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G' with  $x' \in S'$  and  $w' \notin S'$  as in Lemma 3. Then  $S = S' \cup \{a_1, b_2, c_1, c_2, c\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, first,  $b_2 \to d_2 \to a_2 \to z_1$ . Further,  $a_1 \to d_1 \to b_1 \to y_1 \to x_1 \to a \to d \to b$ . The forcing chains in G' with respect to S' remain valid in G except in the following cases. If  $x' \to w'$  occurs in G', then  $\{y', z'\} \subset S'$  by Lemma 3. Then  $x' \to w'$  will be omitted in G and all remaining forcing chains in G' with respect to S'

will remain valid in G. Thus, S is a semitotal forcing set of G. If  $y' \to w'$  occurs in G', then  $z' \in S'$  or  $w' \to z'$ . In this case, either  $y' \to w'$  will be omitted in G or  $y' \to w' \to z'$  will be replaced by  $y' \to z'$  in G. Finally, all remaining forcing chains in G' with respect to S' will remain valid in G and S is a semitotal forcing set. Lastly, if  $z' \to w'$ , a similar argument to that used above shows that S is a semitotal forcing set. Thus,  $F_{t2}(G) \leq F_{t2}(G') + 5 \leq \frac{3}{8}n' + \frac{1}{4} + 5 = \frac{3}{8}(n-14) + \frac{1}{4} + 5 < \frac{3}{8}n + \frac{1}{4}$ .

By Claim 9 and Claim 11, we may assume that every diamond-unit is adjacent to two distinct triangle-units. Now G must contain a diamond-chain  $L_1$ . Using our earlier notation, let  $D_1$  be the diamond in  $L_1$ , where  $V(D_1) = \{a_1, b_1, c_1, d_1\}$ and where  $a_1b_1$  is the missing edge in  $D_1$ . Let  $T_1$  and  $T_2$  be the two triangles in  $L_1$ , where  $V(T_i) = \{x_i, y_i, z_i\}$  for  $i \in [2]$  and  $x_1b_1, x_2a_1$  are edges. We note that  $T_1$  and  $T_2$  are two distinct triangle-units of G. If  $y_1$  or  $z_1$  is adjacent to  $y_2$  or  $z_2$ , then G contains an induced subgraph isomorphic to H shown in Figure 9 by  $n \geq 14$ .



Figure 9. An induced subgraph H of G.

**Claim 12.** If G contains an induced subgraph H shown in Figure 9, then  $F_{t2}(G) < \frac{3}{8}n + \frac{1}{4}$ .

**Proof.** Let G' be the graph obtained from G by removing the vertices in  $V(D) \cup \{x_2, y_2\}$  and adding the edges  $z_2x_1$  and  $z_2y_1$ . Since G is a connected claw-free cubic graph, so also is the graph G'. We note that the subgraph of G' induced by  $\{x_1, y_1, z_1, z_2\}$  is a diamond-unit where  $z_1z_2$  is the missing edge in this unit. Let G' have order n'. Then  $n' = n - 6 \ge 8$ .

If  $G' = N_2$ , then  $G = G_{14,2}$  shown in Figure 10(a), and  $F_{t2}(G) \leq 5 < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_k$  with  $k \geq 3$ , then G contains a diamond-chain  $L_2$ , a contradiction. Now assume that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \geq 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$ . Let S' be a minimum semitotal forcing set of G'with  $x_1 \in S'$  and  $y_1 \notin S'$  as in Lemma 3. If  $|S' \cap \{z_1, z_2\}| = 2$ , it is clear that  $(S' \setminus \{x_1\}) \cup \{a_1, b_1, c_1\}$  is a semitotal forcing set of G. Now consider  $|S' \cap \{z_1, z_2\}| \in$  $\{0, 1\}$ . We claim that  $S = S' \cup \{c_1, x_2\}$  is a semitotal forcing set of G. If  $|S' \cap \{z_1, z_2\}| = 0$ , without loss of generality, assume that  $z_1$  becomes black before  $z_2$ . Then, the same forcing chains in G' with respect to S' remain valid in G until  $z_1$  becomes black. Further,  $z_1 \to y_1 \to y_2 \to z_2$  and  $x_1 \to b_1 \to d_1 \to a_1$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. If  $|S' \cap \{z_1, z_2\}| = 1$ , without loss of generality, assume that  $z_1 \in S'$  and  $z_2 \notin S'$ . Then  $x_1$  is not a forcing vertex in G' by Lemma 3. Next we consider two cases. In the case of  $z_1 \to y_1 \to z_2$  in G', we replace with two forcing chains when we apply the color-change rule to S in  $G: z_1 \to y_1 \to y_2 \to z_2$ ,  $x_1 \to b_1 \to d_1 \to a_1$ . In the case of  $z_2$  is forced by its neighbor different from  $x_1$  and  $y_1$ , then the same forcing chains in G' with respect to S' remain valid in G until  $z_2$  becomes black. Further,  $z_2 \to y_2 \to y_1$  and  $x_1 \to b_1 \to d_1 \to a_1$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G'. Thus, S is a semitotal forcing set of G. Therefore,  $F_{t2}(G) \leq F_{t2}(G') + 2 \leq \frac{3}{8}n' + \frac{1}{4} + 2 = \frac{3}{8}(n-6) + \frac{1}{4} + 2 < \frac{3}{8}n + \frac{1}{4}$ .



Figure 10. Two connected claw-free cubic graphs  $G_{14.2}$  and  $G_{18.3}$ , where the black vertices form a semitotal forcing set of their respective graphs.

By Claim 12, we assume that G contains no induced subgraph H. Now, G contains a diamond-chain  $L_1$  with two distinct triangle-units  $T_1$  and  $T_2$ , and neither  $y_1$  nor  $z_1$  is adjacent to  $y_2$  or  $z_2$ . Let u and v be the neighbors of  $y_1$  and  $z_1$ , respectively, not in  $T_1$ . If  $uv \in E(G)$ , then G contains a double triangle-chain  $DT_2$ , a contradiction. Hence,  $uv \notin E(G)$ .

Let G' be the graph obtained from G by removing all vertices in  $V(T_1) \cup V(D_1)$ , subdividing the edge  $y_2z_2$  and denoting the resulting new vertex by w, and adding the edges  $x_2w$  and uv. Since G is a claw-free cubic graph, so also is the graph G'. Let G' have order n'. Then n' = n - 6. Since G is connected, either G' is connected of order  $n' \geq 10$  (note that  $\{u, v\} \cap \{y_2, z_2\} = \emptyset$ ), or G' has two components  $C_1$  and  $C_2$ , where  $C_1$  contains the vertex u and  $n_1 = |V(C_1)| \geq 4$ ,  $C_2$  contains the vertex  $x_2$  and  $n_2 = |V(C_2)| \geq 8$ ,  $n' = n_1 + n_2$ .

In the case where G' is connected. If  $G' = N_3$ , then  $G = G_{18.3}$  shown in Figure 10(b), and  $F_{t2}(G) \le 6 < \frac{3}{8}n + \frac{1}{4}$ . If  $G' = N_k$  with  $k \ge 4$ , then G contains

a  $L_2$ , a contradiction. Hence, suppose that  $G' \notin \mathcal{N}_{cubic}$  and  $n' \geq 10$ . Applying the inductive hypothesis to G',  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{4}$ .

In the case where G' is disconnected. Recall that  $n_1 \geq 4$  and  $n_2 \geq 8$ . If  $C_2 = N_k$  with  $k \geq 2$ , then G may contain a  $B_1$ , or a  $B_2$ , or a  $L_2$ , a contradiction. Hence, suppose that  $C_2 \notin \mathcal{N}_{cubic}$  and  $n_2 \geq 10$ . Applying the inductive hypothesis to  $C_2$ ,  $F_{t2}(C_2) \leq \frac{3}{8}n_2 + \frac{1}{4}$ . If  $C_1 = K_4$ , then G contains a  $B_1$ , a contradiction. If  $C_1 = N_k$  with  $k \geq 2$ , then G may contain a  $B_2$  or a  $L_2$ , a contradiction. If  $C_1 = C_3 \Box K_2$ , then  $C_2$  is the graph obtained from G by removing all vertices in  $V(C_1) \cup V(T_1) \cup V(D_1)$ , subdividing the edge  $y_2 z_2$  and denoting the resulting new vertex by w, and adding the edges  $x_2w$  and uv. Now  $n_2 = n - 12$ . Let  $S_2$  be a minimum semitotal forcing set of  $C_2$  with  $x_2 \in S_2$  and  $w \notin S_2$  as in Lemma 3. We can easily get that  $S_2 \cup \{a_1, c_1, y_1, u_1\}$  is a semitotal forcing set of G, where  $u_1$  is a neighbor of u in G different from  $y_1$ . Thus,  $F_{t2}(G) \leq F_{t2}(C_2) + 4 \leq \frac{3}{8}n_2 + \frac{1}{4} + 4 = \frac{3}{8}(n - 12) + \frac{1}{4} + 4 < \frac{3}{8}n + \frac{1}{4}$ . Hence, suppose that  $C_1 \notin \mathcal{N}_{cubic}$  and  $n_1 \geq 10$ . Applying the inductive hypothesis to  $C_1$ ,  $F_{t2}(C_1) \leq \frac{3}{8}n_1 + \frac{1}{4}$ . Therefore,  $F_{t2}(G') = F_{t2}(C_1) + F_{t2}(C_2) \leq \frac{3}{8}n_1 + \frac{1}{4} + \frac{3}{8}n_2 + \frac{1}{4} = \frac{3}{8}n' + \frac{1}{2}$ .

Now, we have  $F_{t2}(G') \leq \frac{3}{8}n' + \frac{1}{2}$  whether G' is connected or not. Let S' be a minimum semitotal forcing set of G' with  $x_2 \in S'$  and  $w \notin S'$  as in Lemma 3. Let  $G'' = G' - w + y_2 z_2$ . Then S' is also a semitotal forcing set of G''. If  $|S' \cap \{u, v\}| = 0$ , without loss of generality, assume that u becomes black before v. Then,  $S = S' \cup \{a_1, c_1\}$  is a semitotal forcing set of G. This is because if we apply the color-change rule to S in G, then the same forcing chains in G''with respect to S' remain valid in G until u becomes black. Further,  $u \to y_1$ and  $a_1 \rightarrow d_1 \rightarrow b_1 \rightarrow x_1 \rightarrow z_1 \rightarrow v$ . Finally, all remaining white vertices in G will eventually become black using the same forcing chains as in G''. If  $|S' \cap \{u, v\}| = \{1, 2\}$ , without loss of generality, assume  $u \in S'$ . If u is not a forcing vertex during the forcing process with respect to S' in G'', then  $S' \cup \{z_1, c_1\}$  is a semitotal forcing set of G. The same forcing chains in G'' with respect to S'remain valid in G. Finally,  $u \to y_1 \to x_1 \to b_1 \to d_1$ . It follows that all vertices in G become black. Now assume that u is a forcing vertex during the forcing process with respect to S' in G''. If  $u \to v$ , then  $S' \cup \{z_1, c_1\}$  is also a semitotal forcing set of G. This is because  $u \to v$  in G'' will be replaced by two forcing chains in  $G: u \to y_1 \to x_1 \to b_1 \to d_1 \to a_1, z_1 \to v$ . If  $u \to u_1$ , then  $u_2 \in S'$  by Lemma 2, where  $\{u_1, u_2\} = N_G(u) \setminus \{y_1\}$ . And in this time step, v is black. One can easily verify that  $S = (S' \setminus \{u\}) \cup \{y_1, z_1, c_1\}$  is a semitotal forcing set of G. Therefore,  $F_{t2}(G) \leq |S'| + 2 = F_{t2}(G') + 2 \leq \frac{3}{8}n' + \frac{1}{2} + 2 = \frac{3}{8}(n-6) + \frac{1}{2} + 2 = \frac{3}{8}n + \frac{1}{4}$ . This completes the proof.

Combining  $F_{t2}(C_3 \Box K_2) = 3 < \frac{3}{8}n + 1$  with Theorem 5 and Theorem 8, we get the following theorem immediately.

**Theorem 13.** If  $G \neq K_4$  is a connected claw-free cubic graph of order n, then  $F_{t2}(G) \leq \frac{3}{8}n+1$ , with equality if and only if  $G = N_k$  and k is even.

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