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CHROMATIC RAMSEY NUMBERS OF GENERALISED MYCIELSKI GRAPHS

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Abstract

We revisit the Burr-Erdős-Lovász conjecture on chromatic Ramsey numbers. We show that it admits a proof based on the Lovász ϑ parameter in addition to the proof of Xuding Zhu based on the fractional chromatic number. However, there are no proofs based on topological lower bounds on chromatic numbers, because the chromatic Ramsey numbers of generalised Mycielski graphs are too large. We show that the 4-chromatic generalised Mycielski graphs other than K_4 all have chromatic Ramsey number 14, and that the n-chromatic generalised Mycielski graphs all have chromatic Ramsey number at least $2^{n/4}$.

Keywords: chromatic Ramsey numbers, fractional chromatic numbers, Lovász ϑ parameter, box complexes, generalised Mycielski graphs.

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1. Introduction

The chromatic Ramsey number $R_\chi(G)$ of a graph G is the least integer n such that if the edges of the complete graph K_n are coloured in red and blue, then G admits a homomorphism (that is, an edge-preserving map) to a subgraph of K_n spanned by edges of the same colour. The concept of chromatic Ramsey numbers is due to Burr, Erdős and Lovász [1], though our presentation is a characterisation rather than the original definition.

The standard Ramsey number $R(G)$ of G is the least n such that any red-blue colouring of the edges of K_n contains a monochromatic *copy* of G. In particular this implies $R(G) \geq |V(G)|$, while we can have $R_\chi(G) < |V(G)|$ since homomorphisms can identify nonadjacent vertices. The definition of Ramsey numbers extends to families of graphs: $R(\mathcal{F})$ is the least n such that any red-blue colouring of K_n contains a monochromatic copy of a graph in $\mathcal{F}$. If we take $\mathcal{H}(G)$ to be the family of homomorphic images of G (obtained by identifying independent sets of G to single vertices), we get $R_\chi(G) = R(\mathcal{H}(G))$. This is our presentation above, as formulated in [1].

The original definition of R_χ uses the notation $H \to (G, G)$, which means that when the edges of H are coloured in red and blue, there is a monochromatic copy of G. Thus

$$R(G) = \min\{n \mid K_n \to (G, G)\} = \min\{|V(H)| \mid H \to (G, G)\}.$$

Burr, Erdős and Lovász considered optimising a graph parameter $\iota(H)$ other than the number of its vertices. In this way, we can define

$$R_\iota(G) = \min\{\iota(H) \mid H \to (G, G)\},$$

and in particular

$$R_\chi(G) = \min\{\chi(H) \mid H \to (G, G)\}.$$

This is indeed the first definition of the chromatic Ramsey number, and Theorem 1 of [1] proves its equivalence with our definition given above.

The chromatic Ramsey number is homomorphism-monotone. That is, if G admits a homomorphism to H, then $R_\chi(G) \leq R_\chi(H)$. In particular, if $\chi(G) = n$, then $R_\chi(G) \leq R_\chi(K_n) = R(K_n) \equiv R(n)$, the n-th Ramsey number. Burr, Erdős and Lovász [1] also proved the lower bound $R_\chi(G) \geq (\chi(G) - 1)^2 + 1$. They conjectured that this bound was best possible in the following sense.

Conjecture 1 (The Burr-Erdős-Lovász conjecture [1]). *For every integer $n \geq 1$, there exists a graph G such that $\chi(G) = n$ and $R_\chi(G) = (n-1)^2 + 1$.*

A tentative proof was sketched, based on what turned out to be an independent formulation of the conjecture of Hedetniemi [5] on the chromatic number of a categorical product of graphs. In recent years, Shitov [9] has refuted Hedetniemi's conjecture. This implies that any proof of the Burr-Erdős-Lovász conjecture along the suggested lines has to have a "flavour" involving some well-behaved lower bound on the chromatic number.

In particular, the "fractional version" of Hedetniemi's conjecture was proved some time ago by Zhu [17], and this was sufficient to prove the Burr-Erdős-Lovász conjecture. We review the argument in the next section, and show that it has a parallel "vectorial version" based on the Lovász ϑ-parameter of the complement of a graph.

This opens the question as to whether other proofs of the Burr-Erdős-Lovász conjecture could be obtained through the topological lower bounds on the chromatic number and topological versions of Hedetniemi's conjecture. This is where the "generalised Mycielski graphs" enter the picture. These are defined in Section 3, and their connection with a topological lower bound on the chromatic number is explained. We compute the chromatic Ramsey number of every generalised Mycielski graph with chromatic number at most 4. In general, chromatic Ramsey numbers of generalised Mycielski graphs are too high to even come close to a topological proof of the Burr-Erdős-Lovász conjecture. However, their determination may be an interesting problem in its own right.

2. PROOFS OF THE BURR-ERDŐS-LOVÁSZ CONJECTURE

2.1. The original argument

The value $(n-1)^2+1$ in Conjecture 1 is Turan's "beautiful lower bound" on the Ramsey number $R(n)$, discovered while facing the threat of deportation during World War II. The vertices of $K_{(n-1)^2}$ can be arranged in a square, and then the horizontal edges can be coloured red, and all other edges blue. The resulting colouring has no monochromatic copy of K_n, which proves that $R(n)$ is at least $(n-1)^2+1^\dagger$. More significantly, both red and blue edges span graphs that are $(n-1)$-colourable. Therefore any graph G with $R_\chi(G) \leq (n-1)^2$ must be $(n-1)$-colourable.

However, when the edges of $K_{(n-1)^2+1}$ are coloured in red and blue, one of the two classes of edges spans a graph with chromatic number at least n. Let $\mathcal{C}$ be the set of all red-blue colourings of the edges of $K_{(n-1)^2+1}$. For each $c \in \mathcal{C}$, let G_c be a monochromatic subgraph with chromatic number at least n. The categorical product $\Pi_{c\in\mathcal{C}} G_c$ is defined by

$$\begin{aligned} V(\Pi_{c\in\mathcal{C}} G_c) &= \Pi_{c\in\mathcal{C}} V(G_c), \\ E(\Pi_{c\in\mathcal{C}} G_c) &= \{\{u,v\} \mid \{\pi_c(u), \pi_c(v)\} \in E(G_c) \text{ for all } c \in \mathcal{C}\}, \end{aligned}$$

where π_c is the projection on $V(G_c)$. (We write $G_1 \times G_2$ for $\Pi_{i\in\{1,2\}} G_i$.) By construction, for each red-blue colouring c of $E(K_{(n-1)^2+1})$, π_c is a homomorphism from $\Pi_{i\in\mathcal{C}} G_i$ to G_c. Thus $R_\chi(\Pi_{c\in\mathcal{C}} G_c) \leq (n-1)^2+1$, and if $\chi(\Pi_{c\in\mathcal{C}} G_c) \geq n$, then $R_\chi(\Pi_{c\in\mathcal{C}} G_c) > (n-1)^2$, so that $\Pi_{c\in\mathcal{C}} G_c$ witnesses the validity of Conjecture 1.

The remaining question is why the bound $\chi(\Pi_{c\in\mathcal{C}} G_c) \geq n$ should hold. Burr, Erdős, and Lovász thought it would follow from a seemingly natural general identity that they could not prove.

Conjecture 2 (Conjecture 2 of [1]). $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

†At that point Turan believed $R(n) = (n-1)^2+1$; in [15] he writes "I had no other support for the truth of this conjecture than the symmetry and some dim feeling of beauty; perhaps the ugly reality was what made me believe in the strong connection of beauty and truth".

This was the first appearance in a journal article of a conjecture that had been formulated ten years earlier by Hedetniemi in the technical report [5]. The conjecture gained popularity in the eighties, with strong partial results being proved while the general case remained seemingly intractable. Early references attribute the conjecture to Burr, Erdős and Lovász [1]. Duffus, Sands and Woodrow [2] called it "Hedetniemi's conjecture", the name under which it became widely known. The conjecture was finally refuted in 2019 by Shitov [9].

However, along the way, it had been noticed that Hedetniemi's conjecture is much stronger than what is needed to prove the Burr-Erdős-Lovász Conjecture 1. As we will see below, variants of Hedetniemi's conjecture for suitable lower bounds on the chromatic number can be used.

2.2. The fractional chromatic number

Recall our assertion that for any red-blue colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph G_c such that $\chi(G_c) \geq n$. More precisely, the following holds.

(i) If there is a red copy of K_n, we can select G_c as this copy.

(ii) Otherwise, the blue edges span a graph G_c with $(n-1)^2+1$ vertices and indepencence number at most $n-1$.

Thus we have the following.

Remark 3. For any red-blue colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph G_c such that $\chi(G_c) \geq |V(G_c)|/\alpha(G_c) > n-1$.

For any graph H that admits a homomorphism to G, the value $|V(H)|/\alpha(H)$ is a lower bound on $\chi(G)$. For a fixed G, the maximum of all such lower bounds $|V(H)|/\alpha(H)$ is the *fractional chromatic number* $\chi_{\mathrm{f}}(G)$ of G. Again, this is a characterisation rather than the standard definition. Usually, $\chi_{\mathrm{f}}(G)$ is defined as the smallest possible sum of nonnegative weights given to the independent sets of G such that for each vertex of G, the sum of weights of independent sets containing it is at least 1 (see [17]). This formulation allows to use standard linear programming duality. With this, Zhu proved the fractional version of Hedetniemi's conjecture.

Theorem 4 [17]**.** *For any graphs G, H, $\chi_{\mathrm{f}}(G \times H) = \min\{\chi_{\mathrm{f}}(G), \chi_{\mathrm{f}}(H)\}$.*

Applying this result to the product of graphs G_c as selected above completes the first proof of the Burr-Erdős-Lovász conjecture

$$\begin{aligned}\chi(\Pi_{c\in\mathcal{C}} G_c) \;&\geq\; \chi_{\mathrm{f}}(\Pi_{c\in\mathcal{C}} G_c) = \min\{\chi_{\mathrm{f}}(G_c) \mid c \in \mathcal{C}\} \\ &\geq \min\left\{\frac{|V(G_c)|}{\alpha(G_c)} \mid c \in \mathcal{C}\right\} > n-1.\end{aligned}$$

2.3. The Lovász ϑ function

Let $\overline{G}$ denote the complement of a graph G. If f and g are proper vertex-colourings of G and $\overline{G}$ respectively, then (f, g) is a proper vertex-colouring of the complete graph with $V(G)$ as vertex-set. Therefore $\chi(G) \cdot \chi(\overline{G}) \geq |V(G)|$. In particular, if G and $\overline{G}$ are spanned, respectively, by the red and blue edges in a red-blue colouring of the edges of $K_{(n-1)^2+1}$, then $\chi(G) \cdot \chi(\overline{G}) \geq (n-1)^2 + 1$, hence $\max\{\chi(G), \chi(\overline{G})\} > n-1$. This is an alternate argument showing that one of the two classes of edges spans a graph with chromatic number at least n.

More generally, if ι is any graph invariant satisfying $\iota(G)\cdot\iota(\overline{G}) \geq |V(G)|$, then for any red-blue colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph G_c such that $\iota(G_c) > n-1$. One such invariant is the classical Lovász ϑ function. The inequality $\vartheta(G) \cdot \vartheta(\overline{G}) \geq |V(G)|$ is one of its many well-known properties. However, $\vartheta(G)$ is not a lower bound for $\chi(G)$, but rather for $\chi(\overline{G})$. In other words, $\vartheta(\overline{G}) \leq \chi(G)$. In [3], Godsil, Roberson, Šámal and Severini use the notation $\overline{\vartheta}(G)$ to denote $\vartheta(\overline{G})$, and call $\overline{\vartheta}(G)$ the *strict vector chromatic number* of G. They prove the following.

Theorem 5 [3]**.** *For any graphs G, H, $\overline{\vartheta}(G \times H) = \min\{\overline{\vartheta}(G), \overline{\vartheta}(H)\}$.*

This is the final ingredient needed for a second proof of the Burr-Erdős-Lovász conjecture. For any red-blue colouring c of the edges of $K_{(n-1)^2+1}$, one the two classes of edges spans a graph G_c such that $\overline{\vartheta}(G_c) > n-1$. We then have

$$\chi(\Pi_{c\in\mathcal{C}} G_c) \quad \geq \quad \overline{\vartheta}(\Pi_{c\in\mathcal{C}} G_c) = \min\{\overline{\vartheta}(G_c) \mid c \in \mathcal{C}\} > n-1.$$

Note that this proof only uses some properties of the strict vector chromatic number, rather than its explicit definition. We now proceed with definitions of the vector chromatic number and the strict vector chromatic number.

Let $\mathcal{S}^d$ denote the unit sphere in the vector space $\mathbb{R}^{d+1}$. A map $\phi : V(G) \to \mathcal{S}^d$ is called a *vector k-colouring* if whenever $\{u, v\}$ is an edge of G, we have

$$\phi(u)^T\phi(v) \leq \frac{-1}{k-1}. \tag{1}$$

The *vector chromatic number* $\chi_{\text{vec}}(G)$ of G is the infimum of real numbers $k \in (1, \infty)$ such that G admits a vector k-colouring. When the inequality in (1) is replaced by an equality, the vector k-colouring ϕ is called *strict*. Thus, the *strict vector chromatic number* $\overline{\vartheta}(G)$ of G is the infimum of real numbers $k \in (1, \infty)$ such that G admits a strict vector k-colouring.

By definition, we have $\chi_{\text{vec}}(G) \leq \overline{\vartheta}(G)$. Both χ_{vec} and $\overline{\vartheta}$ are clearly homomorphism-monotone, and it can be shown that $\chi_{\text{vec}}(K_n) = \overline{\vartheta}(K_n) = n$, thus both parameters are lower bounds on the chromatic number (see [3]). Similar semidefinite programming duality methods are available to deal with the vector chromatic number and the strict vector chromatic number, but the Hedetniemi-type identity for the vector chromatic number was proved just recently.

Theorem 6 [4]**.** *For any graphs G, H, $\chi_{\text{vec}}(G \times H) = \min\{\chi_{\text{vec}}(G), \chi_{\text{vec}}(H)\}$.*

However, there is no proof of the Burr-Erdős-Lovász conjecture based on the vector chromatic number, because the inequality $\chi_{\text{vec}}(G) \cdot \chi_{\text{vec}}(\overline{G}) \geq |V(G)|$ fails in general. For instance, let G be the Johnson graph $J(6,3)$. That is, the vertices of G are the twenty subsets of $\{1,2,3,4,5,6\}$ of cardinality 3, and two of these are joined by an edge if they intersect in a set of cardinality 2. Then G is a 9-regular graph which is both vertex-transitive and edge-transitive. By Lemma 5.2 of [3], we then have

$$\chi_{\text{vec}}(G) = \overline{\vartheta}(G) = 1 - \tfrac{9}{\tau},$$

where τ is the least eigenvalue of G. The spectrum of a Johnson graph is well-known; in particular $\tau = -3$ thus $\chi_{\text{vec}}(G) = 4$.

Now the edges of $\overline{G}$ join pairs $\{A, B\}$ of 3-sets such that $|A \cap B| \leq 1$. A vector 4-colouring ϕ of $\overline{G}$ can be defined by

$$\phi(A) = \tfrac{1}{\sqrt{6}} \cdot (2 \cdot \mathbb{1}_A - \mathbb{1}),$$

where $\mathbb{1}_A \in \mathbb{R}^6$ is the characteristic vector of A, and $\mathbb{1}$ is the all ones vector. Thus $\chi_{\text{vec}}(\overline{G}) \leq 4$, and we then have $\chi_{\text{vec}}(G) \cdot \chi_{\text{vec}}(\overline{G}) \leq 16 < 20 = |V(G)|$. This shows that the edges of K_{20} can be split in two parts which each spans a graph with vector chromatic number at most 4, even though one part is guaranteed to span a graph with chromatic number 5, since $20 \geq (5-1)^2 + 1$.

2.4. Summary

Proving the Burr-Erdős-Lovász conjecture amounts to finding lower bounds ι on the chromatic number satisfying the two properties

(i) Whenever the edges of $K_{(n-1)^2+1}$ are coloured in red and blue, one colour class spans a graph G such that $\iota(G) > n-1$,

(ii) $\iota(G \times H) = \min\{\iota(G), \iota(H)\}$.

A long time ago, Turan briefly thought that the clique number satisfied the first property, but the inequality $R(n) \gg (n-1)^2 + 1$ is now very well known. Later, Burr, Erdős and Lovász conjectured that the chromatic number itself satisfied the second property, but this "Hedetniemi's conjecture" has now been disproved. With time, interesting bounds such as the fractional chromatic number and the strict vector chromatic number have been well studied, and both of these satisfy properties (i) and (ii). The case of the vector chromatic number is interesting. Even though it satisfies property (ii), the following remains unknown.

Problem 7. For a given n, what is the least value of m such that when the edges of K_m are coloured in red and blue, there is a monochromatic subgraph G such that $\chi_{\mathrm{vec}}(G) > n - 1$?

The only known upper bound for such m is the Ramsey number $R(n)$. Finding a smaller upper bound may be of independent interest.

In another direction, some "topological" lower bounds on the chromatic number have been developed. Through "generalised Mycielski graphs", we present some of them in the next section, and show that they cannot be used for other proofs of the Burr-Erdős-Lovász conjecture.

3. GENERALISED MYCIELSKI GRAPHS

For $k \geq 1$, let $\mathbb{P}_k$ denote the path with vertices $0, 1, \ldots, k$ linked consecutively, with a loop at 0. For a graph G, the *k-th generalised Mycielskian* $M_k(G)$ of G is defined by

$$M_k(G) = (G \times \mathbb{P}_k) / \sim_k,$$

where $\sim_k$ is the equivalence which identifies all vertices whose second coordinate is k. Figure 1 shows the graphs $M_1(C_5)$, $M_2(C_5)$ and $M_3(C_5)$, where C_5 is the 5-cycle. In general, $M_1(G)$ is the graph obtained from G by adding a new vertex adjacent to all vertices of G, while $M_2(G)$ is the standard Mycielskian over G. It is well known that $\chi(M_1(G)) = \chi(M_2(G)) = \chi(G) + 1$. However, there are graphs G for which $\chi(M_3(G)) = \chi(G)$.

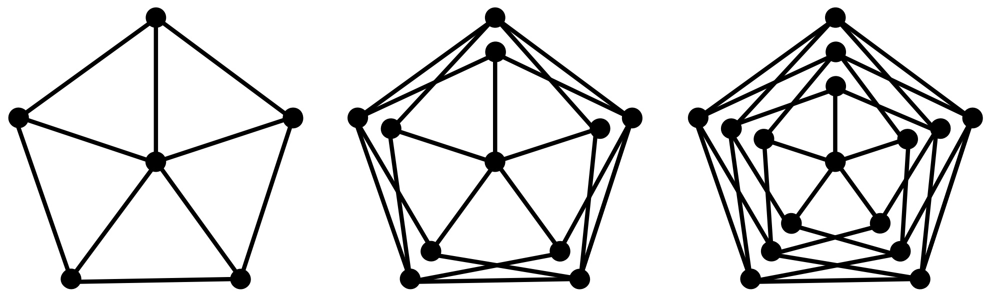

Figure 1. Generalised Mycielskians of C_5.

The classes $\mathcal{K}_n$ of *generalised Mycielski graphs* are defined recursively as follows: $\mathcal{K}_2 = \{K_2\}$, and for $n \geq 3$,

$$\mathcal{K}_n = \{M_k(G) : G \in \mathcal{K}_{n-1}, k \geq 1\}.$$

Chromatic numbers of generalised Mycielski graphs were determined by Stiebitz.

Theorem 8 [14]**.** *For every $G \in \mathcal{K}_n$, $\chi(G) = n$.*

Stiebitz proved this result using the neighbourhood complex introduced by Lovász [6] to prove Kneser's conjecture. However, as the field of topological bounds on chromatic numbers evolved, other simplicial complexes were considered. In particular, some bounds are derived from the so-called "box complex" $B(G)$ of a graph G.

$$\omega(G) \leq \operatorname{coind}(B(G)) + 2 \leq \operatorname{ind}(B(G)) + 2 \leq \chi(G). \tag{2}$$

We refer the reader to [12] and references therein for definitions of the box complex $B(G)$ of G, its index $\operatorname{ind}(B(G))$ and coindex $\operatorname{coind}(B(G))$. Alternative characterisations of these parameters, which do not rely on simplicial complexes, were eventually found. In particular, we have the following characterisation.

Theorem 9 [10]**.** *For any graph H, $\operatorname{coind}(B(H)) + 2$ is the largest n such that there exists a G in $\mathcal{K}_n$ admitting a homomorphism to H.*

The Hedetniemi-type identity for the coindex was proved by Simonyi and Zsbán.

Theorem 10 [12]**.** *For any graphs G, H,*

$$\operatorname{coind}(B(G \times H)) = \min\{\operatorname{coind}(B(G)), \operatorname{coind}(B(H))\}.$$

This would be an ingredient in a topological proof of the Burr-Erdős-Lovász conjecture based on the coindex of the box complex. What is missing is a proof that given a red-blue colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph G_c such that $\operatorname{coind}(B(G_c)) + 2 = n$. By Theorem 9, this is equivalent to the existence of a graph $H \in \mathcal{K}_n$ that admits a homomorphism to G_c. In fact, the same $H_n \in \mathcal{K}_n$ should work for every c, since for every $H, H' \in \mathcal{K}_n$, there exists $H'' \in \mathcal{K}_n$ which admits a homomorphism to both H and H'. Thus this topological proof of the Burr-Erdős-Lovász conjecture can be completed if and only if for every n, there is a graph H_n in $\mathcal{K}_n$ such that $R_\chi(H_n)$ is the minimum possible value $(n-1)^2 + 1$.

For $n = 3$, everything works. Indeed $\mathcal{K}_3$ consists of the odd cycles, and it is easy to see that the odd cycles $\{C_{2k+1} = M_k(K_2) | k \geq 1\}$ satisfy $R_\chi(C_3) = 6$ and $R_\chi(C_{2k+1}) = 5$ for all $k > 1$. We look next at $\mathcal{K}_4$.

3.1. Generalised Mycielskians of odd cycles

The graphs $M_1(M_k(K_2))$, $k \geq 1$ are the odd wheels. In [8], it is shown that every odd wheel has chromatic Ramsey number 14, except for $K_4 = M_1(M_1(K_2))$, which has chromatic Ramsey number $18 = R(4)$. This implies that

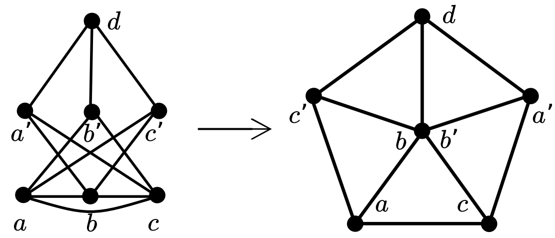

Figure 2. Homomorphism from $M_2(M_1(K_2)))$ to $M_1(M_2(K_2)))$.

$R_\chi(M_{k'}(M_k(K_2))) \leq 14$ for all $k \geq 2$ and $k' \geq 1$, since $M_{k+1}(G)$ admits a homomorphism to $M_k(G)$ for every k and G. Now, $M_2(M_1(K_2))$ admits a homomorphism to $M_1(M_2(K_2))$, as shown in Figure 2.
Therefore every element of $\mathcal{K}_4$ other than K_4 has chromatic Ramsey number at most 14. We will prove the following.

Theorem 11. *Every element of* $\mathcal{K}_4$ *other than* K_4 *has chromatic Ramsey number* 14.

Proof. Since every element of $\mathcal{K}_4$ other than K_4 has chromatic Ramsey number at most 14, it only remains to show that there exists a red-blue colouring of the edges of K_{13} such that no element of $\mathcal{K}_4$ admits a homomorphism to a subgraph spanned by monochromatic edges. We use the Cayley graph (or circulant) $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1, \pm2, \pm4\})$ depicted in Figure 3. Its vertices are the elements of the cyclic group $\mathbb{Z}_{13}$, and there is an edge joining x to y if and only if $y - x \in \{\pm1, \pm2, \pm4\}$. The vertices of $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1, \pm2, \pm4\})$ can be identified to those of K_{13}, and its edges to the edges coloured red. The blue edges are then those of $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm3, \pm5, \pm6\})$. But the two graphs $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1, \pm2, \pm4\})$ and $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm3, \pm5, \pm6\})$ are isomorphic, since the group automorphism ϕ of $\mathbb{Z}_{13}$ defined by $\phi(x) = 5x$ maps $\{\pm1, \pm2, \pm4\}$ to $\{\pm3, \pm5, \pm6\}$. Therefore it suffices to show that there is no homomorphism from any element of $\mathcal{K}_4$ to $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1, \pm2, \pm4\})$.

Let C be an odd cycle and suppose that for some k there exists a homomorphism ψ from $M_k(C)$ to $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1, \pm2, \pm4\})$. Each edge $\{x, y\}$ of $M_k(C)$ corresponds to two opposite arcs (x, y), (y, x). We define the *label* $\ell(x, y)$ of the arc (x, y) to be the element of $\mathbb{Z}_2$ given by

$$\ell(x, y) = \begin{cases} 1 \text{ if } \psi(y) - \psi(x) \in \{1, -2, 4\}, \\ 0 \text{ if } \psi(y) - \psi(x) \in \{-1, 2, -4\}, \end{cases}$$

where $\psi(y) - \psi(x)$ is computed in $\mathbb{Z}_{13}$.

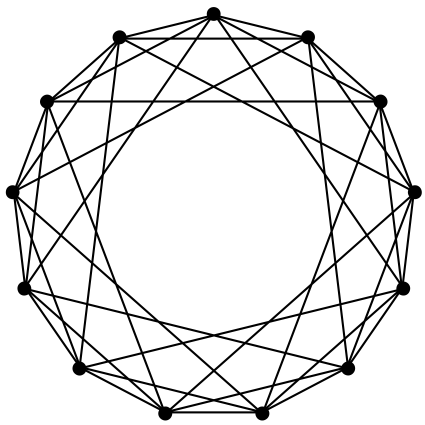

Figure 3. The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm 1, \pm 2, \pm 4\})$.

A 4-*crown* in a graph is a set of four distinct arcs $\{(a,b),(c,b),(c,d),(a,d)\}$. We use the following.

Fact. *If $\{(a,b),(c,b),(c,d),(a,d)\}$ is a 4-crown in $M_k(C)$, then*

$$\ell(a,b)+\ell(c,b)+\ell(c,d)+\ell(a,d)=0$$

in $\mathbb{Z}_2$.

Proof. Clearly, if ψ is not bijective on the 4-cycle $\{a,b,c,d\}$ of $M_k(C)$, then it identifies arcs of the 4-crown in pairs with the same label. Thus we can suppose that ψ maps $\{a,b,c,d\}$ to a 4-cycle $\{x, x+i, x+i+j, x+i+j+k\}$ of $\mathbb{Z}_{13}$. We then have i,j,k and $i+j+k$ in $\{\pm 1, \pm 2, \pm 4\}$, and to have four distinct values we must have $i \neq -j \neq k$. We can have $k=-i$, yielding a "commutative" 4-cycle $\{x, x+i, x+i+j, x+j\}$ of $\mathbb{Z}_{13}$. In this case, opposite arcs of the 4-crown have labels that sum to 1, hence the sum on the four arcs is 0. The other 4-cycles of $\mathbb{Z}_{13}$ either have three edges corresponding to a difference of ± 4, and have the form $\{x, x+4, x+8, x+12\}$ or have one edge corresponding to a difference of ± 4 and have one of the forms $\{x, x+1, x+2, x+4\}$, $\{x, x+1, x+3, x+4\}$ or $\{x, x+2, x+3, x+4\}$. In all these cases the 4-crown has two arcs with label 0 and two with label 1, except for a cycle of the form $\{x, x+1, x+3, x+4\}$, in which case all four arcs have label 0 or all four have label 1. $\square$

Recall that $M_k(C) = (C \times \mathbb{P}_k)/\sim_k$; we denote π_2 the projection of an element of $C \times \mathbb{P}_k$ on $\mathbb{P}_k$. The *base* B of $C \times \mathbb{P}_k$ consists of its arcs (x,y) such that $\pi_2(x) = \pi_2(y) = 0$. The *upward layer* $L_{i,i+1}$ is the set of arcs (x,y) such that $\pi_2(x) = i$ and $\pi_2(y) = i+1$, and the *downward layer* $L_{i+1,i}$ is the set of arcs

(x, y) such that $\pi_2(x) = i+1$ and $\pi_2(y) = i$. The sum of the labels of the arcs of B is 1, because C has an odd number of edges, each of which corresponding to two arcs with different labels. A 4-crown $\{(a,b),(c,b),(c,d),(a,d)\}$ with $\pi_2(a) = \pi_2(b) = \pi_2(c) = 0$ and $\pi_2(d) = 1$ has $(a,b),(c,b)$ in B and $(c,d),(a,d)$ in $L_{0,1}$. By the Fact, we have $\ell(a,b)+\ell(c,b) = \ell(c,d)+\ell(a,d)$. Using all such 4-crowns, we see that the sum of the labels of the arcs of $L_{0,1}$ is also 1. Similarly, a 4-crown $\{(a,b),(c,b),(c,d),(a,d)\}$ with $\pi_2(b) = i-1$, $\pi_2(a) = \pi_2(c) = i$ and $\pi_2(d) = i+1$ has $(a,b),(c,b)$ in $L_{i,i-1}$ and $(c,d),(a,d)$ in $L_{i,i+1}$, with $\ell(a,b)+\ell(c,b) = \ell(c,d)+\ell(a,d)$. Thus the sum of labels of the arcs of $L_{i,i+1}$ is the same as that of $L_{i,i-1}$. Also, the sum of labels of the arcs of $L_{i+1,i}$ is the same as the sum of labels of the arcs of $L_{i,i+1}$, since there is an even number of edges $\{x,y\}$ with $\pi_2(x) = i$, $\pi_2(y) = i+1$, each contributing 1 to one of the two sums. Thus each layer has arc labels that sum to 1.

We now reach a contradiction. The sum of labels of the arcs of $L_{k-1,k}$ should then be 1, but the equivalence $\sim_k$ identifies arcs of $C \times \{k-1,k\}$ pairwise, so that their labels cancel out. The sum should be 0. Therefore the homomorphism ψ from $M_k(C)$ to $\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1,\pm2,\pm4\})$ cannot exist. ∎

It can also be shown that $\mathrm{ind}(B(\mathrm{Cay}(\mathbb{Z}_{13}, \{\pm1,\pm2,\pm4\}))) + 2 = 3$, hence there is no topological proof of the Burr-Erdős-Lovász conjecture based on the index of the box complex either. Also note that the Hedetniemi-type identity

$$\mathrm{ind}(B(G \times H)) = \min\{\mathrm{ind}(B(G)), \mathrm{ind}(B(H))\}$$

is not proved. It is discussed in [7] and [16], where it is shown that it would follow from Hedetniemi's conjecture, if the latter were true.

3.2. Asymptotic bounds on $R_\chi(\mathcal{K}_n)$

In general, $R_\chi(\mathcal{K}_n) \equiv \min\{R_\chi(G) | G \in \mathcal{K}_n\}$ is much greater than $(n-1)^2+1$. In fact, we will prove the following.

Theorem 12. $R_\chi(\mathcal{K}_n) > 2^{n/4}$.

Proving Theorem 12 is equivalent to showing that there is a red-blue colouring of the edges of $K_{\lfloor 2^{n/4} \rfloor}$ such that the graphs $G, \overline{G}$ spanned respectively by the red and blue edges satisfy $\mathrm{coind}(B(G)) < n-2$ and $\mathrm{coind}(B(\overline{G})) < n-2$. For this purpose we will use the "zig-zag theorem" of Simonyi and Tardos [11].

Theorem 13 [11]**.** *Let G be a graph such that $\mathrm{coind}(B(G)) + 2 \geq n$. Suppose that the vertices of G are labeled with integers $1, \ldots, m$. Then there is an increasing sequence $\ell_1, \ell_2, \ldots, \ell_n$ of labels such that G contains a copy of the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$, with one part consisting of the vertices labeled ℓ_i with i odd and the other of the vertices labeled ℓ_i with i even.*

***Proof of Theorem* 12.** We use a probabilistic argument similar to the one proving $R(n) > 2^{n/2}$. The vertices of $K_{\lfloor 2^{n/4} \rfloor}$ are labeled $1, 2, \ldots, \lfloor 2^{n/4} \rfloor$, and let its edges be coloured randomly red or blue. For an increasing sequence $\ell_1, \ell_2, \ldots, \ell_n$ of labels, the probability that all edges $\{\ell_{2i+1}, \ell_{2j}\}$ have the same colour is at most $\frac{2}{2^{n^2/4}}$. There are $\binom{\lfloor 2^{n/4} \rfloor}{n}$ such sequences, therefore the probability P that one sequence has all edges between labels of different parity of the same colour satisfies

$$P < \binom{\lfloor 2^{n/4} \rfloor}{n} \cdot \frac{2}{2^{n^2/4}} < \frac{\lfloor 2^{n/4} \rfloor^n}{2^{n^2/4}} \leq 1.$$

Therefore there exists a red-blue colouring of the edges of $K_{\lfloor 2^{n/4} \rfloor}$ for which the conclusion of Theorem 13 does not old for either of the graphs $G, \overline{G}$ spanned respectively by the red and blue edges. These graphs then satisfy $\mathrm{coind}(B(G)) + 2 < n$ and $\mathrm{coind}(B(\overline{G})) + 2 < n$. ■

In [13], the hypothesis of Theorem 13 are weakened to $\mathrm{ind}(B(G)) + 2 \geq n$. This indicates that it may be possible to strengthen the bound of Theorem 12. In any case, we have

$$2^{n/4} < R_\chi(\mathcal{K}_n) \leq R(n) < 2^{2n}.$$

In particular, $R(n)$ is bounded polynomially in terms of $R_\chi(\mathcal{K}_n)$, that is,

$$R(n) \leq R_\chi(\mathcal{K}_n)^8.$$

It would be interesting to know whether this bound can be improved.

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