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DISJOINT MAXIMAL INDEPENDENT SETS IN GRAPHS AND HYPERGRAPHS

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Abstract

In this paper, we consider the question of the existence of disjoint maximal independent sets (MIS) in graphs and hypergraphs. The question was raised in the 1970's independently by Berge and Payan. They considered the question of characterizing the graphs that admit disjoint MIS, and in particular whether regular graphs do. In this paper, we are interested in the existence of disjoint MIS in a graph or in its complement, motivated by the fact that most constructions of graphs that do not admit disjoint MIS are such that their complement does. We prove that there are disjoint MIS in a graph or its complement whenever the graph has diameter at least three or has chromatic number at most four. We also define a graph of chromatic number 5 and diameter 2 which does not admit disjoint MIS nor its complement.

As our work was first motivated by a more recent work on disjoint MIS in hypergraphs by Acharya (2010), we also consider the question of the existence of disjoint MIS in hypergraphs. We answer a question by Jose and Tuza (2009), proving that there exists balanced k-connected hypergraphs admitting no disjoint MIS.

Keywords: maximal independent set, clique, disjoint sets.

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1. Introduction

We first introduce standard definitions to graphs and hypergraphs that will be used throughout the rest of the paper. Specificities of the definitions for hypergraphs are discussed in Section 3

In a graph or a hypergraph $\mathcal{H} = (V, E)$, the neighborhood $N(x)$ of a vertex x of V is the set of all the vertices that belong to a common edge with x. More generally, the neighborhood of a subset $S \subseteq V$ of vertices is defined as $N(S) = \bigcup_{x \in S} N(x)$. A subset S of V is an independent set of $\mathcal{H}$ if no two vertices of S belong to a same edge. An independent set S is maximal (inclusion-wise) if there exist no superset $S' \supsetneq S$ that is also independent. Observe that a maximal independent set of $\mathcal{H}$ is also a dominating set of $\mathcal{H}$, i.e., every vertex in $\mathcal{H}$ has a neighbor in the set S. Maximal independent sets are also known as independent dominating sets, which have been extensively studied in the literature, we refer to the survey by Goddard and Henning [5].

In this paper, we are interested in the existence of disjoint maximal independent sets (disjoint MIS) in graphs and hypergraphs. Observe first that all graphs do not contain disjoint MIS. An example of a graph with no disjoint MIS can be made by adding an isolated vertex to any connected graph. A connected example is depicted in Figure 1: any MIS of G_2 contains at least three among the four degree 1 vertices, so any two MIS intersect in at least two degree 1 vertices.

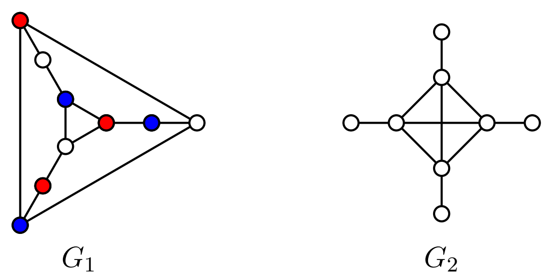

Figure 1. Red and blue vertices form disjoint MIS of G_1. The graph G_2 does not admit disjoint MIS.

Another known infinite family of graphs admitting no disjoint MIS is formed by the coronas of odd cycles. Those graphs are obtained by adding a degree one vertex adjacent to every vertex of an odd cycle (see Figure 2). In these graphs, any maximal independent set contains at most half the vertices of the odd cycle, and thus more than half the degree one vertices. Therefore any two MIS intersect in at least one of the degree one vertices.

Based on this observation, Schaudt [11] gave a polynomial time algorithm that either computes disjoint MIS in a graph or returns an induced corona of an odd cycle. Note however that having no induced corona of a cycle is not a

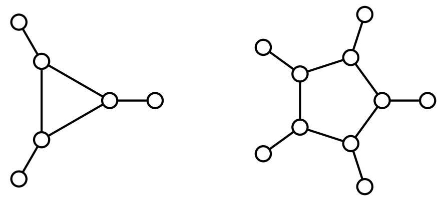

Figure 2. Coronas of odd cycle have no disjoint MIS.

necessary condition to have disjoint MIS. For example, the graph G_1 in Figure 1 does contain an induced corona of a 3-cycle, as well as disjoint MIS. Actually, deciding whether a graph admits disjoint MIS is an NP-complete problem, as shown by Henning *et al.* in [6].

Berge proposed in conferences (as mentioned in [2, 3]) the following conjecture, which was independently proposed by Payan in [9].

Conjecture 1 (Berge and Payan, 1970's)**.** *Every nonempty regular simple graph contains disjoint* MIS*.*

Though, Payan [10] has found a family of counter-examples to the above conjecture, the smallest graph of which contains 630 vertices, all of degree 280.

However, the conjecture was proven to be true for $(n-k)$-regular graphs where $1 \leq k \leq 7$ (see [3]), $k \leq 10$ (see [9]) and for $k < 2\sqrt{2n}-2$ (see [4]). Morever Payan proved also the conjecture in particular for claw-free regular graphs [8].

2. About Disjoint MIS or Disjoint Maximal Cliques

We consider here the question of the existence of disjoint MIS in a graph or in its complement, or equivalently the question of a graph admitting disjoint MIS or disjoint maximal cliques (i.e., disjoint maximal complete subgraphs). Observe that the simple examples of graphs not admitting disjoint MIS shown above all admit disjoint maximal cliques. First, if a nontrivial graph contains an isolate vertex v, then v itself is a maximal complete subgraph disjoint from any other clique in the graph. In the family of coronas of odd cycles, there are many maximal cliques formed by a vertex of degree one with its single neighbor, and all are disjoint. Similarly, we do not detail Payan's construction here, but it does contain disjoint maximal cliques.

The study of existence of disjoint MIS in a graph and in its complement was already initiated by Cockayne and Hedetniemi in [3], though with a different perspective, since Conjecture 1 was not disproved yet. They conjectured that if a graph admits disjoint MIS, then its complement does too, which is disproved by the examples mentioned above.

This section is thus motivated by the following question, which we explore further.

Problem 2. Let G be a nontrivial graph. Is it true that either G or $\overline{G}$ admits disjoint MIS? Equivalently, is it true that either G admits disjoint MIS, or G admits disjoint maximal cliques?

2.1. About the diameter

One first direction to consider the question is in regard of the diameter of the graph. Note that the only graphs of diameter one are complete graphs, where each vertex is a MIS in the graph. Thus nontrivial graphs with diameter one admit disjoint MIS. For larger diameters, we get the following:

Theorem 3. *If G is a graph of diameter at least 3, then G contains disjoint maximal cliques.*

Proof. Let G be a graph of diameter at least 3. Consider two vertices u and v at distance at least three in G, and two maximal cliques, one containing u and the other containing v. Since u and v are at distance more than two, they have no common neighbors, and the two cliques may not share a vertex. They are thus disjoint. ■

Observe in particular that the above theorem covers the (easy) case when the graph is not connected.

Corollary 4. *If a graph admits neither disjoint MIS, nor disjoint maximal cliques, it must be of diameter two.*

2.2. About the chromatic number

Our next result uses the chromatic number of the graph as a parameter. Recall that a proper coloring of a graph G is an assignment of colors to the vertices so that no adjacent vertices receive the same color. The chromatic number $\chi(G)$ of the graph G is the minimum number of colors in a proper coloring of G. We consider the following greedy algorithm for producing a proper coloring of a graph: take the vertices in some order, and assign to each vertex the least color not used among its already colored neighbors.

Observation 5. *For every graph, there exists an ordering of the vertices so that the greedy algorithm produces an optimal coloring.*

Proof. Let G be a graph with chromatic number k, and c be an optimal coloring of G, with colors 1 to k. Consider any order of the vertices which begins with all vertices receiving color 1 in c, then all vertices receiving color 2, and so on until color k.

Let c' be the coloring of G obtained by applying the above greedy algorithm with this ordering. Note that for a vertex v, the color $c'(v)$ may differ from $c(v)$ (for example if $c(v) = 2$ but v has no neighbor colored 1). However, we can ensure inductively that for all v, $c'(v) \leq c(v)$: by induction, all the already colored neighbors of a vertex v received colors less than $c(v)$, so the least available color is at most $c(v)$. Therefore, the coloring obtained by the greedy algorithm from that ordering uses at most k colors. ■

We proceed with a series of lemmas that will allow us to show that any graph with chromatic number at most 4 admits disjoint MIS or disjoint cliques.

Lemma 6. *If G is a connected graph with chromatic number $\chi(G) = 2$, then G contains disjoint* MIS.

Proof. Consider a two-coloring of G, and denote by V_1 and V_2 the set of vertices colored 1 and 2, respectively (in other words, V_1 and V_2 are the partite sets of the bipartite graph G). Obviously, V_1 and V_2 are independent sets. Since the graph is connected, every vertex colored 2 has a neighbor colored 1, and reciprocally. So V_1 and V_2 are also maximal, they thus are disjoint MIS. ■

Lemma 7. *If G is a connected graph with chromatic number $\chi(G) = 3$, then G contains disjoint* MIS *or disjoint maximal cliques.*

Proof. Consider an ordering of the vertices so that the greedy coloring of G uses three colors, and denote V_1, V_2 and V_3 the sets of vertices attributed the corresponding colors by the greedy algorithm. Note that by definition, every vertex in V_2 or V_3 has a neighbor in V_1, so V_1 is necessarily a MIS. Suppose first that every vertex in V_1 is adjacent to at least one vertex in V_2. Then V_2 is a dominating set in G, and thus V_1 and V_2 are disjoint MIS, as required. Suppose next that every vertex in V_1 is adjacent to at least one vertex in V_3. Then V_3 dominates V_1, and it may be completed into a MIS S using only vertices from V_2. Then V_1 and S are disjoint MIS, as required.

We now assume that both the above assumptions are false, i.e., there exists a vertex u in V_1 that has no neighbor in V_2, and a vertex w in V_1 that has no neighbor in V_3.

Since neither u nor w are isolated, there exists a vertex $u' \in V_3$ adjacent to u and a vertex $w' \in V_2$ adjacent to w. Since u has no neighbor in $V_1 \cup V_2$ and u' has no neighbor in V_3, the complete subgraph $\{u, u'\}$ is maximal.

Similarly, since w has no neighbor in $V_1 \cup V_3$ and w' has no neighbor in V_2, the set $\{w, w'\}$ is a maximal clique. Therefore, the sets $\{u, u'\}$ and $\{w, w'\}$ form disjoint maximal cliques. ∎

Lemma 8. *If G is a connected graph with chromatic number $\chi(G) = 4$, then G contains disjoint* MIS *or disjoint maximal cliques.*

Proof. Consider an ordering of the vertices so that the greedy coloring of G uses four colors, and denote V_1, V_2, V_3 and V_4 the sets of vertices attributed the corresponding colors by the greedy algorithm.

Similarly as in the proof of the previous lemma, if there exists $t \in \{2, 3, 4\}$, such that every vertex of V_1 has a neighbor in V_t, then V_1 and a MIS containing V_t form disjoint MIS. We thus assume that for every $t \in \{2, 3, 4\}$, there exists a vertex of V_1 that has no neighbor in V_t.

Suppose some vertex $u \in V_1$ has no neighbor in two other partite sets (e.g. V_2 and V_3), and denote by V_t the only partite set where u has neighbors (in the example, V_4). Since u is not isolated in G, u necessarily has a neighbor u' in V_t. Since u has neighbors only in V_t, and V_t is an independent set, $\{u, u'\}$ is a maximal clique in G. Let $w \in V_1$ denote a vertex with no neighbor in V_t. Then any maximal clique C_w containing w contains neither u nor u', and so C_w and $\{u, u'\}$ are disjoint maximal cliques.

Now suppose that neither of the above conditions hold, and thus that there are three distinct vertices u, v and w in V_1 that do not have neighbors in V_2, V_3 and V_4 respectively. Let C_u, C_v and C_w denote three maximal cliques containing u, v and w respectively. Note that by our assumption, each of C_u, C_v and C_w contain at most three vertices. If any two such cliques are disjoint, the graph G contains disjoint maximal cliques, so let us assume they pairwise intersect.

Since u has no neighbor in V_2 and v has no neighbor in V_3, C_u and C_v may only intersect in some vertex from V_4, say w'. Similarly, C_u and C_w must intersect in some vertex v' in V_3, and C_v and C_w intersect in some vertex u' in V_2. A picture of the current identified subgraph is provided in Figure 3.

Note that since u has no neighbor in V_2, v has no neighbor in V_3 and w has no neighbor in V_4, we necessarily have $C_u = \{u, v', w'\}$, $C_v = \{u', v, w'\}$ and $C_w = \{u', v', w\}$. Moreover, $\{u, u'\}$, $\{v, v'\}$ and $\{w, w'\}$ are independent sets. Let I_u be a maximal independent set containing both u and u', and I_v a maximal independent set containing both v and v'. If I_u and I_v are disjoint, the lemma holds, thus assume by way of contradiction that there exist a vertex z in the intersection $I_u \cap I_v$, and let C_z be a maximal clique containing z.

If C_z does not contain w', then since u and v' are not incident to z, C_z and C_u are disjoint, and the lemma holds. Thus assume C_z contains w'. Then necessarily, $w \notin C_z$. But then since the vertices u' and v' are not adjacent to z either, C_w and C_z are disjoint, and the lemma holds. ∎

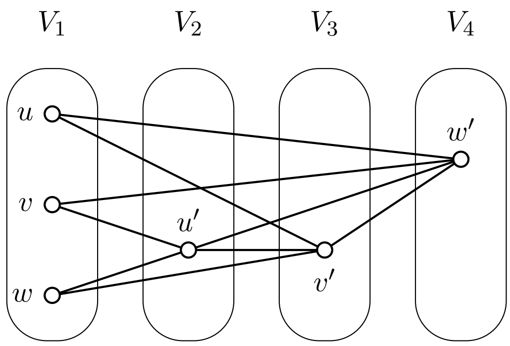

Figure 3. Situation in the proof of Lemma 8.

As a corollary, we get the following result.

Theorem 9. *If G is a connected graph with chromatic number $\chi(G) \leq 4$, then G contains disjoint* MIS *or disjoint cliques.*

So we now can infer that the only graphs that may have neither disjoint MIS nor disjoint maximal cliques have diameter two and chromatic number at least 5. Actually, we found a family of split graphs that have this property, which we describe now.

Let K_p be a complete graph with vertex set $\{x_1, x_2, \ldots, x_p\}$, where $p > 4$. G_p is a graph obtained from K_p by connecting any vertex z_{ij} of $Z = \{z_{ij} \mid i, j \in \{1, 2, \ldots, p\}\ i < j\}$ to any vertex x_r of K_p, where $r \notin \{i, j\}$. The graph G_p has $p + \binom{p}{2}$ vertices, diameter 2 and chromatic number p (see Figure 4 for $p = 5$).

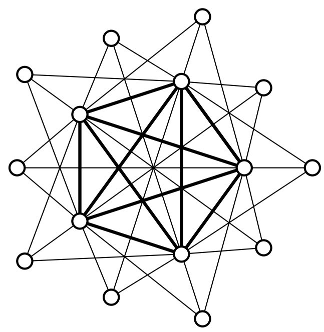

Figure 4. The graph G_5 which has neither disjoint MIS nor disjoint maximal cliques.

Proposition 10. *For $p > 4$, G_p has neither disjoint* MIS *nor disjoint maximal cliques.*

Proof. In the graph G_p, a maximal independent set may contain at most one vertex of the complete subgraph K_p. It is thus either the set Z, or a set $S_i =$

$\{x_i\} \cup \{z_{kl} \mid i \in \{k, l\}\}$ for some $1 \le i \le p$. No two of these independent sets are disjoint for $p \ge 4$. In particular, S_i and S_j intersect in $z_{i,j}$. Thus the graph G_p admits no disjoint MIS.

On the other hand, a maximal clique in G_p contains at most one vertex in Z, and thus is either K_p or a clique C_{ij} of the form $\{z_{ij}\} \cup \{x_k \mid k \notin \{i, j\}\}$ for $1 \le i < j \le p$. Any two such cliques intersect in at least $p - 4$ vertices, and thus no two are disjoint for $p \ge 4$. This concludes the proof. ∎

This implies that both above results are best possible in the way they are defined. However, the question of characterization of graphs with no disjoint MIS nor disjoint maximal cliques remains open.

3. Hypergraphs

The study of disjoint MIS in hypergraphs was started by Acharya in [1] where he asks for a characterization of hypergraphs having disjoint MIS [1, Problem 2]. Recall that we here use for definition of an independent set in a hypergraph a set where no two vertices belong to the same edge. With this definition, a set in a hypergraph is independent if and only if it corresponds to an independent set in the graph made from the hypergraph by replacing every edge by a complete subgraph. Though, we are interested in a conjecture from Jose and Tuza [7] about balanced hypergraphs, for which the equivalence may not be used, for the definition of balanced hypergraphs as given next requires edges of size more than 2.

A cycle in a hypergraph is an alternating sequence of vertices and edges $(x_1, E_1, x_2, E_2, \ldots, x_k, E_k)$, with all edges and vertices distinct, such that $x_i, x_{i+1} \in E_i$ for all $1 \le i < k$, with the special case of E_k containing both x_k and x_1. A hypergraph is balanced if every odd cycle uses an edge containing at least three vertices in the cycle (see Figure 5 for an example of balanced odd cycle with seven hyperedges).

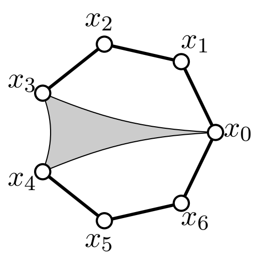

Figure 5. A balanced odd cycle in a hypergraph.

As for graphs, the existence of disjoint MIS in hypergraphs is not verified in general. Acharya conjectured that every balanced hypergraph admits disjoint MIS. Jose and Tuza disproved that conjecture [7], and suggested that the question might still be relevant for k-connected balanced hypergraphs [7, Problem 4], that is hypergraphs where there are no subset of $k-1$ vertices whose removal would disconnect the hypergraph. Here, we answer to this question by the negative proposing for any k an infinite family of balanced k-connected hypergraphs $\mathcal{H}_{k,t}$ that does not have disjoint MIS.

The hypergraphs $\mathcal{H}_{k,t} = (V, \mathcal{E})$ are constructed as follows: let $V = \{u_{i,j} \mid 1 \leq i \leq k+1, 1 \leq j \leq t\}$ and $\mathcal{E} = \{X_i \mid 1 \leq i \leq k+1, i \neq 2\} \cup \{Y_j, Z_j \mid 1 \leq j \leq t\}$ where $X_i = \{u_{i,1}, \ldots, u_{i,t}\}$, $Y_j = \{u_{2,j}, \ldots, u_{k+1,j}\}$ and $Z_j = \{u_{1,j}, u_{2,j}\}$. Figure 6 illustrates that construction.

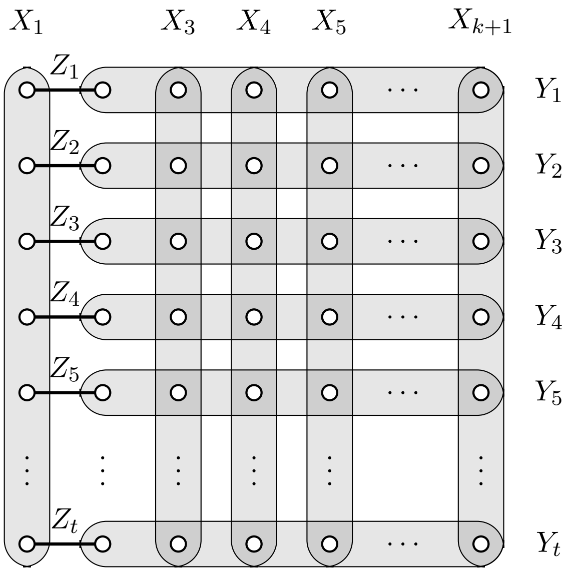

Figure 6. The hypergraph $\mathcal{H}_{k,t}$.

Proposition 11. *Let $k \in \mathbb{N}, t \geq 2k+1$. The hypergraph $\mathcal{H}_{k,t}$ is balanced, k-connected, and does not have disjoint* MIS.

Proof. We first prove that $\mathcal{H}_{k,t}$ is balanced. Note that any vertex of $\mathcal{H}_{k,t}$ belongs to precisely two edges, one in $E_1 = \{Y_j \mid 1 \leq j \leq t\} \cup \{X_1\}$ and one in $E_2 = \{Z_j \mid 1 \leq j \leq t\} \cup \{X_i \mid 3 \leq i \leq k+1\}$. Therefore, any cycle C of $\mathcal{H}_{k,t}$ uses alternating edges among the two sets E_1 and E_2. Thus $|C \cap E_1|$ is equal to $|C \cap E_2|$ and C is an even cycle.

We now prove that the hypergraph $\mathcal{H}_{k,t}$ is k-connected, by giving at least k vertex disjoint walks between any two non-adjacent vertices $u_{i,j}$ and $u_{i',j'}$ $(1 \leq$

$i \leq i' \leq k$, $1 \leq j, j' \leq t$). Note first that if $i = i'$, then both are equal to 2 (or the corresponding vertices would be adjacent). For each case, we propose the (at least k) following vertex disjoint walks:

- if $i = i' = 2$, $\big\{(u_{2,j}, u_{\alpha,j}, u_{\alpha,j'}, u_{2,j'}), 1 \leq \alpha \leq k+1, \alpha \neq 2\big\}$,
- if $i \neq 2$ and $i' \neq 2$, $\big\{(u_{i,j}, u_{i,\beta}, u_{i',\beta}, u_{i',j'}), 1 \leq \beta \leq t\big\}$,
- if $i = 2, i' \geq 3$, $\big\{(u_{i,j}, u_{\alpha,j}, u_{\alpha,j'}, [u_{2,j'},]u_{i',j'}), 1 \leq \alpha \leq k+1, \alpha \neq 2\big\}$,
- if $i = 1, i' = 2$, $\big\{(u_{1,j}, u_{1,\alpha}, u_{2,\alpha}, u_{\alpha,\alpha}, u_{\alpha,j'}, u_{2,j'}) \mid 1 \leq \alpha \leq k+1, \alpha \neq 2\big\}$.

Let us finally show that $\mathcal{H}_{k,t}$ has no disjoint MIS. We denote by U_2 the set of vertices $\{u_{2,j} \mid 1 \leq j \leq t\}$. Let S be a maximal independent set of $\mathcal{H}_{k,t}$. Note first that $S \setminus U_2$ contains at most k vertices, or two would be in a same edge X_i. For any j such that $\{u_{i,j} \mid 1 \leq i \leq k+1\} \cap (S \setminus U_2) = \emptyset$, by maximality of S, $u_{2,j} \in S$. Therefore, S contains at least $t-k$ vertices in U_2. Now, since $t \geq 2k+1$, any two maximal independent sets intersect in some vertex $u_{2,j}$, and $\mathcal{H}_{k,t}$ contains no disjoint MIS. ■

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