

## ON THE VERTEX POSITION NUMBER OF GRAPHS

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### Abstract

In this paper we generalise the notion of visibility from a point in an integer lattice to the setting of graph theory. For a vertex  $x$  of a graph  $G$ , we say that a set  $S \subseteq V(G)$  is an  $x$ -*position set* if for any  $y \in S$  the shortest  $x, y$ -paths in  $G$  contain no point of  $S \setminus \{y\}$ . We investigate the largest and

smallest orders of maximum  $x$ -position sets in graphs, determining these numbers for common classes of graphs and giving bounds in terms of the girth, vertex degrees, diameter and radius. Finally we discuss the complexity of finding maximum vertex position sets in graphs.

**Keywords:** geodesic, vertex position set, vertex position number, general position number.

**2020 Mathematics Subject Classification:** 05C12, 05C69.

## 1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple. For a graph  $G$  we will denote the subgraph induced by a subset  $S \subseteq V(G)$  by  $G[S]$ . The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $u, v$ -path in  $G$ ; any such path is called a *geodesic*. The distance is a metric on the vertex set  $V(G)$ . The diameter  $\text{diam}(G)$  of a connected graph  $G$  is the length of any longest geodesic. For any vertex  $u$  of  $G$ , the *eccentricity* of  $u$  is  $e(u) = \max\{d(u, v) : v \in V\}$ . A vertex  $v$  of  $G$  such that  $d(u, v) = e(u)$  is called an *eccentric vertex* of  $u$ .

If a vertex  $u$  of  $G$  is adjacent to a vertex  $v$ , we write  $u \sim v$ . The *open neighbourhood* (or simply *neighbourhood*) of a vertex  $u$  in a graph  $G$  is the set  $N_G(u) = \{v \in V(G) : u \sim v\}$  consisting of all vertices  $v$  which are adjacent to  $u$ , whilst the *closed neighbourhood* of  $u$  is defined by  $N_G[u] = \{u\} \cup N(u)$ . The *degree* of  $u$  is  $\deg(u) = |N(u)|$ . The *second neighbourhood*  $N^2(u)$  is the set  $\{v \in V(G) : d(u, v) = 2\}$  of vertices at distance two from  $u$ . If it is clear in which graph the neighbourhood is taken we will omit the subscript.

The *complement*  $\overline{G}$  is the graph on the same vertex set  $V(G)$  as  $G$ , but a pair of vertices  $u, v$  of  $\overline{G}$  are adjacent in  $\overline{G}$  if and only if they are nonadjacent in  $G$ . A vertex  $u$  is *simplicial* if the subgraph induced by its neighborhood  $N(u)$  is complete; we will denote the number of simplicial vertices of a graph  $G$  by  $s(G)$  and the set of all simplicial vertices of  $G$  by  $\text{Ext}(G)$ . A set of vertices in a graph is *independent* if no two vertices in the set are adjacent; the independence number  $\alpha(G)$  is the number of vertices in a largest independent set of  $G$ . A graph is a *block graph* if every maximal 2-connected component is a clique. The graph  $G$  is *vertex-transitive* if the automorphism group  $\text{Aut}(G)$  acts transitively on  $V(G)$ . The *join*  $G \vee H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ , i.e.,  $G \vee H$  is obtained from the disjoint union  $G \dot{\cup} H$  by adding all possible edges between  $G$  and  $H$ . For other basic graph theoretic terminology not defined here we refer to [4, 6].

Visibility and illumination problems are among the most attractive and interesting research topics in combinatorics, geometry and number theory [3]. Such problems have been studied intensively in the context of the integer lattice; a set

$\Lambda$  of points of the lattice is *visible* from a point  $x$  if for any  $y \in \Lambda$  the straight line segment from  $x$  to  $y$  contains no other points of  $\Lambda$ . A well-known result from elementary analytic number theory first proved by Sylvester [20] states that the density of the set of points in the integer lattice that are visible from the origin is  $\frac{6}{\pi^2}$  [2]. In particular, in Chapter III of [14], it is shown how to place a set of points with positive integer coordinates  $(i, j)$ ,  $j \leq i$ , in such a way that each point is visible from the origin  $(0, 0)$ , by also maximising the number of points with the same abscissa; this construction is related to the Farey series and Euler's totient function  $\phi$ . Other interesting visibility problems in integer lattices can be found in [9, 13].

In recent years the algorithmic component of visibility problems has attracted great attention under the name *art gallery* or *watchman* problems, which lie in the intersection of combinatorial and computational geometry [19]. Art gallery problems, theorems and algorithms are so named after the following celebrated question of V. Klee, which he posed 50 years ago in 1973: 'What is the minimum number of guards sufficient to cover the interior of an  $n$ -wall gallery?'. This problem was solved by Chvátal and subsequently by Fisk. By creating idealised situations such as obstacles, guards, etc., the theory succeeds in abstracting the algorithmic essence of many visibility problems.

Taking our inspiration from the result of Sylvester [20], in this paper we consider a generalisation of 'local visibility problems' to the context of the general position problem in graph theory. The general position problem originated in Dudeney's no-three-in-line problem [8] and the general position subset selection problem [10, 18] from discrete geometry. These problems were generalised to graphs independently in [5] and [15]. A set  $S$  of vertices of a graph  $G$  is in *general position* if for any  $u, v \in S$  any  $u, v$ -geodesic does not intersect  $S \setminus \{u, v\}$ . The *general position number*  $\text{gp}(G)$  of  $G$  is the number of vertices in a largest general position set in  $G$ . We refer the reader to some recent papers [1, 16, 17, 21–23] for more information on the general position problem.

In a recent paper Di Stefano [7] introduced the concept of a *mutual visibility set* in a graph; a set  $S$  of points in a graph  $G$  are *mutually visible* if for any  $u, v \in S$  there exists a shortest  $u, v$ -path in  $G$  that does not pass through  $S \setminus \{u, v\}$ ; the *mutual visibility number*  $\mu(G)$  of  $G$  is the number of vertices in a largest mutual visibility set in  $G$ . In [7] the mutual visibility number of some classes of graphs are determined and it is shown that the problem of finding a largest mutual visibility set is NP-complete for general graphs.

We now study a 'local' version of these problems using a parameter that we call the *vertex position number* of a graph. The plan of this paper is as follows. In Section 2 we provide some bounds on the vertex position numbers of a graph. In Section 3 the vertex position numbers of some common classes of graphs are determined. We characterise the graphs with very large or small vertex

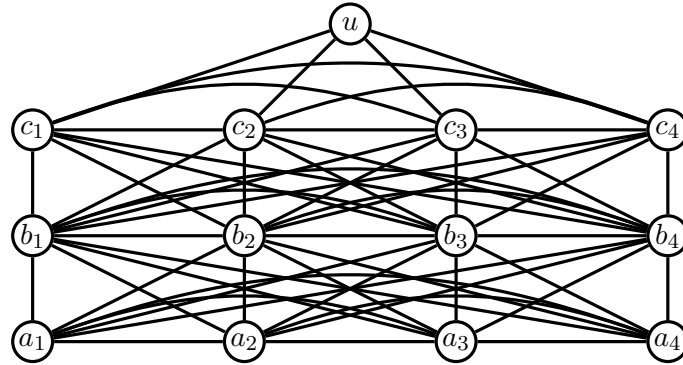


Figure 1.

position numbers in Section 4. Finally in Section 5 we consider the computational complexity of finding the vertex position number of a graph.

## 2. VERTEX POSITION SETS IN GRAPHS

In this section we derive bounds for the vertex position numbers of a graph in terms of the minimum and maximum degrees, radius and diameter. First we formally define the vertex position numbers.

**Definition 1.** For any graph  $G$  and a fixed vertex  $x \in V(G)$ , a set  $S_x \subseteq V(G)$  is an  $x$ -position set if for any  $y \in S_x$  no vertex of  $S_x \setminus \{y\}$  lies on any  $x, y$ -geodesic in  $G$ . The  $x$ -position number of  $G$  is defined to be the maximum cardinality of an  $x$ -position set and is denoted by  $p_x(G)$  or simply  $p_x$ . An  $x$ -position set of cardinality  $p_x(G)$  is called a  $p_x$ -set. The maximum value of  $p_x(G)$  among all vertices  $x$  of  $G$  is called the vertex position number  $\text{vp}(G)$  of  $G$ . Similarly, the minimum value of  $p_x(G)$  among all vertices  $x$  of  $G$  is called the minimum vertex position number  $\text{vp}^-(G)$  of  $G$ .

To illustrate these concepts, consider the graph  $G$  in Figure 1. We give the  $x$ -position numbers of  $G$  for representative vertices in Table 1, together with a (not necessarily unique)  $x$ -position set. We see from the table that  $\text{vp}^-(G) = 4$  and  $\text{vp}(G) = 11$ .

Unless stated otherwise (for example in Theorem 13) we assume all graphs to be connected. However, Definition 1 also applies to disconnected graphs; if  $x$  belongs to a component  $C$  of a disconnected graph  $G$ , then any vertex  $y$  from another component  $D$  of  $G$  can be included in an  $x$ -position set, as there is no  $x, y$ -path in  $G$ . Hence in this case  $p_x(G) = (n - |C|) + p_x(C)$ .

For any vertex  $x \in V(G)$  the set  $\{x\}$  is an  $x$ -position set; however, by the

Vertex	$p_x$ -set	$p_x(G)$
$u$	$\{c_1, c_2, c_3, c_4\}$	4
$c_1$	$\{u, c_2, c_3, c_4, b_1, b_2, b_3, b_4\}$	8
$b_1$	$\{c_1, c_2, c_3, c_4, b_2, b_3, b_4, a_1, a_2, a_3, a_4\}$	11
$a_1$	$\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$	7

Table 1.

convention in Definition 1, if  $G$  is connected, then  $x$  is not contained in any  $x$ -position set of order  $\geq 2$ . Hence for any connected graph  $G$  with order  $n \geq 2$  we have  $1 \leq p_x(G) \leq n - 1$  for any  $x \in V(G)$  and more generally a (not necessarily connected) graph  $G$  has  $\text{vp}(G) = n$  if and only if  $G$  has an isolated vertex. These bounds are sharp: for any path  $P_n$  of length  $\geq 1$  we have  $p_x(P_n) = 1$  for either terminal vertex, whilst for  $n \geq 2$  we have  $p_x(K_n) = n - 1$  for every vertex of a complete graph  $K_n$ . In this section we derive several bounds for the vertex position numbers in terms of various graph parameters. First we compare the vertex position number with the general position number.

**Lemma 2.** *The vertex position number and general position number of a graph are related by  $\text{vp}(G) \geq \text{gp}(G) - 1$ .*

**Proof.** Let  $S$  be a gp-set of  $G$  with  $|S| = \text{gp}(G)$ . Choose a vertex  $x \in S$ . Then  $S \setminus \{x\}$  is an  $x$ -position set, implying that  $\text{vp}(G) \geq p_x(G) \geq \text{gp}(G) - 1$ . ■

The bound in Lemma 2 is met by the complete graph  $K_n$ . However, we now give an example to show that the numbers  $\text{vp}^-(G)$ ,  $\text{gp}(G)$  and  $\text{vp}(G)$  can be arbitrarily far apart. For  $r \geq 2$  we define the vertex set of the graph  $G(r)$  to be  $\{u_{i,j} : 1 \leq i \leq 7, 1 \leq j \leq r\} \cup \{u\}$ . Let  $H(r)$  be the graph on the same vertex set as  $G(r)$  with adjacencies defined as follows.

- $u \sim u_{1,j}$  for  $1 \leq j \leq r$ ,
- $u_{i,j} \sim u_{i,j'}$ ,  $1 \leq i \leq 7$ ,  $1 \leq j, j' \leq r$  and  $j \neq j'$ , and
- $u_{i,j} \sim u_{i+1,j'}$  for  $1 \leq i \leq 6$  and  $1 \leq j, j' \leq r$ .

Now define  $G(r)$  to be the graph formed by deleting all edges to the vertices  $u_{3,1}$ ,  $u_{4,1}$  and  $u_{5,1}$  except for the path  $u_{2,1}, u_{3,1}, u_{4,1}, u_{5,1}, u_{6,1}$ . See Figure 2 for an example. The minimum vertex position number of  $G(r)$  is  $r$  (attained at the vertex  $u$ ) and the vertex position number is  $6r - 4$  (attained at the vertex  $u_{4,1}$ ).

**Lemma 3.** *For  $r \geq 3$ , we have  $\text{vp}^-(G(r)) = r$ ,  $\text{gp}(G(r)) = 2r$ ,  $\text{vp}(G(r)) = 6r - 4$ .*

In Theorem 21 we will see that  $\text{vp}^-(G)$  can be larger than  $\text{gp}(G)$ . Lemma 3 raises the question of how far apart the numbers  $\text{vp}^-(G)$  and  $\text{vp}(G)$  can be.

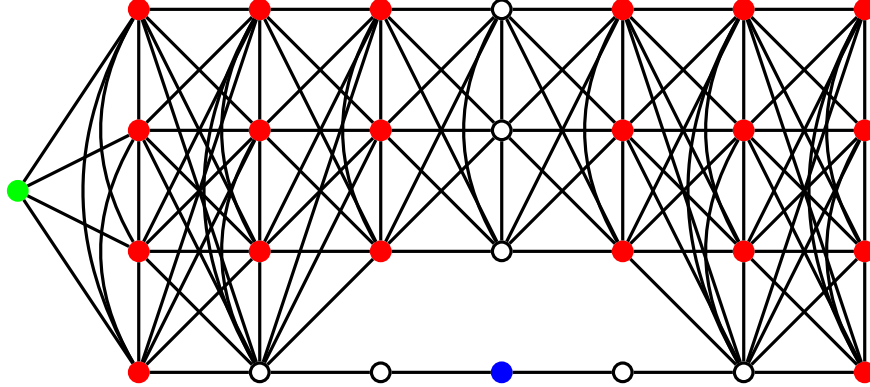


Figure 2. A graph  $G(r)$  with  $vp^- = r$  (green vertex) and  $vp = 6r - 4$  (blue vertex). A largest  $vp$ -set of the blue vertex is shown in red. Here  $r = 4$ .

**Problem 4.** Is the ratio  $\frac{vp(G)}{vp^-(G)}$  bounded for connected graphs?

We now bound the vertex position numbers in terms of the vertex degrees.

**Lemma 5.** *Let  $G$  have minimum degree  $\delta$  and maximum degree  $\Delta$ . Then  $vp^-(G) \geq \delta$  and  $vp(G) \geq \Delta$ .*

**Proof.** It follows from Definition 1 that for any vertex  $x$  of  $G$  the neighbourhood  $N(x)$  is an  $x$ -position set of  $G$ . Therefore for all vertices  $x \in V(G)$  we have  $p_x(G) \geq \deg(x)$ . This implies that all  $p_x$ -sets have order at least  $\delta$  and there exists a  $p_x$ -set with order at least  $\Delta$ . ■

We now generalise this result to sets of vertices at given distance from a fixed vertex; this leads to bounds on  $p_x(G)$  in terms of the order  $n$  of  $G$  and the eccentricity  $e(x)$  of the vertex  $x$ .

**Lemma 6.** *For any vertex  $x$  of  $G$  with eccentricity  $e(x)$  the vertex position number satisfies  $p_x(G) \geq \frac{n-1}{e(x)}$ . Thus  $vp^-(G) \geq \frac{n-1}{\text{diam}(G)}$  and  $vp(G) \geq \frac{n-1}{\text{rad}(G)}$ .*

**Proof.** For  $1 \leq t \leq e(x)$ , let  $V_t$  be the set of vertices at distance exactly  $t$  from  $x$  in  $G$ . Each of the sets  $V_t$  is an  $x$ -position set and one of them must have order at least  $\frac{n-1}{e(x)}$ , so that  $p_x(G) \geq \frac{n-1}{e(x)}$ . Since  $\text{rad}(G) \leq e(x) \leq \text{diam}(G)$  for all  $x \in V(G)$ , the result follows. ■

This bound immediately characterises the graphs with vertex position number one.

**Corollary 7.** *A graph  $G$  satisfies  $vp^-(G) = 1$  if and only if  $G$  is a path. The only connected graphs with  $vp(G) = 1$  are  $K_1$  and  $K_2$ .*

**Proof.** If  $x$  is the terminal vertex of a path  $P_n$ , then  $p_x(P_n) = 1$ . Conversely, if  $\text{vp}^-(G) = 1$ , then  $G$  is connected and by Lemma 6 we must have  $\text{diam}(G) = n - 1$ , which implies that  $G$  is a path. By Lemma 5 any graph  $G$  with  $\text{vp}(G) = 1$  must have maximum degree  $\Delta \leq 1$ , which proves the latter statement. ■

The argument of Lemma 6 also easily yields the vertex position number of the join of two graphs.

**Corollary 8.** *If  $G_i$  has order  $n_i$  and maximum degree  $\Delta_i$  for  $i = 1, 2$ , then the vertex position number of the join  $G_1 \vee G_2$  is*

$$\text{vp}(G_1 \vee G_2) = \max\{n_1 + \Delta_2, n_2 + \Delta_1\}.$$

**Proof.** If both  $G_1$  and  $G_2$  are complete, then the result is trivial, so we can assume that the diameter of  $G_1 \vee G_2$  is two. For any vertex  $x$  with degree  $\Delta_1$  in  $G_1$  the set  $V(G_2) \cup N_{G_1}(x)$  is an  $x$ -position set with order  $n_2 + \Delta_1$  by Lemma 5, with a similar result for vertices  $y$  of maximum degree in  $G_2$ . Hence  $\text{vp}(G_1 \vee G_2) \geq \max\{n_1 + \Delta_2, n_2 + \Delta_1\}$ .

Suppose without loss of generality that the largest vertex position set is attained at a vertex  $x \in V(G_1)$ . Then we can assume that a  $p_x$ -set contains a vertex of  $G_2$ , for otherwise  $\text{vp}(G_1 \vee G_2) = p_x(G_1 \vee G_2) \leq n_1 < p_y(G_1 \vee G_2)$  for any  $y \in V(G_2)$ , a contradiction. As a maximum  $p_x$ -set  $S_x$  contains a vertex of  $G_2$ ,  $S_x$  cannot contain any vertex of  $V(G_1) \setminus N_{G_1}[x]$ , so that  $p_x(G_1 \vee G_2) \leq n_2 + \Delta_1$ . The result follows. ■

**Theorem 9.** *For any connected graph  $G$  we have  $\text{vp}^-(G) \geq \lceil \frac{\Delta+1}{3} \rceil$ . If  $G$  is bipartite, then  $\text{vp}^-(G) \geq \lceil \frac{\Delta}{2} \rceil$ .*

**Proof.** Let  $G$  be a connected graph with maximum degree  $\Delta$  and let  $x$  be a vertex of  $G$  with this degree. Let  $y$  be any vertex of  $G$ . If  $y = x$ , we have  $p_y(G) \geq \Delta$ , so suppose that  $y \neq x$  and let  $r \geq 0$  be the length of the shortest path from  $y$  to  $N(x)$ . Then it can be seen that the distance from  $y$  to any vertex of  $N[x]$  is one of  $r$ ,  $r + 1$  or  $r + 2$ . It follows that one of the level sets of  $G$  in the distance partition with respect to  $y$  must have order at least  $\frac{\Delta+1}{3}$ . Hence  $p_y(G) \geq \frac{\Delta+1}{3}$  and our proof is complete. If  $G$  is bipartite, then the distance from  $y$  to any vertex of  $N(x)$  is either  $r$  or  $r + 2$ , so we can improve the bound to  $\text{vp}^-(G) \geq \frac{\Delta}{2}$  in this case. The constructions in Figure 3 show that both of these bounds are tight (in both cases  $x$  is a vertex with maximum degree and  $p_y(G) = \text{vp}^-(G)$ ). ■

Now we give an upper bound for the vertex position number in terms of vertex eccentricity.

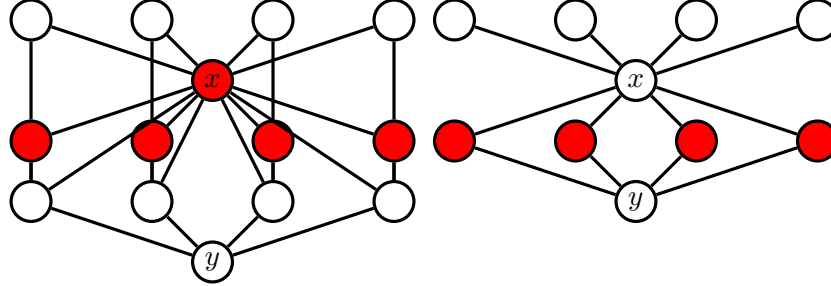


Figure 3. Graphs with  $\text{vp}^-(G) = \lceil \frac{\Delta+1}{3} \rceil$  (left) and  $\text{vp}^-(G) = \frac{\Delta}{2}$  (right) with a  $p_y$ -set in red.

**Lemma 10.** *For any vertex  $x$  of  $G$  with eccentricity  $e(x)$ , the  $x$ -vertex position number of  $G$  is bounded above by  $p_x(G) \leq n - e(x)$ . Thus  $\text{vp}(G) \leq n - \text{rad}(G)$  and  $\text{vp}^-(G) \leq n - \text{diam}(G)$ .*

**Proof.** Fix a vertex  $x$  of  $G$  with eccentricity  $e(x)$ . Let  $y$  be an eccentric vertex of  $x$ , i.e.  $d(x, y) = e(x)$ . Let  $x = u_0, u_1, \dots, u_{e(x)} = y$  be an  $x, y$ -geodesic in  $G$  and  $S_x$  be an  $x$ -position set of  $G$  with order  $p_x(G)$ . Suppose that  $u_i, u_j \in S_x$  for some  $i, j$  with  $0 \leq i < j \leq e(x)$ ; then  $u_i$  lies on an  $x, u_j$ -geodesic, contradicting the fact that  $S_x$  is an  $x$ -position set. Hence any  $x$ -position set contains at most one vertex from the set  $\{x, u_1, u_2, \dots, u_{e(x)}\}$ . Thus  $p_x(G) = |S_x| \leq n - 1 - e(x) + 1 = n - e(x)$ . Thus  $\text{vp}(G) \leq n - \text{rad}(G)$  and  $\text{vp}^-(G) \leq n - \text{diam}(G)$ . ■

The following theorem improves the upper bound for  $\text{vp}(G)$  from Lemma 10.

**Theorem 11.** *For any graph  $G$  with  $\text{rad}(G) \geq 3$  we have  $\text{vp}(G) \leq n - \text{rad}(G) - 1$ .*

**Proof.** Suppose that  $G$  has radius  $\text{rad}(G) \geq 3$  and meets the upper bound in Lemma 10. Then the largest value of the vertex position number is achieved by a central vertex, call it  $u$ . Let  $S_u$  be any  $u$ -position set of order  $n - \text{rad}(G)$ . By the argument of Lemma 10 there is a path  $P$  from  $u$  to one of its eccentric vertices such that  $P$  contains just one vertex of  $S_u$  and all vertices of  $V(G) \setminus V(P)$  belong to  $S_u$ . As the shortest path from  $u$  to any vertex  $x \in V(G) \setminus V(P)$  cannot pass through another vertex of  $V(G) \setminus V(P)$ , the shortest path from  $u$  to  $x$  consists of a section of  $P$  followed by an edge from  $P$  to  $x$ . Hence each vertex of  $V(G) \setminus V(P)$  has an edge to  $P$ ; however, this contradicts our supposition that  $u$  is a central vertex. ■

A vertex  $v$  in a connected graph  $G$  is a *boundary vertex* of a vertex  $u$  if  $d(u, w) \leq d(u, v)$  for each neighbour  $w$  of  $v$ . The set of all boundary vertices of  $u$  is denoted by  $\partial(u)$ .



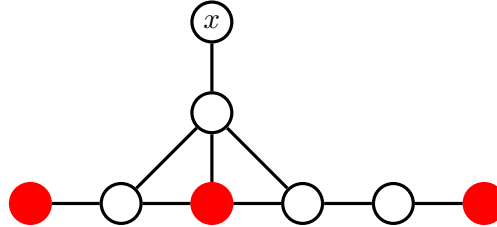


Figure 4. The boundary  $\partial(x)$  (shown in red) is a  $p_x$ -set.

**Proposition 12.** *For any connected graph  $G$  and any vertex  $x \in V(G)$ , the boundary  $\partial(x)$  is an  $x$ -position set of  $G$ .*

**Proof.** Assume to the contrary that  $\partial(x)$  is not an  $x$ -position set; hence there must be a geodesic  $x = u_0, u_1, \dots, u_i = z, u_{i+1}, \dots, u_k = y$  such that  $z, y \in \partial(x)$  and  $z \neq y$ . This shows that  $d(x, u_{i+1}) > d(x, z)$ , contradicting the fact that  $z$  is a boundary vertex of  $x$ . Hence  $\partial(x)$  is an  $x$ -position set. ■

It follows from Proposition 12 that for any  $x \in V(G)$  the set  $\text{Ext}(G) \setminus \{x\}$  is an  $x$ -position set. The bound in Proposition 12 is tight for the vertex  $x$  in the graph in Figure 4, but this is not true in general. In fact for any  $s \geq t$  if we take a vertex  $x$  in the partite set of order  $t + 1$  in the complete bipartite graph  $K_{s,t+1}$ , then  $|\partial(x)| = t$  but  $p_x(K_{s,t+1}) = s$ .

Finally we present a Nordhaus-Gaddum relation for the vertex position number. In the following we do not assume that both  $G$  and  $\overline{G}$  are connected.

**Theorem 13.** *For any graph  $G$  we have  $n - 1 \leq \text{vp}(G) + \text{vp}(\overline{G}) \leq 2n - 1$ . Both bounds are tight.*

**Proof.** Let  $x$  be any vertex of a graph  $G$  with degree  $\deg(x)$ . In the complement  $\overline{G}$  the vertex  $x$  has degree  $\deg'(x) = n - 1 - \deg(x)$ . By Lemma 5 we thus have

$$(1) \quad \text{vp}(G) + \text{vp}(\overline{G}) \geq p_x(G) + p_x(\overline{G}) \geq \deg(x) + n - 1 - \deg(x) = n - 1.$$

To show that equality can hold, consider the cycle  $C_n$  for  $n \geq 5$ . If  $n = 5$ , the result is simple, as  $\overline{C_5} \cong C_5$ , so take  $n \geq 6$ . Label the vertices of the cycle  $x_0, x_1, \dots, x_{n-1}$ , where  $x_0 \sim x_{n-1}$  and  $x_i \sim x_{i+1}$  for  $0 \leq i \leq n - 2$ . As will be shown in Corollary 23, we have  $\text{vp}(C_n) = 2$ . Consider the vertex  $x_0$  (as  $\overline{C_n}$  is vertex-transitive the choice is arbitrary) and let  $S$  be a largest  $x_0$ -position set in  $\overline{C_n}$ . The degree of  $x_0$  is  $\deg(x_0) = n - 3$ , so that by Lemma 5 we have  $\text{vp}(\overline{C_n}) \geq n - 3$ . Suppose that a vertex  $x_i$ ,  $3 \leq i \leq n - 3$ , belongs to  $S$ ; then as  $x_0, x_i, x_1$  and  $x_0, x_i, x_{n-1}$  are shortest paths we must have  $x_1, x_{n-1} \notin S$ , so that  $|S| \leq n - 3$ . Furthermore  $S$  cannot contain both vertices  $x_{n-2}$  and  $x_1$  and

likewise cannot contain both  $x_2$  and  $x_{n-1}$ , so that if  $S \cap \{x_3, \dots, x_{n-3}\} = \emptyset$ , we have  $|S| \leq 2 < n-3$ . Thus  $\text{vp}(\overline{C}_n) = n-3$  and we have  $\text{vp}(C_n) + \text{vp}(\overline{C}_n) = n-1$ .

Applying this argument to a vertex of maximum degree  $\Delta$  and a vertex with minimum degree  $\delta$  gives the stronger bound  $\text{vp}(G) + \text{vp}(\overline{G}) \geq n-1 + \Delta - \delta$ , so we see that we have equality in equation 1 if and only if  $G$  is regular and both  $G$  and  $\overline{G}$  have vertex position number equal to their maximum degree.

Trivially for  $n \geq 2$  we have  $\text{vp}(G) \leq n$ , with equality if and only if  $G$  has an isolated vertex. The graphs  $G$  and  $\overline{G}$  cannot both contain an isolated vertex. Therefore we do not have equality in both  $\text{vp}(G) \leq n$  and  $\text{vp}(\overline{G}) \leq n$ , so it follows that  $\text{vp}(G) + \text{vp}(\overline{G}) \leq 2n-1$ . Equality holds if and only if  $G$  contains an isolated vertex or a universal vertex. ■

### 3. VERTEX POSITION NUMBERS OF CERTAIN CLASSES OF GRAPHS

In this section, we determine the  $x$ -position number of certain standard classes of graphs.

**Lemma 14.** *Let  $x$  be a vertex of a connected graph  $G$  and  $S_x \subseteq V(G)$  be an  $x$ -position set of  $G$ . If  $C_1, C_2, \dots, C_k$  are the components of  $G[S_x]$ , then there exist  $r_1, r_2, \dots, r_k$  such that  $d(x, y) = r_i$  for all  $y \in C_i$ .*

**Proof.** Suppose that there is a component  $C$  of  $G[S_x]$  and  $y, y' \in V(C)$  such that  $d(x, y) \neq d(x, y')$ . Then, considering a shortest path from  $y$  to  $y'$  in  $G[C]$ , we see that there is a pair  $z, z' \in V(C)$  such that  $z \sim z'$  and  $d(x, z') = d(x, z) + 1$ . However, this implies that an  $x, z$ -geodesic followed by the edge  $z \sim z'$  is a shortest  $x, z'$ -path that passes through  $z$ , a contradiction. ■

**Theorem 15.** *If  $G$  is a bipartite graph, then  $\text{vp}(G) \leq \alpha(G)$ .*

**Proof.** Let  $S_x$  be an  $x$ -position set of  $G$  and suppose for a contradiction that  $S_x$  is not an independent set. Then there are  $y, z \in S_x$  such that  $y \sim z$  in  $G$ . By Lemma 14 we have  $d(x, y) = d(x, z) = r$  for some  $r \geq 1$ . A shortest  $x, y$ -path, a shortest  $x, z$ -path and the edge  $y \sim z$  together constitute an odd circuit, implying the existence of an odd cycle; since  $G$  is bipartite, this is impossible and it follows that  $p_x(G) \leq \alpha(G)$ . ■

**Theorem 16.** *For  $r \geq 2$ , let  $K_{n_1, n_2, \dots, n_r}$  be the complete multipartite graph with partite sets  $V_1, V_2, \dots, V_r$ , where  $n_i = |V_i|$  and  $n_1 \geq n_2 \geq \dots \geq n_r$ . Set  $n = \sum_{i=1}^r n_i$ . Then if the vertex  $x$  lies in  $V_i$ , the  $x$ -position number is given by*

$$p_x(K_{n_1, n_2, \dots, n_r}) = \max\{n - n_i, n_i - 1\}.$$

*Thus  $\text{vp}(K_{n_1, n_2, \dots, n_r}) = n - n_r$ .*

**Proof.** Let  $x \in V_i$  and let  $S$  be a maximum  $x$ -position set of the graph. Set  $V = \bigcup_{i=1}^r V_i$ . Suppose that  $S$  contains a vertex  $y \in V_i \setminus \{x\}$ ; for any vertex  $z \in V \setminus V_i$  the path  $x, z, y$  is a geodesic, so that in this case  $S \cap (V \setminus V_i) = \emptyset$ . Thus either  $S \subseteq V \setminus V_i$  or  $S \subseteq V_i \setminus \{x\}$ . Conversely, both of these sets are  $x$ -position sets by the argument of Lemma 6, which yields the claimed bounds. ■

Theorem 16 shows that equality holds in the bound of Theorem 15 for all complete bipartite graphs.

**Lemma 17.** *Let  $G$  be a connected graph of order  $n$ . Then for each  $x \in V(G)$  there is a maximum  $x$ -position set without cut-vertices.*

**Proof.** Suppose that there is a maximum  $x$ -position set  $S$  containing a cut-vertex  $v \neq x$  of  $G$ . Let  $C_1, C_2, \dots, C_k$  be the components of  $G \setminus \{v\}$ , where  $k \geq 2$ . Without loss of generality we may assume that  $x \in V(C_1)$ . Then it follows that  $S \cap V(C_i) = \emptyset$  for all  $i = 2, 3, \dots, k$ . Let  $u_i$  be any vertex in  $C_i$  for all  $i = 2, 3, \dots, k$ . If  $k \geq 3$ , then the set  $S' = (S \setminus \{v\}) \cup \{u_2, u_3, \dots, u_k\}$  is an  $x$ -position set with order greater than  $S$ , a contradiction to the maximality of  $S$ . Hence  $k = 2$ . Let  $u$  be a farthest vertex from  $v$  in  $C_2$ . Then  $u$  is not a cut-vertex in  $G$ . Moreover,  $S' = (S \setminus \{v\}) \cup \{u\}$  is a maximum  $x$ -position set containing fewer cut-vertices than  $S$ ; this implies the existence of a maximum  $x$ -position set without cut-vertices in  $G$ . ■

**Theorem 18.** *For any connected block graph  $G$ ,*

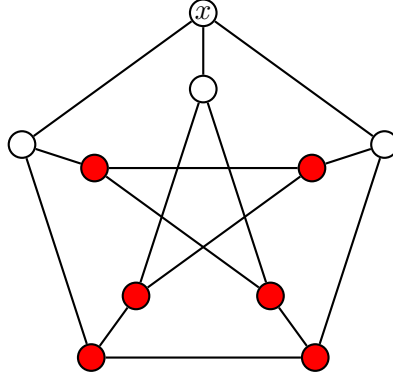
$$p_x(G) = \begin{cases} s(G) - 1, & \text{if } x \text{ is a simplicial vertex,} \\ s(G), & \text{otherwise.} \end{cases}$$

**Proof.** Let  $x$  be a vertex of  $G$  and  $S_x$  a maximum  $x$ -position set. By Lemma 17 we can assume that  $S_x$  contains no cut-vertices; every vertex of a block graph is either simplicial or a cut-vertex, so it follows that  $p_x(G) \leq s(G)$  and, if  $x$  is itself simplicial, then  $p_x(G) \leq s(G) - 1$ . Conversely, by Proposition 12 the set  $\text{Ext}(G)$  is an  $x$ -position set if  $x \notin \text{Ext}(G)$  and  $\text{Ext}(G) \setminus \{x\}$  is an  $x$ -position set if  $x \in \text{Ext}(G)$ , so that we have equality. ■

**Corollary 19.** *For any tree  $T$  with  $\ell$  leaves we have*

$$p_x(T) = \begin{cases} \ell - 1, & \text{if } x \text{ is a leaf,} \\ \ell, & \text{otherwise.} \end{cases}$$

Corollary 19 implies the following bound for the vertex position numbers in terms of the girth of the graph.

Figure 5. A largest  $x$ -position set (red vertices) in the Petersen graph.

**Theorem 20.** *If a graph  $G$  has girth  $g$  and minimum degree  $\delta \geq 2$  and there are  $N$  vertices at distance at most  $\left\lfloor \frac{g-1}{2} \right\rfloor - 1$  from a vertex  $u$ , then  $p_u(G) \leq n - N$ .*

**Proof.** Set  $r = \left\lfloor \frac{g-1}{2} \right\rfloor$ . Fix a vertex  $u$  of  $G$  and consider the subgraph  $G'$  induced by the vertices at distance at most  $r$  from  $u$ .  $G'$  is isomorphic to a tree, possibly with some edges added between the vertices at distance  $r$  from  $u$ . It follows from Corollary 19 that the largest number of vertices from  $G'$  that can belong to a  $u$ -position set is the number of vertices at distance exactly  $r$  from  $u$ ; hence there are at least  $N$  vertices missing from any  $u$ -position set and  $p_u(G) \leq n - N$ . ■

Theorem 20 is tight for the Petersen and Hoffman-Singleton graphs by the argument in Lemma 6. Finally, we note that Lemma 5 gives the vertex position number of sufficiently large Kneser graphs. The Kneser graph  $K(n, k)$  is the graph with vertex set equal to all  $k$ -subsets of  $\{1, 2, \dots, n\}$  with an edge between any two such subsets if and only if they are disjoint.

**Theorem 21.** *For sufficiently large  $n$  we have  $\text{vp}(K(n, k)) = \binom{n-k}{k}$ .*

**Proof.** For  $n \geq 3k$  the Kneser graph  $K(n, k)$  has diameter two. Note that  $K(n, k)$  is vertex-transitive, so  $\text{vp}(K(n, k)) = \text{vp}^-(K(n, k))$  and we can without loss of generality consider the vertex  $\{1, 2, \dots, k\}$ ; let  $S$  be a largest  $\{1, 2, \dots, k\}$ -position set. Lemma 5 gives  $|S| = \text{vp}(K(n, k)) \geq \binom{n-k}{k}$ . Suppose that  $S$  contains a vertex at distance two from  $\{1, 2, \dots, k\}$ , say  $\{1, 2, \dots, i, j_{i+1}, j_{i+2}, \dots, j_k\}$ , where

$$\{1, 2, \dots, k\} \cap \{1, 2, \dots, i, j_{i+1}, j_{i+2}, \dots, j_k\} = \{1, 2, \dots, i\}.$$

As  $\{1, 2, \dots, k\}$  and  $\{1, 2, \dots, i, j_{i+1}, j_{i+2}, \dots, j_k\}$  have exactly  $\binom{n-2k+i}{k}$  common neighbours, we would have  $|S| \leq \binom{n}{k} - \binom{n-2k+i}{k}$ , which is a polynomial in  $n$  of degree  $k-1$ , whereas the vertex degree is a polynomial function of  $n$  with degree

$k$ . Thus for sufficiently large  $n$  compared to  $k$  the bound in Lemma 5 is best possible. ■

Interestingly Theorem 21 implies that for large  $n$  the vertex position number of  $K(n, k)$  is significantly larger than the general position number, as given in [11]; hence, since the Kneser graphs are vertex-transitive,  $\text{vp}^-(K(n, k))$  is larger than  $\text{gp}(K(n, k))$  for sufficiently large  $n$ . For small orders Lemma 5 is not optimal; for example, as previously noted for the Petersen graph  $P$  (isomorphic to  $K(5, 2)$ ) the second neighbourhood of a vertex is a largest vertex position set and  $\text{vp}(P) = \text{gp}(P) = 6$ .

#### 4. CHARACTERISATION RESULTS

We now make use of the bounds derived in Section 2 to characterise graphs with very large or very small vertex position numbers.

**Corollary 22.** *A connected graph  $G$  with order  $n$  satisfies  $\text{vp}(G) = n - 1$  if and only if  $G$  contains a universal vertex, whilst  $\text{vp}^-(G) = n - 1$  if and only if  $G$  is a complete graph.*

**Proof.** By Lemma 5 any universal vertex  $u$  has vertex position number  $p_u(G) = n - 1$ . Conversely, by Lemma 10 any vertex  $u$  with  $p_u(G) = n - 1$  has eccentricity one and hence is universal. If  $\text{vp}^-(G) = n - 1$ , it follows that  $G$  is complete. ■

**Corollary 23.** *A connected graph  $G$  satisfies  $\text{vp}^-(G) = \text{vp}(G) = 2$  if and only if  $G$  is a cycle. Similarly  $\text{vp}(G) = 2$  only for cycles and paths of length  $\geq 2$ .*

**Proof.** Let  $C_n$  be a cycle for some  $n \geq 3$  with vertex set identified with  $\mathbb{Z}_n$ . As  $C_n$  is 2-regular, by Lemma 5 we have  $\text{vp}^-(C_n) \geq 2$ . We now show that  $\text{vp}(C_n) \leq 2$ . If  $n$  is odd, then this follows immediately by Theorem 20, so suppose that  $n$  is even. The  $x$ -vertex position number of any graph  $G$  is bounded above by the number of geodesics with  $x$  as initial vertex needed to cover all vertices of  $V(G)$ , since each of these paths can contain at most one vertex of any  $x$ -position set. (essentially this is Theorem 3.3 in [15]). Taking  $x = 0$  in  $C_n$ , the shortest paths  $0, 1, \dots, \frac{n}{2}$  and  $0, n-1, n-2, \dots, \frac{n}{2}+1$  cover  $V(C_n)$ , so that  $\text{vp}(C_n) = p_0(C_n) \leq 2$ .

Conversely, suppose that  $\text{vp}(G) = 2$ ; it follows from Lemma 5 that  $G$  has maximum degree  $\Delta = 2$ , so that  $G$  is either a path or a cycle. As  $\text{vp}^-(P_n) = 1$  for paths by Corollary 7, it follows that if also  $\text{vp}^-(G) = 2$ , then  $G$  is a cycle. ■

Now we characterise some graphs with very large vertex position number.

**Lemma 24.** *A vertex  $u$  of a connected graph  $G$  with order  $n$  has  $p_u(G) = n - 2$  if and only if either (i)  $u$  has degree  $\deg(u) = n - 2$  or (ii)  $\deg(u) \leq n - 3$  and there is a  $v \in N(u)$  such that for any vertex  $w \notin N[u]$  the vertex  $v$  is the unique common neighbour of  $u$  and  $w$ .*

**Proof.** Let  $G$  be a graph with order  $n$  and let  $u \in V(G)$  satisfy  $p_u(G) = n - 2$ , with  $S_u$  a largest  $u$ -position set. By Lemma 5 the vertex  $u$  is not universal, so by Lemma 10 the eccentricity of  $u$  is two. Suppose that  $\deg(u) \leq n - 3$  and  $|N^2(u)| \geq 2$ .

Let  $x, y$  be any vertices in  $N^2(u)$ . Suppose that one of these vertices, say  $x$ , has at least two common neighbours with  $u$ . If  $x \in S_u$ , then no vertex of  $N(u) \cap N(x)$  can belong to  $S_u$ ; as  $u \notin S_u$ , this accounts for at least three vertices missing from  $S_u$ . Thus  $S_u = V(G) \setminus \{u, x\}$ , so that  $\{y\} \cup N(u) \subseteq S_u$ ; however, as the shortest  $u, y$ -paths pass through  $N(u)$  this is impossible. Therefore every vertex in  $N^2(u)$  has just one neighbour in  $N(u)$ . Similarly, suppose that  $N(u) \cap N(x) = \{v\}$  and  $N(u) \cap N(y) = \{v'\}$ , where  $v \neq v'$ . Then  $|S_u \cap \{v, x\}| \leq 1$  and  $|S_u \cap \{v', y\}| \leq 1$ , which, together with  $u$ , again accounts for at least three vertices missing from  $S_u$ . Thus we must have  $v = v'$  and there is a vertex  $v \in N(u)$  that is the unique neighbour in  $N(u)$  of every vertex in  $N^2(u)$ .

Conversely, it is evident that any vertex  $u$  satisfying either condition (i) or (ii) has  $p_u(G) = n - 2$ . In Case (ii) our argument shows that  $V(G) \setminus \{u, v\}$  is the unique maximum  $u$ -position set. In Case (i) if the unique vertex  $w \notin N[u]$  has degree  $\geq 2$ , then  $V(G) \setminus \{u, w\}$  is the unique maximum  $u$ -position set, whereas if  $w$  is a leaf with support vertex  $w'$ , then  $V(G) \setminus \{u, w\}$  and  $V(G) \setminus \{u, w'\}$  are the two maximum  $u$ -position sets. ■

**Theorem 25.** *For  $n \geq 4$ , a graph  $G$  with order  $n$  satisfies  $\text{vp}^-(G) = \text{vp}(G) = n - 2$  if and only if  $G$  is isomorphic to an even clique with a perfect matching deleted, i.e., if and only if  $G \cong K_{2,2,\dots,2}$ .*

**Proof.** Assume that  $G$  is a graph such that  $p_u(G) = n - 2$  for every  $u \in V(G)$ . Suppose that  $G$  contains a vertex  $u$  with degree  $\leq n - 3$ , so that by Lemma 24 the eccentricity of  $u$  is two and  $u$  has a neighbour  $v$  such that for any vertex  $w \notin N[u]$  the vertex  $v$  is the unique common neighbour of  $u$  and  $w$ . As  $\text{vp}(G) = n - 2$ ,  $G$  contains no universal vertex, so that there is a neighbour  $z$  of  $u$  such that  $v \not\sim z$ . Hence if  $x \in N^2(u)$ , we have  $d(x, z) \geq 3$  and  $x$  has eccentricity at least three, so that by Lemma 10 we have  $p_x(G) \leq n - 3$ , a contradiction. Therefore every vertex of  $G$  has degree  $n - 2$  and  $G$  is isomorphic to a  $K_{2r}$  with a perfect matching deleted. Conversely, by Lemma 24 in such a graph every vertex has vertex position number  $n - 2$ . ■

**Theorem 26.** *For  $n \geq 4$ , a graph  $G$  has  $\text{vp}^-(G) = n - 2$  and  $\text{vp}(G) = n - 1$  if and only if (i)  $G$  is isomorphic to a clique with a non-empty, non-perfect matching deleted or (ii)  $G$  is the join of  $K_1$  with a disjoint union of cliques.*

**Proof.** Let  $G$  be a graph with  $\text{vp}^-(G) = n - 2$  and  $\text{vp}(G) = n - 1$ . We can assume that  $G$  contains  $r \geq 1$  universal vertices as well as at least two vertices with degree  $\leq n - 2$ . If every vertex has degree either  $n - 1$  or  $n - 2$ , then  $G$

is isomorphic to a clique with a matching deleted. To avoid the graph having  $\text{vp}^-(G) = \text{vp}(G) = n - 1$  the matching must be non-empty and to avoid having  $\text{vp}^-(G) = \text{vp}(G) = n - 2$  the matching is not perfect by Theorem 25.

Suppose that  $G$  contains a vertex  $u$  with  $p_u(G) = n - 2$  and degree  $\deg(u) \leq n - 3$ ; then  $u$  has a neighbour  $v$  satisfying condition (ii) in Lemma 24. By Lemma 10 every vertex of  $G$  has eccentricity at most two, so, considering the vertices in  $N^2(u)$ , we see that  $v$  is a universal vertex; in fact, as  $v$  is a cut-vertex, it is the unique universal vertex of  $G$ , so that every other vertex  $w$  of  $G$  must have  $p_w(G) = n - 2$ . Consider the components  $C_1, C_2, \dots, C_t$  of  $G \setminus v$ , where  $t \geq 2$ . Suppose that there is a component  $C_i$  that is not a clique and choose vertices  $z, z' \in C_i$  such that  $z \not\sim z'$ . Then  $z$  has degree at most  $n - 3$  and must also satisfy condition (ii) of Lemma 24. Then as  $d(z, z') \geq 2$  condition (ii) of Lemma 24 implies that every  $z, z'$ -path passes through  $v$ ; however, this contradicts the fact that  $z$  and  $z'$  lie in the same component of  $G \setminus v$ . Therefore  $G$  is the join of  $K_1$  with a disjoint union of cliques. Conversely, any such join has the required parameters. ■

## 5. COMPUTATIONAL COMPLEXITY

In this section we show that, for a given graph  $G$ , the vertex position number  $p_x(G)$  can be computed in polynomial time for any vertex  $x \in V(G)$ . In particular, we will show that  $p_x(G)$ , for each  $x \in V(G)$ , can be computed as an independent set calculated on a graph obtained as a transformation of  $G$ . To this aim we need the following definitions.

**Definition 27.** A graph is a *comparability graph* if the edges connect pairs of elements that are comparable to each other in a partial order.

**Definition 28.** Given a graph  $G$  and a vertex  $x \in V(G)$ , the *reduced graph*  $\tilde{G}_x$  is the graph on the same vertex set  $V(G)$  obtained from  $G$  by removing all the edges connecting vertices at the same distance from  $x$ .

**Definition 29.** Given a graph  $G$  and a vertex  $x \in V(G)$ , the graph  $G_x^*$  is the graph on the same vertex set  $V(G)$  obtained from the reduced graph  $\tilde{G}_x$  by adding an edge between any two vertices of any geodesic to  $x$ .

See Figure 6 for a visualisation of  $\tilde{G}_x$  and  $G_x^*$ , starting from a graph  $G$  and a vertex  $x$ .

**Lemma 30.** *Given a graph  $G$  and a vertex  $x \in V(G)$ ,  $G_x^*$  is a comparability graph.*

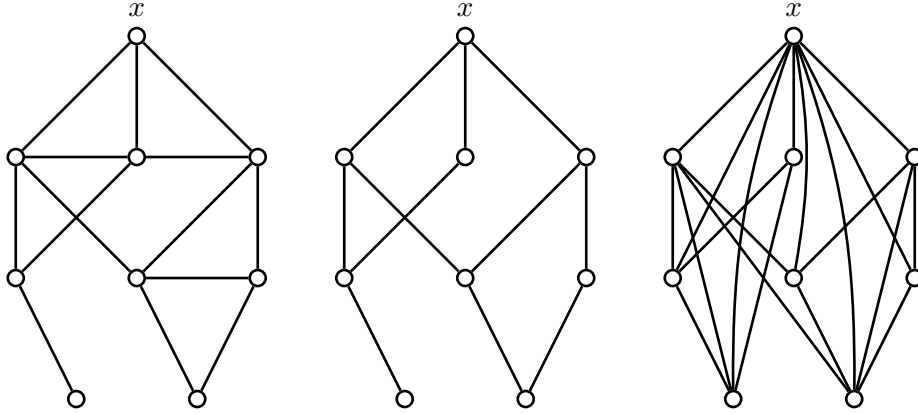


Figure 6. From left to right: a graph  $G$  with a vertex  $x$ , the graph  $\tilde{G}_x$ , and the graph  $G_x^*$ . The graphs  $G$  and  $\tilde{G}_x$  are drawn by placing the vertices at the same distance from  $x$  on a common horizontal level.

**Proof.** We define a partial order  $<$  on  $V(G_x^*)$  by setting  $u < v$  for  $u, v \in V(G_x^*)$  if and only if  $u$  and  $v$  are on the same geodesic to  $x$  and  $d(u, x) < d(v, x)$ . Since  $G_x^*$  has been built from  $\tilde{G}_x$  by adding an edge between every pair of vertices on each geodesic to  $x$ , it follows that any two comparable vertices are adjacent in  $G_x^*$  and hence  $G_x^*$  is a comparability graph. ■

**Lemma 31.** *Given a graph  $G$  and a vertex  $x \in V(G)$ ,  $S_x$  is an  $x$ -position set of  $G$  if and only if  $S_x$  is an  $x$ -position set for  $\tilde{G}_x$ . Hence  $p_x(G) = p_x(\tilde{G}_x)$ .*

**Proof.** Note that any geodesic to  $x$  in  $G$  is also a geodesic to  $x$  in  $\tilde{G}_x$ . The set  $S_x$  is a  $x$ -position set for  $G$  if and only if no two vertices of  $S_x$  lie on a common geodesic to  $x$  in  $G$ ; as the geodesics to  $x$  in  $G$  coincide with the geodesics to  $x$  in  $\tilde{G}_x$ , it follows that  $S_x$  is an  $x$ -position set for  $G$  if and only if it is an  $x$ -position set for  $\tilde{G}_x$ . Consequently, any maximum vertex position set of  $G_x$  is a maximum vertex position set for  $\tilde{G}_x$  and  $p_x(G) = p_x(\tilde{G}_x)$ . ■

Given a graph  $G$ , let us denote the graph induced by vertices in  $V(G) \setminus \{x\}$  by  $G - x$ .

**Lemma 32.** *Given a graph  $G$  and a vertex  $x \in V(G)$ ,  $S_x$  is an  $x$ -position set of  $G$  if and only if  $S_x$  is an independent set of  $G_x^* - x$ . Hence  $p_x(G) = \alpha(G_x^* - x)$ .*

**Proof.** Let  $S_x$  be an  $x$ -position set in  $G$ . By Lemma 31,  $S_x$  is also an  $x$ -position set in  $\tilde{G}_x$ . Assume that  $S_x$  is not an independent set in  $G_x^* - x$ . Then there are two adjacent vertices  $u, v \in V(G_x^* - x) \cap S_x$ . By definition of  $G_x^*$  and Lemma 31,  $u$  and  $v$  would lie on the same geodesic to  $x$  in  $G$ , a contradiction.



For the converse, assume that  $S_x$  is an independent set of  $G_x^* - x$ . Let  $u \in S_x$  and let  $P$  be any  $u - x$  geodesic in  $G$ . Then by the construction of  $G_x^*$  the vertex  $u$  is adjacent to all the vertices of  $V(P - u)$  in  $G_x^*$ . This immediately shows that  $V(P) \cap S_x = \{u\}$ . Consequently,  $S_x$  is an  $x$ -position set in  $G$ . Therefore  $p_x(G) = \alpha(G_x^* - x)$ . ■

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**Algorithm A:**

**Input:** A connected graph  $G$ , a vertex  $x \in V(G)$

**Output:** A maximum  $x$ -position set  $S_x$ , and  $p_x(G)$

```

1 Let  $D[u] := d(u, x)$ , for each  $u \in V(G)$ ;
2 for each  $uv \in E(G)$  do
3   if  $D[u] = D[v]$  then
4      $\quad$  remove  $uv$  from  $E(G)$ ;
5 for each  $u \in V(G - x)$  do
6   Let  $Q$  be a queue and  $R := \{u\}$ ;
7    $Q.enqueue(u)$ ;
8   while  $Q$  is not empty do
9      $v := Q.dequeue()$ ;
10    for each  $w$  in  $N_G(v) \setminus R$  such that  $D[w] > D[v]$  do
11       $Q.enqueue(w)$ ;
12       $R := R \cup \{w\}$ ;
13      if  $D[w] > D[u] + 1$  then
14         $\quad$  add  $uv$  to  $E(G)$ ;
15 Let  $S$  be a maximum independent set of  $G - x$ ;
16 return  $S, |S|$ 
```

---

Figure 7. Algorithm A to compute a maximum  $x$ -position set  $S_x$  of a graph  $G$  and  $p_x(G)$ , for a given  $x \in V(G)$ .

---

**Theorem 33.** *Given a graph  $G$  and a vertex  $x \in V(G)$ , a maximum  $x$ -position set can be computed in  $O(nm \log(n^2/m))$  time, where  $n = |V(G_x^*)|$  and  $m = |E(G_x^*)|$ .*

**Proof.** Algorithm A in Figure 7 computes the distances of each vertex  $v \in V(G)$  from  $x$  at Line 1. This requires  $O(n + m)$  time. With the loop at Line 2,  $\tilde{G}_x$  is computed from  $G$  by removing edges between vertices at the same distance from  $x$ . This requires  $O(m)$  time. The loop at Line 5 add edges to the graph in order to build  $G_x^*$ . This requires  $O(n^2 + nm)$  time, since Lines from 6 to 14 codify for

a breadth-first visit of the vertices on a geodesic to  $x$  passing through a vertex  $u$ . This visit, based on a queue  $Q$  and a set  $R$  of the visited vertices, requires  $O(n + m)$  time and since it is repeated for each vertex  $u$  in  $V(G - x)$ , the total time is  $O(n^2 + nm)$ . Finally, at Line 15 an independent set  $S$  of the resulting comparability graph  $G^* - x$  is computed. According to [12], the computation of an independent set for a comparability graph requires  $O(nm \log(n^2/m))$  time; the last step determines the computational time of the whole algorithm. By Lemma 32, the set  $S$  is also an  $x$ -position set of  $G$ , so that Algorithm A correctly returns  $S$  and its order. ■

**Corollary 34.** Given a connected graph  $G$ ,  $\text{vp}^-(G)$  and  $\text{vp}(G)$  can be computed in  $O(n^4 \log(n))$  time, where  $n = |V(G)|$ .

**Proof.** Given a graph  $G$ , by calling Algorithm A for each vertex  $x$  of  $G$ ,  $\text{vp}^-(G)$  and  $\text{vp}(G)$  can be easily computed. By Theorem 33 each call requires  $O(nm \log(n^2/m))$  steps, where  $n = |V(G_x^*)|$  and  $m = |E(G_x^*)|$ . Considering that  $n = |V(G_x^*)| = |V(G)|$  and  $n - 1 \leq m \leq \binom{n}{2}$ , each call requires at most  $O(n^3 \log(n))$  time, yielding a total of  $O(n^4 \log(n))$  time. ■

### Acknowledgements

The third author gratefully acknowledges funding support from EPSRC grant EP/W522338/1 and London Mathematical Society grant ECF-2021-27. The authors also thank the anonymous reviewers for their helpful feedback.

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Received 14 September 2022

Revised 23 February 2023

Accepted 23 February 2023

Available online 19 March 2023