AN ANALOGUE OF QUASI-TRANSITIVITY FOR EDGE-COLOURED GRAPHS

Christopher Duffy

School of Mathematics and Statistics
University of Melbourne, Australia

e-mail: christopher.duffy@unimelb.edu.au

AND

Todd Mullen

School of Mathematics and Computing Sciences
University of Prince Edward Island, Canada

e-mail: toddmullen26@outlook.com

Abstract

We extend the notion of quasi-transitive orientations of graphs to 2-edge-coloured graphs. By relating quasi-transitive 2-edge-colourings to an equivalence relation on the edge set of a graph, we classify those graphs that admit a quasi-transitive 2-edge-colouring. As a contrast to Ghouilá-Houri's classification of quasi-transitively orientable graphs as comparability graphs, we find quasi-transitively 2-edge-colourable graphs do not admit a forbidden subgraph characterization. Restricting the problem to comparability graphs, we show that the family of uniquely quasi-transitively orientable comparability graphs is exactly the family of comparabilty graphs that admit no quasi-transitive 2-edge-colouring.

Keywords: oriented graph, quasi-transitivity, edge colouring, uniquely quasi-transitively colourable graphs.

2020 Mathematics Subject Classification: 05C15.

1. Introduction and Preliminaries

Using the definition of graph colouring based on homomorphism, one may define a notion of proper vertex colouring for oriented graphs and 2-edge-coloured graphs that respectively takes into account the orientation of the arcs and the colours of

the edges. In surveying the literature on *oriented colourings* and *2-edge-coloured colourings*, one finds seemingly endless instances of similar techniques and results. For example, in both cases, one obtains an upper bound of 80 for the respective chromatic number of oriented and 2-edge-coloured planar graphs by constructing a universal target for homomorphisms of orientations/2-edge-colourings of planar graphs [1, 14]. For further details of oriented colouring and 2-edge-coloured colourings see [15] and [13].

A key feature of oriented colourings is that vertices at the end of a directed 2-path (i.e., *a 2-dipath*) must receive different colours. So one may bound the oriented chromatic number of an oriented graph $\overrightarrow{G}$ from below by computing the *2-dipath chromatic number*, the chromatic number of $\overrightarrow{G}^2$, where $\overrightarrow{G}^2$ is the undirected graph formed from $\overrightarrow{G}$ by first adding an edge between vertices at the end of an induced 2-dipath and then removing the orientation from the arcs of $\overrightarrow{G}$ [12, 16]. From this, one observes that if $\overrightarrow{G}$ has no induced 2-dipaths, then the 2-dipath chromatic number of $\overrightarrow{G}$ is equal to the chromatic number of the simple graph underyling $\overrightarrow{G}$. Oriented graphs with no induced 2-dipaths are studied in the literature under the name *quasi-transitive digraphs* [2].

In [2], a recursive method is used to characterize the graphs that can have their edges oriented into a quasi-transitive digraph. In [10], a related term is defined, *k-quasi-transitive digraphs*, in which vertices must be adjacent if there exists a directed path of length k connecting them. Much of the research on quasi-transitive digraphs and k-quasi-transitive digraphs have focused on *strong* digraphs. A *strong* digraph is a graph in which, for every pair of vertices, there exists a directed path containing those two vertices. A characterization of strong 3-quasi-transitive digraphs is given in [7].

We will opt for the name *quasi-transitive orientations* instead of *quasi-transitive digraphs* to relate more closely to the following two terms. A graph G is *quasi-transitively orientable* if there exists a quasi-transitive orientation for which G is the underlying graph. A graph G is *uniquely quasi-transitively orientable* if there exist exactly two quasi-transitive orientations for which G is the underlying graph. We use the word "unique" to describe the case in which there are two possible orientations because one can be created by reversing the direction of every arc in the other.

Quasi-transitive orientations arise only as certain orientations of comparability graphs, which are graphs formed from making adjacent related elements of a partial order [8]. We include this theorem now for ease of reference.

Theorem 1 [8]. *A graph G is quasi-transitively orientable if and only if G is a comparability graph.*

Unsurprisingly, one may follow a similar line of thought for proper colourings of 2-edge-coloured graphs. In doing so, one finds that vertices at the ends of a

2-path in which one edge is red and the other is blue (i.e., an alternating 2-path) must receive different colours. To date, however, no work has been done in the study of 2-edge-coloured graphs with no induced alternating 2-path. In this work we undertake the first such study of these objects and find a full classification of those graphs that admit a 2-edge-colouring so as to have no induced alternating 2-path. For this end, we define the following.

A *quasi-transitive* 2-*edge-colouring* c of a graph G is a mapping $c : E(G) \to \{R, B\}$ so that for all pairs $xy, yz \in E(G)$, with $c(xy) \neq c(yz)$, we have $xz \in E(G)$. Quasi-transitive 2-edge colourings are introduced in [5] as "2-$\mathcal{N}\mathcal{M}$-closed" colourings. A quasi-transitive 2-edge-colouring is *trivial* when $c(e) = c(f)$ for all $e, f \in E(G)$. In other words, the edge colouring is *monochromatic*. As trivial quasi-transitive 2-edge-colourings exist for all graphs, all graphs are quasi-transitively colourable. A graph G is *properly quasi-transitively colourable* if there exists a nontrivial quasi-transitive 2-edge-colouring of G. A graph G is *uniquely quasi-transitively colourable* if there exist exactly 2 nontrivial quasi-transitive 2-edge-colourings of G. We use the term unique for graphs with exactly two colourings because the existence of a single nontrivial quasi-transitive 2-edge-colouring implies the existence of a second, created by interchanging all of the colours. Examples of quasi-transitive 2-edge-colourings are shown in Figure 1.
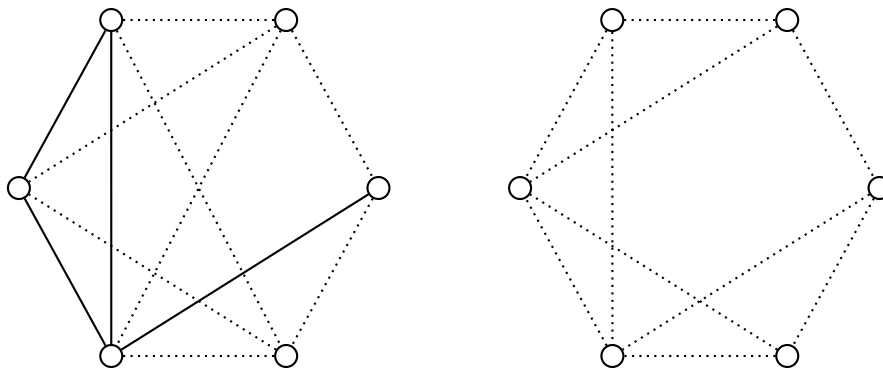

Figure 1. Quasi-transitive 2-edge-colourings shown on two graphs with R edges shown as dotted lines and B edges shown as full lines (note that the second graph only has trivial quasi-transitive 2-edge-colourings).

Given a set of edges E, we denote by $V(E)$ the set of endpoints of edges in E. Also, we denote by $G[E]$ the graph with edge set E and vertex set $V(E)$. For other graph theoretic terminology not defined herein, we refer the reader to [4].

Our work proceeds as follows. In Section 2 we introduce an equivalence relation on the set of edges of a graph that partitions the edges into subsets of edges that must be assigned the same colour in any quasi-transitive 2-edge-colouring. With the aim of classifying those graphs that are properly quasi-transitively

colourable, we study the structure of the subgraphs induced by the equivalence classes. Our work yields a full classification of properly quasi-transitively colourable graphs and uniquely quasi-transitively colourable graphs. In doing so, we provide an infinite family of uniquely quasi-transitively colourable graphs.

In Section 3 we provide a characterization of those graphs for which the equivalence relation yields exactly three equivalence classes. In doing so, we provide examples of infinite families of such graphs.

In Section 4 we explore uniquely quasi-transitively orientable graphs. By restricting our work to comparability graphs, we find that the equivalence relation introduced for the study of properly quasi-transitively colourable graphs yields insight into those comparability graphs for which there are exactly two quasi-transitive orientations. In particular, we show that the family of uniquely quasi-transitively orientable graphs is exactly the family of comparability graphs that admit only the trivial quasi-transitive 2-edge-colouring.

2.   Edge Partitions and Quasi-Transitive 2-Edge-Colourings

In this section, we show the respective sets of red and blue edges in a quasi-transitive 2-edge-colouring arise as unions of equivalence classes under an equivalence relation on the edges of a graph. From this observation, we show the family of properly quasi-transitively colourable graphs are exactly those graphs with at least two equivalence classes with respect to this equivalence relation. We begin with a number of results leading up to the definition of this equivalence relation.

Let G be a graph and let $e \in E(G)$. Let $\mathcal{S}_e$ be the set comprising all subsets S of $E(G)$ with the following properties:

1. $e \in S$; and

2. for all $d \notin S$, there is no induced copy of P_3 in G comprising d and an edge of S.

Theorem 2. *The set $\mathcal{S}_e$ is closed with respect to intersection.*

Proof. Let G be a graph and let $e \in E(G)$. Toward a contradiction, suppose S and T are two sets in $\mathcal{S}_e$ and $S \cap T$ is not in $\mathcal{S}_e$. Since e is in $S \cap T$ and $S \cap T$ is not in $\mathcal{S}_e$, there must exist some induced copy of P_3 containing an edge d in $S \cap T$ and an edge f in $E(G) \setminus (S \cap T)$. Therefore, any set in $\mathcal{S}$ that contains every edge in $S \cap T$ must contain f as well. This is a contradiction, as f is in at most one of S and T. Therefore, the intersection of any two sets in $\mathcal{S}_e$ is a set in $\mathcal{S}_e$.   ∎

Corollary 3. *The element of $\mathcal{S}_e$ with the fewest number of edges is unique.*

Proof. Let S and T be distinct elements of $\mathcal{S}_e$. Toward a contradiction, suppose that no set in $\mathcal{S}_e$ contains fewer edges than either S or T. The set $S \cap T$ contains fewer edges than either S or T. By Theorem 2, $S \cap T$ is in $\mathcal{S}_e$. This is a contradiction. Therefore, the element of $\mathcal{S}_e$ with the fewest number of edges is unique. ∎

Denote by S_e the unique set of least order in $\mathcal{S}_e$.

Corollary 4. *The graph $G[S_e]$ is connected.*

With Theorem 6, we establish that these sets of least order form a partition on the edges.

Lemma 5. *For every pair of edges $uv, xy \in E(G)$, if $uv \in S_{xy}$, then $S_{uv} \subseteq S_{xy}$.*

Proof. Let $uv, xy \in E(G)$ and let $uv \in S_{xy}$. The intersection, $S = S_{xy} \cap S_{uv}$, contains the edge uv. If there exists any edge s in S such that s is in an induced copy of P_3 with an edge d in $E(G) \setminus S$, then d is in both S_{xy} and S_{uv}. This however, cannot be the case since d would then be in the intersection S. Thus S is the smallest set of edges that both contains uv and is such that for all edges e in $E(G) \setminus S$, the edge e does not exist in an induced copy of P_3 with any edge in S. Therefore, $S = S_{uv}$ and $S_{uv} \subseteq S_{xy}$. ∎

Theorem 6. *For every pair of edges xy and uv in $E(G)$, we have $S_{xy} = S_{uv}$ if and only if $xy \in S_{uv}$.*

Proof. One direction of this statement is clear because xy is, by definition, in S_{xy}. Thus if $S_{xy} = S_{uv}$, then $xy \in S_{uv}$.

Consider now $xy \in S_{uv}$ with $xy \neq uv$. By Lemma 5, $S_{xy} \subseteq S_{uv}$. Toward a contradiction, suppose $S_{uv} \neq S_{xy}$. So $S_{uv} \nsubseteq S_{xy}$. No edge in $S_{uv} \setminus S_{xy}$ exists in an induced copy of P_3 with an edge in S_{xy}. Thus, by Lemma 5, for any edge ab in S_{xy}, the set S_{ab} does not contain any edge in $S_{uv} \setminus S_{xy}$. This implies that $uv \in S_{uv} \setminus S_{xy}$. However, since $uv \in S_{uv} \setminus S_{xy}$ and no edge in $S_{uv} \setminus S_{xy}$ exists in an induced copy of P_3 with an edge in S_{xy}, we conclude that xy is not in S_{uv}, which is a contradiction.

Therefore $S_{uv} = S_{xy}$. ∎

Corollary 7. *Let G be a connected graph and let $uv, vw \in E(G)$. If $S_{uv} \neq S_{vw}$, then $uw \in E(G)$.*

By Theorem 6, for $e, f \in E(G)$ it follows that either $S_e = S_f$ or $S_e \cap S_f = \emptyset$. Thus there exists a subset of edges $E'(G) \subset E(G)$ so that

- $\bigcup_{e \in E'(G)} S_e = E(G)$; and

- $S_e \cap S_f = \emptyset$ for all $e, f \in E'(G)$ with $e \neq f$.

Using this partition of the edges, we study quasi-transitive 2-edge-colourings of graphs.

Let $\mathcal{C}_G$ be the relation on $E(G)$ so that $e \sim f$ when $c(e) = c(f)$ for every quasi-transitive 2-edge-colouring c of G. The relation $\mathcal{C}_G$ will serve as a partition on the edges, so $\mathcal{C}_G$ is an equivalence relation. Let $[e]_{\mathcal{C}}$ denote the equivalence class of e with respect to this relation.

Theorem 8. *Let G be a connected graph. For every $e \in E(G)$, we have $[e]_{\mathcal{C}} = S_e$.*

Proof. Let G be a connected graph and let $e \in E(G)$. Let $S = [e]_{\mathcal{C}} \cap S_e$. Since $e \in [e]_{\mathcal{C}}$ and $e \in S_e$, necessarily S is non-empty. Toward a contradiction, suppose $[e]_{\mathcal{C}} \neq S_e$. Thus since G is connected, there exists an edge in $E(G)$ that is incident with an edge in S but not itself in S.

Let d be an edge in $E(G) \setminus S$ that is incident with an edge f in S.

Since the edge d is not in S, either d is not in $[e]_{\mathcal{C}}$ or d is not in S_e. We will show that, in both cases, there does not exist an induced copy of P_3 comprising d and f. Label the vertices so that $d = v_1v_2$ and $f = v_2v_3$.

Case 1. d is not in $[e]_{\mathcal{C}}$. This implies there exists a quasi-transitive 2-edge-colouring c of G in which $c(f)$ and $c(d)$ are not equal. Thus the path $v_1v_2v_3$ is not an induced copy of P_3 because that would imply every proper quasi-transitive 2-edge-colouring c of G has $c(d)$ equal to $c(f)$.

Case 2. d is not in S_e. This implies there does not exist an induced copy of P_3 containing d and an edge in S_e. Thus $v_1v_2v_3$ is not an induced copy of P_3.

So no edge in S is contained in an induced copy of P_3 with the edge d. Therefore, since S is a subset of S_e, and S contains e, we have that $S = S_e$. However in addition, with no edges in S existing in an induced copy of P_3 with an edge in $E(G) \setminus S$, there exist quasi-transitive 2-edge-colourings of G in which every edge in $E(G) \setminus S$ maps to R and every edge in S maps to B. So no edges in $E(G) \setminus S$ are in $[e]_{\mathcal{C}}$.

Therefore, since S is a subset of $[e]_{\mathcal{C}}$, we have $S_e = S = [e]_{\mathcal{C}}$. ∎

Corollary 9. *If G is a connected graph such that $\mathcal{C}_G$ has k equivalence classes, then there are 2^k quasi-transitive 2-edge-colourings of G.*

With our next theorem, we show that there exist graphs with any positive integral number of equivalence classes. This theorem will rely on some graph theory terminology that we must first define.

A *threshold graph* is a graph constructed from a single vertex by repeatedly performing one of two operations: adding a universal vertex or adding a vertex that is adjacent to no previously added vertices. A *split graph* is a graph such that the vertices can be partitioned into a complete graph and an independent set.

Theorem 10. *For every integer $k \geq 1$, there exists a graph G such that $\mathcal{C}_G$ has exactly k equivalence classes.*

Proof. Let G_n be a threshold graph with n vertices, where n is an even integer, constructed by alternating the two possible actions, beginning with adding a universal vertex to a copy of K_1 (see Figure 2). G_n is a split graph in which the complete graph and vertex set have the same number of vertices.
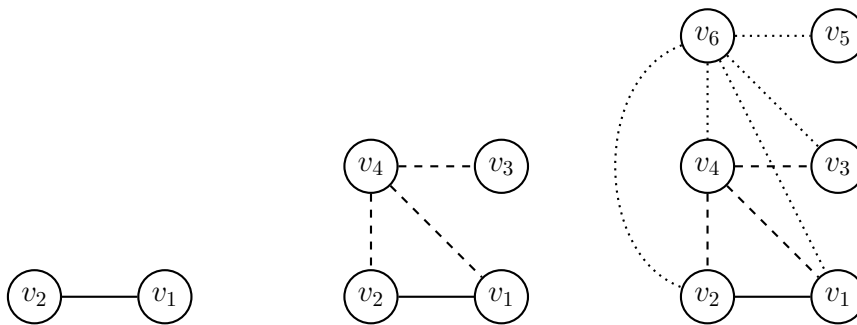

Figure 2. Threshold graphs constructed by alternating the two possible actions. Equivalence classes are shown by solid, dashed, and dotted lines.

Let e be an edge in G_n. In G_{n+k}, for all even integers $k \geq 2$, S_e will be the same set as it is in G_n (assuming the vertices receive the same label in both graphs, as pictured in Figure 2) because every added vertex is either adjacent to every vertex in $V(S_e)$ or none of the vertices in $V(S_e)$. So G_{n+2} has more equivalence classes than G_n for all $n \geq 2$. The two vertices, v_{n+1} and v_{n+2}, added to G_n to create G_{n+2} are adjacent and this edge, $v_{n+1}v_{n+2}$, induces a copy of P_3 with every other edge that is in G_{n+2} and not in G_n. Therefore, G_{n+2} has exactly one more equivalence class than G_n, for all $n \geq 2$. Since G_2 has only one edge and thus only one equivalence class, the graph G_n has exactly $\frac{n}{2}$ equivalence classes for all even positive integers n. ∎

To classify those graphs that admit only trivial quasi-transitive 2-edge-colourings, it suffices to classify those graphs for which $S_e = S_f$ for all $e, f \in E(G)$. Similarly, to classify those graphs that are uniquely quasi-transitively 2-edge-colourable, it suffices to classify those graphs for which there exists a pair of edges $e, f \in E(G)$ so that $E(G) = S_e \cup S_f$ and $S_e \neq S_f$. For these ends, we define the following notation and terminology.

Let G be a graph and let $e \in E(G)$. An S_e-*path* is a path in $G[S_e]$. An S_e-*k-path* is a path of length k in $G[S_e]$. For $v \in V(G)$, denote by $N_{G[S_e]}[v]$ the closed neighbourhood of v in $G[S_e]$.

With an eye towards a classification of those graphs with at least 2 equivalence classes (and thus, also a classification of those graphs with exactly one

equivalence class), these next two lemmas show that in every graph that admits a non-trivial quasi-transitive 2-edge-colouring, there exists an equivalence class S_e for which $V(S_e) \neq V(G)$.

Lemma 11. *Let G be a graph with n vertices. If H is an induced subgraph of G such that*

- *H is connected;*
- *$2 \leq |V(H)| \leq n - 1$; and*
- *for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$;*

then for all edges e in $E(H)$, we have $S_e \subseteq E(H)$.

Proof. Let G be a graph with n vertices and let H be an induced subgraph of G such that the three conditions listed in the statement of the lemma are satisfied. Let $e \in E(H)$. Toward a contradiction, suppose that some edge f is in S_e and not in $E(H)$. This implies that some endpoint, v_f, of f is not in $V(H)$. Since S_e is defined to be the smallest set in $\mathcal{S}_e$, every edge in S_e must exist in some induced copy of P_3 with another edge in S_e. Also, every edge in S_e must be contained in some path containing e in which every pair of incident edges belong to an induced copy of P_3 in G. Without loss of generality, suppose that f is incident with some edge in $E(H)$ along a path containing e in which every pair of incident edges belong to an induced copy of P_3 in G. So v_f is adjacent to some vertex in $V(H)$, but v_f is not adjacent to every vertex in $V(H)$. This is a contradiction. Therefore, $S_e \subseteq E(H)$. ∎

Lemma 12. *Let G be a connected graph and let e and f be edges in $E(G)$. If $S_e \neq S_f$, then $V(S_e) \neq V(S_f)$.*

Proof. Let G be a connected graph and let e and f be edges in $E(G)$ such that $S_e \neq S_f$. Toward a contradiction, suppose that $V(S_e) = V(S_f)$. Let $uv \in S_f$. The set $N_{G[S_e]}[u]$ is not empty since u is incident with at least one edge in S_e. Let $G[S_e] - N_{G[S_e]}[u]$ be the induced subgraph of $G[S_e]$ containing the vertices of $V(S_e) \setminus N_{G[S_e]}[u]$. Let $B_e(u, v)$ be the set of vertices in the component containing v in $G[S_e] - N_{G[S_e]}[u]$.

Let $C_e(u, v)$ be the set of all vertices in $V(G)$ that are in neither $N_{G[S_e]}[u]$ nor $B_e(u, v)$. The set $C_e(u, v)$ may be empty.

Case 1. There exists some choice of edge e and adjacent vertices u and v such that $uv \in S_f$, and the set $B_e(u, v)$ has at least two vertices. We show every vertex in $V(G) \setminus B_e(u, v)$ that is adjacent to a vertex in $B_e(u, v)$ must be adjacent to every vertex in $B_e(u, v)$. Recall that $V(S_e) = V(S_f)$. Since all induced copies of P_3 have both or neither of its edges in S_e, if there exists a vertex in $V(G) \setminus V(S_e)$, then it is either adjacent to every vertex in $V(S_e)$ or none of the vertices in $V(S_e)$.

Every vertex b in $B_e(u,v)$ exists in some S_e-path with endpoints b and v. Every vertex in $B_e(u,v)$ that is adjacent to v with an edge in S_e must be adjacent to u with an edge not in S_e, by Corollary 7. By induction, for some k, suppose that every vertex b in $B_e(u,v)$ that exists in an S_e-k-path with endpoints b and v is adjacent to u with an edge not in S_e. Let q be a vertex that exists in an S_e-$(k+1)$-path with endpoints q and v. Since the neighbour of q in this path is adjacent to u with an edge not in S_e, by Corollary 7, q is adjacent to u with an edge not in S_e. Therefore, for every vertex b in $B_e(u,v)$, b is adjacent to u with an edge that is not in S_e. By Corollary 7, every vertex in $B_e(u,v)$ is adjacent to every vertex in $N_{G[S_e]}[u]$.

There does not exist any edge in S_e connecting a vertex in $B_e(u,v)$ to a vertex in $C_e(u,v)$ because this would imply that these two vertices exist in the same component of $G[S_e] - N_{G[S_e]}(u)$. So for all pairs of adjacent vertices $b \in B_e(u,v)$ and $c \in C_e(u,v)$, the edge bc is not in S_e. By induction suppose that c is adjacent to b and every vertex b' in $B_e(u,v)$ such that b' exists in an S_e-k-path with b as an endpoint, for some $k \geq 1$. Let b'' be a vertex such that there exists an S_e-$(k+1)$-path with endpoints b and b''. Since the neighbour of b'' in this path is adjacent to c with an edge not in S_e, by Corollary 7, the edge $b''c$ is in $E(G)$. Therefore, every vertex in $V(G) \setminus B_e(u,v)$ that is adjacent to a vertex in $B_e(u,v)$ must be adjacent to every vertex in $B_e(u,v)$. So there does not exist any induced copy of P_3 in G such that one edge has both endpoints in $B_e(u,v)$ and the other does not.

Let x be a vertex in $B_e(u,v)$ such that $vx \in S_e$. Since vx has both endpoints in $B_e(u,v)$, by Lemma 11, the equivalence class S_{vx} is a subset of the edges with both endpoints in $B_e(u,v)$. Note that it may be possible that S_{vx} is exactly the set of edges with both endpoints in $B_e(u,v)$. Since $N_{G[S_e]}[u]$ is nonempty, there exists some edge in S_e that does not have both of its endpoints in $B_e(u,v)$. However, since $vx \in S_e$, we have that $S_{vx} = S_e$. This is a contradiction. So $V(S_e) \neq V(S_f)$.

Case 2. For all choices of edge, e, and adjacent vertices u and v such that $uv \in S_f$, the set $B_e(u,v)$ only contains v. Recall that v must have at least one neighbour via an edge in S_e because $V(S_e) = V(S_f)$. Let x be a neighbour of v such that vx is in S_e. Since x is not in $B_e(u,v)$, there does not exist an S_e-path connecting x to v that contains no vertices in $N_{S_e}[u]$. Since the edge xv is in S_e, and v is not in $N_{S_e}[u]$, the vertex x must be in $N_{G[S_e]}[u]$ (see Figure 3).

The vertex v is a neighbour of x via an edge that is not in S_f, and v is a neighbour of u via an edge in S_f. This implies that u is in $N_{G[S_f]}[x]$ since otherwise, there would exist an S_f-path from u to v that contains no vertices in $N_{G[S_f]}[x]$, contradicting our supposition that $B_f(x,v)$ only contains v (see Figure 4). However, the edge xu is in S_e, which is distinct from S_f. This is a contradiction. So $V(S_e) \neq V(S_f)$.
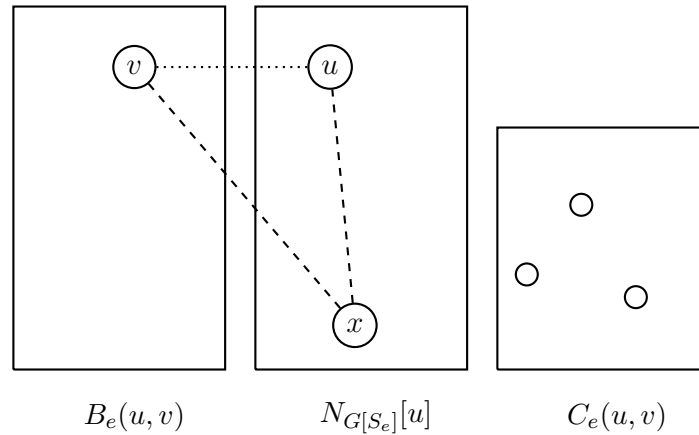
Figure 3. Graph G shown with vertices partitioned into sets $B_e(u,v)$, $N_{G[S_e]}[u]$, and $C_e(u,v)$. S_e is represented by dashed edges and S_f is represented by dotted edges.
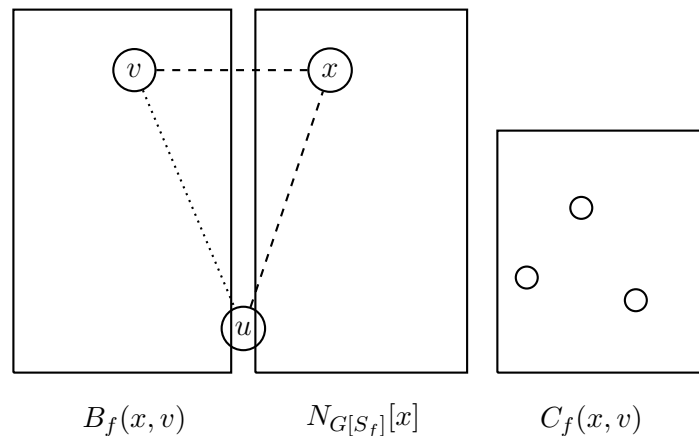

Figure 4. Graph G shown with vertices partitioned into sets $B_f(x,v)$, $N_{G[S_f]}[x]$, and $C_f(x,v)$. S_e is represented by dashed edges and S_f is represented by dotted edges. As u must be in both $N_{G[S_f]}[x]$ and $B_f(x,v)$, a contradiction arises.   ■

Corollary 13. *Let G be a connected graph. If $\mathcal{C}_G$ has exactly two equivalence classes, S_e and S_f, then either $V(S_e) \subset V(S_f)$ or $V(S_f) \subset V(S_e)$.*

Using Lemma 12 we provide our main result of this section: a classification of those connected graphs that admit a non-trivial quasi-transitive 2-edge-colouring.

Theorem 14. *Let G be a connected graph with n vertices. Then G is properly quasi-transitively colourable if and only if there exists an induced subgraph H of G such that*

- *H is connected;*
- $2 \leq |V(H)| \leq n - 1$; *and*
- *for every vertex v in $V(G) \setminus V(H)$, if v is adjacent to a vertex in $V(H)$, then v is adjacent to every vertex in $V(H)$.*

Proof. Let G be a connected graph with n vertices. Suppose first that there exists an induced subgraph H of G such that

- H is connected;
- $2 \leq |V(H)| \leq n - 1$; and
- for every vertex v in $V(G) \setminus V(H)$, if v is adjacent to a vertex in $V(H)$, then v is adjacent to every vertex in $V(H)$.

From the definition of H, it follows that there does not exist an induced copy of P_3 in G containing exactly one edge from $E(H)$. Therefore, each of $E(H)$ and $E(G) \setminus E(H)$ arise as the union of some number of equivalence classes under equivalence relation $\mathcal{C}_G$. Therefore, the equivalence relation $\mathcal{C}_G$ contains at least two equivalence classes, which means that G is properly quasi-transitively colourable by Theorem 8.

Suppose now that G is properly quasi-transitively colourable. Thus by Theorem 8, the equivalence relation $\mathcal{C}_G$ has at least two equivalence classes. By Lemma 12, there exists some equivalence class S_e of $\mathcal{C}_G$ such that $V(S_e) \neq V(G)$. Let H be the subgraph induced by $V(S_e)$. By Corollary 4, H is connected. Since S_e is an equivalence class and there exists at least one vertex in $V(G) \setminus V(S_e)$, it follows that $2 \leq |V(H)| \leq n - 1$.

Let v be a vertex in $V(G) \setminus V(S_e)$ that is adjacent to a vertex u in $V(S_e)$. Since v is in $V(G) \setminus V(S_e)$, the edge uv is not in S_e. The vertex u has some neighbour w such that uw is in S_e. By Corollary 7, since v is in $V(G) \setminus V(S_e)$, the edge vw exists. Use this as a base case and induct on the distance from u along the shortest S_e-path to show that v is adjacent to every vertex in H. Suppose that for some $k \geq 1$, every vertex x that exists in an S_e-k-path that also contains u is such that vx is in $E(G)$. Let y be a vertex which does not exist in any S_e-k-path that contains u, but does exist in an S_e-$(k + 1)$-path that contains u. The vertex y is adjacent to a vertex z that exists in an S_e-k-path that contains u. Thus vz is in $E(G)$ and, by Corollary 7, vy is in $E(G)$. Therefore, v is adjacent to every vertex in H. So for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$. Thus, our conclusion holds. ∎

Contrasting this result with the analogous result for quasi-transitively orientable graphs (Theorem 1), we see a significant difference in the resulting classification. We return to quasi-transitively orientable graphs in Section 4. For

now, however, we highlight the following difference between quasi-transitively orientable graphs and properly quasi-transitively colourable graphs.

Corollary 15. *The family of quasi-transitively colourable graphs admits no forbidden subgraph characterization.*

Proof. For all graphs J, there exists a connected graph H that contains J as a subgraph. Let G be the graph created by joining H and K_1. By Theorem 14, this graph G is properly quasi-transitively colourable. Thus every graph is a subgraph of some quasi-transitively colourable graph. Therefore, the family of quasi-transitively colourable graphs admits no forbidden subgraph characterization. ∎

Theorem 14 gives a classification of properly quasi-transitively colourable graphs based on the existence of an induced subgraph with particular properties. So the lack of existence of such a subgraph, and the existence of a unique such subgraph give rise to the following classifications of those graphs which admit only the trivial quasi-transitive 2-edge-colouring and those which admit a unique quasi-transitive 2-edge-colouring.

Corollary 16. *G is not properly quasi-transitively colourable if and only if for every induced proper connected subgraph H of G such that $E(H) \neq \emptyset$, there exists some vertex u in $V(G) \setminus V(H)$ that is adjacent to at least one vertex in $V(H)$, but not adjacent to all vertices in $V(H)$.*

Theorem 17. *A connected graph G with n vertices is uniquely quasi-transitively colourable if and only if there exists exactly one induced subgraph H of G such that*

- *H is connected;*
- *$2 \leq |V(H)| \leq n - 1$; and*
- *for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$.*

Proof. Suppose that a graph G with n vertices is uniquely quasi-transitively colourable. By Theorem 8, there are exactly two equivalence classes of $\mathcal{C}_G$, call them S_e and S_f for edges e and f in $E(G)$. By Corollary 13, either $V(S_e) \subset V(S_f)$ or $V(S_f) \subset V(S_e)$. By Theorem 14, there exists an induced subgraph H of G such that

- H is connected;
- $2 \leq |V(H)| \leq n - 1$; and
- for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$.

Since every induced copy of P_3 has either both or neither of its edges in $E(H)$, the edges of H make up either an equivalence class or a union of equivalence classes. Since we know that there are only two equivalence classes in G, the edges of H are one equivalence class, call it S_e, and the rest of the edges comprise the other equivalence class, call it S_f. Thus if there is a second induced subgraph H' satisfying our list of requirements, then it must be for the subgraph induced by $V(S_f)$ to make up H'. However, by Theorem 14, $V(H')$ must be a proper subset of $V(G)$, and we know that either $V(S_e)$ or $V(S_f)$ is equal to $V(G)$. Therefore, there can be no second induced subgraph satisfying our list of requirements.

Now suppose there exists a unique proper induced subgraph H of G such that

- H is connected;
- $2 \le |V(H)| \le n - 1$; and
- for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$.

By Theorem 14, there exists a nontrivial quasi-transitive 2-edge-colouring of G. Thus since $E(H)$ is either an equivalence class or some union of equivalence classes, there are at least two equivalence classes of $\mathcal{C}_G$. Toward a contradiction, suppose that $\mathcal{C}_G$ has more than two equivalence classes. This yields two cases. Either $E(H)$ contains multiple equivalence classes of $\mathcal{C}_G$, or $E(G) \setminus E(H)$ contains multiple equivalence classes of $\mathcal{C}_G$.

Case 1. $E(H)$ *contains multiple equivalence classes of $\mathcal{C}_G$.* Let e be an edge such that $S_e \subset E(H)$. Since S_e is an equivalence class, every induced copy of P_3 in G is such that either both or neither of its edges belong to S_e. The graph $G[S_e]$ is connected by Corollary 4. We know $2 \le |V(G[S_e])| \le n - 1$ since S_e contains at least one edge and $S_e \subset E(H)$. Since S_e is an equivalence class, every vertex u in $V(G) \setminus V(S_e)$ is such that if u is adjacent to a vertex in $V(S_e)$, then u is adjacent to every vertex in $V(S_e)$. Thus the choice of H is not unique since S_e satisfies all of the necessary requirements. This is a contradiction.

Case 2. $E(G) \setminus E(H)$ *contains multiple equivalence classes of $\mathcal{C}_G$.* By Lemma 12, if any two equivalence classes, S_e and S_f, are such that $V(S_e) = V(S_f)$, then $S_e = S_f$. Therefore, since $E(G) \setminus E(H)$ contains multiple equivalence classes, some equivalence class, $S_e \subset E(G) \setminus E(H)$, must be such that $V(S_e) \ne V(G)$. Let S_d be an equivalence class such that $S_d \subset E(G) \setminus E(H)$ and $V(S_d) \ne V(G)$. The graph $G[S_d]$ is connected, by Corollary 4, and $V(S_d)$ has between 2 and $n-1$ vertices. Also since every induced copy of P_3 either has both or neither of its edges in S_d, we have that for every vertex u in $V(G) \setminus V(S_d)$, if u is adjacent to a vertex in $V(S_d)$, then u is adjacent to every vertex in $V(S_d)$. Thus the choice of H is not unique, and this is a contradiction. Therefore, G is uniquely quasi-transitively colourable. $\blacksquare$

Using Theorem 17, we provide the following example of an infinite family of graphs that admit a unique quasi-transitive 2-edge-colouring.

Let $P_k = w_0 w_1 w_2 \cdots w_{k-1}$. Construct G from P_k by adding vertices u and v and edges uv, uw_0 and vw_0. By observation we have $S_{uv} = \{uv\}$ and $S_e = \{uw_0, vw_0\} \cup \{w_i w_{i-1} \mid 0 \leq i \leq k-1\}$ for all $e \neq uv$. Therefore $E(G)/\mathcal{C} = \{S_{uv}, S_{uw_0}\}$.

By Corollary 9, it follows that G is uniquely quasi-transitively colourable. In the statement of Theorem 17, the edge uv plays the role of H.

In this construction, each of the subgraphs induced by the two equivalence classes satisfy the criteria of Corollary 16. That is, the equivalence classes induce an partition of the graph into two subgraphs that admit only trivial quasi-transitive 2-edge-colourings. Such a partition exists in general.

Theorem 18. *Let G be a graph. If S_e is an equivalence class of $\mathcal{C}_G$, then $G[S_e]$ is not properly quasi-transitively colourable.*

Proof. Let G be a graph, let n be the number of vertices in G, and let e be an edge in $E(G)$. Toward a contradiction, suppose that $G[S_e]$ is properly quasi-transitively colourable. By Theorem 14, there exists an induced subgraph H of $G[S_e]$ such that

- H is connected;
- $2 \leq |V(H)| \leq n - 1$; and
- for every vertex u in $V(G) \setminus V(H)$, if u is adjacent to a vertex in $V(H)$, then u is adjacent to every vertex in $V(H)$.

Since S_e is an equivalence class of $\mathcal{C}_G$, for every vertex u in $V(G) \setminus V(S_e)$, if u is adjacent to a vertex in $V(S_e)$, then u is adjacent to every vertex in $V(S_e)$. So every vertex in $V(G) \setminus V(H)$ that is adjacent to some vertex in $V(H)$ must be adjacent to every vertex in $V(H)$. Thus for all h in $E(H)$, the subset S_h of $E(G)$ must equal $E(H)$. Therefore, $E(H)$ is an equivalence class of $\mathcal{C}_G$. This is a contradiction because S_e is an equivalence class of $\mathcal{C}_G$ and $E(H)$ is a proper subset of S_e. Therefore, $G[S_e]$ is not properly quasi-transitively colourable. ∎

We conclude this section with two more results that further aid in understanding the structure of graphs that contain multiple equivalence classes.

Theorem 19. *Let G be a connected graph and let x and y be in $V(G)$. Every shortest path between x and y in G must only contain edges from a single equivalence class of $\mathcal{C}_G$.*

Proof. Let G be a graph and let x and y be in $V(G)$. Toward a contradiction, suppose there exists a shortest path $P = v_1 \cdots v_k$ from x to y, with $v_1 = x$, $v_k = y$, and that some pair of incident edges in P, $v_{i-1}v_i$ and $v_i v_{i+1}$, are such that $S_{v_{i-1}v_i}$

is not equal to $S_{v_i v_{i+1}}$. By Corollary 7, the edge $v_{i-1}v_{i+1}$ must exist because otherwise an induced P_3 with edges from two different equivalence classes would result. The contradiction arises because the shortest path from x to y, P, can be made shorter by replacing the edges $v_{i-1}v_i$ and $v_i v_{i+1}$ by the edge $v_{i-1}v_{i+1}$. Therefore, every shortest path between x and y in G must only contain edges from a single equivalence class of $\mathcal{C}_G$. ∎

Corollary 20. *Let G be a connected graph. There is at most one equivalence class, S_e, of $\mathcal{C}_G$ such that $V(S_e)$ contains a vertex incident only with edges in S_e.*

Proof. Let v be a vertex incident only with edges in S_e, for some edge e in $E(G)$. Toward a contradiction, suppose that u is in $V(S_f)$ for some edge f in $E(G)$ that is not equal to e, and that u is incident only with edges in S_f. Let P be a shortest path connecting u to v. By Theorem 19, P only contains edges from a single equivalence class. This is a contradiction because u and v are not incident with any pair of respective edges in the same equivalence class. Therefore, there is at most one equivalence class, S_e, of $\mathcal{C}_G$ such that $V(S_e)$ contains a vertex incident only with edges in S_e. ∎

3. Intersection of Equivalence Classes

In Section 2, we characterized graphs with at least two equivalence classes (Theorem 14) and graphs with exactly two equivalence classes (Theorem 17). In this section, we characterize graphs with exactly three equivalence classes. We approach our study by considering the structure of the subgraph induced by a pair of equivalence classes.

By Corollary 13, when graphs have exactly two equivalence classes, S_e and S_f, either $V(S_e) \subset V(S_f)$ or $V(S_f) \subset V(S_e)$. However, when a graph has more than two equivalence classes, this subset property does not necessarily hold. Let S_e and S_f be distinct equivalence classes of $\mathcal{C}_G$. It is not necessary for there to be any vertices in $V(S_e) \cap V(S_f)$. However if G is a connected graph with at least two equivalence classes, then there must exist some vertex v and some pair of equivalence classes, S_e and S_f, such that v is in both $V(S_e)$ and $V(S_f)$. For the purposes of this section, we will suppose, without loss of generality, the set $V(S_e) \cap V(S_f)$ is nonempty (see Figure 5).

Every edge with endpoints in both $V(S_e) \setminus V(S_f)$ and $V(S_f) \setminus V(S_e)$ must be in neither S_e nor S_f. There exist cases in which a single vertex is in the intersection of two equivalence classes and cases where the intersection contains two non-adjacent vertices (both cases arise in $K_4 \setminus \{e\}$, see Figure 6).

For the purposes of this section, we will suppose that $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are non-empty. We will prove a number of results

regarding the types of graphs in which these three vertex sets are nonempty, culminating in a classification of graphs with exactly three equivalence classes (Theorem 28). We begin with a rather specific result, but one that will be useful in proving Theorem 22.
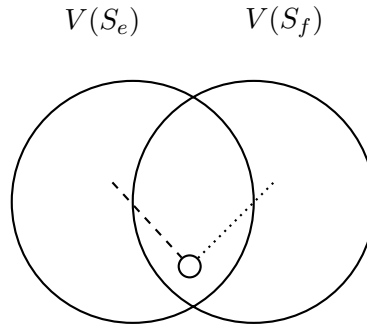
$$V(S_e) \qquad\qquad V(S_f)$$

Figure 5. The vertex sets of two equivalence classes S_e (represented by dashed edges) and S_f (represented by dotted edges) shown with a nonempty intersection.
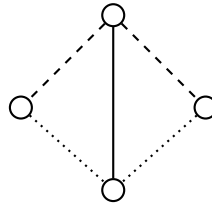

Figure 6. $K_4 \setminus \{e\}$ shown with the three equivalence classes represented by full, dashed, and dotted lines.

Lemma 21. *Let G be a connected graph and let $e, f \in E(G)$ be such that S_e and S_f are equivalence classes of $\mathcal{C}_G$. If*

- $V(S_e) \setminus V(S_f)$ *is nonempty;*
- $u, v, y \in V(S_e) \cap V(S_f)$ *and* $x \in V(S_f) \setminus V(S_e)$;
- $uv, vx \in S_f$, $vy \notin S_f$, *and* $uy \in S_e$;

then $ux \in E(G)$.

Proof. Let G be a connected graph and let $e, f \in (G)$ be such that S_e and S_f are equivalence classes of $\mathcal{C}_G$. Suppose

- $V(S_e) \setminus V(S_f)$ is nonempty;
- $u, v, y \in V(S_e) \cap V(S_f)$ and $x \in V(S_f) \setminus V(S_e)$;
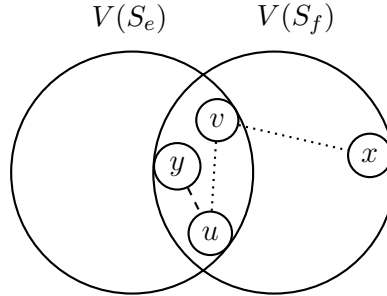- $uv, vx \in S_f$, $vy \notin S_f$, and $uy \in S_e$.

Figure 7. Theoretical intersection $V(S_e) \cap V(S_f)$ containing both endpoints of an edge in S_f and both endpoints of an edge in S_e. Dotted edges are in S_f and dashed edges are in S_e.

By Corollary 7, since vy and vx belong to different equivalence classes, the edge xy must exist (See Figure 7). Since $x \in V(S_f) \setminus V(S_e)$, the edge xy is not in S_e. Therefore by Corollary 7, since xy and uy belong to different equivalence classes, the edge ux must exist. ∎

Theorem 22. *Let G be a connected graph and let $e, f \in E(G)$ be such that S_e and S_f are equivalence classes of $\mathcal{C}_G$. If $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then $V(S_e) \cap V(S_f)$ does not contain any pair of vertices u and v such that uv is in either S_e or S_f.*

Proof. Let G be a connected graph and let $e, f \in E(G)$ be such that $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty. Toward a contradiction, suppose that u and v are in $V(S_e) \cap V(S_f)$ and that uv is in S_f. Since $V(S_f) \setminus V(S_e)$ is nonempty, S_f must contain some other edge. In order for S_f to not be a singleton set, uv must exist in some induced copy of P_3 with another edge from S_f. Without loss of generality, suppose xv is in S_f and that uvx is an induced copy of P_3. Every vertex in $V(S_e) \cap V(S_f)$ must be incident with an edge in S_e.

Case 1. There does not exist a vertex in $V(S_e) \cap V(S_f)$ that is adjacent to v with an edge in S_f and also adjacent to a vertex in $V(S_e) \setminus V(S_f)$ with an edge in S_e. So every edge in S_e that is incident with u must have both endpoints in $V(S_e) \cap V(S_f)$. Let uy be such an edge. By Corollary 7, since uv and uy belong to different equivalence classes, the edge vy must exist. By Lemma 21, if $vy \notin S_f$, then $ux \in E(G)$, contradicting uvx being an induced copy of P_3. So $vy \in S_f$. However, if every neighbour of u via S_e edges is a vertex in $V(S_e) \cap V(S_f)$ that is adjacent to v via an S_f edge, then there does not exist an S_e edge with one endpoint in $V(S_e) \setminus V(S_f)$ and the other endpoint in $V(S_e) \cap V(S_f)$. Therefore since $V(S_e) \setminus V(S_f)$ is nonempty, $G[S_e]$ is not connected. This contradicts Corollary 4.

Case 2. *Some vertex in* $V(S_e) \cap V(S_f)$ *that is adjacent to* v *with an edge in* S_f *is also adjacent to a vertex in* $V(S_e) \setminus V(S_f)$ *with an edge in* S_e. Without loss of generality, suppose that u is such a vertex. Let y be a vertex in $V(S_e) \setminus V(S_f)$ such that $uy \in S_e$. By Corollary 7, since uv and uy belong to different equivalence classes, the edge vy must exist. The edge vy is not in S_f since $y \in V(S_e) \setminus V(S_f)$. So by Corollary 7, since vx and vy belong to different equivalence classes, the edge xy must exist. The edge xy is not in S_e since $x \in V(S_f) \setminus V(S_e)$. Finally by Corollary 7, since uy and xy belong to different equivalence classes, the edge ux must exist. This contradicts uvx being an induced copy of P_3.

Therefore, there does not exist a pair of adjacent vertices $u, v \in V(S_e) \cap V(S_f)$ such that uv is in either S_e or S_f.   ■

The following four results will build upon each other and be used to justify Theorem 27, and then finally yield our classification of those graphs that have exactly three equivalence classes (Theorem 28).

Lemma 23. *Let G be a connected graph and let S_e and S_f be two equivalence classes of $\mathcal{C}_G$. If $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then every vertex in $V(S_e) \setminus V(S_f)$ is adjacent to every vertex in $V(S_f)$ and every vertex in $V(S_f) \setminus V(S_e)$ is adjacent to every vertex $V(S_e)$.*
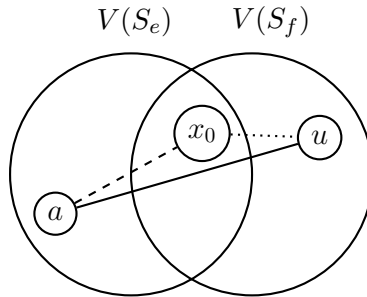

Figure 8. The vertex sets of two equivalence classes with nonempty intersection. Dotted edges are in S_f and dashed edges are in S_e. The full edge is from a third equivalence class, S_{au}.

Proof. Let $x_0 \in V(S_e) \cap V(S_f)$, and $a \in V(S_e) \setminus V(S_f)$ such that $ax_0 \in S_e$ (see Figure 8). Such a pair of vertices must exist because otherwise, with no edge with one endpoint in $V(S_e) \cap V(S_f)$ and one endpoint in $V(S_e) \setminus V(S_f)$, the graph $G[S_e]$ would be disconnected, contradicting Corollary 4. Since x_0 is in $V(S_e) \cap V(S_f)$, there exists a vertex u such that ux_0 is in S_f. The vertex u must be in $V(S_f) \setminus V(S_e)$ by Theorem 22. By Corollary 7, since ax_0 and ux_0 are in different equivalence classes, the edge au must exist. The vertex a is not incident with any edges in S_f and the vertex u is not incident with any edges

in S_e. We will use induction to show that every vertex in $V(S_f)$ is adjacent to a (and equivalently, every vertex in $V(S_e)$ is adjacent to u), inducting on the length of the shortest S_f-path connecting a vertex to x_0. By Theorem 22, every vertex that is adjacent to x_0 with an edge in S_f must be in $V(S_f) \setminus V(S_e)$. Thus by Corollary 7, every vertex that is adjacent to x_0 with an edge in S_f must also be adjacent to a with an edge in neither S_e nor S_f. Now suppose that for every vertex $v \in V(S_f)$ such that the shortest S_f-path connecting v to x_0 is length k, v is adjacent to a. If there does not exist some vertex v such that the shortest S_f-path connecting v to x_0 is length $k+1$, then a is adjacent to every vertex in $V(S_f)$. Otherwise, let $P = x_0 \cdots x_{k+1}$ be an S_f-$(k+1)$-path. Then by Corollary 7, since a is adjacent to x_k with an edge that is not in S_f, the edge ax_{k+1} must exist. Therefore, by induction, the vertex a is adjacent to every vertex in S_f (and equivalently, every vertex in $V(S_e)$ is adjacent to u).

We now prove that every other vertex in $V(S_e) \setminus V(S_f)$ is adjacent to every vertex in $V(S_f)$ (and equivalently, that every other vertex in $V(S_f) \setminus V(S_e)$ is adjacent to every vertex in $V(S_e)$). Let z be a vertex in $V(S_e) \setminus V(S_f)$. By Corollary 7, since the vertex z is adjacent to the vertex u, the vertex z is also adjacent to every vertex which is adjacent to u with an edge in S_f. We will induct on the distance from u along S_f-paths. Suppose that every vertex y, such that the shortest S_f-path connecting y to u is length k, is adjacent to z. If there does not exist some vertex y such that the shortest S_f-path connecting y to u is length $k+1$, then z is adjacent to every vertex in $V(S_f)$. Otherwise, let $P = uy_1 \cdots y_{k+1}$ be an S_f-$(k+1)$-path. Then by Corollary 7, since z is adjacent to y_k with an edge that is not in S_f, the edge zy_{k+1} must exist. Therefore, by induction, the vertex z is adjacent to every vertex in S_f. Thus every vertex in $V(S_e) \setminus V(S_f)$ is adjacent to every vertex in $V(S_f)$. Equivalently, every vertex in $V(S_f) \setminus V(S_e)$ is adjacent to every vertex in $V(S_e)$. ∎

Corollary 24. *Let G be a connected graph, and let S_e and S_f be equivalence classes of $\mathcal{C}_G$. If $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then every edge in S_e and every edge in S_f has an endpoint in $V(S_e) \cap V(S_f)$.*

Proof. Let G be a connected graph, and let S_e and S_f be equivalence classes of $\mathcal{C}_G$. Toward a contradiction, suppose that f has both endpoints in $V(S_f) \setminus V(S_e)$ and that both $V(S_e) \cap V(S_f)$ and $V(S_e) \setminus V(S_f)$ are nonempty. By Lemma 23, the vertices in $V(S_e) \cap V(S_f)$ are joined to the vertices in $V(S_f) \setminus V(S_e)$. So there does not exist an induced copy of P_3 containing an edge with both endpoints in $V(S_f) \setminus V(S_e)$ and an edge with one endpoint in $V(S_f) \cap V(S_e)$. So S_f does not contain any edge with an endpoint in $V(S_e) \cap V(S_f)$. This is a contradiction though as $V(S_e) \cap V(S_f)$ is nonempty. Therefore, if $V(S_e) \cap V(S_f)$ is nonempty, then every edge in S_f (and thus, S_e) has an endpoint in $V(S_e) \cap V(S_f)$. ∎

For a graph G, a subset A of $V(G)$ *induces a join* if the subgraph of G induced by A is a join of some pair of subgraphs H and J of G.

Lemma 25. *Let G be a connected graph, and let S_e and S_f be equivalence classes of $\mathcal{C}_G$. If $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then none of those three vertex sets induces a join.*

Proof. We will first prove that $V(S_e) \setminus V(S_f)$ (and thus, $V(S_f) \setminus V(S_e)$) does not induce a join. By Corollary 24, every edge in S_e has an endpoint in $V(S_e) \cap V(S_f)$. Toward a contradiction, suppose that the subgraph induced by $V(S_e) \setminus V(S_f)$ is a join. Call the two joined vertex sets A and B, so that $A \cup B = V(S_e) \setminus V(S_f)$. By Theorem 22, e does not have both endpoints in $V(S_e) \cap V(S_f)$. By Theorem 22 and Corollary 24, e has an endpoint in $V(S_e) \setminus V(S_f)$. Suppose, without loss of generality, that e has an endpoint in A. We know that $G[S_e]$ is a connected graph, by Corollary 4, and that B is joined not only to A, but also to $A \cup (V(S_e) \cap V(S_f))$. Hence there cannot exist an induced copy of P_3 with one edge having both endpoints in $A \cup (V(S_e) \cap V(S_f))$ and the other edge having one endpoint in B. Thus, no edge with an endpoint in B is in S_e. This is a contradiction. Neither $V(S_e) \setminus V(S_f)$ nor $V(S_f) \setminus V(S_e)$ induces a join.

Now we prove that $V(S_e) \cap V(S_f)$ does not induce a join using a similar argument. By Corollary 24, every edge in S_e has an endpoint in $V(S_e) \cap V(S_f)$. Toward a contradiction, suppose that the subgraph induced by $V(S_e) \cap V(S_f)$ is a join. Call the two joined vertex sets A' and B', so that $A' \cup B' = V(S_e) \cap V(S_f)$. Suppose, without loss of generality, that e has an endpoint in A'. We know that $G[S_e]$ is a connected graph, by Corollary 4, and that B' is joined to not only A', but also to $A' \cup (V(S_e) \setminus V(S_f))$. Hence there cannot exist an induced copy of P_3 with one edge having both endpoints in $A' \cup (V(S_e) \setminus V(S_f))$ and the other edge having one endpoint in B'. Thus, no edge with an endpoint in B' is in S_e. This is a contradiction. Therefore, $V(S_e) \cap V(S_f)$ does not induce a join. ∎

Lemma 26. *Let G be a connected graph and let S_e and S_f be equivalence classes of $\mathcal{C}_G$. If $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then every edge with one endpoint in $V(S_e) \setminus V(S_f)$ and one endpoint in $V(S_f) \setminus V(S_e)$ belongs to the same equivalence class of $\mathcal{C}_G$, call it S_d.*

Proof. Let G be a connected graph and let S_e and S_f be equivalence classes of $\mathcal{C}_G$. Suppose that $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty. We know from Lemma 23 that $V(S_e) \setminus V(S_f)$ is joined to $V(S_f) \setminus V(S_e)$ and that none of these edges belong to S_e or S_f. Let $v \in V(S_e) \setminus V(S_f)$. It will be sufficient to prove that every vx, where x is a vertex in $V(S_f) \setminus V(S_e)$, belongs to the same equivalence class. Suppose vw_0 is in some equivalence class S_d for some w_0 in $V(S_f) \setminus V(S_e)$. By Lemma 25, we know that $V(S_f) \setminus V(S_e)$ is not a join. Therefore, there exists some vertex w_1 in $V(S_f) \setminus V(S_e)$ that is not adjacent to

w_0. Since $w_0 v w_1$ is an induced copy of P_3, both vw_0 and vw_1 belong to the same equivalence class. Thus, vw_1 is in S_d. By induction, suppose that v is adjacent to k different vertices in $V(S_f) \setminus V(S_e)$, $w_0, \ldots, w_k$, with edges in S_d. Let G_k be the subgraph induced by $\{w_0, \ldots, w_k, v\}$. Since $V(S_f) \setminus V(S_e)$ is not a join, if there exists a vertex in $V(S_f) \setminus V(S_e)$ that is not in G_k, then there exists such a vertex that is not adjacent to every vertex in G_k. Let w_{k+1} be such a vertex and let w_k be a vertex in G_k not adjacent to w_{k+1}. Since w_k is not adjacent to w_{k+1}, the edges vw_k and vw_{k+1} belong to the same equivalence class, S_d. Therefore for all x in $V(S_f) \setminus V(S_e)$, vx belongs to S_d. Therefore, every edge with one endpoint in $V(S_e) \setminus V(S_f)$ and one endpoint in $V(S_f) \setminus V(S_e)$ belongs to the same equivalence class. ∎

Theorem 27. *If G is a connected graph, $\mathcal{C}_G$ has exactly three equivalence classes, S_e and S_f are distinct equivalence classes of $\mathcal{C}_G$, and $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ are all nonempty, then G is complete tripartite.*

Proof. Let G be a connected graph, $\mathcal{C}_G$ have exactly three equivalence classes, S_e and S_f be distinct equivalence classes of $\mathcal{C}_G$, and $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, and $V(S_e) \cap V(S_f)$ all be nonempty. By Lemma 23, if a is a vertex in $V(S_e) \setminus V(S_f)$ and b is a vertex in $V(S_f) \setminus V(S_e)$, then ab is an edge in $E(G)$. By Lemma 26, every such edge belongs to the same equivalence class, call it S_{ab}. By Lemma 23, every vertex in $V(S_e) \cap V(S_f)$ is adjacent to every vertex in $V(S_e) \setminus V(S_f)$ and every vertex in $V(S_f) \setminus V(S_e)$. Now all that remains to be shown is that there are no edges in $E(G)$ such that both endpoints are in $V(S_e) \setminus V(S_f)$, $V(S_f) \setminus V(S_e)$, or $V(S_e) \cap V(S_f)$. Toward a contradiction, suppose that an edge, say yz, exists with both endpoints in $V(S_e) \setminus V(S_f)$. If yz does not exist in an induced copy of P_3, then yz is itself a fourth equivalence class and this would be a contradiction. So yz exists in some induced copy of P_3. Both endpoints of yz are adjacent to every vertex in $V(S_f) \setminus V(S_e)$ and $V(S_e) \cap V(S_f)$. Therefore, any induced copy of P_3 which contains yz must only contain edges in $V(S_e) \setminus V(S_f)$. Thus no edge with both endpoints in $V(S_e) \setminus V(S_f)$ is in S_e, S_f, or S_{ab}. This is a contradiction since G has exactly three equivalence classes. Therefore, no edge exists with both endpoints in $V(S_e) \setminus V(S_f)$ (or equivalently, $V(S_f) \setminus V(S_e)$).

Now toward a contradiction, suppose that an edge wx exists with both endpoints in $V(S_e) \cap V(S_f)$. If wx does not exist in an induced copy of P_3, then wx is itself a fourth equivalence class and this would be a contradiction. So wx exists in some induced copy of P_3. Both endpoints of wx are adjacent to every vertex in $V(S_f) \setminus V(S_e)$ and $V(S_e) \setminus V(S_f)$. Therefore, any induced copy of P_3 which contains wx must only contain edges in $S_e \cap S_f$. Thus no edge with both endpoints in $V(S_e) \cap V(S_f)$ is in S_e, S_f, or S_{ab}. This is a contradiction since G has exactly three equivalence classes. Therefore, no edge exists with both endpoints in $V(S_e) \cap V(S_f)$. So G is complete tripartite. ∎

Graphs with exactly three equivalence classes are classified as follows.

Theorem 28. *Let G be a connected graph such that $\mathcal{C}_G$ has exactly three equivalence classes, S_d, S_e, and S_f. If G is not complete tripartite, then there exists $e \in E(G)$ so that $V(S_e) = V(G)$.*

Proof. Let G be a connected graph such that $\mathcal{C}_G$ has exactly three equivalence classes. Since G is connected, every equivalence class S_x must be such that $V(S_x) \cap V(S_y) \neq \emptyset$ for some other equivalence class S_y. Suppose that for every pair of equivalence classes S_x and S_y, either $V(S_x) \setminus V(S_y)$, $V(S_y) \setminus V(S_x)$, or $V(S_x) \cap V(S_y)$ is empty, because otherwise by Theorem 27, G is complete tripartite. This implies that for every pair of equivalence classes, S_x and S_y, either the intersection of their vertex sets is empty or one vertex set is a subset of the other. Since G is connected, some pair of equivalence classes, S_e and S_f, are such that $V(S_f) \subseteq V(S_e)$. Let S_d be the other equivalence class. Since G is connected, either $V(S_d)$ is a subset of $V(S_e)$ or $V(S_e)$ is a subset of $V(S_d)$. In either case, there exists an equivalence class for which the vertex set contains the vertex sets of all other equivalence classes. This implies there exists some equivalence class such that the vertex set contains every vertex in G.   ∎

We provide the following infinite family of graphs that are not complete tripartite and have exactly three equivalence classes. Let $P_k = v_0, v_1, \ldots, v_{k-1}$ and $P_k' = v_0', v_1', \ldots, v_{k-1}'$ be disjoint paths. Let G be the graph formed from P_k and P_k' by adding a universal vertex v. We claim G has exactly three equivalence classes.

For each $0 \leq i, j \leq k - 1$, the edge $v_i v_j'$ does not exist. Therefore the edges incident with v are in the same equivalence class because of the induced copies of P_3 that exist.

Since every vertex not in P_k is adjacent to either every vertex in P_k or no vertex in P_k, by Lemma 11, the equivalence class of any edge in P_k contains only edges of P_k. Therefore $S_{v_i v_{i+1}} = E(P_k)$ for all $0 \leq i \leq k - 2$. A similar argument implies $S_{v_i' v_{i+1}'} = E(P_k')$ for all $0 \leq i \leq k - 2$. Therefore $E(G)/\mathcal{C} = \{S_{v_0 v_1}, S_{v_0' v_1'}, S_{uv_0}\}$.

Notice that in the statement of Theorem 28, vv_0 plays the role of e.

4.   Uniquely Quasi-Transitively Orientable Graphs

The statement of Corollary 15 hints at a significant difference between the family of quasi-transitively orientable graphs and properly quasi-transitively colourable graphs. Recall that the family of quasi-transitively orientable graphs is equal to the family of comparability graphs (Theorem 1). Such graphs admit a forbidden subgraph characterisation [11] and thus can be identified in polynomial time. On
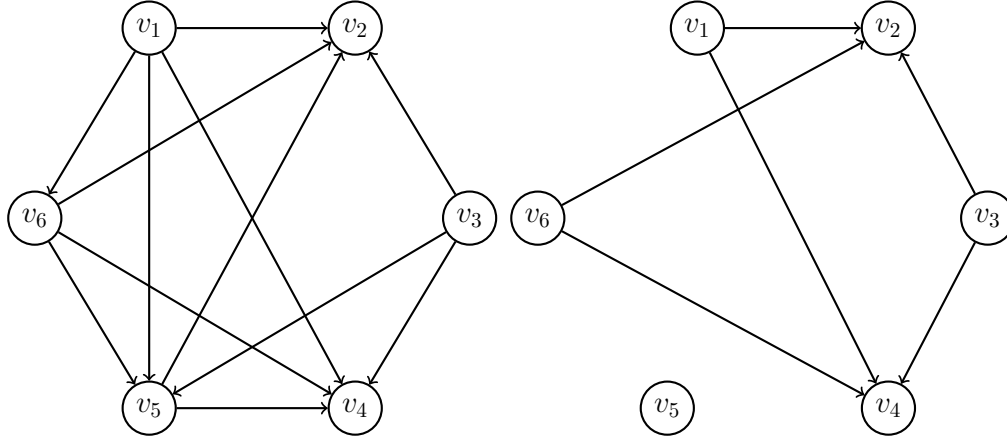
Figure 9. Quasi-transitive orientation Q of a graph G and the partial quasi-transitive orientation of Q generated by the arc $\overrightarrow{v_1 v_2}$.

the other hand, by Theorem 14 and Corollary 15, no such forbidden subgraph characterisation of properly quasi-transitively colourable graphs exists. However, one may verify that the conditions of Theorem 14 can be checked in polynomial time.

In this section we restrict our attention to the family of comparability graphs. Using techniques similar to those in Section 2 we find that sets of the form S_e arise as equivalence classes in a relation related to quasi-transitive orientations of comparability graphs.

Let G be a comparability graph and consider $uv \in E(G)$. As G is a comparability graph, there exists at least one quasi-transitive orientation of G for which the edge uv is oriented to have its head at v. Note that if $vw \in E(G)$ and $uw \notin E(G)$, then necessarily, the edge vw is oriented with its head at v whenever uv is oriented with its head at v.

A *partial quasi-transitive orientation generated by* $\overrightarrow{uv}$ *in* G is a quasi-transitive orientation of a largest subgraph (most edges) of G that is uniquely quasi-transitively orientable, in which $\overrightarrow{uv}$ is an arc. In Figure 9, we see the graph G from Figure 1 with a quasi-transitive orientation of the edges and we also see a partial quasi-transitive orientation generated by an edge in G.

Theorem 29. *For all comparability graphs* G, *if* $uv \in E(G)$, *then*

- *there exists a unique partial quasi-transitive orientation generated by* $\overrightarrow{uv}$ *and*

- *the set of edges that are directed in the unique partial quasi-transitive orientation generated by* $\overrightarrow{uv}$ *is the equivalence class* S_{uv} *of* $\mathcal{C}_G$.

Proof. Suppose G is a comparability graph and that uv is chosen to be directed $\overrightarrow{uv}$ in a quasi-transitive orientation X of G. Every induced P_3 of the form uvw is

such that $\overrightarrow{wv}$ is an arc in X. Also for every P_3 that contains either uv or vw, the direction of the other edge is the same in all partial quasi-transitive orientations generated by $\overrightarrow{uv}$. Let S be a minimal subset of $E(G)$ which contains uv and is such that every induced P_3 either has both or neither of its edges in S. By Corollary 3, S is the only smallest subset of $E(G)$ which contains uv and is such that every induced P_3 either has both or neither of its edges in S. No partial quasi-transitive orientation generated by $\overrightarrow{uv}$ can contain any edges not contained in S because there would exist a partition of the edge set into two sets such that no induced copy of P_3 includes an edge from both sets. Thus S is unique, and by Theorem 8, the set S is equal to the equivalence class S_{uv} of $\mathcal{C}_G$.   ∎

We will denote the unique partial quasi-transitive orientation generated by $\overrightarrow{uv}$ in G by $\Gamma_{\overrightarrow{uv}}$. Let $\mathcal{O}_G$ be the relation on $E(G)$ so that $uv \sim xy$ when any of the following are true:

- $\Gamma_{\overrightarrow{uv}} = \Gamma_{\overrightarrow{xy}}$ • $\Gamma_{\overrightarrow{uv}} = \Gamma_{\overrightarrow{yx}}$ • $\Gamma_{\overrightarrow{vu}} = \Gamma_{\overrightarrow{xy}}$ • $\Gamma_{\overrightarrow{vu}} = \Gamma_{\overrightarrow{yx}}$.

Given a graph G, by Theorem 29, O_G is an equivalence relation. One can see this by noting that the second part of Theorem 29 guarantees that the equivalence classes of C_G are exactly those of O_G.

Corollary 30. *Let G be a comparability graph. If S_e is an equivalence class of $\mathcal{O}_G$, then $G[S_e]$ is uniquely quasi-transitively orientable.*

Corollary 31. *For all integers $k \geq 1$, there exists a graph G with k equivalence classes under the relation $\mathcal{O}_G$.*

Proof. By Theorem 10 and Theorem 29, for all integers $k \geq 1$, there exists a graph G with k equivalence classes under the relation $\mathcal{C}_G$ and both $\mathcal{C}_G$ and $\mathcal{O}_G$ partition the edges of G in the same manner. Therefore, for all integers $k \geq 1$, there exists a graph G with k equivalence classes under the relation $\mathcal{O}_G$.   ∎

Corollary 32. *If G is a comparability graph so that $\mathcal{C}_G$ has k equivalence classes, then there are 2^k quasi-transitive orientations of G.*

Proof. For all graphs G, the number of equivalence classes of $\mathcal{C}_G$ is equal to the number of equivalence classes of $\mathcal{O}_G$. For every arc $\overrightarrow{uv}$, there exists a unique partial quasi-transitive orientation generated by $\overrightarrow{uv}$ by Theorem 29. Therefore, there exist 2^k quasi-transitive orientations of G, where k is the number of equivalence classes.   ∎

By the definitions of S_{uv} and $[\overrightarrow{uv}]_{\mathcal{O}}$, Theorem 29 tells us S_{uv} is the largest set of edges such that for every quasi-transitive orientation X of G and every edge $ab \in S_{uv}$, if $\overrightarrow{ab}$ and $\overrightarrow{uv}$ are arcs in X, then $\overrightarrow{ab}$ is an arc in every quasi-transitive orientation of G in which $\overrightarrow{uv}$ is an arc. In Figure 9, the image on the right shows the set of oriented edges as exactly those of the set $S_{v_1 v_2}$.

By Theorem 8 and Theorem 29, we have $|E(G)/\mathcal{C}| = |E(G)/\mathcal{O}|$. So we have the following result.

Theorem 33. *A comparability graph G is uniquely quasi-transitively orientable if and only if G admits only the trivial quasi-transitive 2-edge-colouring.*

5. Conclusions and Future Work

Though the respective literature yields significant examples of commonalities in the study of oriented graphs and 2-edge-coloured graphs, our work here highlights a fundamental difference between these two relational structures. As with many concepts under study in oriented and 2-edge-coloured graphs, the difference between the definitions of quasi-transitivity for oriented graphs and for 2-edge-coloured graphs is only slight. However in this case, the resulting classification is markedly different.

Such a phenomenon also occurs with the study of chromatic polynomials of oriented and 2-edge-coloured graphs. In [6], Cox and Duffy fully classified those oriented graphs whose oriented chromatic polynomial was identical to the chromatic polynomials of the underlying simple graph as quasi-transitive orientations of co-interval graphs, notably a class of graphs for which the chromatic polynomial can be computed in polynomial time [9]. In the sequel for 2-edge-coloured graphs, Beaton, Cox, Duffy and Zolkavich [3] obtain only a partial classification of 2-edge-coloured graphs whose 2-edge-coloured chromatic polynomial is identical to the chromatic polynomial of the underlying simple graph. Here, work was stymied by a paucity of research results on quasi-transitive 2-edge-coloured graphs. However, their preliminary results suggest that those 2-edge-coloured graphs for which the 2-edge-coloured chromatic polynomial is identical to the chromatic polynomial of the underlying simple graph will comprise a family of graphs for which computing the chromatic polynomial is NP-hard. We expect that our results, particularly Theorem 14, will provide the insight needed to complete this classification.

References

[1] N. Alon and T.H. Marshall, *Homomorphisms of edge-colored graphs and Coxeter groups*, J. Algebraic Combin. **8** (1998) 5–13.
https://doi.org/10.1023/A:1008647514949

[2] J. Bang-Jensen and J. Huang, *Quasi-transitive digraphs*, J. Graph Theory **20** (1995) 141–161.
https://doi.org/10.1002/jgt.3190200205

[3] I. Beaton, D. Cox, C. Duffy and N. Zolkavich, *Chromatic polynomials of 2-edge coloured graphs* (2020).
arXiv:2007.13710

[4] J.A. Bondy and U.S.R. Murty, Graph Theory, in: Grad. Texts in Math. **244** (Springer, New York, 2008).

[5] A. Contreras-Balbuena, H. Galeana-Sánchez and I.A. Goldfeder, *Alternating Hamiltonian cycles in* 2-*edge-colored multigraphs*, Discrete Math. Theor. Comput. Sci. **21(1)** (2019) 12.
https://doi.org/10.23638/DMTCS-21-1-12

[6] D. Cox and C. Duffy, *Chromatic polynomials of oriented graphs*, Electron. J. Combin. **26(3)** (2019) #P3.55.
https://doi.org/10.37236/8240

[7] H. Galeana-Sánchez, I.A. Goldfeder and I. Urrutia, *On the structure of strong* 3-*quasi-transitive digraphs*, Discrete Math. **310** (2010) 2495–2498.
https://doi.org/10.1016/j.disc.2010.06.008

[8] Ghouilá-Houri, *Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C.R. Acad. Sci. Paris **254** (1962) 1370–1371
https://www.sciencedirect.com/science/article/pii//0012365X9190108E.

[9] M.C. Golumbic, *The complexity of comparability graph recognition and coloring*, Computing **18** (1977) 199–208.
https://doi.org/10.1007/BF02253207

[10] C. Hernández-Cruz and H. Galeana-Sánchez, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530.
https://doi.org/10.1016/j.disc.2012.05.005

[11] C.T. Hoàng, *Efficient algorithms for minimum weighted colouring of some classes of perfect graphs*, Discrete Appl. Math. **55** (1994) 133–143.
https://doi.org/10.1016/0166-218X(94)90004-3

[12] G. Macgillivray and K. Sherk, *A theory of* 2-*dipath colourings*, Australas. J. Combin. **60** (2014) 11–26.

[13] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud and É. Sopena, *Homomorphisms of* 2-*edge-colored graphs*, Discrete Appl. Math. **158** (2010) 1365–1379.
https://doi.org/10.1016/j.dam.2009.09.017

[14] A. Raspaud and É. Sopena, *Good and semi-strong colorings of oriented planar graphs*, Inform. Process. Lett. **51** (1994) 171–174.
https://doi.org/10.1016/0020-0190(94)00088-3

[15] É. Sopena, *Homomorphisms and colourings of oriented graphs: An updated survey*, Discrete Math. **339** (2016) 1993–2005.
https://doi.org/10.1016/j.disc.2015.03.018

[16] K. Young, 2-Dipath and Proper 2-Dipath Colouring (Master's Thesis, University of Victoria, 2009).