TOTAL VERTEX PRODUCT IRREGULARITY STRENGTH OF GRAPHS

Marcin Anholcer

Poznań University of Economics and Business
Institute of Informatics and Quantitative Economics
Al. Niepodległości 10, 61-875 Poznań, Poland

e-mail: m.anholcer@ue.poznan.pl

Azam Sadat Emadi

and

Doost Ali Mojdeh

Department of Mathematics, Faculty of Mathematical Sciences
University of Mazandaran, Babolsar, Iran

e-mail: math_emadi2000@yahoo.com
damojdeh@umz.ac.ir

Abstract

Consider a simple graph G. We call a labeling $w : E(G) \cup V(G) \to \{1, 2, \ldots, s\}$ (*total vertex*) *product-irregular*, if all product degrees $pd_G(v)$ induced by this labeling are distinct, where $pd_G(v) = w(v) \times \prod_{e \ni v} w(e)$. The strength of w is s, the maximum number used to label the members of $E(G) \cup V(G)$. The minimum value of s that allows some irregular labeling is called the *total vertex product irregularity strength* and denoted $tvps(G)$. We provide some general bounds, as well as exact values for chosen families of graphs.

Keywords: product-irregular labeling, total vertex product irregularity strength, vertex-distinguishing labeling.

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1. Introduction

Let G be a simple undirected graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. Let us assign a label (positive

integer) $w(x)$ to every $x \in E(G) \cup V(G)$. For each $v \in V(G)$, its *product degree* is defined by the formula

$$pd_G(v) = w(v) \times \prod_{e \ni v} w(e),$$

where $d_G(v)$ is the degree of v in G and $e \ni v$ means that the vertex v is incident to the edge e. In particular, in the case of an isolated vertex v we have $pd_G(v) = w(v)$.

We call w (*total vertex*) *product-irregular* when for each couple of vertices $u, v \in V(G), u \neq v$, we have $pd_G(u) \neq pd_G(v)$. The (*total vertex product*) *strength* of the labeling w is defined as

$$tvps_w(G) = \max\{w(x) \,|\, x \in E(G) \cup V(G)\}.$$

Consequently, the *total vertex product irregularity strength* of G is described with the formula

$$tvps(G) = \min\{tvps_w(G) \,|\, w \text{ is product-irregular}\}.$$

This concept is a variant of the product irregularity strength, introduced by Anholcer [3] and studied in [4] and [9]. This time the graphs under consideration have no isolated edges and at most one isolated vertex and the product degree is defined as the product of the labels of the incident edges only:

$$pd_G(v) = \begin{cases} \prod_{e \ni v} w(e) & \text{if } d_G(v) > 0, \\ 0 & \text{if } d_G(v) = 0. \end{cases}$$

Similarly as in the definition above, one defines the (product) strength of the labeling $ps_w(G)$ as the maximum label used and the respective graph invariant (*product irregularity strength*, $ps(G)$) is the minimum strength among all the product-irregular labelings.

The motivation to study this kind of problems were the well-known *irregularity strength* and *total vertex irregularity strength*, where the vertex weighted degrees are defined as the sums of labels (instead of products). The irregularity strength was defined in [8] and then analyzed by many authors (see e.g. [1, 2, 6, 11–13, 15, 18]). The best known general upper bound is $s(G) \leq 6n/\delta$ for graphs with $\delta \geq 6$ (see [14]). Majerski and Przybyło [16] improved it for dense graphs of sufficiently large order (they proved that in such case $s(G) \leq (4 + o(1))n/\delta + 4$). The total vertex irregularity strength was in turn introduced by Bača *et al.* in [7] and then also attracted some attention (see e.g. [6, 19]). The best general result $tvs(G) \leq 3n/\delta$ is due to Anholcer *et al.* [5]. Majerski and Przybyło [17] showed that for dense graphs of sufficiently large order the bound is $tvs(G) \leq (2 + o(1))n/\delta + 4$.

Let us focus again on the product version of the problem. Anholcer [3] proved several lower bounds for general and regular graphs. In particular, it was shown that for every graph G with n_d vertices of degree d,

$$ps(G) \geq \max_{\delta(G) \leq d \leq \Delta(G)} \left\{ \left\lceil \frac{d}{e} n_d^{1/d} - d + 1 \right\rceil \right\},$$

while for r-regular graphs on n vertices, the inequality takes the form

$$ps(G) \geq \left\lceil \frac{r}{e} n^{1/r} - r + 1 \right\rceil.$$

In the same paper it was also shown that $ps(G) \leq p(|E(G)|)$, where $p(k)$ denotes the kth prime number. A better bound for sufficiently large graphs follows from the results of Pikhurko [20]: $ps(G) \leq |E(G)|$. A substantial improvement to this bounds was proved by Darda and Hujdurović [9]. Namely, they showed that for most graphs

$$ps(G) \leq |V(G)| - 1.$$

In the case of cycles the above results can be improved. In particular, the exact values for short cycles have been presented in [3]. In the same paper it was shown that the lower bound for general cycles is given by the inequality

$$ps(C_n) \geq \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.$$

On the other hand, it was proved in [3] that for every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$

$$ps(C_n) \leq \left\lceil (1 + \varepsilon)\sqrt{2n} \ln n \right\rceil.$$

The two last results hold also for paths P_n and all the Hamiltonian graphs of order n. In the same paper it was proved that given any $\varepsilon > 0$, one can find $n_j^{(0)}, j = 1, \ldots, k$ such that for every k-tuple $(n_1, n_2, \ldots, n_k)$, $n_j \geq n_j^{(0)}$, $j = 1, 2, \ldots, k$, the upper bounds for toroidal grids and grids are

$$ps\big(T_{n_1 \times n_2 \times \cdots \times n_k}\big) \leq \left\lceil (1 + \varepsilon)\sqrt{2} \left(\sum_{j=1}^{k} \sqrt{n_j} \right) \ln \left(\sum_{j=1}^{k} n_j \right) \right\rceil$$

and

$$ps\big(G_{n_1 \times n_2 \times \cdots \times n_k}\big) \leq \left\lceil (1 + \varepsilon)\sqrt{2} \left(\sum_{j=1}^{k} \sqrt{n_j} \right) \ln \left(\sum_{j=1}^{k} n_j \right) \right\rceil,$$

respectively.

Skowronek-Kaziów considered the local versions of both problems (with vertex labels allowed or not). In both cases the goal was to distinguish only the neighboring vertices. In particular she proved that if the vertex labels are allowed, then 3 colors are enough to properly color any graph [21] and if one cannot label the vertices, then it is enough to use at most 4 colors [22]. Note that the respective lower bounds are 2 and 3, so the obtained results are almost optimal. In the case of complete graphs Skowronek-Kaziów obtains these lower bounds, so we can see that

$$ps(K_n) = 3$$

and

$$tvps(K_n) = 2.$$

In [4] Anholcer presented, among others, some results for bipartite graphs. In particular, if $2 \leq m \leq n$, then

$$ps(K_{m,n}) = 3$$

if and only if $n \leq \binom{m+2}{2}$, while otherwise $ps(K_{m,n}) \geq 4$. In the same paper it was shown that given arbitrary integer $D \geq 3$, for almost all forests F such that $\Delta(F) = D$, $n_2 = 0$, $n_0 \leq 1$ and if one removes all the pendant edges, then in the resulting forest F', $n_2 = 0$, the product irregularity strength equals to

$$ps(F) = n_1.$$

Darda and Hujdurović [9] generalized some of those results. In particular, they proved that for positive integers $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_k$, and $K_{m_1,m_2,\ldots,m_k}$ being the complete multipartite graph such that $m_k \leq m_1 + m_2 + \cdots + m_{k-1}$, the equality $ps(K_{m_1,m_2,\ldots,m_k}) = 3$ holds.

In this paper, some bounds on $tvps(G)$ have been presented. In Section 2 we show some general results, as well as the results for regular graphs. In Section 3 we consider cycles and paths. Then we generalize these results for grids and toroidal grids in Section 4. Finally some bounds for complete multipartite graphs are presented in Section 5. We conclude the paper with some open problems.

2. General Bounds

Since the goal is to obtain distinct product degrees, a unique multiset of labels must be used to label the edges incident to every vertex (although, of course, it does not need to be a sufficient condition). Recall that n_d denotes the number of vertices of degree d (note that $d + 1$ labels are present in the product degree

of such vertex). Let s be the strength of the labeling (i.e., the maximum label used). Then

$$n_d \leq \binom{s+d}{s-1} = \binom{s+d}{d+1} < \left(\frac{e(s+d)}{d+1}\right)^{d+1},$$

so for every d such that $\delta(G) \leq d \leq \Delta(G)$ the following holds:

$$s \geq \left\lceil \frac{d+1}{e} n_d^{1/(d+1)} - d \right\rceil.$$

It means that the following two observations are true.

Proposition 2.1. *For every graph G,*

$$tvps(G) \geq \max_{\delta(G) \leq d \leq \Delta(G)} \left\{ \left\lceil \frac{d+1}{e} n_d^{1/(d+1)} - d \right\rceil \right\}.$$

Proposition 2.2. *For every r-regular graph G such that $|V(G)| = n$,*

$$tvps(G) \geq \left\lceil \frac{r+1}{e} n^{1/(r+1)} - r \right\rceil.$$

Now let us consider the upper bounds. Obviously, if one assigns the label 1 to every vertex, then the obtained product degrees are equal to the ones in the case when vertex labels are not allowed. It follows that given any G for which $ps(G)$ is defined, $tvps(G) \leq ps(G)$ holds. In particular, it holds that $ps(G) \leq |V(G)| - 1$. However, if one considers a general graph, the best we can obtain is the order of G. In fact, after labeling all the edges in an arbitrary way, one can compute the temporary product degrees assuming that all the vertex labels equal to 1. Now the final labels of vertices, being distinct numbers from the set $\{1, 2, \ldots, |V(G)|\}$, are assigned in increasing order consistent with non-decreasing order of the temporary product degrees. Obviously the resulting sequence of product degrees is strictly increasing. So we obtain the following.

Proposition 2.3. *For every graph G of order n, $tvps(G) \leq n$.*

This bound cannot be improved in the general case, as the example of the empty graph shows.

In the next section we provide some results for cycles and paths.

3.   CYCLES AND PATHS

Denote by C_k a cycle of length k. Then it holds.

Fact 3.1. $tvps(C_3) = tvps(C_4) = 2$,
$tvps(C_5) = tvps(C_6) = tvps(C_7) = tvps(C_8) = tvps(C_9) = tvps(C_{10}) = 3$,
$tvps(C_{11}) = tvps(C_{12}) = tvps(C_{13}) = tvps(C_{14}) = tvps(C_{15}) = tvps(C_{16}) = 4$,
$tvps(C_n) \geq 5$ if $n \geq 17$.

Proof. Let us start with proving the fact that the given values of $tvps(C_n)$ are the minimum possible.

By using label 1, only one product can be obtained: 1, so it is impossible to label C_3 with just this label. Thus, one has to apply at least two labels to label any cycle C_n, where $n \geq 3$.

Similarly we can observe that when using labels 1 and 2, one can obtain four distinct product degrees: 1, 2, 4 and 8, so at least three labels must be applied for C_n, where $n \geq 5$.

Using labels 1, 2 and 3 it is possible to obtain ten products: 1, 2, 3, 4, 6, 8, 9, 12, 18 and 27, so we have to use at least four labels to obtain a product-irregular labeling of C_{11} and any longer cycle.

The labels 1, 2, 3 and 4 can result in at most sixteen product degrees: 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48 and 64, so at least 5 colors are necessary for C_n, where $n \geq 17$.

Now we are going to show that the values in the theorem are enough to obtain product-irregular labelings. For every cycle C_n, $3 \leq n \leq 16$, let the sequence of labels be

$$S_n = ([w(v_1)], w(v_1 v_2), [w(v_2)], \ldots, w(v_{n-1} v_n), [w(v_n)], w(v_n v_1))$$

(the labels of vertices are in the square brackets). Sample sequences of labels minimizing the value of $tvps(C_n)$ are listed below:

$S_3 = ([1], 1, [2], 2, [2], 2)$,
$S_4 = ([1], 1, [1], 1, [2], 2, [2], 2)$,
$S_5 = ([1], 1, [2], 2, [3], 3, [3], 3, [2], 2)$,
$S_6 = ([1], 1, [1], 1, [2], 2, [3], 3, [3], 3, [2], 2)$,
$S_7 = ([1], 1, [1], 2, [2], 2, [2], 3, [3], 3, [3], 2, [2], 1)$,
$S_8 = ([1], 1, [3], 1, [1], 2, [2], 2, [2], 3, [3], 3, [3], 2, [2], 1)$,
$S_9 = ([1], 1, [1], 3, [2], 1, [1], 2, [2], 2, [2], 3, [3], 3, [3], 2, [2], 1)$,
$S_{10} = ([1], 1, [1], 3, [1], 3, [2], 1, [1], 2, [2], 2, [2], 3, [3], 3, [3], 2, [2], 1)$,
$S_{11} = ([1], 3, [1], 3, [1], 1, [1], 1, [1], 2, [2], 3, [3], 3, [3], 4, [4], 4, [4], 3, [3], 2)$,
$S_{12} = ([1], 3, [1], 3, [1], 1, [4], 1, [1], 1, [1], 2, [2], 3, [3], 3, [3], 4, [4], 4, [4], 3, [3], 2)$,
$S_{13} = ([4], 1, [2], 1, [2], 3, [4], 2, [4], 2, [3], 3, [1], 3, [4], 4, [2], 4, [4], 4, [3], 3, [3], 3, [2], 2)$,

$$S_{14} = ([4], 1, [2], 1, [4], 1, [2], 3, [4], 2, [4], 2, [3], 3, [1], 3, [4], 4, [2], 4, [4], 4, [3],$$
$$3, [3], 3, [2], 2),$$
$$S_{15} = ([4], 1, [2], 1, [3], 1, [4], 1, [2], 3, [4], 2, [4], 2, [3], 3, [1], 3, [4], 4, [2], 4, [4],$$
$$4, [3], 3, [3], 3, [2], 2),$$
$$S_{16} = ([4], 1, [2], 1, [1], 1, [3], 1, [4], 1, [2], 3, [4], 2, [4], 2, [3], 3, [1], 3, [4], 4, [2],$$
$$4, [4], 4, [3], 3, [3], 3, [2], 2).$$

This completes the proof. $\blacksquare$

Now we are going to show the bounds for arbitrary n.

Proposition 3.2. *For every $n > 2$,*

$$tvps(C_n) \geq \left\lceil \sqrt[3]{6n} - 1 \right\rceil.$$

Proof. In the case of C_n, the inequality

$$n_d \leq \binom{s+d}{d+1}$$

takes the form

$$s(s+1)(s+2) \geq 6n,$$

which implies

$$(s+1)^3 \geq 6n,$$

and finally

$$s \geq \sqrt[3]{6n} - 1.$$

Since s must be an integer, the last inequality is equivalent to

$$s \geq \left\lceil \sqrt[3]{6n} - 1 \right\rceil.$$

$\blacksquare$

Next, let us prove some upper bounds on $tvps(C_n)$. Recall that in the label sequences, the vertex labels are surrounded with square brackets.

Lemma 3.3. *For every $s \geq 3$ there exist product-irregular labelings of C_{3s-2}, C_{3s-1} and C_{3s} with the label sequence containing the subsequence*

$$s - 1, [s - 1], s, [s], s, [s], s - 1.$$

Proof. For C_7, C_8 and C_9, see the sequences given in the proof of Fact 3.1. Now, assume that the statement is true for some s. Take any n, $3s - 2 \leq n \leq 3s$. Without loss of generality we can assume that $w(v_{n-2}v_{n-1}) = s - 1$, $w(v_{n-1}) = s-1$, $w(v_{n-1}v_n) = s$, $w(v_n) = s$, $w(v_nv_1) = s$, $w(v_1) = s$, $w(v_1v_2) = s-1$. As one can see, the three largest product degrees are $w(v_n - 1) = s(s-1)^2$, $w(v_n) = s^3$ and $w(v_1) = s^2(s-1)$. We now extend the labeling to C_{n+3} adding only one label $s+1$ in the following way (of course the edge v_nv_1 is removed together with its label): $w(v_nv_{n+1}) = s$, $w(v_{n+1}) = s$, $w(v_{n+1}v_{n+2}) = s+1$, $w(v_{n+2}) = s+1$, $w(v_{n+2}v_{n+3}) = s+1$, $w(v_{n+3}) = s+1$,$w(v_{n+3}v_1) = s$. As one can easily check, the new labeling contains the sequence

$$s, [s], s+1, [s+1], s+1, [s+1], s.$$

Moreover it preserves the product degrees for all v_i, $1 \leq i \leq n$, while for the three new vertices we have $pd(v_{n+1}) = s^2(s+1)$, $pd(v_{n+2}) = (s+1)^3$, $pd(v_{n+3}) = s(s+1)^2$. Clearly, the new labeling satisfies the subsequence condition and is product-irregular, so the proof follows by induction. ∎

Lemma 3.4. *For every $s \geq 4$ there exist product-irregular labelings of C_{4s-3}, C_{4s-2}, C_{4s-1} and C_{4s} with the label sequence containing the subsequence*

$$s - 1, [s], s, [s-2], s, [s], s, [s-1], s-1.$$

Proof. For C_{13}, C_{14}, C_{15} and C_{16}, see the sequences given in the proof of Fact 3.1. Now, assume that the statement is true for some s. Take any n, $4s - 3 \leq n \leq 4s$. Without loss of generality we can assume that $w(v_{n-2}v_{n-1}) = s - 1$, $w(v_{n-1}) = s$, $w(v_{n-1}v_n) = s$, $w(v_n) = s-2$, $w(v_nv_1) = s$, $w(v_1) = s$, $w(v_1v_2) = s$, $w(v_2) = s - 1$, $w(v_2v_3) = s - 1$. As one can see, the four largest product degrees are $w(v_{n-1}) = s^2(s-1)$, $w(v_n) = s^2(s-2)$, $w(v_1) = s^3$ and $w(v_2) = s(s-1)^2$. Similarly as in the proof of Lemma 3.3, we extend the labeling to C_{n+4} adding only one label $s+1$ by assigning $w(v_nv_{n+1}) = s$, $w(v_{n+1}) = s+1$, $w(v_{n+1}v_{n+2}) = s+1$, $w(v_{n+2}) = s - 1$, $w(v_{n+2}v_{n+3}) = s + 1$, $w(v_{n+3}) = s + 1$, $w(v_{n+3}v_{n+4}) = s + 1$, $w(v_{n+4}) = s$, $w(v_{n+4}v_1) = s$. As one can easily check, the new labeling contains the sequence

$$s, [s+1], s+1, [s-1], s+1, [s+1], s+1, [s], s.$$

Moreover it preserves the product degrees for all v_i, $1 \leq i \leq n$, while for the four new vertices we have $pd(v_{n+1}) = s(s+1)^2$, $pd(v_{n+2}) = (s-1)(s+1)^2$, $pd(v_{n+3}) = (s+1)^3$, $pd(v_{n+4}) = s^2(s+1)$. Clearly, the new labeling satisfies the subsequence condition and is product-irregular, so the proof follows by induction. ∎

The Lemmas 3.3 and 3.4 imply the following.

Proposition 3.5.

$$tvps(C_n) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } 7 \leq n \leq 12, \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \geq 13. \end{cases}$$

The following theorem gives the upper bound for the cycles of sufficiently large order.

Theorem 3.6. *For every $\varepsilon > 0$ there exists n_0 such that for every $n \geq n_0$*

$$tvps(C_n) \leq \left\lceil (1 + \varepsilon) 3 \sqrt[3]{2} (1 + \varepsilon) n^{1/3} \ln n \right\rceil.$$

Proof. Let $s = \left\lceil (1 + \varepsilon) 3 \sqrt[3]{2} (1 + \varepsilon) n^{1/3} \ln n \right\rceil$ and let p be the greatest prime that satisfies the condition $p \leq \lfloor 2/3\pi(s) \rfloor$, where $\pi(s)$ is the value of the prime-counting function at s, that is, the number of primes less than or equal to s. From Bertrand-Chebyshev theorem [23], $p \geq \lfloor 2/3\pi(s) \rfloor / 2 + 1$.

We start with finding the edge labels.

For every q, $1 \leq q < p/2$, we define the sequence: $0, q \mod p, 2q \mod p, \ldots,$ $(p-1)q \mod p, pq \mod p = 0$. We will refer to such a sequence as "the chain".

In any fixed chain, each number a, $0 \leq a \leq p-1$, appears exactly once, as q and p are relatively prime, and so the order of q in the additive group Z_p is equal to $p-1$. Thus, if we join all those chains together and form a multichain (0's being common members of every two neighboring chains), every pair of numbers from the considered set will appear at most once as a pair of consecutive numbers (here we use the assumption that $q < p/2$). Next, let $p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on, where the following terms of the sequence are consecutive primes. Replace every number a in the multichain constructed above by p_a. None of the pairs repeats, so the same holds for the products. Moreover, since any two consecutive numbers are always distinct, there is no product equal to the square of a natural number. Now we join r identical multichains together in the same way as we joined chains to obtain the multichain and close the cycle, where $1 \leq r \leq \lceil \pi(s) \rceil / 3$ will be chosen later (in particular it may happen that $r = \lceil \pi(s) \rceil / 3$). We are going to show that it is possible to label at least n edges in this way.

In every chain p labels have been used and each chain is based on distinct natural number smaller than $p/2$, so one can label

$$m \geq r \frac{p(p-1)}{2} \geq \lceil \pi(s)/3 \rceil \frac{(2/3\lfloor \pi(s) \rfloor)^2}{8} \geq \frac{1}{54} \left\lfloor \frac{s}{\ln s} \right\rfloor^3$$

edges. Because of the choice of s, for sufficiently large n, the inequality $m > n$ holds.

We use the maximum number of multichains and then the maximum number of chains that allows to label at most n edges. It means that the number of labeled edges is at least $n - (p-1)r/2 + 1$ but not greater than n. Let t denote the number of unlabeled edges. Of course, $0 \leq t < (p-1)r/2 - 1$. Moreover, exactly p labels have been applied and no squares of natural numbers have occured so far. Now let us choose t edges from the last chains of as many multichains as necessary (i.e., choose all the edges in the last chains of $\lfloor t/(p-1) \rfloor$ multichains and $t - (p-1)\lfloor t/(p-1) \rfloor$ in any other multichain). Then, replace each of those edges with two new edges, assigned the same label. No product degree changes, as the only change is that some sequences of the form p_i, p_j, p_k become p_i, p_j, p_j, p_k. Eventually only some new products of the form p_j^2 can appear. So we obtain edge labeling of the cycle of length n. Note that every product degree appears at most r times, as in every multichain all the degrees are distinct. Now for every set of vertices with equal product degree we use the primes greater than any label used so far (there are exactly r of them) to distinguish the product degrees of those vertices. This results with a product-irregular labeling of C_n. ∎

It is straightforward to see that the above results for cycles C_n can be used also to prove upper bounds on the $tvps(G)$, where G is a path (it is enough to remove any edge with label 1), Hamiltonian graph (one labels with 1 all the edges not belonging to the Hamiltonian cycle) or semi-Hamiltonian graph (one labels with 1 all the edges not belonging to the Hamiltonian path).

4. Grids and Toroidal Grids

Given k paths P_j ($j = 1, 2, \ldots, k$) with vertex sets V_j ($j = 1, 2, \ldots, k$), where $|V_j| = n_j$, the grid $G_{n_1 \times n_2 \times \cdots \times n_k}$ is the Cartesian product of those paths: $V = V(G_{n_1 \times n_2 \times \cdots \times n_k}) = V_1 \times V_2 \times \cdots \times V_k$ and two vertices $(u_1, u_2, \ldots, u_k)$, $(v_1, v_2, \ldots, v_k) \in V$ are adjacent if and only if all their coordinates but one, say j^{th}, are the same, and $(u_j, v_j) \in E(P_j)$. By taking cycles instead of paths, one obtains the toroidal grid $T_{n_1 \times n_2 \times \cdots \times n_k}$.

Lemma 4.1. *Let $n_1, n_2, \ldots, n_k$ be natural numbers, $n_j \geq 3$ for all $j = 1, 2, \ldots, k$.*

1. Let $p_1, p_2, \ldots, p_k$ and $r_1, r_2, \ldots, r_k$ be $2k$ natural numbers (not necessarily distinct) such that p_j primes as the edge labels and r_j primes as vertex labels are enough to obtain a product-irregular labeling of C_{n_j} ($j = 1, 2, \ldots, k$). Then $\sum_{j=1}^{k} (p_j + r_j)$ primes are enough to obtain a product-irregular labeling of the toroidal grid $T_{n_1 \times n_2 \times \cdots \times n_k}$.

2. Let $p_1, p_2, \ldots, p_k$ and $r_1, r_2, \ldots, r_k$ be $2k$ natural numbers (not necessarily distinct) such that p_j primes as the edge labels and r_j primes as vertex labels are enough to obtain a product-irregular labeling of P_{n_j} ($j = 1, 2, \ldots, k$). Then

$\sum_{j=1}^{k} (p_j + r_j)$ *primes are enough to obtain a product-irregular labeling of the grid* $G_{n_1 \times n_2 \times \cdots \times n_k}$.

Proof. First consider the toroidal grids. We will use the induction on k. For $k = 1$ it is trivially true. Assume that we have managed to label $T_{n_1 \times n_2 \times \cdots \times n_{k-1}}$ using $\sum_{j=1}^{k-1} p_j$ primes. To construct $T_{n_1 \times n_2 \times \cdots \times n_k}$, join n_k copies of $T_{n_1 \times n_2 \times \cdots \times n_{k-1}}$ in such a way that for every vertex v of $T_{n_1 \times n_2 \times \cdots \times n_{k-1}}$, all copies of v are joined with the same copy of C_{n_k}. The sets of labels applied to the edges incident with each copy of v are distinct, as they consist of constant set of labels used to label $T_{n_1 \times n_2 \times \cdots \times n_{k-1}}$ and distinct pairs of labels used on C_{n_k} (taken from a disjoint set of numbers). We define the vertex label as the product of the labels of the corresponding vertices in the factors (i.e., in $T_{n_1 \times n_2 \times \cdots \times n_{k-1}}$ and C_{n_k}). It results in a product-irregular labeling of $T_{n_1 \times n_2 \times \cdots \times n_k}$ using $p_k + r_k$ new primes, which is the desired number of labels.

In the case of $G_{n_1 \times n_2 \times \cdots \times n_k}$ we use P_{n_j} instead of C_{n_j}.   ∎

Note that the maximum label used in the above labeling is not greater than $\max\{P, R\}$, where P is the maximum edge label among all cycles (paths) and R the product of the maximum vertex labels of all cycles (paths). Now we can formulate the main result of this section.

Theorem 4.2. *For every $k \geq 2$ and every $\varepsilon > 0$ there exist $n_j^{(0)}$, $j = 1, \ldots, k$ such that for every k-tuple $(n_1, n_2, \ldots, n_k)$, $n_j \geq n_j^{(0)}$, $j = 1, 2, \ldots, k$,*

1. $tvps\,(T_{n_1 \times n_2 \times \cdots \times n_k}) \leq \left\lceil (1+\varepsilon)k^{k-2} \left(\sum_{j=1}^{k} n_j^{k/(2k+1)}\right) \sum_{j=1}^{k} \ln n_j \right\rceil$;

2. $tvps\,(G_{n_1 \times n_2 \times \cdots \times n_k}) \leq \left\lceil (1+\varepsilon)k^{k-2} \left(\sum_{j=1}^{k} n_j^{k/(2k+1)}\right) \sum_{j=1}^{k} \ln n_j \right\rceil$.

Proof. In order to find an irregular labeling of a cycle C_n^j it is enough to use p_j labels (primes or 1) to label the edges and

$$r_j = \left\lceil \frac{2n}{p_j(p_j - 1)} \right\rceil$$

labels for vertices (see the proof of Theorem 3.6). In particular, if $p_j = \left\lfloor n_j^{k/(2k+1)} \right\rfloor$, then $r_j \leq \left\lceil n_j^{1/(2k+1)} \right\rceil$. This means that we need at most

$$\sum_{j=1}^{k} \left(\left\lfloor n_j^{k/(2k+1)} \right\rfloor + \left\lceil n_j^{1/(2k+1)} \right\rceil \right)$$

distinct labels (all of them being primes or 1). Let

$$s = \left\lceil (1+\varepsilon)k^{k-2} \left(\sum_{j=1}^{k} n_j^{k/(2k+1)} \right) \sum_{j=1}^{k} \ln n_j \right\rceil$$

and

$$\varepsilon' = (1 + \varepsilon)^{1/k} - 1.$$

We assign

$$\sum_{j=1}^{k} \left\lceil n_j^{1/(2k+1)} \right\rceil$$

smallest primes to the vertices (1 will be used as edge label). This means that the maximum vertex label R_j in every cycle satisfies the condition

$$R_j < (1 + \varepsilon') \left(\sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \ln \left(\sum_{j=1}^{k} n_j^{1/(2k+1)} \right)$$

for sufficiently large s_j (i.e., n_j), $j = 1, \ldots, n$. Consequently, the maximum vertex label R in $T_{n_1 \times n_2 \times \cdots \times n_k}$ is not greater than

$$R = \prod_{j=1}^{k} R_j < (1 + \varepsilon) \left(\left(\sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \ln \left(\sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \right)^k$$

for sufficiently large n_j, $j = 1, \ldots, k$. For sufficiently large n_j we have

$$\ln \left(\sum_{j=1}^{k} n_j^{1/(2k+1)} \right) \le \sum_{j=1}^{k} \ln \left(n_j^{1/(2k+1)} \right).$$

From Hölder's Inequality it follows that

$$\sum_{j=1}^{k} n_j^{1/(2k+1)} \le \left(\sum_{j=1}^{k} n_j^{k/(2k+1)} \right)^{1/k} k^{1-1/k}$$

and

$$\sum_{j=1}^{k} \ln \left(n_j^{1/(2k+1)} \right) \le \left(\sum_{j=1}^{k} \left(\ln \left(n_j^{1/(2k+1)} \right) \right)^k \right)^{1/k} k^{1-1/k}$$

$$< \left(\sum_{j=1}^{k} \ln \left(k n_j^{1/(2k+1)} \right) \right)^{1/k} k^{1-1/k}$$

$$< \left(\sum_{j=1}^{k} \ln \left(n_j^{k/(2k+1)} \right) \right)^{1/k} k^{1-1/k}.$$

This implies

$$R < (1+\varepsilon)k^{2k-2}\left(\sum_{j=1}^{k} n_j^{k/(2k+1)}\right)\sum_{j=1}^{k}\ln n_j \le s.$$

On the other hand, for sufficiently large n_j, $j = 1, \ldots, n$, the highest edge label is not greater than

$$(1+\varepsilon)\left(\sum_{j=1}^{k}\left(\left\lfloor n_j^{k/(2k+1)}\right\rfloor + \left\lceil n_j^{1/(2k+1)}\right\rceil\right)\right)\ln\left(\sum_{j=1}^{k}\left(\left\lfloor n_j^{k/(2k+1)}\right\rfloor + \left\lceil n_j^{1/(2k+1)}\right\rceil\right)\right)$$

$$< (1+\varepsilon)\left(2\sum_{j=1}^{k} n_j^{k/(2k+1)}\right)2\sum_{j=1}^{k}\ln n_j^{k/(2k+1)} < s.$$

Since in every cycle 1 appears as an edge label, the same reasoning proves the theorem for the grid $G_{n_1 \times n_2 \times \cdots \times n_k}$. ∎

5.   Complete Multipartite Graphs

In this section we prove two simple results for complete multipartite graphs. Let us start with the bipartite case.

Proposition 5.1. *Let m and n be two integers such that $3 \le m \le n \le \binom{m+2}{2}$. Then*

$$tvps(K_{m,n}) = 3.$$

Proof. The fact that the labels 1, 2 and 3 are enough follows from the equality $ps(K_{m,n}) = 3$ (it is enough to label all vertices with 1 and we are done). On the other hand, if we use only labels 1 and 2, then the product degrees could take at most $n+2$ values, namely $1, 2, 4, \ldots, 2^{n+1}$, while there are at least $n+3$ vertices, so at least third label is necessary. ∎

By using a similar argument, we can prove the exact value for complete multipartite graphs.

Proposition 5.2. *Let $m_1, m_2, \ldots, m_k$ be integers such that $3 \le m_1 \le m_2 \le m_3 \le \cdots \le m_k \le m_1 + m_2 + \cdots + m_{k-1}$. Then*

$$tvps\big(K_{m_1,m_2,\ldots,m_k}\big) = 3.$$

Proof. Let us recall that Darda and Hujdurović [9] proved that if $K_{m_1,m_2,\ldots,m_k}$ is the complete multipartite graph such that $0 < m_1 \le m_2 \le m_3 \le \cdots \le m_k \le m_1 + m_2 + \cdots + m_{k-1}$, then the equality $ps(K_{m_1,m_2,\ldots,m_k}) = 3$ holds. By putting

label 1 on every vertex we get $tvps(K_{m_1,m_2,\ldots,m_k}) \leq 3$. In order to show that two labels are not enough, observe that $\Delta(K_{m_1,m_2,\ldots,m_k}) = m_2 + \cdots + m_k$, so using labels 1 and 2 one can obtain at most $2 + m_2 + \cdots + m_k$ different weights, while the number of vertices is $m_1 + m_2 + \cdots + m_k \geq 3 + m_2 + \cdots + m_k$. ∎

6. Conclusion

In this paper, a new graph invariant, total vertex product irregularity strength, has been introduced. It is a parameter similar to the total vertex irregularity strength, where the weighted degrees are computed as products. We proved several results for general and regular graphs, as well as for some specific families of graphs, like cycles, paths and grids.

Most of our results show that for any d-regular graph G of order n it holds that $c_1 n^{1/(d+1)} \leq g \leq c_2 n^{1/(d+1)} \ln n$ for some constants c_1 and c_2. Erdös [10] proved that in the case of integers not greater than n, the cardinality $s(n)$ of a subset resulting in distinct pairwise products (multiplicative Sidon set) cannot be much greater than the number of primes not greater than n:

$$s(n) = \pi(n) + \Theta\left(\frac{n^{3/4}}{\ln^{3/2} n}\right).$$

This makes us believe that the upper bounds presented in this paper are closer to the exact value of $tvps(G)$ than the lower and the effort of the further research should be focused on the improvement of the latter ones. For that reason we formulate the two following open problems.

Problem 6.1. Is there a constant c such that for every d-regular graph G of order n

$$tvps(G) \geq cn \ln \ln n?$$

Problem 6.2. Is there a constant c such that for every d-regular graph G of order n

$$tvps(G) \geq \frac{cn \ln n}{\ln \ln n}?$$

We also believe that the logarithmic factor is enough to guarantee the existence of a product-irregular labeling of any graph. This encourages us to pose the following conjecture.

Conjecture 6.3. *There exists a constant c such that for every d-regular graph G of order n*

$$tvps(G) \leq cn^{1/(d+1)} \ln n.$$

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