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GLOBAL DOMINATED COLORING OF GRAPHS

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Abstract

In this paper, we initiate a study of global dominated coloring of graphs as a variation of dominated colorings. A global dominated coloring of a graph G is a proper coloring such that for each color class there are at least two vertices, one of which is adjacent to all the vertices of this class while the other one is not adjacent to any vertex of the class. The global dominated chromatic number of G is the minimum number of colors used among all global dominated colorings of G. In this paper, we establish various bounds on the global dominated chromatic number of a graph in terms of some graph invariants including the order, dominated chromatic number, domination number and total domination number. Moreover, characterizations of extremal graphs attaining some of these bounds are provided. We also discuss the global dominated coloring in trees and split graphs.

Keywords: global dominated coloring, dominated coloring, dominated chromatic number.

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1. Introduction

By a graph $G = (V, E)$, we mean a graph with no loops and multiple edges with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = |V(G)|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N_G(v) = \{u \in V : v \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of v is the cardinality of its open neighborhood. The *maximum degree* of G is denoted by $\Delta = \Delta(G)$. A vertex of degree one is called a *leaf.* A *tree* is a connected acyclic graph. A *star* S_q is a tree of order $q + 1$ with at least q vertices of degree 1. A *bistar* $B_{p,q}$ is a graph formed by two stars S_p and S_q by adding an edge between their center vertices. We write P_n, C_n and K_n for the *path*, *cycle* and *complete graph* of order n, respectively. The complete graph of order 3 is called a triangle. If F is a graph, then a graph G is F*-free* if it has no induced subgraph isomorphic to F. A *path* joining two vertices x and y is called a (x, y)-path. The *distance* between two vertices x and y in a connected graph G is the number of edges in a shortest (x, y)-path. The *diameter* of a connected graph G is the maximum distance between two vertices of G. A set S is *independent* if no two vertices in S are adjacent. For standard definitions and terminologies on basic graph theory, the reader is referred to [3].

Graph coloring and domination are two important areas in graph theory. A k*-coloring* of G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that any two adjacent vertices u and v have different colors, that is $f(u) \neq f(v)$. Also, the set of vertices having the same color will be called a color class, thereby forming an independent set. Consequently, a k-coloring of G is equivalent to a partition of the vertex set $V(G)$ into k-independent sets. The smallest integer k for which G has a k-coloring is the *chromatic number* $\chi(G)$ of G.

A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V \setminus D$ has a neighbor in D and is a *total dominating set*, if every vertex in V has a neighbor in D. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) is the minimum cardinality of a dominating set (respectively, total dominating set) of G. A total dominating set D of G is a *global total dominating set* [10] if D is also a total dominating set of the complement graph $\overline{G}$ of G. The *global total domination number* $\gamma_{gt}(G)$ is the minimum cardinality of a global total dominating set of G. For more details on global domination and its variations we refer the reader to the book chapter [2].

In recent years, several coloring problems have been defined with additional properties that involve the concept of domination. As examples we can cite dominator coloring [5], total dominator coloring [7], and dominated coloring [11]. A survey of selected results on dominator and total dominator colorings can be found in the book chapter [8]. Consider a k-coloring C of G with color classes $V_1, V_2, \ldots, V_k$. A color class V_i is said to be *dominated* by a vertex u (or u is a *dominating vertex* of V_i) if u is adjacent to all vertices of V_i. The color class V_i is said to be *anti-dominated* by a vertex v (or v is an *anti-dominating vertex* of V_i) if v is not adjacent to any vertex of V_i. Now, if each vertex of G is a dominating vertex for some color class (possible its own class if it is alone), then C is called a *dominator coloring*, while if each color class of C has a dominating vertex, then C is called a *dominated coloring.* For the latter, the minimum number of colors used among all dominated colorings of G is the *dominated chromatic number*, denoted by $\chi_{dom}(G)$. It is worth mentioning that dominated colorings of graphs offer some interesting applications in social networks and genetic theory (see [4, 9]). In 2019, Sahul Hamid and Rajeswari [6] introduced the concept of global dominator coloring of a graph G defined a k-coloring of G with color classes $V_1, V_2, \ldots, V_k$ satisfying the property that for every vertex v, there exist V_i, V_j with for $i \neq j$ such that v is a dominating vertex of V_i and an anti-dominating vertex of V_j. For the sake of notation, we will write $C = (V_1, V_2, \ldots, V_k)$ any k-coloring with color classes $V_1, V_2, \ldots, V_k$.

In this paper, we are interested in the study of the global version of the dominated colorings. A *global dominated coloring* of G is a coloring such that every color class has both dominating and anti-dominating vertices. The minimum number of colors used among all global dominated colorings of G is called the *global dominated chromatic number* of G, denoted by $\chi_{gdom}(G)$. It is quite obvious that not all graphs admit a global dominated coloring, for example, the complete graph does not. The existence of such a coloration in a graph G of order n is subject to that G has no isolated vertex and the maximum degree in G is at most $n-2$, that is $\gamma(G) \geq 2$. Therefore, let $\mathcal{L}$ denote the family of all graphs without isolated vertices having domination number at least two. A global dominated coloring of G with $\chi_{gdom}(G)$ number of colors will be called a χ_{gdom}*-coloring* of G, and likewise a χ_{dom}*-coloring* of G is defined.

In this paper, we first show that the global dominated chromatic number of a graph $G \in \mathcal{L}$ of order n is bounded below by 3 and above by n, where the extremal graphs attaining each bound are given. Moreover, various bounds in terms of some graph invariants including the dominated chromatic number, global total domination number, total domination number, domination number and maximum degree are presented. For the class of trees T different from stars, it is shown that $\gamma_t(T) \leq \chi_{gdom}(T) \leq \gamma_t(T) + 2$, where a characterization of all trees T such that $\chi_{gdom}(T) = \gamma_t(T) + i$ is provided for any $i \in \{0, 1, 2\}$.

We close this section by recalling two results given in [11] that will be useful in our investigations.

Theorem 1 [11]**.** *The dominated chromatic number of a triangle-free graph G is equal to its total domination number.*

Theorem 2 [11]**.** *Let G be a split graph such that its maximum clique is of order k. Then $\chi_{dom}(G) = \chi(G) = \omega(G) = k$.*

2. Existence Result and First Bounds

The purpose of this section is to present at first a result dealing with the existence of graphs with prescribed value χ_{gdom}, and then a lower bound and an upper bound on the global dominated chromatic number. Moreover, in the remainder of this section we will focus on the graphs reaching these bounds.

We begin with a realization result of graphs G of order n and $\chi_{gdom}(G) = k$, where $n \geq k \geq 4$. For the case $k = 3$, the order n has to be at least 6 which is discussed in Theorem 6.

Theorem 3. *For any integers k and n with $n \geq k \geq 4$, there exists a connected graph $G \in \mathcal{L}$ of order n with $\chi_{gdom}(G) = k$.*

Proof. We consider several cases depending on whether $n = k, k+1$ or $n > k+1$.

Case 1. $n = k$.

Subcase 1.1. n is even. Assume that $n = 2m$, and consider the cocktail party graph $K_{m\times 2}$ with vertex set $V(K_{m\times 2}) = \{u_i, v_j : 1 \leq i, j \leq m\}$ and edge set $E(K_{m\times 2}) = \{u_iu_j, u_iv_j, v_iv_j :$ for all i, j and $i \neq j\}$. Clearly the graph induced by the u_i's as well as the one induced by the v_i's is a complete graph on m vertices. So let the color of vertex u_i be i. Now each vertex v_j has to be given a new color in order to achieve global dominated coloring of $K_{m\times 2}$. Therefore $\chi_{gdom}(K_{m\times 2}) = 2m = k$. (For an example, see the graph G_1 of order 6 illustrated in Figure 1, where $\chi_{gdom}(G_1) = 6$).

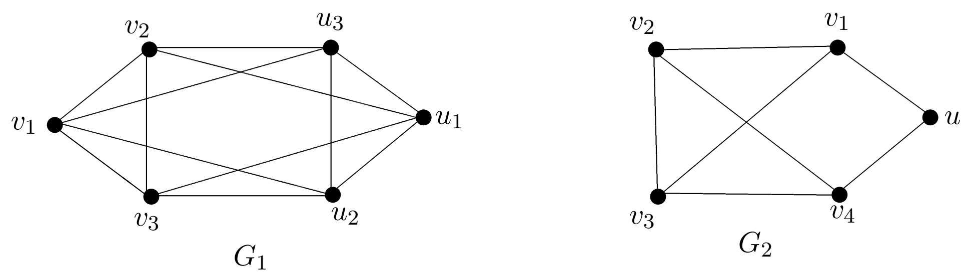

Figure 1. Graphs G_1 and G_2 of order 6 and 5, respectively, with $\chi_{gdom}(G_1) = 6$ and $\chi_{gdom}(G_2) = 5$.

Subcase 1.2. n is odd. Assume that $n = 2m + 1$, and consider the graph $K_{2m} - e$, whose vertices are labelled $v_1, v_2, \ldots, v_{2m}$ and let $e = v_1v_{2m}$. Add a new vertex u and join it by edges to v_1 and v_{2m} to get the resulting graph G. Clearly the graph induced by all v_i's but v_{2m} is a complete graph on $2m - 1$ vertices. Hence at least $2m - 1$ colors are required to color G. Also the vertices v_{2m} and u have to be given new color in order to achieve global dominated coloring of G. Therefore $\chi_{gdom}(G) = k$. (For an example, see the graph G_2 of order 5 illustrated in Figure 1, where $\chi_{gdom}(G_2) = 5$).

Case 2. $n = k + 1$. First assume that k is even, and consider the graph G^* with vertex set $V(G) = V(K_{m\times 2}) \cup \{x\}$ and edge set $E(G) = E(K_{m\times 2}) \cup \{xv_j :$ for all $2 \leq j \leq m\}$. One can see that $\chi_{gdom}(G^*) = k$. (For an example, see the graph of order 7 illustrated in Figure 2, where $\chi_{gdom}(G^*) = 6$).

Assume now that k is odd, and consider the graph G^{**} with vertex set $V(G^{**}) = V(G^*) \cup \{y\}$ and edge set $E(G^{**}) = E(G^*) \cup \{xy, yv_j :$ for all $1 \leq j \leq m\}$. One can see that $\chi_{gdom}(G^{**}) = k$. (For an example, see the graph of order 8 illustrated in Figure 2, where $\chi_{gdom}(G^{**}) = 7$).

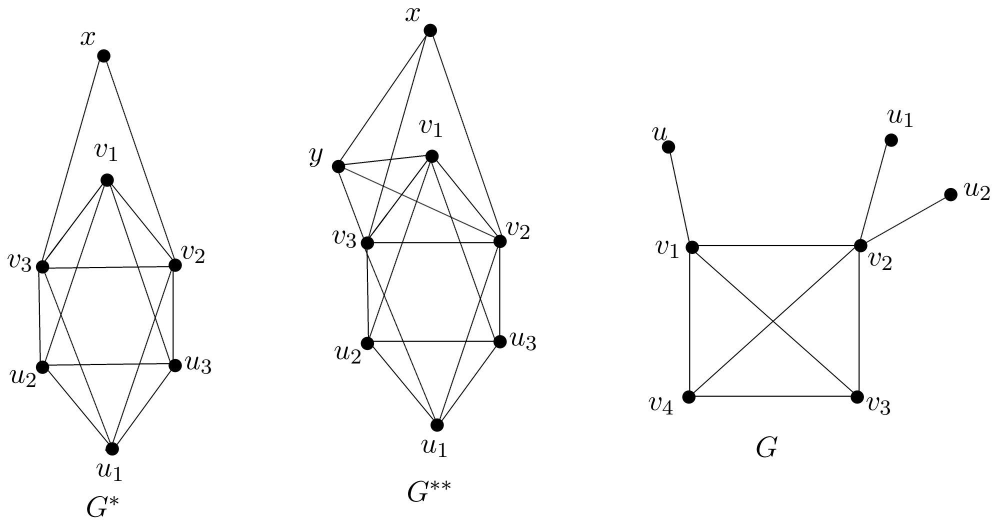

Figure 2. Graphs G^*, G^{**} and G of order 7, 8 and 7, respectively, with $\chi_{gdom}(G^*) = 6$, $\chi_{gdom}(G^{**}) = 7$ and $\chi_{gdom}(G) = 4$.

Case 3. $n > k+1$. Consider the complete graph K_k whose vertices are labelled $v_1, v_2, \ldots, v_k$. Clearly k different colors say $\{1, 2, \ldots, k\}$ are needed to color the vertices of K_k, where the color of vertex v_i is i, for $1 \leq i \leq k$. Now add a new vertex u and the edge v_1u and add $n - k - 1$ other new vertices $u_1, \ldots, u_{n-k-1}$ attached at v_2, and let G denote the resulting graph of order n. Color u with color 3 and the u_j's with color 4. Such a coloring is a global dominated coloring of G and thus $\chi_{gdom}(G) = k$. (For an example, see the graph of order 7 illustrated in Figure 2, where $\chi_{gdom}(G) = 4$). ■

Theorem 4. *For any graph $G \in \mathcal{L}$ of order n, $3 \leq \chi_{gdom}(G) \leq n$.*

Proof. Clearly, $\chi_{gdom}(G) \neq 1$. Now, if $C = (V_1, V_2)$ is a global dominated coloring of G, then V_1 should have a dominating vertex belonging to V_2 leading that the color class V_2 has no anti-dominating vertex, a contradiction. Hence $\chi_{gdom}(G) \geq 3$.

The upper bound follows from the fact that any coloring of G using n colors is a global dominated coloring of G. ∎

Restricted to bipartite graphs, the upper bound in Theorem 4 is improved by the following result.

Theorem 5. *Let $G \in \mathcal{L}$ be a connected bipartite graph of order n and maximum degree Δ. Then $\chi_{gdom}(G) \leq n - \Delta + 2$, and this bound is sharp.*

Proof. Consider a vertex v of maximum degree, and color its neighbors with two colors, and assign to the remaining vertices a new color unused. This coloring is global dominated, and so $\chi_{gdom}(G) \leq n - \Delta + 2$.

That this bound is sharp may be seen for bistars $B_{\Delta-1,1}$, where $\Delta \geq 2$. ∎

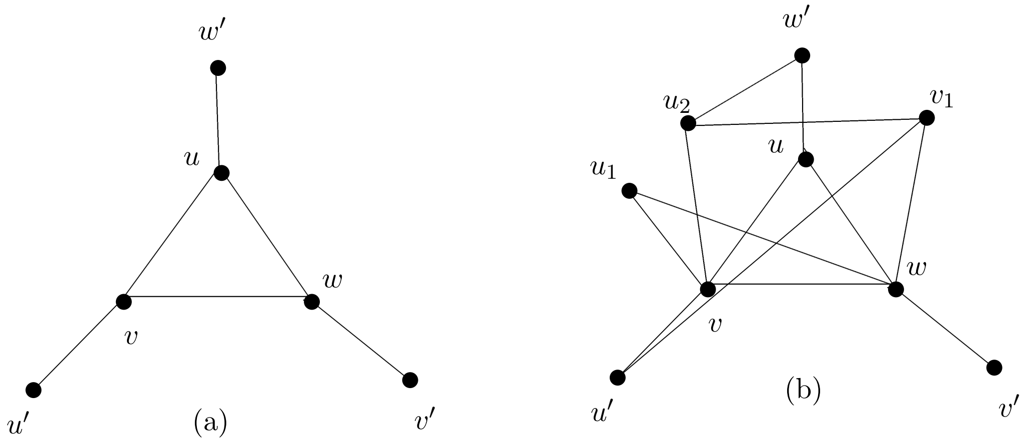

Figure 3. Graphs belonging to family $\Im$.

In the aim to characterize graphs $G \in \mathcal{L}$ such that $\chi_{gdom}(G) = 3$, we define the family $\Im$ of connected graphs G that are obtained from the graph H in Figure 3(a) by adding three sets of vertices X, Y and Z (which may be empty) in the following way.

(1) $X \cup \{u, u'\} = X'$ is an independent set with v and v' as dominating and anti-dominating vertices, respectively;

(2) $Y \cup \{v, v'\} = Y'$ is an independent set with w and w' as dominating and anti-dominating vertices, respectively;

(3) $Z \cup \{w, w'\} = Z'$ is an independent set with u and u' as dominating and anti-dominating vertices, respectively;

(4) Subject to the above conditions, possible edges may exist between the vertices X', Y' and Z'.

For an example of a connected graph belonging to $\Im$, see Figure 3(b).

Theorem 6. *Let G be a graph of $\mathcal{L}$. Then $\chi_{gdom}(G) = 3$ if and only if $G \in \Im$.*

Proof. Assume that $\chi_{gdom}(G) = 3$ and let $C = (V_1, V_2, V_3)$ be a χ_{gdom}-coloring of G. First, G is connected and the connectedness follows from the fact that each V_i has a dominating vertex and an anti-dominating vertex. We claim that dominating and anti-dominating vertices of each V_i belong to a same color class. To see this, assume without loss of generality, that V_1 has a dominating vertex in V_2 and an anti-dominating vertex in V_3. It follows that the dominating vertex of V_3 belongs to V_2. But then V_2 has no anti-dominating vertex, a contradiction, which proves the claim. Now since each V_i cannot contain dominating and anti-dominating vertices for the other two color classes, we deduce that $|V_i| \geq 2$ for every $i \in \{1, 2, 3\}$. Accordingly, let $u, u' \in V_1$, $v, v' \in V_2$ and $w, w' \in V_3$. By the above claim, let V_1 have v and v' as dominating and anti-dominating vertices respectively, V_2 have w and w' as dominating and anti-dominating vertices respectively, and V_3 have u and u' as dominating and anti-dominating vertices respectively. Note that the vertices u, v, w, u', v' and w' induce a subgraph isomorphic to the graph H in Figure 3(a). Moreover, by considering the sets $X = V_1 \setminus \{u, u'\}$, $Y = V_2 \setminus \{v, v'\}$ and $Z = V_3 \setminus \{w, w'\}$, one can see that these three sets fulfill the conditions for the construction of connected graphs of the family $\Im$, and therefore $G \in \Im$.

Conversely, assume that $G \in \Im$. Then G is obtained from the graph H by adding three sets of vertices X, Y and Z (which may be empty) satisfying conditions (1) to (4). Then $(X \cup \{u, u'\}, Y \cup \{v, v'\}, Z \cup \{w, w'\})$ is a global dominated coloring of G, implying that $\chi_{gdom}(G) \leq 3$. The equality follows from Theorem 4. ■

Our next step is to characterize the graphs $G \in \mathcal{L}$ of order n such that $\chi_{gdom}(G) = n$. We begin with the following result showing that any connected graph in $\mathcal{L}$ of order n and diameter at least 5 has a global dominated chromatic number at most $n - 2$.

Proposition 7. *Let $G \in \mathcal{L}$ be a connected graph of order n and diameter at least 5. Then $\chi_{gdom}(G) \leq n - 2$.*

Proof. Let $u, v \in V(G)$ be two vertices at distance at least 5 in G, and let P be a (u, v)-path. Let x and y be two vertices on P at distance two from u and v, respectively. Consider the coloring $C = (V_1, V_2, \ldots, V_{n-2})$, where $V_1 = \{u, x\}, V_2 = \{v, y\}$, and all the remaining vertices of G are spread over the sets $V_3, \ldots, V_{n-2}$ so that each one contains a single vertex. Then C is a global dominated coloring of G, and thus $\chi_{gdom}(G) \leq n - 2$. ■

Recall that the *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G.

Theorem 8. *Let $G \in \mathcal{L}$ be a connected graph of order n. Then $\chi_{gdom}(G) = n$ if and only if $\alpha(G) = 2$.*

Proof. Let G be a connected graph of $\mathcal{L}$ such that $\chi_{gdom}(G) = n$. Clearly, $\alpha(G) \geq 2$, since $\gamma(G) \geq 2$ for every graph $G \in \mathcal{L}$. Now, assume that $\alpha(G) \geq 3$. Then there exist a pair of non-adjacent vertices v_1, v_2 in G such that $\{v_1, v_2\}$ is not a dominating set of G. If v_1 and v_2 have a common neighbor in G, then coloring v_1, v_2 with the same color and each of the remaining vertices with a new color provides a global dominated coloring of G using $n-1$ colors, a contradiction. Hence we can assume that v_1 and v_2 have no common neighbor. Therefore, G has diameter at least three. Since G is connected, there is a path between v_1 and v_2. By Proposition 7, $d(v_1, v_2) \in \{3, 4\}$. Consider the following two situations.

Case 1. $d(v_1, v_2) = 4$. Let $v_1v_3v_4v_5v_2$ be a (v_1, v_2)-path of length four in G. Then as before coloring v_1, v_4 with a same color and each of the remaining vertices with a new color provides a global dominated coloring of G using $n-1$ colors, a contradiction.

Case 2. $d(v_1, v_2) = 3$. Let $v_1v_3v_4v_2$ be a (v_1, v_2)-path of length three in G. Since $\{v_1, v_2\}$ is not a dominating set of G, there exists a vertex u which is not adjacent to either v_1 or v_2. Now, if u is adjacent to v_3, then coloring v_1, u with a same color and each of the remaining vertices with a new color is a global dominated coloring of G, a contradiction. By symmetry, u is not adjacent to v_4, and therefore $N(u) \cap \{v_1, v_2, v_3, v_4\} = \emptyset$. But then assigning to v_1, v_4 a same color and to each of the remaining vertices a new color is a global dominated coloring of G, a contradiction.

Therefore, we conclude that $\alpha(G) = 2$.

Conversely, if $\alpha(G) = 2$, then every set of two non-adjacent vertices in G is a dominating set of G, and thus every vertex of G has to be assigned a new color, that is $\chi_{gdom}(G) = n$. ∎

Proposition 9. *Let $G \in \mathcal{L}$ be a disconnected graph of order n. Then $\chi_{gdom}(G) = n$ if and only if every component of G is a complete graph.*

Proof. Let $G \in \mathcal{L}$ be a disconnected graph and let Q be any component of G. If Q is not a complete graph, then there are two non-adjacent vertices x and y having a common neighbor. In this case, assigning x, y a same color and each of the remaining vertices of G a new color is a global dominated coloring of G, contradicting $\chi_{gdom}(G) = n$. Hence Q is a complete graph, and the proof is complete. ∎

3. Relations Involving χ_{gdom} and χ_{dom}

In this section, we consider relations between the global dominated chromatic and the dominated chromatic numbers. We shall then be interested in the extremal graphs reaching these bounds.

Theorem 10. *For any graph $G \in \mathcal{L}$, $\chi_{dom}(G) \leq \chi_{gdom}(G) \leq 2\chi_{dom}(G)$.*

Proof. The lower bound follows from the fact that any global dominated coloring of G is also a dominated coloring. To prove the upper bound, let $C = (V_1, V_2, \ldots, V_k)$ is any χ_{dom}-coloring of G. Note that if $|V_j| = 1$ for some j, then since $G \in \mathcal{L}$, the single vertex in V_j has a non-neighbor that can be considered as an anti-dominating vertex. From each V_i with $|V_i| \geq 2$, pick a vertex, say x_i, and color it with an unused color. Observe that such a vertex x_i is an anti-dominating vertex for $V_i \setminus \{x_i\}$. Hence the resulting coloring is a global dominated coloring of G, and thus $\chi_{gdom}(G) \leq 2\chi_{dom}(G)$. ■

Theorem 11. *Let G be a graph of $\mathcal{L}$. Then $\chi_{gdom}(G) = \chi_{dom}(G)$ if and only if there exists a χ_{dom}-coloring of G such that none of the color class is a dominating set of G.*

Proof. Assume that G is a graph of $\mathcal{L}$ such that $\chi_{gdom}(G) = \chi_{dom}(G)$, and let C be a χ_{gdom}-coloring of G. Then C is also a χ_{dom}-coloring, since $\chi_{gdom}(G) = \chi_{dom}(G)$. Moreover, by the definition of global dominated colorings, every class has anti-dominating set, and so no color class is a dominating set of G.

The converse is straightforward, and we omit the details. ■

From Theorem 11, we derive the following sufficient condition for graphs of $\mathcal{L}$ to have equal global dominated and dominated chromatic numbers.

Corollary 12. *Let G be a graph of $\mathcal{L}$ with domination number $\gamma(G) = \gamma$. If G is S_γ-free, then $\chi_{gdom}(G) = \chi_{dom}(G)$.*

Proof. Let C be a χ_{dom}-coloring of G. Since G is S_γ-free, each color class of C contains less than γ vertices, and therefore none of the color classes of C is a dominating set of G. By Theorem 11, $\chi_{gdom}(G) = \chi_{dom}(G)$. ■

Corollary 13. *For every disconnected graph $G \in \mathcal{L}$, $\chi_{gdom}(G) = \chi_{dom}(G)$.*

Proof. Let C be any χ_{dom}-coloring of G. Since each color class must have a dominating vertex, no two vertices from different components of G can have a same color. Therefore none of the color classes in C is a dominating set of G, and the desired result follows from Theorem 11. ■

The next result is a consequence of Theorem 11 and gives the exact value of the global dominated chromatic number of paths and cycles of order at least four. Recall that for $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Corollary 14. *For $n \geq 4$,*

(1) $\chi_{gdom}(P_4) = \chi_{gdom}(P_5) = 4$ *and* $\chi_{gdom}(P_n) = \gamma_t(P_n)$ *for* $n \geq 6$.

(2) $\chi_{gdom}(C_4) = 4, \chi_{gdom}(C_5) = 5$ *and* $\chi_{gdom}(C_n) = \gamma_t(C_n)$ *for* $n \geq 6$.

Proof. The result can be easily checked for $n \in \{4, 5\}$. Hence assume that $n \geq 6$. Since for such an order, paths and cycles are triangle-free graphs, we have by Theorem 1, $\chi_{dom}(P_n) = \gamma_t(P_n)$ and $\chi_{dom}(C_n) = \gamma_t(C_n)$. Furthermore, by consider the χ_{dom}-coloring C as defined in [11] for the proof of Theorem 1, and since $n \geq 6$, it is easy to note that none of the color class of C is a dominating set neither for paths nor for cycles. Hence by Theorem 11, $\chi_{gdom}(P_n) = \chi_{dom}(P_n)$ and $\chi_{gdom}(C_n) = \chi_{dom}(C_n)$ yielding the desired result. ■

Theorem 15. *Let G be a graph of $\mathcal{L}$. If $\chi_{gdom}(G) = 2\chi_{dom}(G)$, then for every χ_{dom}-coloring C of G, the following conditions are satisfied.*

(i) *Each color class of C contains at least two vertices, and*

(ii) *Each color class of C is a dominating set of G.*

Proof. Assume that G is a graph of $\mathcal{L}$ such that $\chi_{gdom}(G) = 2\chi_{dom}(G)$, and let $C = (V_1, V_2, \ldots, V_{\chi_{dom}(G)})$ be a χ_{dom}-coloring of G. Suppose that there exists a color class of C, say V_1, such that $|V_1| = 1$. We note the argument we use for V_1 remains valid for any color class with a single vertex. First, observe that since $G \in \mathcal{L}$, $\Delta(G) \leq n - 2$ and thus there is at least one vertex in G having no neighbor in V_1. Also, since G has no isolated vertex, there is a vertex in G having a neighbor in V_1. Hence V_1 has a dominating vertex as well as an anti-dominating vertex. Now, for each V_i with $|V_i| \geq 2$, we select a vertex that we put alone in a set denoted by V_i'. Note that since V_i is dominated, $V_i \setminus V_i'$ and V_i' also remain dominated. In addition, as observed before for V_1, set V_i' has an anti-dominating vertex. Therefore V_1 together with all $V_i \setminus V_i'$ and V_i' for every $i \neq 1$, is a global dominated coloring of G using at most $2\chi_{dom}(G) - 1$, a contradiction. Hence (i) holds. To prove (ii), assume that some color class, say V_1, does not dominate $V(G)$. Then there is a vertex $v \in V_j$ for some $j \neq 1$, say $j = 2$, such that v has no neighbor in V_1. Hence vertex v is an anti-dominating vertex for V_1. Since each color class contains at least two vertices, by item (i), then as before, from any color class V_i with $i \notin \{1, 2\}$, we choose a vertex that will be alone in the new set V_i'. In this case, V_1, $V_2 \setminus \{v\}, \{v\}$ and all $V_i \setminus V_i'$ and V_i' for every $i \notin \{1, 2\}$ is a global dominated coloring of G using $2\chi_{dom}(G) - 1$, a contradiction. Hence (ii) follows. ■

4. Relations to Other Graph Invariants

In this section, we present some results relating the global dominated chromatic number to some graph parameters including the domination and total domination numbers and the clique number. We first provide a result that relates the global dominated chromatic number and the global total domination number. Recall that the global total domination number of a graph G, denoted by $\gamma_{gt}(G)$, is the minimum cardinality of a set that total dominates both G and its complement graph $\overline{G}$.

Theorem 16. *For any graph $G \in \mathcal{L}$, we have $\chi_{gdom}(G) \geq \frac{1}{2}\gamma_{gt}(G)$.*

Proof. Let $C = (V_1, V_2, \ldots, V_{\chi_{gdom}})$ be a χ_{gdom}-coloring of G, and let D be a set of vertices formed as follows: for each color class V_i we put in D exactly two vertices, one of its dominating vertices and the other amongst its anti-dominating vertices. Clearly, $|D| \leq 2\chi_{gdom}(G)$. We shall show that D is a global total dominating set of G. Since each vertex of D belongs to some color class which is dominated by at least one dominating vertex, set D is accordingly a total dominating set of G. Moreover for any vertex $u \in V(G)$, u belongs to some color class which is anti-dominated by some vertex of D. Hence D is a global total dominating set of G, and thus $\gamma_{gt}(G) \leq 2\chi_{gdom}(G)$. ■

Restricted to triangle-free graphs, we shall see that the global dominated chromatic number is bounded by twice the domination number plus one. We use the following result due to Bollobás and Cockayne [1].

Theorem 17 (Bollobás and Cockayne [1])**.** *If G is a graph without isolated vertices, then G has a minimum dominating set D such that for all $d \in D$, there exists a neighbor $f(d) \in V \setminus D$ of d such that $f(d)$ is not a neighbor of any vertex $x \in D \setminus \{d\}$.*

Theorem 18. *Let $G \in \mathcal{L}$ be a triangle-free graph of order n. Then $\chi_{gdom}(G) \leq 2\gamma(G) + 1$, and the bound is sharp.*

Proof. Let D be a minimum dominating set of G satisfying the property of Theorem 17. Recall that since $G \in \mathcal{L}$, G has no isolated vertices, $\triangle(G) \leq n - 2$ and $\gamma(G) = |D| \geq 2$. We also note that since G is triangle free, the set of neighbors of every vertex of D is an independent set. However, two non-adjacent vertices of D may have common neighbors. Now, consider the coloring C of the vertices of G defined as follows: each vertex x of D is assigned a new color c_x; for each vertex $x \in D$ color its neighbors in $V \setminus D$ not colored by a new color c'_x, and let C_x denote such a color class. It is straightforward to see that C is a dominated coloring of G. Moreover, since every vertex x of D has at least one vertex $f(x)$ in $V \setminus D$ for which x is the only neighbor in D, the number of colors used by

C is exactly $2|D| = 2\gamma(G)$. In the following we will discuss on the globality of the dominated coloring C. First, since $|D| \geq 2$ and each vertex $x \in D$ has at least one neighbor $f(x) \in V \setminus D$ such that $f(x)$ is not a neighbor of any vertex $x \in D \setminus \{x\}$, the color class $\{x\}$ of C has both a dominating vertex and an anti-dominating vertex. Hence it remains to discuss the color classes formed by the vertices of $V \setminus D$. Observe that each of such classes has a dominating vertex, for instance, their neighbors in D. Now, if an addition these color classes have anti-dominating vertices, then we are done and C is a global dominated coloring of G with $2|D| = 2\gamma(G)$ colors. Hence assume that some color class C_v of the vertices in $V \setminus D$ using color c'_v, does not have an anti-dominating vertex. Recall that according to the above notation, the vertices of C_v have v as a neighbor in D. We proceed with the following two claims.

Claim 1. *v has no neighbor in D.*

Proof. Suppose not, and let $u \in D$ be a neighbor of v. Since G is triangle-free, vertex u has no neighbor in C_v, and thus u becomes an anti-dominating vertex for the color class C_v, contradicting our assumption that C_v has no anti-dominating vertex.

Claim 2. *C_v is the unique color class with no anti-dominating vertex.*

Proof. Suppose not, and let C_u be a color class of the vertices of $V \setminus D$ with no anti-dominating vertex. Then by Claim 1, the neighbor $u \in D$ of the vertices of C_u has no neighbor in D. Moreover, no neighbor of u belongs to C_v (else such a vertex becomes an anti-dominating vertex for C_u). Likewise, no neighbor of v is in C_u. Therefore v and u are anti-dominating vertices of C_u and C_v, respectively, a contradiction. This proves the claim.

For the sequel, we will see that $|C_v| \geq 2$. Suppose to the contrary, that $|C_v| = 1$, and let $w \in V \setminus D$ be the unique vertex in C_v. Since C_v has no anti-dominating vertex, vertex w is adjacent to all vertices of G, but then w would be of degree $n-1$, contradicting the fact that $G \in \mathcal{L}$. Hence $|C_v| \geq 2$. Let $f(v)$ denote the vertex of C_v such that $f(v)$ is not a neighbor of any vertex $x \in D \setminus \{v\}$. Then recoloring $f(v)$ with a new color, $f(v)$ becomes an anti-dominating vertex for the color class $C_v \setminus \{f(v)\}$, thereby providing a global dominated coloring of G with $2\gamma(G) + 1$.

The bound is sharp for the cycle C_5 (by Corollary 14), and the proof is complete. ■

For the class of trees different from stars, we give an upper bound of the global dominated chromatic number in terms of the total domination number which improves the upper bound of Theorem 18.

Theorem 19. *Let T be a tree different from a star. Then $\gamma_t(T) \leq \chi_{gdom}(T) \leq \gamma_t(T) + 2$, and the bounds are sharp.*

Proof. The lower bound follows from Theorems 1 and 10. To prove the upper bound, let D_t be a minimum total dominating set of T. We will use the same coloring defined in the proof of Theorem 1 (see [11]), and for proving purposes we shall describe it as follows. Constructing pairs of adjacent vertices in D_t (some vertices may be single after that). For every obtained pair (a, b), we give a new color c_b to b and the neighborhood of a, and give another new color c_a to a and the neighborhood of b. For any single vertex $d \in D_t$ non associated with a pair, we assign a new color c_d to the neighbors of d belonging to $V \setminus D_t$. The coloring C as defined is a dominated coloring using $|D_t|$ colors. Now, if for each pair of adjacent vertices (u_1, v_1) in D_t neither $N(u_1)$ nor $N(v_1)$ is a dominating set of T, then the dominated coloring C defined above is global and thus $\chi_{gdom}(T) \leq \gamma_t(T)$. Hence we can assume that there is a pair of adjacent vertices (u_1, v_1) in D_t such that either $N(u_1)$ or $N(v_1)$, say $N(u_1)$, is a dominating set of T. We claim that no other pair of adjacent vertices distinct from (u_1, v_1) exists in D_t. Suppose to the contrary that there is another distinct pair of adjacent vertices (u_2, v_2) in D_t. Clearly, u_2 and v_2 cannot have a common neighbor in $N(u_1)$, for otherwise u_2, v_2 and such a common neighbor induce a triangle, a contradiction. Hence let u_2 and v_2 be dominated by two different vertices x and y in $N(u_1)$, respectively. Then the vertices u_1, x, u_2, v_2 and y would induce a cycle, a contradiction too. Whence the claim. Now considering the coloring C defined above, pick a vertex from $N(u_1)$ and another vertex from $N(v_1)$ (if $N(v_1)$ is also a dominating set of T) and assign to them two new colors. The resulting coloring remains dominated and it is global. Therefore, $\chi_{gdom}(T) \leq |D_t| + 2$.

Clearly, the lower bound is attained for all path graphs on at least 6 vertices (by Corollary 14) while the upper bound is attained for bistars $B_{r,s}$ with $r, s \geq 1$. ■

Our next aim is to characterize the trees T different from stars such that $\chi_{gdom}(T) = \gamma_t(T) + k$ for all $k \in \{0, 1, 2\}$. Let $\Im_1$ be the family of trees T that can be obtained from a star $K_{1,t}$ ($t \geq 2$), by adding r vertices and s disjoint stars, each of order at least two with $s \geq 1$ and $r \geq 0$ by attaching the r vertices as well as the centers of the s stars to one leaf of the star $K_{1,t}$. An example of a tree in $\Im_1$ is shown in Figure 4.

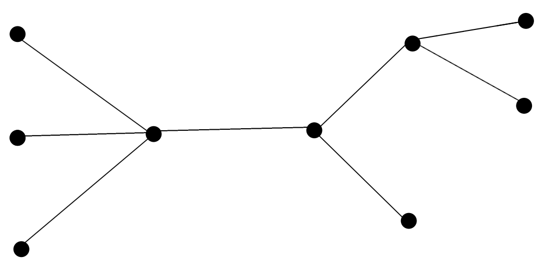

Figure 4. $G \in \Im_1$.

Theorem 20. *Let T be a tree different from a star. Then*

$$\chi_{gdom}(T) = \begin{cases} \gamma_t(T) + 2 & \text{if and only if } T = B_{p,q}, \text{ where } p, q \geq 1, \\ \gamma_t(T) + 1 & \text{if and only if } T \in \Im_1, \\ \gamma_t(T) & \text{if and only if } T \notin \Im_1 \cup \{B_{p,q}\}. \end{cases}$$

Proof. Let D_t be the minimum total dominating set of G. Assume first that $\chi_{gdom}(T) = |D_t| + 2$. Then according to the argument used in the proof of Theorem 19, there exists a pair of adjacent vertices (u, v) in D_t such that both $N(u)$ and $N(v)$ are dominating sets of T. Therefore $\gamma_t(T) = 2$ and thus $D_t = \{u, v\}$. Moreover, since T is different from a star, we deduce that T is a bistar $B_{p,q}$ where $p, q \geq 1$.

Assume now that $\chi_{gdom}(T) = |D_t| + 1$. Then again according to the proof of Theorem 19, there exist a pair of adjacent vertices (u, v) in D_t such that either $N(u)$ or $N(v)$ but not both is a dominating set of T. Therefore $T \in \Im_1$. Following the two situations considered before, we deduce that if $\chi_{gdom}(T) = |D_t|$, then $T \notin \Im_1 \cup \{B_{p,q}\}$.

The converse is obvious. ∎

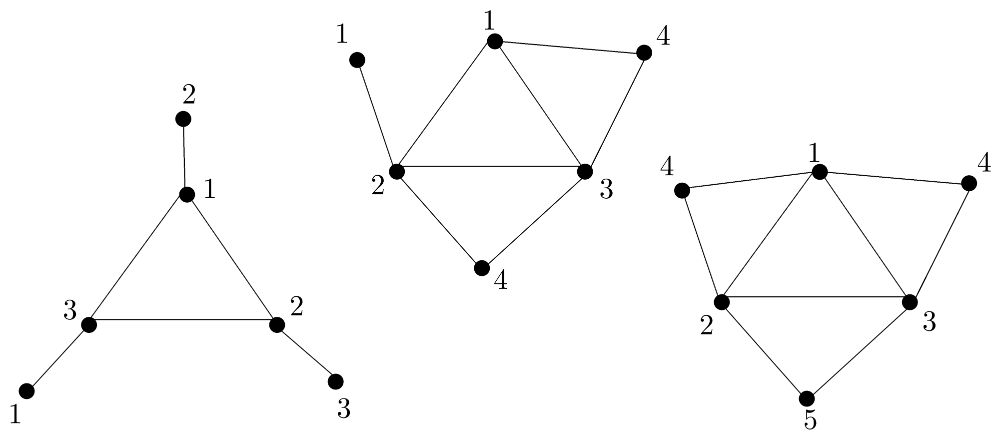

Figure 5. χ_{gdom}-coloring of split graphs on 6 vertices with χ_{gdom} number $\omega, \omega + 1$, and $\omega + 2$.

Theorem 21. *Let $G \in \mathcal{L}$ be a split graph with clique Q of maximum order ω. Then $\omega \leq \chi_{gdom}(G) \leq \omega + 2$, and these bounds are sharp.*

Proof. The lower bound is straightforward. To prove the upper bound, let the vertex set of Q be $\{v_1, v_2, \ldots, v_\omega\}$ and let I be the independent set of G. We first observe that no vertex of I is adjacent to all vertices Q (else G has a clique of order $\omega + 1$). Also, no vertex of Q is adjacent to all vertices of I (otherwise, $\Delta(G) = n - 1$). Moreover, since $\gamma(G) \geq 2$ because of $G \in \mathcal{L}$, $|I| \geq 2$. From the previous facts, we conclude that any vertex of Q has a non-neighbor in I. For coloring the vertices of G, we follow the strategy used in the proof of Theorem 2 (see [11]). Every vertex v_i is assigned the color i. Thus ω colors are used for the vertices of Q. For a vertex $v \in V \setminus Q$, the color $c(v)$ of v is given by $c(v) = \min\{i : 1 \leq i \leq \omega, v_i \notin N(v), v_{i+1(\mathrm{mod}\omega)} \in N(v)\}$. This coloring has been shown in [11] to be dominated whereby Theorem 2 was obtained. For each $i \in \{1, \ldots, \omega\}$, let V_i denote the color class with color i. If this dominated coloring is global, then we are done. For otherwise, let $j = \min\{i : V_i$ has no anti-dominating vertex$\}$. Note that each color class with no anti-dominating vertex contains at least two vertices (else, it contains a single vertex and such a vertex belongs to Q which has necessarily a non-neighbor in I). In addition, all vertices of a color class with no anti-dominating vertex belong to I except one that belongs to Q). Now for each color class (if any) V_k with $k \neq j$ having no anti-dominating pick a single vertex belonging to I and color all these picked vertices with color $\omega + 1$. Clearly, since V_j has no anti-dominating vertex, vertex v_j having already the color j is adjacent to all the vertices placed in the new color class $V_{\omega+1}$. Hence the color class $V_{\omega+1}$ is dominated by v_j and every vertex of $V_j \setminus \{v_j\}$ becomes an anti-dominating vertex of $V_{\omega+1}$. In that case, we color v_j with color $\omega + 2$ and let $V_{\omega+2} = \{v_j\}$. Clearly $V_{\omega+2}$ is dominated and it has all vertices of $V_j \setminus \{v_j\}$ as anti-dominating vertices. This new coloring is a global dominated coloring of G with at most $\omega + 2$ number of colors. The graphs on 6 vertices shown in Figure 5 have global dominated chromatic numbers $\omega, \omega + 1, \omega + 2$. ■

5. OPEN PROBLEMS

We conclude this paper by a list some open problems.

1. Characterize the graphs G with $\chi_{gdom}(G) = n - 1$.
2. Characterize the graphs G with $\chi_{gdom}(G) = 4$.
3. Give a structural characterization of graphs G with $\chi_{gdom}(G) = 2\chi_{dom}(G)$ and $\chi_{gdom}(G) = \chi_{dom}(G)$.
4. Characterize the triangle-free graphs G with $\chi_{gdom}(G) = 2\gamma(G) + 1$.
5. Characterize the split graphs with χ_{gdom} number ω, $\omega + 1$ and $\omega + 2$.
6. We know that $\chi_{dom}(G) \leq \chi_{gdom}(G) \leq 2\chi_{dom}(G)$ for every graph $G \in \mathcal{L}$. One can attempt to find a realization result to get a graph G with $\chi_{gdom}(G) = \chi_{dom}(G) + k$, where k is an integer such that $0 \leq k \leq \chi_{dom}(G)$.

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