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STRONG CHROMATIC INDEX OF CLAW-FREE GRAPHS WITH EDGE WEIGHT SEVEN[1]

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Abstract

Let G be a graph and k a positive integer. A strong k-edge-coloring of G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that for any two edges e and e' that are either adjacent to each other or adjacent to a common edge, $\phi(e) \neq \phi(e')$. The strong chromatic index of G is the minimum integer k such that G has a strong k-edge-coloring. The edge weight of G is defined to be $\max\{d(u) + d(v) : uv \in E(G)\}$, where $d(v)$ denotes the degree of v in G. In this paper, we prove that every claw-free graph with edge weight at most 7 has strong chromatic index at most 9, which is sharp.

Keywords: strong edge coloring, strong chromatic index, claw-free graph, edge weight.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

We consider only undirected simple graphs in this paper. Let $G = (V(G), E(G))$ be a graph. For $v \in V(G)$, let $N(v) = \{u \in V(G) : uv \in E(G)\}$ (respectively, $N[v] = N(v) \cup \{v\}$) denote the open (respectively, closed) neighborhood of v and let $d(v) = |N(v)|$ be the degree of v. Denote by $\Delta(G)$ the maximum degree of G. The weight of an edge uv in G is defined as $d(u) + d(v)$, and the *edge weight* of G is defined to be the maximum weight among all its edges. For convenience, we use the abbreviation $[1, n]$ for $\{1, 2, \ldots, n\}$, where n is any positive integer.

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Let e and e' be two edges of G. If e and e' are adjacent to each other, we say that the distance between e and e' is 1, and if they are not adjacent but both of them are adjacent to a common edge, we say they are at distance 2. Given a positive integer k, a *strong k-edge-coloring* of G is a mapping $\phi : E(G) \to [1, k]$ such that for any two edges e and e' that are at distance 1 or 2, $\phi(e) \neq \phi(e')$. The *strong chromatic index* of G, denoted by $\chi'_s(G)$, is the minimum integer k such that G has a strong k-edge-coloring.

The concept of strong edge coloring, first introduced by Fouquet and Jolivet [8], can be used to model the conflict-free channel assignment problem in radio networks [15, 16]. In 1985, Erdős and Nešetřil [6, 7] proposed the following conjecture about the upper bound of $\chi'_s(G)$ in terms of $\Delta(G)$, which if proven, would be tight.

Conjecture 1 (Erdős and Nešetřil [6, 7]). *If G is a graph with maximum degree $\Delta(G)$, then*

$$\chi'_s(G) \leq \begin{cases} 5\Delta(G)^2/4, & \text{if } \Delta(G) \text{ is even,} \\ (5\Delta(G)^2 - 2\Delta(G) + 1)/4, & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

The conjecture is clearly true for $\Delta(G) \leq 2$. The case $\Delta(G) = 3$ was verified by Andersen [1] in 1992, and independently by Horák, Qing, and Trotter [11] in 1993. However, this conjecture is still open for $\Delta(G) \geq 4$. In 1990, Horák [10] first established an upper bound of 23 for $\Delta(G) = 4$ and then Cranston [4] improved this bound to 22 in 2006. More recently, Huang, Santana and Yu [12] proved the upper bound of 21, which is the best bound so far. For the case $\Delta(G) = 5$, Zang [18] showed that $\chi'_s(G) \leq 37$. This upper bound of 37 is eight larger than the conjectured bound of 29, but it is the only progress as we know. For larger $\Delta(G)$, the problem is widely open.

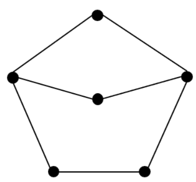
Notice that the maximum degree is a parameter of graphs based on which we are able to form a hierarchy of all simple graphs. Moreover, this hierarchy could be refined by another parameter of graphs, namely the edge weight. For instance, all *subcubic graphs*, i.e., the graphs with maximum degree less than or equal to 3, are properly contained in the class of graphs with edge weight at most 6. In the meantime, any graph with edge weight at most 6 other than the complete bipartite graph $K_{1,5}$ belongs to the class of graphs with maximum degree 4. It is clear that the strong chromatic index of a graph is bounded below by its edge weight minus one. Thus it might be interesting to explore the upper bounds for the strong chromatic indices of graphs in terms of their edge weights.

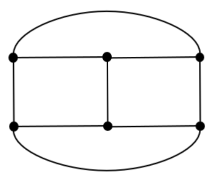
In 2008, Wu and Lin [17] proved that the strong chromatic index of any graph with edge weight at most 5 other than the graph H_0 is at most 6, where H_0 is a graph with strong chromatic index 7, as shown in Figure 1. That is the first solved non-trivial case with respect to the edge weight. Moreover, the upper bound of 6 is the best possible for such graphs. In 2020, inspired by this result, Chen, Huang, Yu and Zhou [3] formulated a conjecture in terms of the edge weight as follows.

Conjecture 2 (Chen, Huang, Yu and Zhou [3]). *If G is a graph with edge weight $W \geq 5$, then*

$$\chi'_s(G) \leq \begin{cases} 5\lceil W/4\rceil^2 - 8\lceil W/4\rceil + 3, & \text{if } W \equiv 1 \pmod 4, \\ 5\lceil W/4\rceil^2 - 6\lceil W/4\rceil + 2, & \text{if } W \equiv 2 \pmod 4, \\ 5\lceil W/4\rceil^2 - 4\lceil W/4\rceil + 1, & \text{if } W \equiv 3 \pmod 4, \\ 5\lceil W/4\rceil^2, & \text{if } W \equiv 0 \pmod 4. \end{cases}$$

In [3], the authors also indicated that the bounds given in Conjecture 2, if true, would be tight. Also, the bounds in Conjecture 2 are precisely the same as the bounds in Conjecture 1 when the edge weights are even. Moreover, they proved that the strong chromatic index of any graph with edge weight at most 6 (respectively, at most 7) is 10 (respectively, 15). In particular, the bound of 10 is tight and the bound of 15 is two larger than the conjectured bound of 13. For graphs with edge weight 8, Chen, Chen, Zhao and Zhou [2] gave an upper bound of 21 (recall that the conjectured bound is 20), which is a natural extension of the current best bound 21 for graphs with maximum degree 4 in [12] as each such graph has edge weight 8.





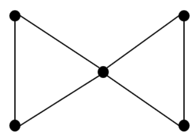


Figure 1. The graph H_0. Figure 2. The 3-prism. Figure 3. The graph H_1.

An induced subgraph of G isomorphic to the complete bipartite graph $K_{1,3}$ is called a *claw* of G. A graph is called *claw-free* if it has no claw. It is well known that claw-free graphs constitute an important superclass of the class of line graphs. In 2020, Dębski, Junosza-Szaniawski and Śleszyńska-Nowak [5] presented the following upper bound for the strong chromatic indices of claw-free graphs.

Theorem 3 (Dębski, Junosza-Szaniawski and Śleszyńska-Nowak [5]). *For any claw-free graph G with maximum degree $\Delta(G)$, $\chi'_s(G) \leq \frac{9}{8}\Delta(G)^2 + \Delta(G)$.*

In 2022, Lv, Li and Zhang [14] proved that, for any claw-free subcubic graph G other than the triangular prism, $\chi'_s(G) \leq 8$. Please see Figure 2 for the triangular prism (also called the 3-prism). Notice that the 3-prism is a claw-free cubic graph with its strong chromatic index being equal to 9. Recently in our manuscript [13], we improved the bound of 8 to 7 for such graphs and constructed infinitely many graphs attaining the upper bound 7. Notice that any connected claw-free graph with edge weight at most 6 is either a subcubic claw-free graph or isomorphic to H_1 (see Figure 3), hence we immediately have the following corollary.

Corollary 4. *Let G be a claw-free graph with edge weight at most 6. If no component of G is isomorphic to the 3-prism, then $\chi'_s(G) \leq 7$.*

In this paper, we study the class of claw-free graphs with edge weight at most 7 and prove the following theorem.

Theorem 5. *Let G be a claw-free graph. If the edge weight of G is at most 7, then $\chi'_s(G) \leq 9$.*

Remark 6. It is easy to verify that the class of claw-free graphs with edge weight at most 7 is properly contained in the class of graphs with maximum degree at most 4. Recall that for the general case when $\Delta(G) = 4$, the upper bound in Conjecture 1 is 20, and the best bound proved by Huang, Santana and Yu [12] is 21. However, the upper bound for the claw-free graphs with maximum degree 4 given in Theorem 3 is 22, which is even weaker than the general case. For graphs with edge weight at most 7, the upper bound in Conjecture 2 is 13, the bound proved by Chen, Huang, Yu and Zhou [3] is 15, while our bound for claw-free graphs in Theorem 5 is 9.

Remark 7. The 3-prism shown in Figure 2 indicated the sharpness of the upper bound 9. Notice that the edge weight of the 3-prism is 6. In fact, there are also some claw-free graphs with edge weight 7 attaining the upper bound 9, please refer to Figure 4.

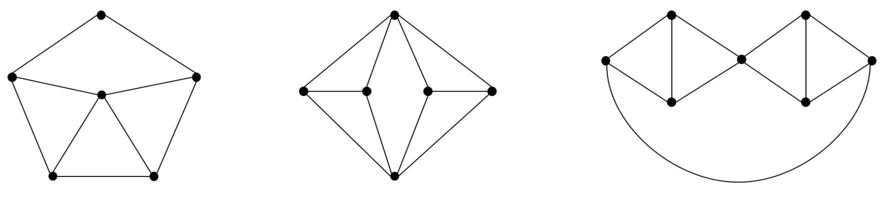


Figure 4. The three graphs with edge weight 7 and strong chromatic index 9.

The rest of this paper is organized as follows. Section 2 introduces some definitions and notation. We investigate the basic properties of a minimal counterexample in Section 3 and then complete the proof of Theorem 5 in Section 4. In the last section, we summarize our results and propose some further research directions.

2. Preliminaries and Notation

For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X. An *i-vertex* is a vertex of degree i in G. For any $uv \in E(G)$ with $d(u) = i$, we say that u is an *i-neighbor* of v. In addition, we denote by K_n the complete graph with n vertices.

We use α, β, γ to denote colors and ϕ, ψ, σ to denote edge colorings. Given two distinct edges e and e' of G, we say that e *sees* e' in G if they are distance 1 or 2 apart. An edge coloring of a graph G is *good*, if it is a strong edge coloring of G using at most 9 colors. A *good partial coloring* of a graph G is a good coloring ϕ of some subgraph H of G such that $\phi(e) \neq \phi(e')$ if e and e' see each other in G.

Let ϕ be a good partial coloring of G. We say that e *sees* a color α in ϕ, if e sees an edge e' for which $\phi(e') = \alpha$. For $e \in E(G)$, let $F_\phi(e)$ denote the set of colors that e sees in ϕ. We denote by $\bar{E}_\phi$ the set of edges of G not already assigned colors by ϕ. For $e \in \bar{E}_\phi$, let $A_\phi(e)$ denote the set of colors that e does not see in ϕ. It is clear that $A_\phi(e) = [1, 9] \setminus F_\phi(e)$ for any $e \in \bar{E}_\phi$.

The key to extending a good partial coloring ϕ of G to the whole graph is to color all edges in $\bar{E}_\phi$ properly. Often we will use Hall's theorem [9] to do that, by which $\{A_\phi(e) : e \in \bar{E}_\phi\}$ has a system of distinct representatives (abbreviated SDR) if and only if $|\bigcup_{e\in M} A_\phi(e)| \geq |M|$ for every $M \subseteq \bar{E}_\phi$. Whenever $\{A_\phi(e) : e \in \bar{E}_\phi\}$ has an SDR, ϕ can be extended to a good coloring of G. In this situation, we will say that we can obtain a good coloring of G by SDR.

3. The Propertities of Minimal Counterexample

Let G be a counterexample to Theorem 5 with $|V(G)|$ being minimized, that is, G is a claw-free graph with edge weight at most 7 and $\chi'_s(G) \geq 10$. According to Corollary 4, the edge weight of G must be equal to 7. By the minimality of G, it is clear that G is a connected graph with at least 10 edges. Recall that the maximum degree of any claw-free graph with edge weight at most 7 does not exceed 4, we have $\Delta(G) \leq 4$.

In the proof of the following lemmas, we will often show a contradiction by extending a strong 9-edge-coloring of a subgraph (or the modified subgraph) of G to the whole graph. It should be pointed out that the subgraphs we consider have the property that any two of its edges at distance greater than 2 must also be at distance greater than 2 in G.

Lemma 8. *G contains no K_4.*

Proof. Suppose that G contains a K_4 with four vertices v_1, v_2, v_3, v_4 (see Figure 5). Since the edge weight of G is 7, exactly one of the four vertices v_1, v_2, v_3, v_4 is a 4-vertex. Without loss of generality, let $d(v_4) = 4$ and v_5 be the fourth neighbor of v_4. It is obvious that $d(v_5) \leq 3$. By the minimality of G, the subgraph

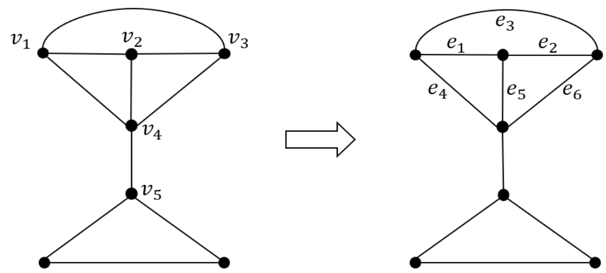


Figure 5. The graph G with a K_4.

$G' = G - \{v_1, v_2, v_3\}$ has a good coloring ϕ, which is also a good partial coloring of G with the six edges $e_1, e_2, \ldots, e_6$ being uncolored. Note that each of the edges e_1, e_2, e_3 sees exactly one colored edge and each of the edges e_4, e_5, e_6 sees at most three colored edges, and thus the coloring ϕ can be easily extended to a good coloring of G, a contradiction. ∎

Lemma 9. *G has no 1-vertices.*

Proof. Suppose that v is a 1-vertex and w is the neighbor of v. Then $G - v$ has a good coloring ϕ. If w is a 4-vertex, then since G is claw-free, w is contained in a K_4, contradicting Lemma 8. Thus the degree of w is at most 3. Again since G is claw-free, it is easy to see that the edge vw sees at most 6 edges in $G - v$. So we can color vw properly to obtain a good coloring of G, a contradiction. ∎

Lemma 10. *G has no 2-vertices.*

Proof. If not, let v_0 be a 2-vertex in G with two neighbors v_1 and v_2. Without loss of generality, assume that $d(v_1) \geq d(v_2)$. We use e_1 and e_2 to denote v_0v_1 and v_0v_2, respectively. By the minimality of G, the graph $G - v_0$ has a good coloring ϕ. If $d(v_1) = 2$, then $d(v_2) = 2$. It is not difficult to see that $|A_\phi(e_1)| \geq 5$ and $|A_\phi(e_2)| \geq 5$. If $d(v_1) = 4$, then we must have $v_1v_2 \in E(G)$ as otherwise v_1 is contained in a K_4, which is a contradiction to Lemma 8. Notice that $d(v_2) \leq 3$ (refer to Figure 6(a)), it is easy to check that $|A_\phi(e_1)| \geq 2$ and $|A_\phi(e_2)| \geq 3$. In both cases, we can extend ϕ to a good coloring of G.

Now we assume that $d(v_1) = 3$. By Lemma 9, $d(v_2) \in \{2, 3\}$. If $d(v_2) = 2$ or $v_1v_2 \in E(G)$, it is straightforward to check that both e_1 and e_2 have at least two available colors and so coloring e_1 and e_2 greedily extends ϕ to a good coloring of G. So we assume that $d(v_2) = 3$ and $v_1v_2 \notin E(G)$. If v_1 and v_2 have at least two common neighbors, as shown in Figure 6(b) and (c), it is easy to check that the number of available colors for e_1 is always at least 2 and the same is true for e_2. Thus we can get a good coloring of G based on ϕ by coloring e_1 and e_2 greedily. We next consider the case $N(v_1) \cap N(v_2) = \{v_0\}$.

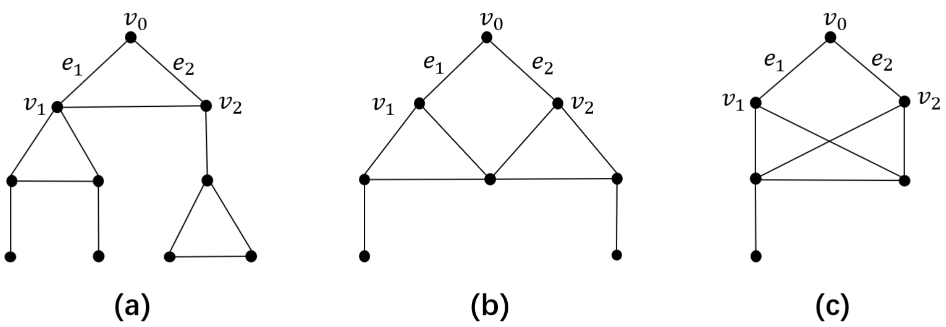


Figure 6. The graph G with a 2-vertex v_0 contained in a small cycle.

Let $N(v_i) = \{v_0, u_i, w_i\}$ for $i = 1, 2$. It is clear that the four vertices u_1, w_1, u_2, w_2 are all distinct. Since G is claw-free, $u_1w_1, u_2w_2 \in E(G)$. Now let $G^* = (G - v_0) \cup \{v_1v_2\}$. Observe that G^* is also a claw-free graph with edge weight 7, by the minimality of G, G^* has a good coloring ψ. Ignoring v_1v_2 in ψ yields a good partial coloring of G with two uncolored edges e_1 and e_2, which we refer to it as σ. Notice that $\psi(v_1v_2) \in A_\sigma(e_1) \cap A_\sigma(e_2)$, we have $|A_\sigma(e_1)| \geq 1$ and $|A_\sigma(e_2)| \geq 1$. If $|A_\sigma(e_1) \cup A_\sigma(e_2)| \geq 2$, then e_1 and e_2 can be colored properly by SDR and we are done. So we assume that $A_\sigma(e_1) = A_\sigma(e_2) = \{\psi(v_1v_2)\}$. It follows from $|A_\sigma(e_1)| = |A_\sigma(e_2)| = 1$ that both v_1 and v_2 must have exactly one 4-neighbor and one 3-neighbor. Suppose that $d(u_1) = d(u_2) = 4$ and $d(w_1) = d(w_2) = 3$. Refer to Figure 7 for the names of the vertices and edges of G. Under our assumptions, it is easily seen that the four edges f_1, f_2, f_3, f_4 receive different colors in σ and $\{\sigma(h_1), \sigma(h_2), \sigma(h_3), \sigma(h_4)\} = \{\sigma(h_5), \sigma(h_6), \sigma(h_7), \sigma(h_8)\} = [1, 9] \setminus \{\psi(v_1v_2), \sigma(f_1), \sigma(f_2), \sigma(f_3), \sigma(f_4)\}$.

Now, if $|N(u_1) \cap N(w_1)| = 2$ (see Figure 7(a)), note that any edge $e \in E(G - v_0) \setminus \{f_1, h_3\}$ sees f_1 in G if and only if it also sees h_3 in G, we can exchange the colors of f_1 and h_3 in σ to get a new partial coloring σ^* of G, in which $A_{\sigma^*}(e_1) = \{\psi(v_1v_2)\}$ and $A_{\sigma^*}(e_2) = \{\psi(v_1v_2), \sigma(f_1)\}$. Consequently, e_1 and e_2 can be colored properly, a contradiction.

So by symmetry, we may assume that $|N(u_1) \cap N(w_1)| = 1$ and $|N(u_2) \cap N(w_2)| = 1$ (see Figure 7(b)). Let $N(u_1) = \{v_1, w_1, x_1, x_2\}$ and $N(w_1) = \{v_1, u_1, y_1\}$. It is obvious that the three vertices x_1, x_2, y_1 are all distinct and $x_1x_2 \in E(G)$. Moreover, by Lemma 8, $d(y_1) \leq 3$. Let $\alpha = \sigma(f_1)$ and $\beta = \sigma(f_2)$. Observe that each of the three edges f_1, f_2, h_3 sees h_1, h_2 and h_4. Erasing the colors of f_2 yields a new partial coloring of G, where the only uncolored edges are f_2, e_1 and e_2. This coloring is still denoted by σ. We next extend σ to a good coloring of G. If $A_\sigma(f_2) = \{\psi(v_1v_2), \beta\}$, then we must have $d(y_1) = 3$ and $\{\sigma(g_3), \sigma(g_4)\} = \{\sigma(f_3), \sigma(f_4)\}$, implying that $\sigma(f_1) \notin \{\sigma(g_3), \sigma(g_4)\}$. Thus we

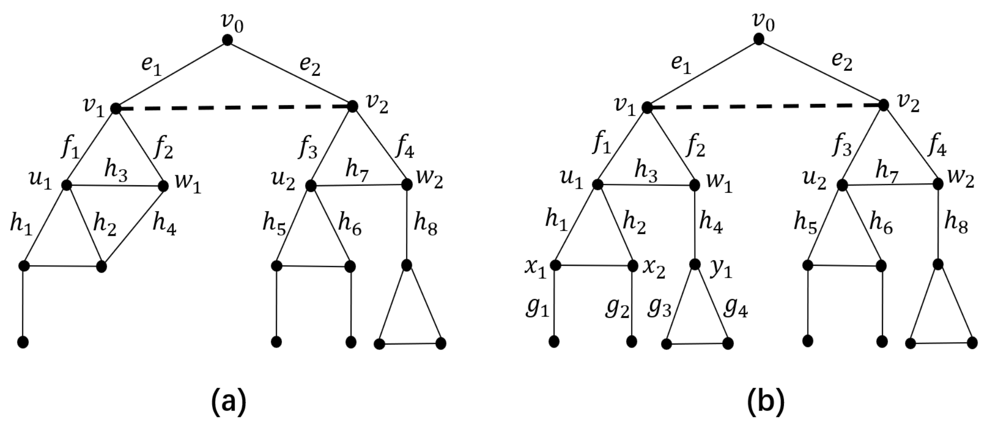


Figure 7. The graph G with a 2-vertex v_0.

can exchange the colors of f_1 and h_3 in σ and then color f_2 with β, which results in a good partial coloring σ^* of G with $A_{\sigma^*}(e_1) = \{\psi(v_1v_2)\}$ and $A_{\sigma^*}(e_2) = \{\psi(v_1v_2), \alpha\}$. Therefore, σ^* can be further extended to a good coloring of G by coloring e_1 and e_2 greedily. If $A_\sigma(f_2) \neq \{\psi(v_1v_2), \beta\}$, then there exists some color $\gamma \in A_\sigma(f_2) \cap \{\sigma(f_3), \sigma(f_4)\}$, we color f_2 with γ to get a new partial coloring σ^* of G, in which $A_{\sigma^*}(e_1) = A_{\sigma^*}(e_2) = \{\psi(v_1v_2), \beta\}$. Therefore, coloring e_1 and e_2 greedily gives a good coloring of G. This completes the proof of the lemma. ■

By Lemmas 9 and 10, each vertex in G is either a 3-vertex or a 4-vertex. Recall that G is a claw-free graph with edge weight 7, for any 4-vertex v in G, each neighbor of v is a 3-vertex and $|E(G[N(v)])| \geq 2$. Also, for any 3-vertex v in G, we have $|E(G[N(v)])| \geq 1$. Moreover, any 3-cycle in G contains at most one 4-vertex. In what follows, we will discuss the number of 4-neighbors of a 3-vertex in G.

Lemma 11. *Each* 3*-vertex in* G *has at least one* 4*-neighbor.*

Proof. Suppose that v_0 is a 3-vertex with three 3-neighbors v_1, v_2, v_3. Let $e_1 = v_0v_1$, $e_2 = v_0v_2$ and $e_3 = v_0v_3$. Since G has at least 10 edges, we may assume that $|E(G[N(v_0)])| \leq 2$. As otherwise, G is isomorphic to K_4, a contradition. By the minimality of G, $G - v_0$ has a good coloring ϕ, which is also a good partial coloring of G with three uncolored edges e_1, e_2, e_3.

If $|E(G[N(v_0)])| = 2$, without loss of generality, assume that $v_1v_2, v_2v_3 \in E(G)$. Let u_1 and u_3 be the third neighbor of v_1 and v_3, respectively. Recall that G is a claw-free graph with at least 10 edges, u_1 and u_3 must be distinct. Futher, by Lemma 8, both u_1 and u_3 are 3-vertices. Then it is easy to verify that

$|A_\phi(e_1)| \geq 3$, $|A_\phi(e_2)| \geq 5$ and $|A_\phi(e_3)| \geq 3$. It follows that ϕ can be extended to a good coloring of G.

If $|E(G[N(v_0)])| = 1$, without loss of generality, assume that $v_1v_2 \in E(G)$. Let u_1 and u_2 be the third neighbor of v_1 and v_2, respectively, and let $N(v_3) = \{v_0, u_3, w_3\}$. The remainder of the proof is divided into the following two cases according to whether vertices u_1 and u_2 are distinct or not.

Case 1. $u_1 = u_2$. Please refer to Figure 8 for this case. It is easy to check that both e_1 and e_2 see at most 7 edges and e_3 sees at most 9 edges in $G - v_0$. Moreover, we have $A_\phi(e_1) = A_\phi(e_2)$. If $|A_\phi(e_1)| = 2$, then we can recolor f_2 with the color $\phi(f_4)$ as f_2 sees only four edges f_1, f_3, h_1, h_2 in $G - v_0$ and the seven edges $f_1, f_2, f_3, f_4, f_5, h_1, h_2$ receive different colors in ϕ. This results in a new partial coloring ψ of G, in which $|A_\psi(e_1)| = 3$, $|A_\psi(e_2)| = 3$ and $|A_\psi(e_3)| \geq 1$. Therefore, greedily coloring e_3, e_2, e_1 in this order gives rise to a good coloring of G.

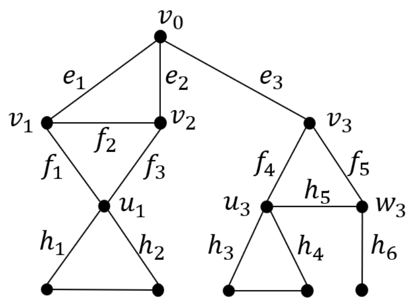


Figure 8. The graph G in *Case* 1.

If $|A_\phi(e_1)| \geq 3$, we may assume that $|A_\phi(e_3)| = 0$ as otherwise ϕ can be easily extended to a good coloring of G by SDR. Thus the nine edges f_1, f_2, f_3, f_4, f_5, h_3, h_4, h_5, h_6 are assigned different colors. This together with $|A_\phi(e_1)| \geq 3$ and the five edges f_1, f_2, f_3, h_1, h_2 receiving different colors in ϕ implies that $\{\phi(h_1), \phi(h_2)\}$ $\cap\{\phi(f_4), \phi(f_5)\} \neq \emptyset$. Now we recolor f_2 with some color $\alpha \in [1, 9] \setminus \{\phi(f_1), \phi(f_2)$, $\phi(f_3), \phi(h_1), \phi(h_2)\}$ to get a new good partial coloring of G, calling it ψ. Notice that $\alpha \in \{\psi(f_4), \psi(f_5), \psi(h_3), \psi(h_4), \psi(h_5), \psi(h_6)\}$, we have $A_\psi(e_3) = \{\phi(f_2)\}$. Moreover, it is also easy to check that $|A_\psi(e_1)| \geq 3$ and $|A_\psi(e_2)| \geq 3$. Thus ψ can be further extended to a good coloring of G.

Case 2. $u_1 \neq u_2$. If $u_1u_2 \in E(G)$, then as G is claw-free we must have $d(u_1) = d(u_2) = 3$. Please see Figure 9 for the names of vertices and edges of G. It is easy to check that $|A_\phi(e_1)| \geq 2$ and $|A_\phi(e_2)| \geq 2$. Notice that the colors of h_1 and h_3 are different, $|F_\phi(e_1) \cap F_\phi(e_2)| \leq 6$. It follows that $|A_\phi(e_1) \cup A_\phi(e_2)| \geq 3$. We may assume that $A_\phi(e_3) = \emptyset$ as otherwise a good coloring of G can be easily obtained by SDR. In this situation, the colors of

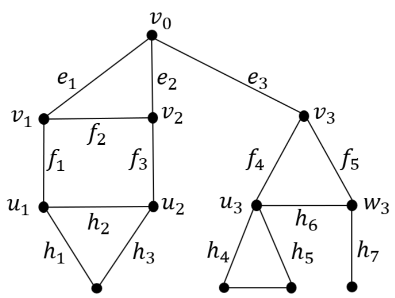


Figure 9. The graph G with $u_1u_2 \in E(G)$ in *Case* 2.

the nine edges $f_1, f_2, f_3, f_4, f_5, h_4, h_5, h_6, h_7$ are different from each other. By recoloring f_2 with some color $\alpha \in [1, 9] \setminus \{\phi(f_1), \phi(f_2), \phi(f_3), \phi(h_1), \phi(h_2), \phi(h_3)\}$, we get a new partial coloring ψ of G, in which $|A_\psi(e_1)| \geq 2$, $|A_\psi(e_2)| \geq 2$ and $|A_\psi(e_3)| = 1$. Observe that $|A_\psi(e_1) \cup A_\psi(e_2)| \geq 3$, we can further extend ψ to a good coloring of G by SDR.

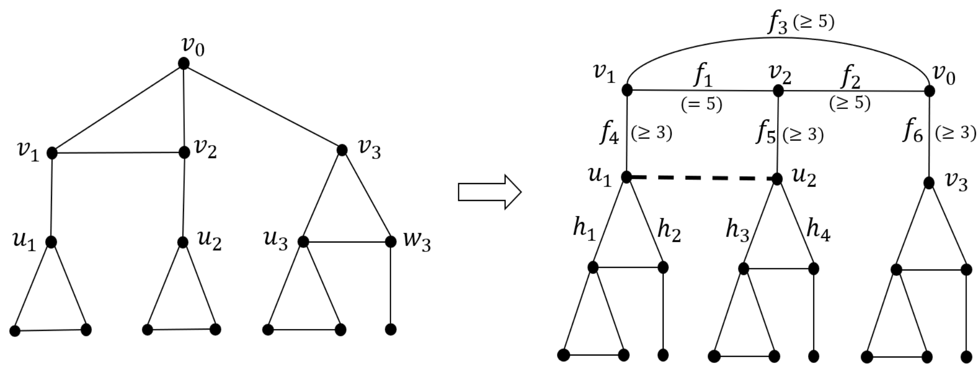


Figure 10. The graph G with $u_1u_2 \notin E(G)$ in *Case* 2.

If $u_1u_2 \notin E(G)$, then, by Lemma 8, $d(u_1) = d(u_2) = 3$. Please refer to Figure 10. Let G^* be the graph obtained from G by deleting the three vertices v_0, v_1, v_2 and adding a new edge u_1u_2. Observe that G^* is indeed a claw-free graph with edge weight at most 7, by the minimality of G, G^* has a good coloring ψ. Ignoring u_1u_2 in ψ yields a good partial coloring of G, where the only uncolored edges are $f_1, f_2, f_3, f_4, f_5, f_6$. This coloring is still called ψ. It is straightforward to check that $|A_\psi(f_1)| = 5$, $|A_\psi(f_2)| \geq 5$, $|A_\psi(f_3)| \geq 5$ and $|A_\psi(f_i)| \geq 3$ for each $i \in \{4, 5, 6\}$. Notice that h_1, h_2, h_3, h_4 receive different colors under ψ, we have $|A_\psi(f_2) \cup A_\psi(f_3)| = 7$, and so the remaining six edges can be colored properly by SDR. As a result, ψ can be extended to a good coloring of G. This finishes the proof. ■

Lemma 12. *For any edge uv of G with $d(u) = 4$ and $d(v) = 3$, $N(u) \cap N(v) \neq \emptyset$.*

Proof. Let uv be an edge of G with $d(u) = 4$ and $d(v) = 3$. If $N(u) \cap N(v) = \emptyset$, then since G is claw-free, u is contained in a K_4, contradiciting Lemma 8. Therefore, $N(u) \cap N(v) \neq \emptyset$ and the lemma holds. ■

Lemma 12 implies directly that each 3-vertex in G has at most two 4-neighbors.

Lemma 13. *Each 3-vertex in G has exactly one 4-neighbor.*

Proof. If not, then by Lemma 11, we may assume that v_0 is a 3-vertex with two 4-neighbors v_1, v_3 and one 3-neighbor v_2. It follows from Lemma 12 that $v_1v_2 \in E(G)$ and $v_2v_3 \in E(G)$. We use u_1, u_2 and u_3, u_4 to denote the other two neighbors of v_1 and v_3, respectively. Clearly, u_1, u_2, u_3, u_4 are all 3-vertices. Since G is claw-free, $u_1u_2 \in E(G)$ and $u_3u_4 \in E(G)$. We denote by w_i the third neighbor of u_i for each $i \in [1, 4]$.

First we assume that w_1, w_2, w_3, w_4 are all 3-vertices. Then it is not difficult to check that $\{w_1, w_2\} \cap \{w_3, w_4\} = \emptyset$. However, it is possible that $w_1 = w_2$ or $w_3 = w_4$. But whether they are equal to each other or not will not affect the following arguments.

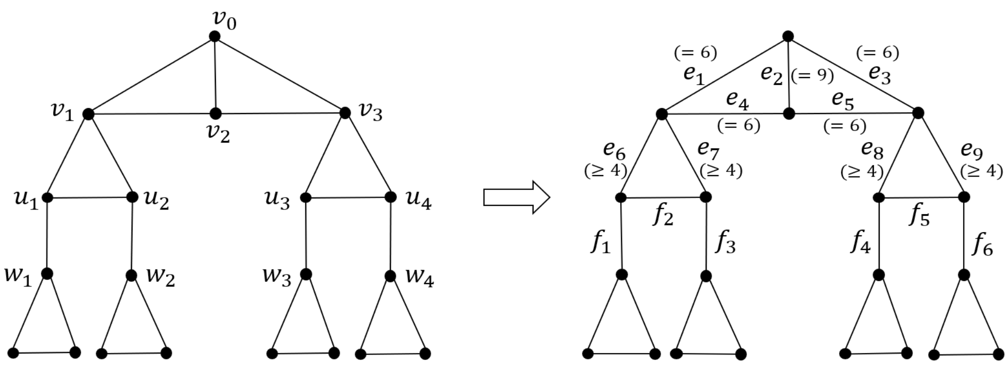


Figure 11. The graph G in Lemma 13.

By the minimality of G, the graph $G - N[v_0]$ has a good coloring ϕ, which is indeed a good partial coloring of G with the nine uncolored edges $e_1, e_2, \ldots, e_9$. As for the names of vertices and edges in G, please refer to Figure 11. It is straightforward to check that $|A_\phi(e_i)| = 6$ for each $i \in \{1, 3, 4, 5\}$, $|A_\phi(e_2)| = 9$ and $|A_\phi(e_j)| \geq 4$ for each $j \in \{6, 7, 8, 9\}$. Observe that $A_\phi(e_1) = A_\phi(e_4)$ and $A_\phi(e_3) = A_\phi(e_5)$.

If $|A_\phi(e_1) \cup A_\phi(e_3)| = 6$, we have $\{\phi(f_1), \phi(f_2), \phi(f_3)\} = \{\phi(f_4), \phi(f_5), \phi(f_6)\}$. Since f_2 sees at most six edges in the graph $G - N[v_0]$, we can always modify ϕ by recoloring f_2 with some color $\alpha \in [1, 9] \setminus (F_\phi(f_2) \cup \{\phi(f_2)\})$ so that there are seven colors available for either e_1 or e_3. Therefore we may assume that $|A_\phi(e_1) \cup A_\phi(e_3)| \geq 7$.

Now, if $A_\phi(e_6) \cap A_\phi(e_8) = \emptyset$, then $|A_\phi(e_6) \cup A_\phi(e_8)| \geq 8$. This implies that the remaining nine edges $e_1, e_2, \ldots, e_9$ can be colored properly by SDR. If $A_\phi(e_6) \cap A_\phi(e_8) \neq \emptyset$, suppose that $\beta \in A_\phi(e_6) \cap A_\phi(e_8)$. Coloring e_6 and e_8 with the same color β results in a new good partial coloring of G, which we refer to it as ψ. It is easy to check that $|A_\psi(e_i)| = 5$ for each $i \in \{1,3,4,5\}$, $|A_\psi(e_2)| = 8$, $|A_\psi(e_7)| \geq 3$ and $|A_\psi(e_9)| \geq 3$. Moreover, it follows from the assumption $|A_\phi(e_1) \cup A_\phi(e_3)| \geq 7$ that $|A_\psi(e_1) \cup A_\psi(e_3)| \geq 6$. As a result, ψ can be further extended to a good coloring of G by SDR.

Now we assume that at least one of the four vertices w_1, w_2, w_3, w_4 are 4-vertices. Notice that if $d(w_1) = 4$, then by Lemma 12, w_1 and u_1 must have the commom neighbor u_2, which implies that $w_1 = w_2$. Similarly, if $d(w_3) = 4$, then $w_3 = w_4$. It is not difficult to see that the above arguments also apply to this case and so we omit the proof here. ■

4. Proof of Theorem 5

***Proof of Theorem* 5.** Choose a 4-vertex v_0 in G with four 3-neighbors v_1, v_2, v_3, v_4. Let e_i denote the edge v_0v_i for each $i \in [1,4]$. Since G is a claw-free graph with at least 10 edges, $2 \leq |E(G[N(v_0)])| \leq 3$.

If $|E(G[N(v_0)])| = 3$, without loss of generality, let $v_1v_2, v_2v_3, v_3v_4 \in E(G)$. We denote by u_1 and u_4 the third neighbor of v_1 and v_4, respectively. Since both v_1 and v_4 have a 4-neighbor v_0, by Lemma 13, $d(u_1) = d(u_4) = 3$. If $u_1 = u_4$, then $G[N[u_1]]$ is a claw. Therefore, we have $u_1 \neq u_4$. By the minimality of G, the graph $G - v_0$ has a good coloring ϕ, which is also a good partial coloring of G with the four uncolored edges e_1, e_2, e_3, e_4. And it is easy to check that $|A_\phi(e_1)| \geq 2$, $|A_\phi(e_2)| \geq 4$, $|A_\phi(e_3)| \geq 4$ and $|A_\phi(e_4)| \geq 2$. Thus greedily coloring e_1, e_4, e_2, e_3 in this order gives a good coloring of G, a contradiction.

Now we assume that $|E(G[N(v_0)])| = 2$ and $v_1v_2, v_3v_4 \in E(G)$. We use u_i to denote the third neighbor of v_i for each $i \in [1,4]$. Because v_1, v_2, v_3, v_4 have the common 4-neighbor v_0, by Lemma 13, u_1, u_2, u_3, u_4 are all 3-vertices. Moreover, for each $i \in [1,4]$, u_i has exactly one 4-neighbor in G, denoted by w_i. Recall that G is claw-free, $u_i \neq u_j$ for any two integers $i \in \{1,2\}$ and $j \in \{3,4\}$. We also have $u_1 \neq u_2$ (respectively, $u_3 \neq u_4$), as otherwise w_1 (respectively, w_3) must be contained in a K_4, contradicting Lemma 8. Therefore u_1, u_2, u_3, u_4 are four distinct vertices. Please refer to Figure 12 for the names of vertices and edges in G. Notice that, for two different integers $i, j \in [1,4]$, it is possible that u_iu_j is an edge of G. However, the existence of these edges will not affect the following arguments.

Recall that u_1, u_2, u_3, u_4 are four 3-vertices and each of them has a 4-neighbor, for each $i \in [1,4]$, u_i has degree at most 1 in $G[\{u_1, u_2, u_3, u_4\}]$. Without loss of generality, we may assume that $u_1u_4 \notin E(G)$ and $u_2u_3 \notin E(G)$. Now let $G^* = (G - N[v_0]) \cup \{u_1u_4, u_2u_3\}$. Clearly, G^* is also a claw-free graph with edge weight 7. By the minimality of G, the graph G^* has a good coloring ϕ. Ignoring the colors of u_1u_4 and u_2u_3 in ϕ yields a good partial coloring of G with ten edges $e_1, e_2, \ldots, e_{10}$ being uncolored. This partial coloring is denoted by ψ. The remainder of the proof is divided into two cases according to whether $\phi(u_1u_4)$ and $\phi(u_2u_3)$ are equal or not.

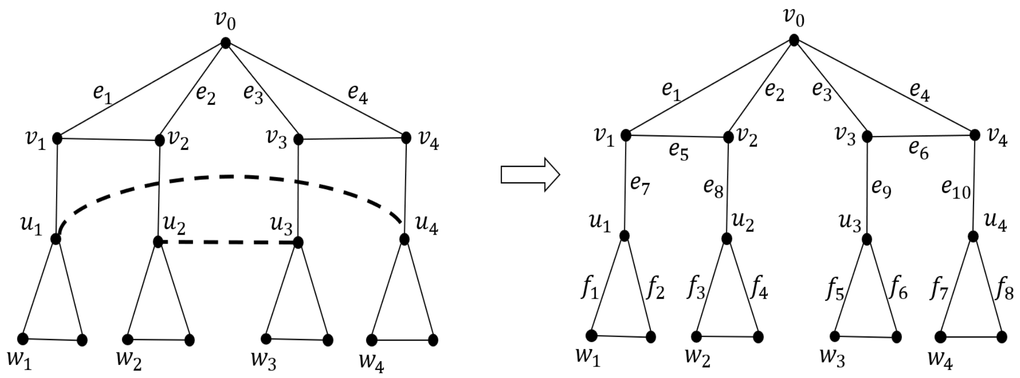


Figure 12. The graph G in the proof of Theorem 5.

If $\phi(u_1u_4) \neq \phi(u_2u_3)$, let $\alpha = \phi(u_1u_4)$ and $\beta = \phi(u_2u_3)$. By coloring e_7 and e_{10} with the same color α and e_8 and e_9 with the same color β, we extend ψ to another good partial coloring σ of G, in which $|A_\sigma(e_i)| \geq 5$ for each $i \in [1,4]$ and $|A_\sigma(e_j)| \geq 3$ for each $j \in \{5,6\}$. Notice that $u_1u_4, f_1, f_2, f_7, f_8$ receive five different colors in ϕ, $F_\sigma(e_1) \cap F_\sigma(e_4) = \{\alpha, \beta\}$. This implies that $|A_\sigma(e_1) \cup A_\sigma(e_4)| = 7$. Therefore, the remaining six edges can be colored properly by SDR and so σ can be further extended to a good coloring of G, a contradiction.

If $\phi(u_1u_4) = \phi(u_2u_3) = \alpha$, we can color e_8 and e_9 with the same color α to obtain another good partial coloring of G, calling it σ. Notice that $\alpha \neq \sigma(f_i)$ for each $i \in [1,8]$. It is easy to check that $|A_\sigma(e_i)| = 6$ for each $i \in [1,4]$, $|A_\sigma(e_j)| \geq 4$ for each $j \in \{5,6\}$, $|A_\sigma(e_7)| \geq 2$ and $|A_\sigma(e_{10})| \geq 2$. Moreover, it is also not difficult to check that $|A_\sigma(e_1) \cup A_\sigma(e_4)| = 8$ and $|A_\sigma(e_2) \cup A_\sigma(e_3)| = 8$. Thus we can extend σ to a good coloring of G by SDR, again a contradiction. This completes the proof of Theorem 5. ■

5. Final Remark

Let G be a connected claw-free graph with edge weight at most 7. This paper proves that $\chi'_s(G) \leq 9$. The three graphs shown in Figure 4 and the 3-prism shown in Figure 2 indicate that this upper bound 9 is sharp. However, we do not find infinitely many claw-free graphs with edge weight 7 and their strong chromatic indices attaining the upper bound 9. Thus, it is natural to ask the following question.

Question 1. *Let G be a connected claw-free graph with edge weight 7 that is not isomorphic to the 3-prism as well as the three graphs shown in Figure 4. Is it true that $\chi'_s(G) \leq 8$?*

It is also interesting to investigate the list version of strong edge coloring.

Question 2. *Is the strong list-chromatic index of any claw-free graph with edge weight 7 at most 9?*

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