# GRAPHS WITH TOTAL MUTUAL-VISIBILITY NUMBER ZERO AND TOTAL MUTUAL-VISIBILITY IN CARTESIAN PRODUCTS

Jing Tian

*School of Science, Zhejiang University of Science and Technology*
*Hangzhou, Zhejiang 310023, PR China, and*
*School of Mathematics*
*Nanjing University of Aeronautics & Astronautics*
*Nanjing, Jiangsu 210016, PR China*

**e-mail:** jingtian526@126.com

and

Sandi Klavžar

*Faculty of Mathematics and Physics*
*University of Ljubljana, Slovenia,*
*Institute of Mathematics, Physics and Mechanics*
*Ljubljana, Slovenia, and*
*Faculty of Natural Sciences and Mathematics*
*University of Maribor, Slovenia*

**e-mail:** sandi.klavzar@fmf.uni-lj.si

## Abstract

If $G$ is a graph and $X \subseteq V(G)$, then $X$ is a total mutual-visibility set if every pair of vertices $x$ and $y$ of $G$ admits a shortest $x, y$-path $P$ with $V(P) \cap X \subseteq \{x, y\}$. The cardinality of a largest total mutual-visibility set of $G$ is the total mutual-visibility number $\mu_{\mathrm{t}}(G)$ of $G$. Graphs with $\mu_{\mathrm{t}}(G) = 0$ are characterized as the graphs in which every vertex is the central vertex of a convex $P_3$. The total mutual-visibility number of Cartesian products is bounded and several exact results proved. For instance, $\mu_{\mathrm{t}}(K_n \,\square\, K_m) = \max\{n, m\}$ and $\mu_{\mathrm{t}}(T \,\square\, H) = \mu_{\mathrm{t}}(T)\mu_{\mathrm{t}}(H)$, where $T$ is a tree and $H$ an arbitrary graph. It is also demonstrated that $\mu_{\mathrm{t}}(G \,\square\, H)$ can be arbitrary larger than $\mu_{\mathrm{t}}(G)\mu_{\mathrm{t}}(H)$.

**Keywords:** mutual-visibility set, total mutual-visibility set, bypass vertex, Cartesian product of graphs, tree.

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## 1. Introduction

Let $G = (V(G), E(G))$ be a graph and $X \subseteq V(G)$. Then two vertices $x$ and $y$ of $X$ are $X$-*visible*, if there exists a shortest $x, y$-path $P$ such that $V(P) \cap X = \{x, y\}$. The set $X$ is a *mutual-visibility set* if its vertices are pairwise $X$-visible. The cardinality of a largest mutual-visibility set is the *mutual-visibility number* $\mu(G)$ of $G$ and a largest mutual-visibility set is called a $\mu$-*set* of $G$.

These concepts were introduced and studied for the first time by Di Stefano in [4]. The study was motivated in many ways, notably by the role that mutual-visibility plays in problems arising in the context of distributed computing by mobile entities, and by the fact that vertices in mutual-visibility may represent entities on some nodes of a network that want to efficiently communicate in such a way that the messages do not pass through other entities. We also mention related concepts in computer science that have been explored: distributed computing by mobile entities [5], mutual-visibility tasks [3], and fat entities modelled as disks in the Euclidean plane [12]. A related graph theory topic is the general position in graphs, introduced in [10, 15] and extensively studied by now, cf. [11, 17]. The general position problem was investigated in detail on Cartesian product graphs [6, 8, 9, 13, 14].

In [1], the mutual-visibility problem was studied on Cartesian products and on triangle-free graphs, while in [2] the focus was on strong products. In these studies, the following tools have proven to be extremely useful. We say that $X \subseteq V(G)$ is a *total mutual-visibility set* of $G$ if every pair of vertices $x$ and $y$ of $G$ is $X$-visible, that is, there exists a shortest $x, y$-path $P$ with $V(P) \cap X \subseteq \{x, y\}$. Note that, by definition, the empty set is a total mutual-visibility set. The cardinality of a largest total mutual-visibility set of $G$ is the *total mutual-visibility number* $\mu_{\mathrm{t}}(G)$ of $G$. Hence, if the empty set is the only total mutual-visibility set of $G$, then $\mu_{\mathrm{t}}(G) = 0$. Further, $X$ is a $\mu_{\mathrm{t}}$-*set* if it is a total mutual-visibility set with $|X| = \mu_{\mathrm{t}}(G)$.

As observed in [1], there exist graphs $G$ with $\mu_{\mathrm{t}}(G) = 0$. Partial results on such graphs were proved, in particular cactus graphs $G$ with $\mu_{\mathrm{t}}(G) = 0$ were characterized. In Section 3 we characterize general graphs $G$ for which $\mu_{\mathrm{t}}(G) = 0$ holds as the graphs that contain no bypass vertices. We introduce the latter concept in Section 2, where we also give further definitions needed, recall some know results, and add a few additional preparatory results. In Section 4 we prove bounds on the total mutual-visibility number of Cartesian product graphs and demonstrate their sharpness by several exact results. For instance, $\mu_{\mathrm{t}}(K_n \,\square\, K_m) = \max\{n, m\}$ and $\mu_{\mathrm{t}}(T \,\square\, H) = \mu_{\mathrm{t}}(T)\mu_{\mathrm{t}}(H)$, where $T$ is a tree. In Section 5 we continue by the investigation of Cartesian products by considering the estimate $\mu_{\mathrm{t}}(G \,\square\, H) \leq \mu_{\mathrm{t}}(G)\mu_{\mathrm{t}}(H)$. It holds in many cases, but on the other hand we show that $\mu_{\mathrm{t}}(G \,\square\, H)$ can be arbitrary larger than $\mu_{\mathrm{t}}(G)\mu_{\mathrm{t}}(H)$. We con-

clude by suggesting some open problems and directions for further investigation.

## 2. PRELIMINARIES AND BYPASS VERTICES

We recall here some definitions, for other undefined graph theory concepts, see [7]. All the graphs in this paper are simple and connected, unless stated otherwise. The order of a graph $G = (V(G), E(G))$ is denoted by $n(G)$, the minimum degree of $G$ is denoted by $\delta(G)$, and the subgraph of $G$ induced by $S \subseteq V(G)$ is denoted by $G[S]$. $S \subseteq V(G)$ is an *independent set* if $G[S]$ is an edgeless graph. The cardinality of a largest independent set is the *independence number* $\alpha(G)$ of $G$. A subgraph $H$ of $G$ is *isometric* if for each vertices $x, y \in V(H)$ the distance between them is the same in $H$ and in $G$. Further, $H$ is *convex* if for each vertices $x, y \in V(H)$, all shortest $x, y$-paths in $G$ lie completely in $H$. The *girth* $g(G)$ of a graph $G$ with a cycle is the length of a shortest cycle of $G$. If $\tau$ and $\tau'$ are two graph invariants, then we say that $G$ is a $(\tau, \tau')$-*graph* if $\tau(G) = \tau'(G)$.

The Cartesian product $G \square H$ of two graphs $G$ and $H$ has the vertex set $V(G \square H) = V(G) \times V(H)$, vertices $(g, h)$ and $(g', h')$ are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. Given a vertex $h \in V(H)$, the subgraph of $G \square H$ induced by the set $\{(g, h) : g \in V(G)\}$ is a *G-layer* and is denoted by $G^h$. *H-layers* ${}^g H$ are defined analogously. Each $G$-layer and each $H$-layer is isomorphic to $G$ and $H$, respectively. Moreover, it is also well-known that each layer of a Cartesian product is its convex subgraph, see [7, Lemma 12.3]. We will use this fact later on many times, sometimes implicitly. More generally, a subgraph $K$ of a Cartesian product $G \square H$ is convex if and only if the projections of $K$ on $G$ and on $H$ are convex [7, Proposition 13.3].

We next recall some known results on the (total) mutual-visibility number. (Recall that a block graph is a graph in which all blocks are complete.)

**Proposition 1** [4, Corollary 4.3]. *Let $T$ be a tree and $L$ the set of its leaves. Then $L$ is a mutual-visibility set and $\mu(T) = |L|$.*

**Proposition 2** [2, Proposition 3.3]. *Block graphs (and hence trees and complete graphs) and graphs containing a universal vertex are all $(\mu, \mu_t)$-graphs.*

**Proposition 3** [2, Proposition 3.1]. *Let $G$ be a graph. If $V(G) = \bigcup_{i=1}^{k} V_i$, where $G[V_i]$ is a convex subgraph of $G$ and $\mu_t(G[V_i]) = 0$ for each $i \in [k]$, then $\mu_t(G) = 0$.*

Here and later on, $[k]$ stands for $\{1, \ldots, k\}$. The following straightforward fact will be used several times later on.

**Proposition 4.** *If $X$ is a total mutual-visibility set of a graph $G$ and $Y \subseteq X$, then $Y$ is also a total mutual-visibility set of $G$.*

To conclude the preliminaries we introduce the following concept which appears essential in the investigation of the total mutual-visibility concept. We say that a vertex $u$ of a graph $G$ is a *bypass vertex* if $u$ is not the middle vertex of a convex $P_3$ in $G$. Otherwise, $u$ is a *non-bypass vertex*. Let $BP(G)$ denote the set of all bypass vertices of $G$ and let $bp(G) = |BP(G)|$. For instance, if $n \geq 1$, then $BP(K_n) = V(K_n)$ because there are no convex paths $P_3$ in a complete graph. Hence $bp(K_n) = n$. Similarly, $bp(K_{n,m}) = n + m$ for $n, m \geq 2$. Indeed, if $u, v, w$ induce a $P_3$ in $K_{n,m}$, then since $n, m \geq 2$, there exists a common neighbor $v'$ of $u$ and $w$, where $v' \neq v$, hence no $P_3$ in $K_{n,m}$ is convex. On the other hand, if $n \geq 5$, then $bp(C_n) = 0$.

The basic fact on bypass vertices is the following.

**Lemma 5.** *If $u$ is a non-bypass vertex of a graph $G$ and $X$ is a total mutual-visibility set of $G$, then $u \notin X$.*

**Proof.** Since $u$ is a non-bypass vertex of $G$, it is the central vertex of a convex $P_3$. If $x$ and $y$ are the neighbors of $u$ on this $P_3$ and $u$ would lie in $X$, then $x$ and $y$ would not be $X$-visible. Hence $u \notin X$. ∎

Lemma 5 implies that

(1) $$\mu_{\mathrm{t}}(G) \leq bp(G).$$

This bound is sharp. If $T$ is a tree with $n(T) \geq 3$, then using Proposition 1 and Proposition 2 we get $\mu_{\mathrm{t}}(T) = bp(T)$. Similarly, $\mu_{\mathrm{t}}(K_n) = bp(K_n) = n$. On the other hand, consider complete bipartite graphs $K_{n,m}$, $n, m \geq 3$. From [2, Corollary 3.6] and [4, Theorem 4.9] we know that $\mu_{\mathrm{t}}(K_{n,m}) = \mu(K_{n,m}) = n + m - 2$, but $bp(K_{n,m}) = n + m$. The graph $G$ from Figure 1 is another sporadic example for which the bound (1) is strict. We have $\mu_{\mathrm{t}}(G) = 1$ and $bp(G) = 2$, where $BP(G) = \{g_6, g_7\}$.
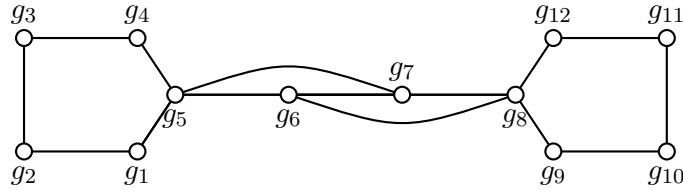


Figure 1. Graph $G$.

## 3.   Graphs with $\mu_{\mathrm{t}} = 0$

In this section we characterize graphs with $\mu_{\mathrm{t}} = 0$ and give several applications of the characterization. We begin by two lemmas, where the first one follows immediately from Proposition 4, and the second one being of independent interest.

**Lemma 6.** *Let $G$ be a graph. Then $\mu_t(G) = 0$ if and only if for every $x \in V(G)$, the set $\{x\}$ is not a total mutual-visibility set of $G$.*

**Lemma 7.** *Let $G$ be a graph with $n(G) \geq 2$ and $u \in V(G)$. Then $\{u\}$ is a total mutual-visibility set of $G$ if and only if $u$ is a bypass vertex.*

**Proof.** First assume that $u$ is a bypass vertex of $G$. Then by definition, $G - u$ is an isometric subgraph of $G$. It follows that in $G$ each pair of vertices is connected by shortest path avoiding $u$, hence $\{u\}$ is a total mutual-visibility set.

The other direction follows by Lemma 5. ∎

Note that Lemma 7 does not extend to two vertices. For instance, two opposite vertices of $C_4$ are bypass vertices, but they do not form a total mutual-visibility set.

The announced characterization now reads as follows.

**Theorem 8.** *Let $G$ be a graph with $n(G) \geq 2$. Then $\mu_t(G) = 0$ if and only if $bp(G) = 0$.*

**Proof.** If $\mu_t(G) = 0$, then $bp(G) = 0$ by Lemmas 6 and 7. Conversely, if $bp(G) = 0$, then Lemma 7 says that $G$ has no singleton total mutual-visibility set, and so by Lemma 6, $\mu_t(G) = 0$. ∎

Clearly, to check whether a vertex is a bypass vertex is algorithmically simple. Hence the characterization of graphs $G$ with $\mu_t(G) = 0$ from Theorem 8 is efficient.

Note that Theorem 8 implies that if $\mu_t(G) = 0$, then $\delta(G) \geq 2$. Another consequence of the theorem is the following.

**Corollary 9.** *Let $G$ be a graph with $g(G) \geq 5$. Then $\mu_t(G) = 0$ if and only if $\delta(G) \geq 2$.*

The Petersen graph applies to Corollary 9. In addition, the corollary implies the characterization of cactus graphs $G$ with $\mu_t(G) = 0$ as given in [2, Proposition 3.2]. For a sporadic example of a graph $G$ with $\mu_t(G) = 0$ see Figure 2.

As another application of Theorem 8 we next determine the theta graphs with $\mu_t = 0$. For any positive integer $k \geq 2$ and $1 \leq p_1 \leq \cdots \leq p_k$, the *theta graph* $\Theta(p_1, \ldots, p_k)$ is the graph consisting of two vertices $a$ and $b$ which are joined by $k$ internally disjoint paths of respective lengths $p_1, \ldots, p_k$, where $p_2 \geq 2$. (We add that several authors use the name theta graph restricted to the case $k = 3$ in our definition, cf. [19].)

**Corollary 10.** *If $1 \leq p_1 \leq \cdots \leq p_k$, $k \geq 2$, $p_2 \geq 2$, then $\mu_t(\Theta(p_1, \ldots, p_k)) = 0$ if and only if the following cases hold:*
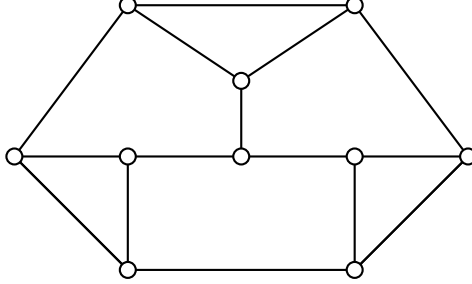
(i) $p_1 = 1$ *and* $p_2 \geq 4$;

Figure 2. A graph with the total mutual-visibility number 0.

(ii) $p_1 = 2$ and $p_2 \geq 3$;

(iii) $p_1 \geq 3$.

**Proof.** Set $\Theta = \Theta(p_1, \ldots, p_k)$ and let $P_1, \ldots, P_k$ be the respective paths of $\Theta$ connecting $a$ and $b$. Then $C^i = P_1 \cup P_i$ is an isometric cycle of $\Theta$ for each $i \in \{2, \ldots, k\}$. From this fact we infer that $\mu_t(\Theta) = 0$ if and only if each of the cycles $C^i$ is of length at least 5. This condition then yields the cases (i)–(iii). ■

We conclude the section with a description of Cartesian products with $\mu_t = 0$.

**Theorem 11.** *If $G$ and $H$ are graphs, then $\mu_t(G \square H) = 0$ if and only if $\mu_t(G) = 0$ or $\mu_t(H) = 0$.*

**Proof.** Assume first that $\mu_t(G \square H) = 0$. Suppose on the contrary that $\mu_t(G) \geq 1$ with a total mutual-visibility set $X$ and $\mu_t(H) \geq 1$ with a total mutual-visibility set $Y$. By Proposition 4, there exist two vertices $x \in X$ and $y \in Y$ such that the sets $\{x\}$ and $\{y\}$ are total mutual-visibility set of $G$ and $H$. Then we claim that $U = \{u\}$ with $u = (x, y)$ is a total mutual-visibility set of $G \square H$.

Let $v, w$ be arbitrary vertices of $G \square H$. We need to show that they are $U$-visible. If $v = u$ or $w = u$, there is nothing to be proved. If $v$ and $w$ lie in the same $G$-layer, then $v$ and $w$ are $U$-visible because their projections onto $G$ are $\{x\}$-visible and since layers in Cartesian product graphs are convex. By the same argument we see that $v$ and $w$ are $U$-visible when $v$ and $w$ lie in the same $H$-layer. The last case to consider is when $v$ and $w$ neither lie in a common $G$-layer or a common $H$-layer. Then $w = (g_1, h_1)$ and $v = (g_2, h_2)$, where $g_1 \neq g_2$ and $h_1 \neq h_2$. Then it is well known that there exist two internally disjoint $v, w$-shortest paths. Since at least one of these two paths does not contain $u$, the vertices $v$ and $w$ are $U$-visible in $G \square H$ also in this case. Hence we conclude that $\mu_t(G \square H) \geq 1$.

To prove the converse, we may assume, without loss of generality, that $\mu_t(G) = 0$. Since $G^h$ is a convex subgraph of $G \square H$ for any $h \in V(H)$, by Proposition 3, we have $\mu_t(G \square H) = 0$. By symmetry, the same result also holds if $\mu_t(H) = 0$, as desired. ■

Theorem 11 extends to an arbitrary number of factors as follows.

**Corollary 12.** *If $G = G_1 \square \cdots \square G_k$, where $k \geq 2$, then $\mu_t(G) = 0$ if and only if $\mu_t(G_i) = 0$ for at least one $i \in [k]$.*

## 4. Total Mutual-Visibility in Cartesian Products

In this section we consider the total mutual-visibility number of Cartesian product graphs. In the previous section we have seen that if $\mu_t(G) = 0$ or $\mu_t(H) = 0$, then $\mu_t(G \square H) = 0$. Hence we may restrict our attention here to factor graphs with the total mutual-visibility number at least 1.

To give general bounds we need the following concept. The *independent total mutual-visibility number* $\mu_{it}(G)$ of $G$ is the cardinality of a largest independent total mutual-visibility set. Setting $\ell(G)$ to be the number of leaves of $G$, it follows from definitions that for any graph $G$, $\ell(G) \leq \mu_{it}(G) \leq \min\{\mu_t(G), \alpha(G)\}$. From Propositions 1 and 2 we know that the leaves set $L$ is a total mutual-visibility set of a tree $T$ and $\mu_t(G) = |L|$. Hence, if $n(T) \geq 3$, then $\mu_t(T) = |L| = \mu_{it}(T)$. Note in addition that $\mu_{it}(K_n) = 1$ while $\mu_t(K_n) = n$.

**Theorem 13.** *If $G$ and $H$ are graphs of order at least 2, $\mu_t(G) \geq 1$, and $\mu_t(H) \geq 1$, then*

$$\max\{\mu_{it}(H)\mu_t(G), \mu_{it}(G)\mu_t(H)\} \leq \mu_t(G \square H) \leq \min\{\mu_t(G)n(H), \mu_t(H)n(G)\}.$$

**Proof.** Let $I_G$ be an independent total mutual-visibility set of $G$ with $|I_G| = \mu_{it}(G)$, and let $X_H$ be a $\mu_t$-set of $H$. Set $U = I_G \times X_H$. We claim that $U$ is a total mutual-visibility set of $G \square H$.

Let $(g, h)$ and $(g', h')$ be arbitrary, different vertices of $G \square H$. If $(g, h)$ and $(g', h')$ lie in the same $G$-layer or the same $H$-layer, then $x$ and $y$ are $U$-visible as layers are convex subgraphs of the product. Hence assume in the rest that $g \neq g'$ and $h \neq h'$.

Let $g = g_0, g_1, \ldots, g_k = g'$ be the consecutive vertices of a shortest $g, g'$-path in $G$ whose internal vertices are not in $I_G$. Similarly, let $h = h_0, h_1, \ldots, h_\ell = h'$ be the consecutive vertices of a shortest $h, h'$-path in $H$ whose internal vertices are not in $X_H$. Assume first that $k = 1$, that is, $gg' \in E(G)$. If $g \notin I_G$, then the path $(g, h) = (g_0, h_0), (g_0, h_1), \ldots, (g_0, h_\ell), (g_1, h_\ell) = (g', h')$ demonstrates that $(g, h)$ and $(g', h')$ are $U$-visible. If $g \in I_G$, then $g' \notin I_G$ and then the path $(g, h) = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell) = (g', h')$ demonstrates that $(g, h)$ and $(g', h')$ are $U$-visible. Assume in the following that $k \geq 2$. Consider now the path $P$ with the consecutive vertices

$$(g, h) = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell), (g_2, h_\ell), \ldots, (g_k, h_\ell) = (g', h').$$

The path $P$ is a shortest $(g, h), (g', h')$-path with no internal vertex in $U$. Note that in all the above cases it is possible that $\ell = 1$ which happens if $hh' \in E(H)$.

This proves the claim which implies that $\mu_t(G \square H) \geq |U| = \mu_{it}(G)\mu_t(H)$. By the commutativity of the Cartesian product, $\mu_t(G \square H) \geq \mu_{it}(H)\mu_t(G)$ and the lower bound follows.

Let $X$ be a total mutual-visibility set of $G \square H$. Since each $G$-layer $G^h$ is convex in $G \square H$ we have $|X \cap V(G^h)| \leq \mu_t(G)$, hence $\mu_t(G \square H) \leq \mu_t(G)n(H)$. Analogously $\mu_t(G \square H) \leq \mu_t(H)n(G)$.  ∎

In the rest of the section we give several exact results on $\mu_t(G \square H)$ which also demonstrate that the bounds of Theorem 13 can be attained. We begin with the following sharpness result for the lower bound.

**Corollary 14.** *If $\mu_t(G \square H) = 1$, then $\mu_t(G) = 1$ and $\mu_t(H) = 1$.*

***Proof.*** By Theorem 11 we have $\mu_t(G) \geq 1$ and $\mu_t(H) \geq 1$. Hence by the lower bound of Theorem 13 we conclude that $\mu_t(G) = 1$ and $\mu_t(H) = 1$.  ∎

The converse of Corollary 14 does not hold. For instance, consider the theta graph $\Theta(2, 2, 4)$ as presented in Figure 3.
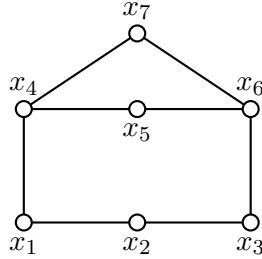


Figure 3. The theta graph $\Theta(2, 2, 4)$.

It is straightforward to see that $\mu_t(\Theta(2, 2, 4)) = 1$ and that $\{x_5\}$ and $\{x_7\}$ are $\mu_t$-sets. In the product $\Theta(2, 2, 4) \square \Theta(2, 2, 4)$ one can see that $\{(x_5, x_5), (x_7, x_7)\}$ is a total mutual-visibility set, hence $\mu_t(\Theta(2, 2, 4) \square \Theta(2, 2, 4)) \geq 2$.

[1, Corollary 3.7] asserts that $\mu(K_n \square K_m) = z(n, m; 2, 2)$, where $z(n, m; 2, 2)$ is the Zarankiewitz's number. To determine $z(n, m; 2, 2)$ is a notorious open problem [16, 18]. Interestingly, the total mutual-visibility number of Cartesian products of complete graphs can be determined as follows, which further demonstrates that the lower bound of Theorem 13 is sharp.

**Proposition 15.** *If $n, m \geq 2$, then $\mu_t(K_n \square K_m) = \max\{n, m\}$.*

**Proof.** Note first that the vertices of a single $K_n$-layer (or a single $K_m$-layer) form a total mutual-visibility set. Hence $\mu_{\mathrm{t}}(K_n \square K_m) \geq \max\{n, m\}$. To prove the other inequality, set $V(K_n) = [n]$ and $V(K_m) = [m]$, so that $V(K_n \square K_m) = [n] \times [m]$. Suppose without loss of generality that $n \geq m$. Let $U$ be an arbitrary total mutual-visibility set of $G \square H$. If each $K_n$-layer contains at most one vertex of $U$ there is nothing to be proved. Assume now that some $K_n$-layer contains (at least) two vertices of $U$. By the symmetry we may assume that $(1, 1) \in U$ and $(2, 1) \in U$. We claim first that $(i, j) \notin U$, where $i, j \geq 2$. Indeed, if $(i, j) \in U$, then the vertices $(1, j)$ and $(i, 1)$ are not $U$-visible. We claim second that $(1, j) \notin U$ for $2 \leq j \leq m$. Indeed, if $(1, j) \in U$ for some $j \geq 2$, then the vertices $(1, 1)$ and $(2, j)$ are not visible. We conclude that if $(1, 1), (2, 1) \in U$, then $U \subseteq V(K_n^1)$ and consequently $|U| \leq n$. Analogously we see that if some $K_m$-layer contains (at least) two vertices of $U$, then $|U| \leq m$. In any case, $\mu_{\mathrm{t}}(K_n \square K_m) \leq \max\{n, m\}$. ■

The next result (when $s \in \{3, 4\}$) also demonstrates sharpness of the lower bound of Theorem 13.

**Proposition 16.** *If $s \geq 3$ and $n \geq 3$, then*

$$\mu_{\mathrm{t}}(C_s \square K_n) = \left\{ \begin{array}{ll} 0; & s \geq 5, \\ n; & \textit{otherwise.} \end{array} \right.$$

**Proof.** If $s \geq 5$, then Corollary 9 and Theorem 11 yield $\mu_{\mathrm{t}}(C_s \square K_n) = 0$. In addition, $\mu_{\mathrm{t}}(C_3 \square K_n) = n$ by Proposition 15. Hence assume that $s = 4$ in the remaining proof.

By Theorem 13 we have $\mu_{\mathrm{t}}(C_4 \square K_n) \geq n$. It remains to demonstrate that $\mu_{\mathrm{t}}(C_4 \square K_n) \leq n$. Let $R$ be a $\mu_{\mathrm{t}}$-set of $C_4 \square K_n$. If each $C_4$-layer contains at most one vertex of $R$, then $\mu_{\mathrm{t}}(C_4 \square K_n) \leq n$ holds clearly. Suppose next that $R$ contains at least two vertices from the some $C_4$-layer. We may without loss of generality assume that $(1, n), (2, n) \in R$. Suppose that there exists another vertex $(i, j) \in R$, where $i \in [4] \backslash [2]$ and $j \in [n-1]$. If $i = 3$, then the two vertices $(2, j)$ and $(3, n)$ are not $R$-visible. Similarly, the vertices $(1, j)$ and $(4, n)$ are not $R$-visible. This would thus mean that $|R| = 2$. We conclude that $\mu_{\mathrm{t}}(C_4 \square K_n) = n$. ■

**Theorem 17.** *If $T$ is tree with $n(T) \geq 3$ and $H$ is a graph with $n(H) \geq 2$, then $\mu_{\mathrm{t}}(T \square H) = \mu_{\mathrm{t}}(T)\mu_{\mathrm{t}}(H)$.*

**Proof.** The lower bound $\mu_{\mathrm{t}}(T \square H) \geq \mu_{\mathrm{t}}(T)\mu_{\mathrm{t}}(H)$ follows by Theorem 13 and the fact that $\mu_{\mathrm{t}}(T) = \mu_{\mathrm{it}}(T)$.

To prove that $\mu_{\mathrm{t}}(T \square H) \leq \mu_{\mathrm{t}}(T)\mu_{\mathrm{t}}(H)$, consider an arbitrary $\mu_{\mathrm{t}}$-set $R$ of $T \square H$. Let $t \in V(T)$ be a vertex with $\deg_T(t) \geq 2$. Then $t$ is a non-bypass vertex of $T$. Hence, if $h \in V(H)$, then $(t, h)$ is a non-bypass vertex of $T \square H$, thus

$(t, h) \notin R$ by Lemma 5. Therefore, $R \cap V(^tH) = \emptyset$. So $R$ contains only vertices in $H$-layers corresponding to the leaves of $T$. Since each such layer can contain at most $\mu_t(H)$ vertices of $R$ we conclude that $\mu_t(T \square H) \leq |R| \leq \mu_t(T)\mu_t(H)$.  ∎

As a consequence of Theorem 17 we obtain the following result which demonstrates sharpness of the upper bound of Theorem 13.

**Corollary 18.** *If $T$ is tree with $n(T) \geq 3$, then $\mu_t(T \square K_n) = n \cdot \mu_t(T)$.*

5.    On the Inequality $\mu_t(G \square H) \leq \mu_t(G)\mu_t(H)$

All the exact results obtained in Section 4 fulfil the bound

(2)                            $$\mu_t(G \square H) \leq \mu_t(G)\mu_t(H).$$

Hence one may wonder whether the upper bound of Theorem 13 can be improved/replaced by (2). Before we answer the question, we prove another result where (2) holds.

A graph $G$ is a *generalized complete graph* if it is obtained by the join of an isolated vertex with a disjoint union of $k \geq 1$ complete graphs [13]. We further say that $G$ is a *non-trivial generalized complete graph* if $k \geq 2$. Note that if $G$ is a non-trivial generalized complete graph, then $\mu_t(G) = n(G) - 1$.

**Theorem 19.** *If $G$ and $H$ are two non-trivial generalized complete graphs, then*

$$\mu_t(G \square H) \leq \mu_t(G)\mu_t(H) = (n(G) - 1)(n(H) - 1).$$

*Moreover, the equality holds if and only if $G$ or $H$ is isomorphic to a star.*

**Proof.** Let $V(G) = \{g_1, \ldots, g_{n(G)}\}$ and $V(H) = \{h_1, \ldots, h_{n(H)}\}$. Let $g_1$ and $h_1$ be the universal vertices of $G$ and $H$, respectively.

Note that $g_1$ is a non-bypass vertex of $G$ and $h_1$ is a non-bypass vertex of $H$. It follows that each vertex from the layer $^{g_1}H$ is a non-bypass vertex of $G \square H$ as well as is each vertex from the layer $G^{h_1}$. Hence $G \square H$ contains at least $n(G) + n(H) - 1$ non-bypass vertices and so by (1),

$$\begin{aligned}
\mu_t(G \square H) &\leq bp(G \square H) \\
&\leq n(G)n(H) - (n(G) + n(H) - 1) \\
&= (n(G) - 1)(n(H) - 1).
\end{aligned}$$

To prove the equality case, by Theorem 17 we know that $\mu_t(G \square H) = (n(G) - 1)(n(H) - 1)$ if $G$ or $H$ is a star. Suppose in the rest that neither $G$ nor $H$ is a star. Then each of them contains an induced subgraph $K_3$. Without loss of generality, assume that $g_1, g_2, g_3$ induce a $K_3$ of $G$, and that $h_1, h_2, h_3$

induce a $K_3$ of $H$. Then the vertices $(g_2, h_2), (g_2, h_3), (g_3, h_3)$, and $(g_3, h_2)$ induce a $C_4$ of $G \square H$. This $C_4$ is convex because it is the Cartesian product of $K_2$ (as a subgraph of $G$) and a $K_2$ (as a subgraph of $H$) and these two $K_2$ are clearly convex in respective factors, cf. [7, Proposition 13.3]. Since at most two vertices of this $C_4$ can lie in a total mutual-visibility set of $G \square H$, we conclude that $\mu_\mathrm{t}(G \square H) < bp(G \square H)$. ∎

In the rest we demonstrate that (2) does not hold in general. For this sake we say that a graph $G$ is *bypass over-visible* if it contains an independent bypass set of vertices $U$ which contains a $\mu_\mathrm{t}$-set $U'$ as a **proper** subset. Note that since $U'$ is an independent set, a bypass over-visible graph is a $(\mu_\mathrm{it}, \mu_\mathrm{t})$-graph.

Before we state the last result of this paper, we construct two families of bypass over-visible graphs. (Another such family will be presented after the proof of the last result.)

If $k \geq 3$, then let $H_k$ be the graph obtained by attaching two pendant vertices to each of the degree $k$ vertices of $K_{2,k}$. See Figure 4 for $H_5$. Let $U_k$ be the set of all the vertices of $H_k$ of degree 1 or 2. Then $U_k$ is an independent bypass set with $|U_k| = k + 4$, while every $\mu_\mathrm{t}$-set of $H_k$ is obtained from $U_k$ by removing one degree 2 vertex. Hence each $H_k$ is a bypass over-visible graphs.
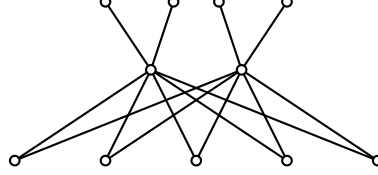


Figure 4. The graph $H_5$.

For another family of bypass over-visible graphs let $\Theta_i$ denote any theta graph $\Theta(p_1, \ldots, p_k)$, where $i \geq 2$, $k \geq 3$, and

$$2 = p_1 = \cdots = p_i < p_{i+1} \leq \cdots \leq p_k.$$

Let $a$ and $b$ be the vertices of $\Theta_i$ of degree $k$. Then $BP(\Theta_i)$ consists of the degree 2 vertices which are adjacent to $a$ and to $b$, so that $bp(\Theta_i) = i$. Note in addition that $BP(\Theta_i)$ is an independent set. On the other hand, $BP(\Theta_i)$ is not a total mutual-visibility set, but becomes such a set if an arbitrary vertex is removed from it. Hence $\mu_\mathrm{t}(\Theta_i) = i - 1$. We conclude that $\Theta_i$ is a bypass over-visible graph.

**Theorem 20.** *If $G$ and $H$ are bypass over-visible graphs, then*

$$\mu_\mathrm{t}(G \square H) > \mu_\mathrm{t}(G)\mu_\mathrm{t}(H).$$

**Proof.** Since $G$ and $H$ are bypass over-visible graphs, there exist independent bypass vertex sets $I_G$ and $I_H$ of $G$ and $H$, respectively, which contain $\mu_t$-sets $S_G$ and $S_H$ as proper subsets. Hence there exist vertices $u \in I_G \setminus S_G$ and $v \in I_H \setminus S_H$. We set $U = (S_G \times S_H) \cup \{(u,v)\}$ and claim that $U$ is a total mutual-visibility set of $G \,\square\, H$.

Consider two arbitrary vertices $x = (g,h)$ and $y = (g',h')$ from $G \,\square\, H$. Suppose first that $g = g'$. If $g \in S_G$, then $x$ and $y$ are $U$-visible because $S_H$ is a total mutual-visibility set of $H$. If $g = u$, then $x$ and $y$ are $U$-visible by Lemma 7 applied to $(u,v)$ and the layer ${}^g H$. In all the other cases $V({}^g H) \cap U = \emptyset$, hence there is nothing to prove. If $h = h'$, then $x$ and $y$ are $U$-visible by the same argument.

Assume in the rest that $g \neq g'$ and $h \neq h'$. Let $P_G : g = g_0, g_1, \ldots, g_k = g'$ be a shortest $g,g'$-path in $G$ whose internal vertices are not in $S_G$. Similar, let $P_H : h = h_0, h_1, \ldots, h_\ell = h'$ be a shortest $h,h'$-path in $H$ whose internal vertices are not in $S_H$. The copy of $P_G$ in the layer $G^w$ will be denoted by $P_G^w$ and the copy of $P_H$ in the layer ${}^z H$ will be denoted by ${}^z P_H$.

Consider first the case $k = 1$, that is, when $gg' \in E(G)$. Assume first that $(g_1, h_0) \notin U$. If $(u,v) \notin V({}^{g_1} P_H)$, then the concatenation of the edge $(g_0, h_0)(g_1, h_0)$ and the path ${}^{g_1} P_H$ is a required $x,y$-path. Suppose next that $(u,v) \in V({}^{g_1} P_H)$. Then, because $g_1 = u \in I_G$ and since $I_G$ is independent, we infer that $g_0 \notin I_G$ and thus also $g_0 \notin S_G$. Consequently, $(g_0, h_\ell) \notin U$ and then the concatenation of the path ${}^{g_0} P_H$ with the edge $(g_0, h_\ell)(g_1, h_\ell)$ is a required $x,y$-path. Assume second that $(g_1, h_0) \in U$. Then by the same argument we see that the path ${}^{g_0} P_H$ followed by the edge $(g_0, h_\ell)(g_1, h_\ell)$ is again a path which ensures that $x$ and $y$ are $U$-visible. Similarly we see that $x$ and $y$ are $U$-visible if $\ell = 1$. Note that the argument also applies when $k = \ell = 1$.

We are left with the case when $k \geq 2$ and $\ell \geq 2$. Assume first that $u \neq g_i$ for $i \in [k-1]$. Then the vertices

$$x = (g_0, h_0), (g_1, h_0), (g_1, h_1), \ldots, (g_1, h_\ell), (g_2, h_\ell), \ldots, (g_k, h_\ell) = y$$

induce a shortest $x,y$-path and the internal vertices of it are not in $U$. Hence $x$ and $y$ are $U$-visible. Similarly we see that $x$ and $y$ are $U$-visible if $v \neq h_j$ for $j \in [\ell-1]$. The remaining case is that $u = g_i$ for some $i \in [k-1]$ and $v = h_j$ for some $j \in [\ell-1]$. Then the vertices

$$x = (g_0, h_0), \ldots, (g_{i-1}, h_0), (g_{i-1}, h_1), \ldots, (g_{i-1}, h_\ell), (g_i, h_\ell), \ldots, (g_k, h_\ell) = y$$

induce a shortest $x,y$-path in $G \,\square\, H$ with no internal vertices in $U$. We conclude that in any case $x$ and $y$ are $U$-visible.                                      ∎

We next present another family of bypass over-visible graphs. Let $m \geq 1$. Then we define the graph $G_m$ as follows. The vertex set is

$$V(G_m) = \{x_0, x_1, \ldots, x_{m+2}\} \cup \{y_1, \ldots, y_m\} \cup \{z_1, \ldots, z_m\}.$$

For any $i \in [m]$ we connect $y_i$ and $z_i$ with $x_i$ and $x_{i+1}$. Finally add the edges $x_0 x_1$ and $x_{m+1} x_{m+2}$. See Figure 5 for $G_5$.
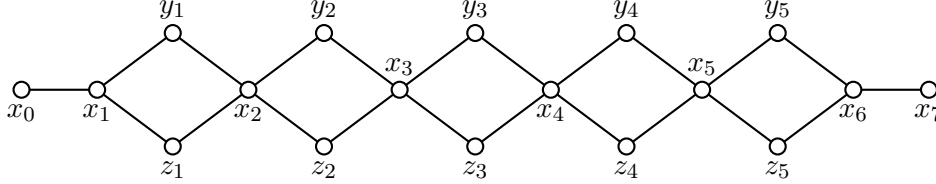


Figure 5. The graph $G_5$.

It is straightforward to see that $BP(G_m) = \{x_0, x_{m+2}\} \cup \{y_1, \ldots, y_m\} \cup \{z_1, \ldots, z_m\}$, hence $bp(G_m) = 2m + 2$ and $BP(G_m)$ is an independent set. In addition, since for any $i$, the vertices $y_i$ and $z_i$ cannot both lie in a total mutual-visibility set, the set $\{x_0, x_{m+2}, y_1, \ldots, y_m\}$ is an independent $\mu_t$-set of $G$. Hence $G_m$ is a bypass over-visible graph. Moreover, $bp(G_m) - \mu_t(G_m) = m$. Now, by a parallel construction as in the proof of Theorem 20 we find out that $\mu_t(G_m \Box G_m) \geq (m + 2)^2 + m$, hence $\mu_t(G \Box H)$ can be arbitrary larger than $\mu_t(G)\mu_t(H)$.

## 6. Concluding Remarks

There are several possibilities how to continue the investigation of this paper, here we emphasize some of them.

We have characterized the graphs $G$ with $\mu_t(G) = 0$. The next step would be to characterize the graphs $G$ with $\mu_t(G) = 1$ (and maybe also with $\mu_t(G) = 2$).

In view of (1) it would be interesting to consider the graphs $G$ with $\mu_t(G) = bp(G)$.

In this paper we had a closer look to the total mutual-visibility number of Cartesian product graphs. Some other graph operations also appear interesting for such investigations, in particular the strong product and the lexicographic product.

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