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A CHVÁTAL-ERDŐS TYPE THEOREM FOR PATH-CONNECTIVITY

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Abstract

For a graph G, let $\kappa(G)$ and $\alpha(G)$ be the connectivity and independence number of G, respectively. A well-known theorem of Chvátal and Erdős says that if G is a graph of order n with $\kappa(G) > \alpha(G)$, then G is Hamilton-connected. In this paper, we prove the following Chvátal-Erdős type theorem: if G is a k-connected graph, $k \geq 2$, of order n with independence number α, then each pair of distinct vertices of G is joined by a Hamiltonian path or a path of length at least $(k-1)\max\left\{\frac{n+\alpha-k}{\alpha}, \left\lfloor\frac{n+2\alpha-2k+1}{\alpha}\right\rfloor\right\}$. Examples show that this result is best possible. We also strength it in terms of subgraphs.

Keywords: connectivity, independence number, Hamilton-connected, Chvátal-Erdős theorem.

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1. Introduction

In this paper, we will consider simple graphs only and generally follow West [12] for notation and terminology not defined here. For a graph G, let $\kappa(G)$ and $\alpha(G)$ be the connectivity and independence number of G, respectively. Two classic results of Chvátal and Erdős are the following.

Theorem 1 (Chvátal and Erdős [2])**.** *If G is a graph of order $n \geq 3$ such that $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Theorem 2 (Chvátal and Erdős [2])**.** *If G is a graph of order $n \geq 3$ such that $\kappa(G) > \alpha(G)$, then G is Hamilton-connected.*

There are infinitely many non-Hamiltonian graphs such that $\alpha \geq k+1$. So it is of interest to get best lower bound lengths of longest cycles when $\alpha \geq k+1$. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle of G if G contains a cycle. For convention, we let $c(G) = 0$ if G is acyclic. Fouquet and Jolivet [3] in 1978 conjectured that if $\alpha \geq k \geq 2$, then $c(G) \geq \frac{k(n+\alpha-k)}{\alpha}$. Until 2007, this conjecture was only verified for the case $k = 2, 3, \alpha - 1$ and $\alpha - 2$ (see [4, 5] and [9]). In 2011, Chen *et al.* [1] showed that $c(G) \geq \frac{k(n+\alpha-k)}{\alpha} - \frac{(k-3)(k-4)}{2}$, which implies that the conjecture of Fouquet and Jolivet is true for $k = 4$. In 2011 the conjecture of Fouquet and Jolivet was confirmed by O, West and Wu in [11]. In [13] we proved the following stronger theorem.

Theorem 3. *Let G be a k-connected graph, $k \geq 2$, of order n and independence number α. Then*

$$c(G) \geq \min\left\{n, k \cdot \max\left\{\frac{n+\alpha-k}{\alpha}, \left\lfloor \frac{n+2\alpha-2k}{\alpha} \right\rfloor\right\}\right\}.$$

It is interesting to ask whether Theorem 2 has a similar extension as that of Theorem 1. The *co-diameter* of a connected graph G, denoted by $d^*(G)$, is the maximum integer t such that every pair of distinct vertices of G is connected by a path of length at least t. For convenience, we let $d^*(G) = 0$ if G is not connected. By West's theorem, we can get the following corollary.

Corollary 4. *Let G be a k-connected graph, $k \geq 2$, of order n and independence number α. Then*

$$d^*(G) \geq \frac{(n-1)(k+1)}{\alpha+k-2}.$$

In this paper, we show the following theorem.

Theorem 5. *Let G be a k-connected graph, $k \geq 2$, of order n and independence number α. Then*

$$d^*(G) \geq \min\left\{n-1, (k-1)\max\left\{\frac{n+\alpha-k}{\alpha}, \left\lfloor \frac{n+2\alpha-2k+1}{\alpha} \right\rfloor\right\}\right\}.$$

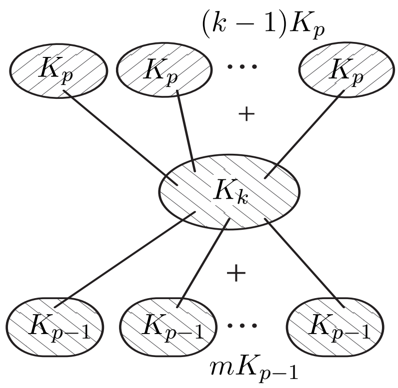

Figure 1. $d^*(G) = (k-1)\left\lfloor \frac{n+2\alpha-2k+1}{\alpha} \right\rfloor$.

The following graphs, depicted in Figure 1, demonstrate that the lower bound in Theorem 5 is sharp.

Let k, m and p be three positive integers with $\min\{k, p\} \geq 2$. For $i \in \{k-1, m\}$ and $j \in \{k, p, p-1\}$, let K_j be the complete graph of order j and iK_j be the graph consists of i disjoint copies of K_j. Note that in Figure 1. $A = K_p$, $B = K_{p-1}$, $C = K_k$, $D = (k-1)K_p$, $E = (m)K_{p-1}$. Let $G = K_k + ((k-1)K_p \cup mK_{p-1})$ be the join of the two graphs K_k and $(k-1)K_p \cup mK_{p-1}$. Clearly, $n = k + p(k-1) + m(p-1)$, $\kappa = k$, $\alpha = k + m - 1$ and $d^*(G) = (k-1)(p+1) = (k-1)\lfloor \frac{n+2\alpha-2k+1}{\alpha} \rfloor$.

For any nonempty graph H, let

$$f(H) = \min\left\{|H| - 1, (k-1)\max\left\{f_1(H), \lfloor f_2(H) \rfloor\right\}\right\}, \tag{1}$$

where $f_i(H) = \frac{(|H|-1)+i(\alpha(H)-k+1)}{\alpha(H)}$, $i = 1, 2$. The function $f(H)$ from the set of graphs to positive real numbers is not monotonic increasing according to the graph inclusion relation, that is, there exists a graph G and a subgraph H of G such that $f(G) < f(H)$. An example will be given after a stronger result is presented below.

Theorem 6. *Let G be a k-connected graph with $k \geq 2$ and let $\mathcal{I}(G)$ be the set of all nonempty induced subgraphs of G. Then*

$$d^*(G) \geq \max\{f(H) \ : \ H \in \mathcal{I}(G)\},$$

where $f(H)$ is defined by (1).

Note that if H' is a spanning subgraph of H, then $f_i(H') \leq f_i(H)$, $i = 1, 2$. The following example, depicted in Figure 2, demonstrates that the lower bound in Theorem 5 may reach the maximum at a proper induced subgraph H.

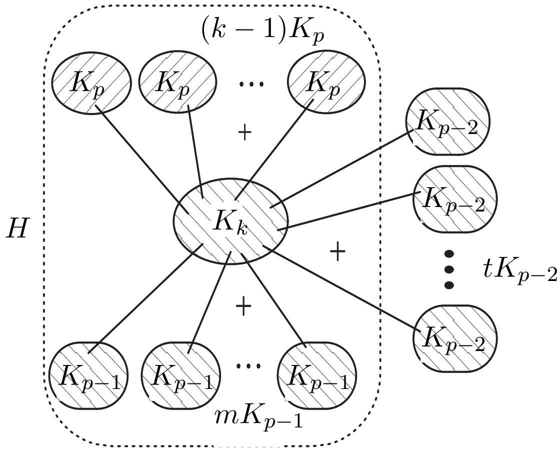

Figure 2. $d^*(G) = k\left\lfloor \frac{|H|+2\alpha(H)-2k+1}{\alpha(H)} \right\rfloor = f(H) > f(G)$.

Let k, m, p and t be four positive integers with $t > k \geq 2$ and $p \geq 3$. Note that in Figure 2. $A = K_p$, $B = K_{p-1}$, $C = K_k$, $D = (k-1)K_p$, $E = (m)K_{p-1}$,$F = K_{p-2}$,$G = tK_{p-2}$. Let $G = K_k + ((k-1)K_p \cup mK_{p-1} \cup tK_{p-2})$ be the join of the two graphs K_k and $(k-1)K_p \cup mK_{p-1} \cup tK_{p-2}$. Clearly, $n = 1 + (k-1)(p+1) + m(p-1) + t(p-2)$, $\kappa = k$, and $\alpha = k + m + t - 1$. Noting that $n + 2\alpha - 2k + 1 = p(k+m+t-1) + (k+m-1) = (p+1)\alpha - t < (p+1)\alpha$, we have $\left\lfloor \frac{n+2\alpha-2k+1}{\alpha} \right\rfloor = p$ and $\frac{n+\alpha-k}{\alpha} = \frac{p\alpha+k-t-1}{\alpha} \leq p$. So, $d^*(G) = k(p+1) > \min\left\{n-1, \max(k-1)\left\{\frac{n+\alpha-k}{\alpha}, \left\lfloor \frac{n+2\alpha-2k+1}{\alpha} \right\rfloor\right\}\right\}$. On the other hand, we have $d^*(G) = (k-1)(p+1) = (k-1)\left\lfloor \frac{|H|+2\alpha(H)-2k+1}{\alpha(H)} \right\rfloor$, where $H = K_k + ((k-1)K_p \cup mK_{p-1})$.

In proofs that follow, we need the the following theorem, which was conjectured in [1].

Theorem 7. *For any graph G, one of the following two statements holds.*

I. *For any two distinct vertices $x, y \in V(G)$, there exists an (x, y)-path P such that $\alpha(G - V(P)) \leq \alpha(G) - 1$.*

II. *There is a non-trivial partition $V_1 \cup V_2$ of $V(G)$ such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$, where $G[V_1]$ and $G[V_2]$ are subgraphs induced by V_1 and V_2, respectively.*

An inductive proof of Theorem 7 can be found in [13]. In order to prove Theorem 7, O, West, and Wu [11] proved Theorem 9, which is a path analogy of Kouider's theorem [8].

Theorem 8 (Kouider [8]). *If H is a subgraph of a k-connected graph G, then either the vertices of H can be covered by one cycle of G or there exists a cycle C of G such that $\alpha(H - V(C)) \leq \alpha(H) - k$.*

Theorem 9 (O, West, and Wu [11]). *Let G be a k-connected graph. If $H \subseteq G$ and x and y are distinct vertices in G, then G contains an (x, y)-path P such that $V(H) \subseteq V(P)$ or $\alpha(H - V(P)) \leq \alpha(H) - (k - 1)$.*

Let $H = G$. Note that if $k \geq 2$, by Theorem 9, we can get G contains an (x, y)-path P such that $V(G) \subseteq V(P)$ or $\alpha(G-V(P)) \leq \alpha(G)-(k-1) \leq \alpha(G)-1$. So when $k \geq 2$ Theorem 7 is a corollary of Theorem 9.

For any two distinct vertices x and y in G, let $d^*(x, y)$ be the length of a longest (x, y)-path in G. By using Theorem 7 and a technical lemma on inserting vertices into a given path, we get a lower bound on $d^*(x, y)$ stated below.

Theorem 10. *Let G be a k-connected graph with $k \geq 2$ and let P be an (x, y)-path in G, where x, y are two distinct vertices of G. Then, for any subgraph H of $G - V(P)$ and any integer s with $s \geq 2$,*

$$d^*(x, y) \geq \min\{(k - 1)s, |P| + |H| - \alpha(H)(s - 2) - 1\}.$$

The remainer of the paper is organized as follows. In Section 2, we prove a technical lemma on inserting vertices into a path — a cycle version of which was proved in [1]. In Section 3, we give an inductive proof of Theorem 7, in Section 4, by using Lemma 11 and Theorem 7, we prove Theorem 10. Finally, in Section 5 we apply Theorems 9 and 10 to prove Theorem 6.

We assume that every path in this paper has an orientation and denote by $P = P[x, y]$ a path from x to y. We also call P an (x, y)-path. The length of P, denoted by $\ell(P)$, is the number of edges in P. For $u, v \in V(P)$, we denote by $u \prec v$ the relationship that u precedes v on P. If $u \prec v$, we denote by $P[u, v]$ (or $u\overrightarrow{P}v$ if the orientation is emphasized) the subpath of P from u to v. The reverse sequence of $u\overrightarrow{P}v$ is denoted by $v\overleftarrow{P}u$. More generally, for any two distinct vertices u and v in a tree T, we let $T[u, v]$ denote the unique path in T from u to v. When R is a path or a tree, we denote $R[u, v] \setminus \{u\}$, $R[u, v] \setminus \{v\}$ and $R[u, v] \setminus \{u, v\}$ by $R(u, v]$, $R[u, v)$ and $R(u, v)$, respectively. We consider them as both paths (or trees)and vertex sets.

Let G be a graph and H_1 and H_2 be two vertex-disjoint subgraphs of G. A path $P = P[x, y]$ in G is called *a path from H_1 to H_2* if $V(P) \cap V(H_1) = \{x\}$ and $V(P) \cap V(H_2) = \{y\}$. A path from $\{x\}$ to a subgraph H of G is also called an *(x, H)-path.* A subgraph F of G is called an *(x, H)-fan of width k* if F is a union of (x, H)-paths $P_1, P_2, \ldots, P_k$, where $V(P_i) \cap V(P_j) = \{x\}$ for $i \neq j$.

In this paper, we let $\mathbb{N}$ denote the set of nonnegative integers. For any two integers $i, j \in \mathbb{N}$ such that $i \leq j$, let $[i, j] = \{\ell \in \mathbb{N} : i \leq \ell \leq j\}$. If $j \geq 1$, let $[j] = [1, j]$.

2. Inserting Vertices Into a Path

Let G be a graph, P be a path of G, and H be a subgraph of $G-V(P)$. A subpath $P[x_1,x_2]$ of P is called an *H-interval of P* if $x_1 \neq x_2$ and there exist two internally vertex disjoint paths P_1 and P_2 from H to P ending at x_1 and x_2, respectively. In addition, if either $V(P_1)\cap V(P_2)=\emptyset$ or $|V(P_1)\cap V(P_2)|=|H|=1$, then $P[x_1,x_2]$ is called a *normal H-interval of P*. An (x,y)-path P is called a *maximal (x,y)-path* if there is no (x,y)-path Q in G such that $V(P)\subsetneq V(Q)$. The following lemma plays a crucial role in the proof of Theorems 5 and 6, and the proof technique has been used in [1, 7] and [6].

Lemma 11. *Let k and s be two integers with $k,s\geq 2$, let G be a k-connected graph, let $P=P[x,y]$ be a maximal (x,y)-path of G and let H be a subgraph of $G-V(P)$ with $|H|\geq s-1$. If every normal H-interval $P[x_1,x_2]$ of P has length at least s, then $\ell(P)\geq (k-1)s$.*

Proof. It suffices to show that there exists a family $\mathcal{I}$ of pairwise edge-disjoint intervals of P such that

$$\sum_{I\in\mathcal{I}}\ell(I)\geq (k-1)s. \tag{2}$$

By the assumption of Lemma 11, we have

$$\text{for every normal } H\text{-interval } I \text{ of } P,\ \ell(I)\geq s. \tag{3}$$

Since P is a maximal (x,y)-path and $G-V(P)\neq\emptyset$, by Menger's Theorem [10], we can deduce that $|V(P)|\geq 2(k-1)\geq k$.

Let $h=|H|$ and $V(H)=\{u_1,\ldots,u_h\}$. Since G is k-connected, for each $i\in[h]$, there exists a (u_i,P)-fan F_i of width k in G. For each $i\in[h]$, let $\{x_{i,1},x_{i,2},\ldots,x_{i,k}\}=V(F_i)\cap V(P)$. In addition, we assume that $x_{i,1},x_{i,2},\ldots,x_{i,k}$ appear in order along $\overrightarrow{P}$. Let $\mathcal{F}_i=\{P[x_{i,j},x_{i,j+1}] : j=1,2,\ldots,k-1\}$. An element I of $\mathcal{F}_i$ is called an *F_i-interval*.

An H-interval $P[x_1,x_2]$ of P is called a *long interval* if $\ell(P[x_1,x_2])\geq s$ and a *short interval* otherwise. Let x_1,x_2 be distinct vertices of P and let $u_i\in V(H)$. A path R in G from x_1 to x_2 is called an *(x_1,u_i,x_2)-arc* if $V(R)\cap V(P)=\{x_1,x_2\}$ and $u_i\in V(R)$. Moreover, we call $P[x_1,x_2]$ a *good u_i-interval* if

(G-1) $\ell(P[x_1,x_2])\leq s-1$,

(G-2) there is an (x_1,u_i,x_2)-arc in G, and

(G-3) for every proper subinterval $P[x_1',x_2']$ of $P[x_1,x_2]$, there is no (x_1',u_i,x_2')-arc in G.

Claim 1. *Every short F_i-interval contains a good u_i-interval, where $i\in[h]$.*

Proof. Since every short F_i-interval satisfies (G-1) and (G-2), (G-3) will be satisfied if we take minimality. □

Claim 2. *Suppose $P[x_1, x_2]$ is a good u_i-interval. Let R be an (x_1, u_i, x_2)-arc and $Q = Q[u, v]$ be a path from H to P. Then,*

(i) $V(R) \cap V(H) = \{u_i\}$,

(ii) $v \notin P(x_1, x_2)$, *and*

(iii) *if $u \neq u_i$, then $V(Q) \cap (V(R) \cup P[x_1, x_2]) = \emptyset$.*

Proof. (i) Assume the contrary that $V(R) \cap V(H) \neq \{u_i\}$. Then, $|V(R) \cap V(H)| \geq 2$. Along the orientation of R from x_1 to x_2, let w be the first vertex of R in H and w' be the last vertex of R in H. Clearly, $w \neq w'$, so that $P[x_1, x_2]$ is a normal H-interval. By (3), we have $\ell(P[x_1, x_2]) \geq s$, contradicting (G-1).

(ii) Assume the contrary that $v \in P(x_1, x_2)$. By (i), $V(R) \cap V(H) = \{u_i\}$. If $V(Q) \cap V(R) = \emptyset$, then $Q[u, v]$ and $R[u_i, x_2]$ are two vertex disjoint paths from H to P. So $P[v, x_2]$ is a normal H-interval. By (3), we have $\ell(P[v, x_2]) \geq s$. Consequently, $\ell(P[x_1, x_2]) \geq s$, a contradiction. Therefore, $V(Q) \cap V(R) \neq \emptyset$. Along the orientation of Q, let z be the last vertex of $V(Q) \cap V(R)$. Then, $z \in R(x_1, u_i] \cup R[u_i, x_2)$. Assume, without loss of generality, $z \in R[u_i, x_2)$. Then, $x_1 \overrightarrow{R} z \overrightarrow{Q} v$ is an (x_1, u_i, v)-arc in G, which contradicts (G-3). Hence, (ii) holds.

(iii) Assume to the contrary there is a vertex $z \in V(Q) \cap (V(R) \cup P[x_1, x_2])$. By (ii), we have $z \notin P(x_1, x_2)$, and hence $z \in V(Q) \cap R[x_1, x_2]$. Let z' be the first vertex of Q on R. Since $u \neq u_i$ and $V(Q) \cap V(H) = u$, $z' \neq u_i$. Assume, without loss of generality, that $z' \in R[x_1, u_i)$. Then, $u \overrightarrow{Q} z' \overleftarrow{R} x_1$ and $u_i \overrightarrow{R} x_2$ are two vertex disjoint paths from H to P in G, so that $P[x_1, x_2]$ is a normal H-interval. By (3), $\ell(P[x_1, x_2]) \geq s$, contrary to (G-1). This completes the proof of Claim 2. □

For each $i \in [h]$, by Claim 1, every short F_i-interval $P[x_{i,j}, x_{i,j+1}]$ contains at least one good u_i-interval. Among all of these good u_i-intervals we specify one as I_{ij}. For each i, let F_i^* denote the set of all such I_{ij}s. For each $I_{ij} \in F_i^*$, let $P[y_{ij}, z_{ij}] = I_{ij}$, that is, we assume that y_{ij} and z_{ij} are two endvertices of I_{ij}; let R_{ij} be an (y_{ij}, u_i, z_{ij})-arc in G.

Claim 3. *For any two intervals $I_{ij} \in F_i^*$ and $I_{i'j'} \in F_{i'}^*$, if $i \neq i'$, then the following three properties hold.*

(i) $V(R_{i'j'}) \cap (V(R_{ij}) \cup V(I_{ij})) = \emptyset$,

(ii) $V(I_{ij}) \cap V(I_{i'j'}) = \emptyset$, *and*

(iii) *there exist at least $s - 1$ vertices on P between I_{ij} and $I_{i'j'}$.*

Proof. (i) Since $I_{i'j'}$ is a good $u_{i'}$-interval, $R_{i'j'} \cap V(H) = \{u_{i'}\}$ by Claim 2(i). Hence $Q = R_{i'j'}[u_{i'}, z_{i'j'}]$ is a path in G from H to P. Since $u_{i'} \neq u_i$, by using Claim 2(iii) with $P[x_1, x_2] = I_{ij}$ and $R = R_{ij}$, we have $V(R_{i'j'}[u_{i'}, z_{i'j'}]) \cap$

$(V(R_{ij}) \cup V(I_{ij})) = \emptyset$. Similarly, we have $V(R_{i'j'}[y_{i'j'}, u_{i'}]) \cap (V(R_{ij}) \cup V(I_{ij})) = \emptyset$. Hence, (i) is true.

(ii) Suppose to the contrary, that $V(I_{ij}) \cap V(I_{i'j'}) \neq \emptyset$. By symmetry, we may assume $\ell(I_{ij}) \geq \ell(I_{i'j'})$. Then, $y_{i'j'} \in I_{ij}$ or $z_{i'j'} \in I_{ij}$, which implies $V(R_{i'j'}) \cap V(I_{ij}) \neq \emptyset$, giving a contradiction to (i).

(iii) By (ii), we may assume $y_{ij}, z_{ij}, y_{i'j'}$ and $z_{i'j'}$ appear on P in the order along P. By Claim 2(i) and Claim 3(i), $u_i \overrightarrow{R_{ij}} z_{ij}$ and $u_{i'} \overleftarrow{R_{i'j'}} y_{i'j'}$ are two vertex-disjoint paths from H to P in G, and hence $P[z_{ij}, y_{i'j'}]$ is a normal H-interval. By (3), $\ell(P[z_{ij}, y_{i'j'}]) \geq s$. □

Claim 4. *For every $i \in [h]$ and $I_{ij} \in F_i^*$, $\ell(I_{ij}) \geq 2$.*

Proof. Assume on the contrary that there is an $I_{ij} = P[y_{ij}, z_{ij}] \in F_i^*$ such that $\ell(I_{ij}) \leq 1$. Then, $V(I_{ij}) = \{y_{ij}, z_{ij}\}$. Set $D = x \overrightarrow{P} y_{ij} \overrightarrow{R_{ij}} z_{ij} \overrightarrow{P} y$. Then $V(D) \supseteq V(P) \cup \{u_i\}$, giving a contradiction to that P is a maximal (x, y)-path. □

For each $i \in [h]$, let t_i be the number of long F_i-intervals and let $t = \max\{t_i : i \in [h]\}$. Assume, without loss of generality, that $t = t_1$. If $t = k - 1$, then $|\mathcal{F}_1| = k - 1$ and $\sum_{I \in \mathcal{F}_1} \ell(I) \geq (k-1)s$. So $\mathcal{I} = \mathcal{F}_1$ satisfies (2). In what follows, we assume $t < k - 1$.

It follows from the definition of t that for each $i \in [h]$, there exists at least $k - t - 1$ short F_i-intervals. This together with the definition of F_i^* implies that $|F_i^*| \geq k - t - 1 > 0$ for each $i \in [h]$. Let $\mathcal{I}_g = \bigcup_{i=1}^{h} F_i^*$. For two distinct intervals I_{ij} and $I_{i'j'}$, $E(I_{ij}) \cap E(I_{i'j'}) = \emptyset$ if $i = i'$ and $j \neq j'$; and $V(I_{ij}) \cap V(I_{i'j'}) = \emptyset$ if $i \neq i'$ (by Claim 3(ii)). So all intervals in $\mathcal{I}_g$ are pairwise edge-disjoint.

Since $|F_i^*| \geq k - t - 1$ for each $i \in [h]$, we have

$$|\mathcal{I}_g| = \sum_{i=1}^{h} |F_i^*| \geq h(k - t - 1) \geq (s - 1)(k - t - 1).$$

This together with Claim 4 implies that

$$\sum_{I \in \mathcal{I}_g} \ell(I) \geq 2|\mathcal{I}_g| \geq s(k - t - 1). \tag{4}$$

If $t = 0$, then by (4), $\mathcal{I} = \mathcal{I}_g$ satisfies (2). So, we assume $t \geq 1$, that is, there is a long interval in F_1.

Claim 5. *Let I be a long F_1-interval and let $I' \in F_i^*$, where $i \in [2, h]$. If $E(I') \cap E(I) \neq \emptyset$, then $E(I') \subseteq E(I)$.*

Proof. Let $I = P[x_{1,p}, x_{1,p+1}]$ and $I' = I_{ij} = P[y_{ij}, z_{ij}]$, for $p, j \in [k-1]$. Since F_1 is an (u_1, P)-fan of width k, $F_1[u_1, x_{1,p}]$ and $F_1[u_1, x_{1,p+1}]$ are two paths from

H to P. By Claim 2(iii), we have $V(F_1[u_1, x_{1,p}]) \cap (V(R_{ij}) \cup V(P[y_{ij}, z_{ij}])) = \emptyset$. Hence $x_{1,p} \notin P[y_{ij}, z_{ij}]$. Similarly, $x_{1,p+1} \notin P[y_{ij}, z_{ij}]$. If $E(I') \cap E(I) \neq \emptyset$, then $E(I') \subseteq E(I)$. This completes the proof of Claim 5. □

Claim 6. *For each long F_1-interval I, there exists a long interval $I' \subseteq I$ such that all intervals in $\mathcal{I}_g \cup \{I'\}$ are pairwise edge disjoint.*

Proof. Denote $I = P[x_{1,p}, x_{1,p+1}]$, where $p \in [k-1]$. Set

$$T = \{i \in [h] : E(I) \cap E(I_{ij}) \neq \emptyset \text{ for some } I_{ij} \in F_i^*\}.$$

If $T = \emptyset$, then $I' = I$ satisfies Claim 6. So, we assume $T \neq \emptyset$. Since every $I_{1j} \in F_1^*$ is contained in a short F_1-interval, we conclude that $T \subseteq [2, h]$.

Since F_1 is an (u_1, P)-fan of width k, $F_1[u_1, x_{1,p}]$ and $F_1[u_1, x_{1,p+1}]$ are two paths from H to P. Choose $I_{ij} \in \mathcal{F}_i^*$, $i \in T$. Note that I_{ij} is a good u_i-interval. Let $P[y_{ij}, z_{ij}] = I_{ij}$ and R_{ij} be an (y_{ij}, u_i, z_{ij})-arc in G. Since $u_1 \neq u_i$, by Claim 2(iii), $V(F_1[u_1, x_{1,p}]) \cap (V(R_{ij}) \cup V(P[y_{ij}, z_{ij}])) = \emptyset$. Then $P[x_{1,p}, y_{ij}]$ is a normal H-interval. By (3), we have $\ell(P[x_{1p}, y_{ij}]) \geq s$, which completes the proof of Claim 6. □

By Claim 6, for each long F_1-interval $P[x_{1,p}, x_{1,p+1}]$, there exists a long interval $I_p \subseteq P[x_{1,p}, x_{1,p+1}]$ such that all intervals in $\mathcal{I}_g \cup \{I_p\}$ are pairwise edge-disjoint. Among all of these I_p's, we specify one as I_p^*. Let $\mathcal{I}_{g'}$ be the set of all such I_p^*s. Clearly, $\mathcal{I}_{g'}$ consists of t pairwise edge disjoint long intervals. By (4), we have

$$\sum_{I \in \mathcal{I}_g} \ell(I) + \sum_{I \in \mathcal{I}_{g'}} \ell(I) \quad \geq \quad s(k-t-1) + ts = (k-1)s.$$

Hence, $\mathcal{I} = \mathcal{I}_g \cup \mathcal{I}_{g'}$ is a set of pairwise edge disjoint intervals of P that satisfies (2). ■

3. PROOF OF THEOREM 10

In this section, we apply Lemma 11 and Theorem 7 to prove Theorem 10, which gives a lower bound on $d^*(x, y)$ in terms of a given path P and an induced subgraph of $G - V(P)$. We first give some definitions.

For any two induced subgraphs H_1 and H_2 of G, we let $H_1 \cup H_2$ denotes the subgraph of G induced by $V(H_1) \cup V(H_2)$. Moreover, we write $H_1 \uplus H_2$ to denote $H_1 \cup H_2$ under the condition $V(H_1) \cap V(H_2) = \emptyset$. For convenience, we allow some of the H_i to be an empty graph in this definition. If H is a graph, we use $G \supseteq_s H$ or $H \subseteq_s G$ to denote that H is a *spanning* subgraph of G, that is, $V(G) = V(H)$

and $E(G) \supseteq E(H)$. Clearly, if $G \supseteq_s H_1 \uplus H_2$, then $\alpha(G) \leq \alpha(H_1) + \alpha(H_2)$. We are interested in the class of graphs for which equality holds. Set

$$\mathcal{G}^* = \{G \ : \ \text{there exist nonempty subgraphs } H_1,\ H_2 \text{ such that } G \supseteq_s H_1 \uplus H_2 \text{ and } \alpha(G) = \alpha(H_1) + \alpha(H_2)\}.$$

For convenience, the following equivalent definition is also used.

$$\mathcal{G}^* = \{G \ : \ \text{there exist nonempty subgraphs } H_1,\ H_2 \text{ such that } G \supseteq_s H_1 \uplus H_2 \text{ and } \alpha(G) \geq \alpha(H_1) + \alpha(H_2)\}.$$

We say that a graph G satisfies *property* $\mathcal{H}_c$ if for every two distinct vertices $u, v \in V(G)$, there exists a path $P = P[u, v]$ in G such that $\alpha(G - V(P)) \leq \alpha(G) - 1$. Let $\mathcal{H}_c^*$ denote the class of graphs satisfying property $\mathcal{H}_c$. Clearly, every Hamilton-connected graph is in $\mathcal{H}_c^*$. The empty graph is not an element of $\mathcal{H}_c^*$.

Lemma 12. *Let G be a graph and α be the independence number of G. Then there are two induced subgraphs H_1 and H_2 such that $G \supseteq_s H_1 \uplus H_2$, $H_2 \in \mathcal{H}_c^*$ and $\alpha(H_1) + \alpha(H_2) \leq \alpha$* ($H_1$ may be an empty graph).

Proof. By Theorem 7, $G \in \mathcal{G}^* \cup \mathcal{H}_c^*$. If $G \in \mathcal{H}_c^*$, then we are done with $H_1 = \emptyset$ and $H_2 = G$. In what follows, we assume $G \in \mathcal{G}^*$. Then, $G \supseteq_s H_1 \uplus H_2$, where H_1, H_2 are induced subgraphs of G with $\alpha(H_1) + \alpha(H_2) \leq \alpha(G)$ and $\alpha(H_1) \geq \alpha(H_2) \geq 1$. Choose (H_1, H_2) such that $\alpha(H_2)$ achieves the minimum. If $H_2 \in \mathcal{H}_c^*$, then we are done. We may assume that $H_2 \notin \mathcal{H}_c^*$. Then, by Theorem 7, we have $H_2 \in \mathcal{G}^*$. So, $H_2 \supseteq_s H_{21} \uplus H_{22}$, where H_{21}, H_{22} are induced subgraphs of H_2 such that $\alpha(H_{21}) + \alpha(H_{22}) \leq \alpha(H_2)$ and $\alpha(H_{2i}) \geq 1$, $i = 1, 2$. Set $H_1' = G[V(H_1) \cup V(H_{21})]$. Then, $\alpha(H_1') + \alpha(H_{22}) \leq (\alpha(H_1) + \alpha(H_{21})) + \alpha(H_{22}) \leq \alpha(H_1) + \alpha(H_2) \leq \alpha(G)$. This together with $G \supseteq_s H_1' \uplus H_{22}$ implies that (H_1', H_{22}) is a pair of induced subgraphs of G that contradicts the choice of (H_1, H_2). This completes the proof of Lemma 12. ■

Before proving Theorem 10, we restate it for reference.

Theorem 10. *Let G be a k-connected graph with $k \geq 2$, let $x \neq y \in V(G)$, and let P be an (x, y)-path in G. Then,*

$$d^*(x, y) \geq \min\{(k-1)s, |P| + |H| - \alpha(H)(s-2) - 1\},$$

where H is any subgraph of $G - V(P)$ and s is any integer with $s \geq 2$.

Proof. Suppose the contrary that

$$\text{for some integer } s \geq 2,\ d^*(x, y) < (k-1)s \tag{5}$$

and there exists an (x, y)-path P in G and a subgraph H of $G - V(P)$ such that

$$d^*(x, y) < |P| + |H| - \alpha(H)(s - 2) - 1. \tag{6}$$

Note that $|H| \neq 0$. Moreover, we choose P and H such that

(i) $|H|$ achieves the minimum, and

(ii) subject to (i), $|P|$ achieves the maximum.

A simple calculation shows that, for any (x, y)-path P' with $V(P') \supseteq V(P)$ and $H' = H - V(P')$, we have $|P'| + |H'| - \alpha(H')(s-2) \geq |P| + |H| - \alpha(H)(s-2)$. So, by (ii), P is a maximal (x, y)-path in G.

It follows from Lemma 12 that there exist two induced subgraphs H_1, H_2 of H (H_1 may be an empty graph) such that $H \supseteq_s H_1 \uplus H_2$, $H_2 \in \mathcal{H}_c^*$, and $\alpha(H_1) + \alpha(H_2) \leq \alpha(H)$. We consider two cases.

Case 1. $|H_2| \leq s - 2$. Since $H_2 \neq \emptyset$, $\alpha(H_1) = \alpha(H) - \alpha(H_2) \leq \alpha(H) - 1$. By the choice of (P, H), Theorem 10 holds for (P, H_1), and hence

$$\begin{aligned} d^*(x, y) &\geq |P| + |H_1| - \alpha(H_1)(s - 2) - 1 \\ &\geq |P| + (|H| - |H_2|) - (\alpha(H) - 1)(s - 2) - 1 \\ &\geq |P| + |H| - \alpha(H)(s - 2) - 1, \end{aligned}$$

contrary to (6).

Case 2. $|H_2| \geq s - 1$. By (5), $\ell(P) \leq d^*(x, y) < (k - 1)s$. By applying Lemma 11 with $H = H_2$, we see that there is a normal H_2-interval $P[x_1, x_2]$ of P with $\ell(P[x_1, x_2]) \leq s - 1$.

Since $P[x_1, x_2]$ is a normal H_2-interval, there exist two internally vertex disjoint paths $P_1 = P_1[u_1, x_1]$, $P_2 = P_2[u_2, x_2]$ in G from H_2 to P such that $|V(H_2) \cap (V(P_1) \cup V(P_2))| = \min\{|H_2|, 2\}$. Since $H_2 \in \mathcal{H}_c^*$, there exists a path $Q = Q[u_1, u_2]$ in H_2 such that $\alpha(H_2 - V(Q)) \leq \alpha(H_2) - 1$. (Note that this is also true if $u_1 = u_2$, which implies $|H| = 1$ and $s = 2$).

Set $P^* = x\overrightarrow{P}x_1\overleftarrow{P_1}u_1\overrightarrow{Q}u_2\overrightarrow{P_2}x_2\overrightarrow{P}y$ and $H^* = H - V(P^*)$. Then, H^* is a subgraph of $G - V(P^*)$ with

$$\begin{aligned} \alpha(H^*) &= \alpha(H[V(H_1) \cup V(H_2) - V(P^*)]) \\ &\leq \alpha(H_1 - V(P^*)) + \alpha(H_2 - V(P^*)) \\ &\leq \alpha(H_1) + \alpha(H_2 - V(Q)) \leq \alpha(H_1) + \alpha(H_2) - 1 = \alpha(H) - 1. \end{aligned} \tag{7}$$

Note that P^* is an (x, y)-path in G such that

$$\begin{aligned} |P^*| + |H^*| &\geq |V(P^*) \cap V(P)| + |V(P^*) \cap V(H)| + |H^*| \\ &= (|P| - |P(x_1, x_2)|) + |H| \geq |P| - (s - 2) + |H|. \end{aligned} \tag{8}$$

Since $|H^*| < |H|$, by the choice of (P, H), we have $d^*(x, y) \geq |P^*| + |H^*| - \alpha(H^*)(s-2) - 1$. This together with (7) and (8) implies that

$$\begin{aligned} d^*(x,y) &\geq |P^*| + |H^*| - (\alpha(H^*)+1)(s-2) - 1 \\ &\geq |P| + |H| - \alpha(H)(s-2) - 1, \end{aligned}$$

contrary to (6). This completes the proof of Theorem 10. ∎

4. Proof of Theorem 6.

It suffices to prove the following equivalent form of Theorem 6.

Theorem 6. *Let G be a k-connected graph, $k \geq 2$, of order n and let V_0 be a nonempty subset of $V(G)$. Then for any two distinct vertices x, y of G,*

$$d^*(x,y) \geq \min\left\{|V_0| - 1, (k-1)\max\left\{f_1(V_0), \lfloor f_2(V_0)\rfloor\right\}\right\},$$

where $f_i(V_0) = f_i(G[V_0]) = \frac{(|V_0|-1)+i(\alpha(G[V_0])-k+1)}{\alpha(G[V_0])}$, $i = 1, 2$.

Proof. Let $s_i = f_i(V_0)$, $i = 1, 2$. By applying Theorem 9 with $H = G[V_0]$, we get an (x, y)-path P in G such that either $V(P) \supseteq V_0$ or $\alpha(G[V_0] - V(P)) \leq \alpha(G[V_0]) - k + 1$. Clearly, Theorem 5 holds if $V(P) \supseteq V_0$. We assume $V_0 \not\subseteq V(P)$ and $\alpha(G[V_0] - V(P)) \leq \alpha(G[V_0]) - k + 1$. Let $H' = G[V_0 - V(P)]$. Then, $\alpha(H') = \alpha(G[V_0] - V(P)) \leq \alpha(G[V_0]) - k + 1$, in particular, $\alpha(G[V_0]) \geq k$. This implies $|V_0| \geq k$ and hence $s_1 = \frac{|V_0|+\alpha(G[V_0])-k}{\alpha(G[V_0])} \geq 1$. We consider the following two cases.

Case 1. $s_1 \geq \lfloor s_2 \rfloor$. If $d^*(x, y) \geq (k-1)s_1$, we are done. Assume $d^*(x, y) < (k-1)s_1$. Set

$$s = \begin{cases} \lceil s_1 \rceil, & \text{if } s_1 > 1 \\ 2, & \text{if } s_1 = 1. \end{cases}$$

Then, s is an integer with $s \geq 2$ and $s - 2 \leq s_1 - 1$. By Theorem 10, we get

$$\begin{aligned} d^*(x,y) &\geq |P| + |H| - \alpha(H)(s-2) - 1 \\ &\geq |V_0| - (\alpha(G[V_0]) - k + 1)(s_1 - 1) - 1 \\ &= (|V_0| + \alpha(G[V_0]) - k) - (\alpha(G[V_0]) - k + 1)s_1. \end{aligned}$$

This together with $d^*(x, y) < (k-1)s_1$ implies that $s_1 > \frac{|V_0|+\alpha(G[V_0])-k}{\alpha(G[V_0])}$, a contradiction.

Case 2. $s_1 < \lfloor s_2 \rfloor$. Then, $\lfloor s_2 \rfloor \geq 2$. If $d^*(x, y) \geq (k-1)\lfloor s_2 \rfloor$, we are done. Assume $d^*(x, y) < (k-1)\lfloor s_2 \rfloor$. By taking $s = \lfloor s_2 \rfloor$ in Theorem 10, we have

$$\begin{aligned} d^*(x,y) &\geq |P| + |H| - \alpha(H)(\lfloor s_2 \rfloor - 2) - 1 \\ &\geq |V_0| - (\alpha(G[V_0]) - k + 1)(\lfloor s_2 \rfloor - 2) - 1 \\ &= (|V_0| + 2\alpha(G[V_0]) - 2k + 1) - (\alpha(G[V_0]) - k + 1)\lfloor s_2 \rfloor. \end{aligned}$$

This together with $d^*(x,y) < (k-1)\lfloor s_2 \rfloor$ implies $\lfloor s_2 \rfloor > \frac{|V_0|+2\alpha(G[V_0])-2k+1}{\alpha(G[V_0])}$, giving a contradiction. ■

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