Discussiones Mathematicae
Graph Theory 44 (2024) 1079–1106
https://doi.org/10.7151/dmgt.2490

THE GENERALIZED 4-CONNECTIVITY OF BALANCED HYPERCUBES

Dongqin Cheng

Department of Mathematics
College of Information Science and Technology
Jinan University, Guangzhou, 510632, China
e-mail: dqcheng168@jnu.edu.cn

Abstract

The balanced hypercube is a kind of highly symmetrical network and possesses many good properties. Generalized connectivity is a new measurement of interconnection networks' fault tolerance. The internally disjoint N-trees are edge-disjoint trees but with intersecting vertex set N. Let $\kappa(N)$ be the maximum number of internally disjoint N-trees and the generalized k-connectivity of G be $\kappa_k(G) = \min\{\kappa(N) \mid N \subset V(G) \text{ and } |N| = k\}$. In this paper, we study the n-dimensional balanced hypercube BH_n and demonstrate that $\kappa_4(BH_n) = 2n - 1$ for $n \geq 1$.

Keywords: interconnection network, balanced hypercube, generalized connectivity, fault tolerance.

2020 Mathematics Subject Classification: 68R10.

1. Introduction

The parallel and distributed system plays a significant role in social networks, cloud computing, Big Data, and so on. Interconnection network as the topological structure of parallel and distributed system has obtained widely studied and applied. An interconnection network (network briefly) is modeled by a graph, where the processors and communication links are corresponding to vertices and edges, respectively. The hypercube [16] is one of the best-known networks. Compared with the hypercube, the balanced hypercube not only keeps many good properties like the hypercube but also has other better properties than the hypercube, including the smaller diameter and that each vertex has a paired vertex which has the same neighborhood [19], so each processor has an alternative processor when processes the same adjacent relationship tasks. Balanced hypercube's other properties have received extensive research [3, 4, 8, 13–15, 17–23].

Connectivity is a traditional way to measure a network's fault tolerance. The connectivity of G is $\kappa(G) = \min\{|N| \mid N \subset V(G)$ and $G - N$ is disconnected or trivial$\}$. For $N \subset V(G)$, the N-tree means that the tree connects each vertex of N. The n internally disjoint N-trees T_is mean that T_is are pairwise edge-disjoint but with intersecting vertex set N, where $1 \leq i \leq n$. The N-trees are important in information transportation in terms of parallel routing design for large-scale networks. The more applications of N-trees in computer communication networks are described in [5]. Let $\kappa(N) = \max\{l \mid T_1, T_2, \ldots, T_l$ are internally disjoint N-trees$\}$. The generalized k-connectivity of G is $\kappa_k(G) = \min\{\kappa(N) \mid N \subset V(G)$ and $|N| = k\}$ [24]. Note that it is equal to connectivity of G when k is 2 [24]. Generalized connectivity [2] uses internally disjoint trees to connect more vertices, which is more important in the application of multi-party computation or communication [17]. So it is a generalization method to determine the fault tolerance of distributed networks.

It is NP-complete to compute $\kappa_k(G)$ [6]. Just a few networks' generalized 4-connectivity were determined, including hypercube [10], hierarchical cubic networks [25], exchanged hypercubes [24], divide-and-swap cube [26], pancake graphs [27], (n, k)-star networks [9], crossed cubes [11], and folded hypercubes [12]. For the n-dimensional balanced hypercube BH_n, it was shown that $\kappa_3(BH_n) = 2n - 1$ when $n \geq 1$ [17]. In our paper, $\kappa_4(BH_n) = 2n - 1$ is further obtained, where $n \geq 1$.

This paper includes four sections. The preliminaries and main results are in the next two sections, respectively, and the conclusion is in last section.

2. Preliminaries

In a graph $G = (V(G), E(G))$, if $(u, v) \in E(G)$ is an edge, then u and v are each other's neighbors. The neighborhood of $u \in V(G)$ is $N_G(u) = \{v \mid (u, v) \in E(G), v \in V(G)\}$ and the degree of $u \in V(G)$ is $d_G(u) = |\{(u, v) \mid v \in V(G)\}|$. Denote $\delta(G)$ as the minimum of all $d_G(u)$ for $u \in V(G)$. Denote $P[x, y] = \langle x_0, x_1, x_2, \ldots, x_l \rangle$ as a path from x to y, where $x_0 = x$, $x_l = y$, x_is $(0 \leq i \leq l)$ are pairwise different, l is the path's length, and the path is l-path. If $x_l = x_0$ and $l \geq 3$, $P[x, y]$ becomes a cycle. For two distinct vertices a and c, the internally disjoint (a, c)-paths are vertex-disjoint paths except for the two common end vertices a and c. For a vertex a and a vertex set B such that $a \notin B$, the (a, B)-paths are vertex-disjoint paths connecting a and each vertex of B except for the only common end vertex a. For two vertex sets $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$, the paired (A, B)-paths are k vertex-disjoint paths $P[a_i, b_i]$s, where $1 \leq i \leq k$. The other terminology and notations not given here can be found in [1]. The BH_n has two methods to define. (Throughout this paper, among the labels of vertices of BH_n, the "$\pm$" and "$+$" are by modulo 4 operation. We omit "(mod) 4" for simplicity.)

Definition 1 [19]**.** $BH_n = (V(BH_n), E(BH_n))$, where $V(BH_n) = V_e \cup V_o$, where $V_e = \{(v_0, v_1, \ldots, v_{n-1}) \mid v_i \in \{0,1,2,3\}$ for $1 \leq i \leq n-1, v_0 \in \{0,2\}\}$ and $V_o = \{(v_0, v_1, \ldots, v_{n-1}) \mid v_i \in \{0,1,2,3\}$ for $1 \leq i \leq n-1, v_0 \in \{1,3\}\}$, and $E(BH_n) = E_0 \cup E_i$, where $E_0 = \{((v_0, v_1, \ldots v_{n-1}), (v_0 \pm 1, v_1, \ldots, v_{n-1}))\}$ and $E_i = \{((v_0, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{n-1}), (v_0 \pm 1, v_1, \ldots, v_{i-1}, v_i + (-1)^{v_0}, v_{i+1}, \ldots, v_{n-1})) \mid 1 \leq i \leq n-1\}$.

BH_n is a bipartite graph, and $|V(BH_n)| = 2^{2n}$. Let V_e be the set of white vertices and V_0 the set of black vertices. Let E_0 be the set of 0-dimensional edges and E_i the set of i-dimensional edges for $1 \leq i \leq n-1$. BH_1 is a 4-cycle, two drawing methods of BH_2 are depicted in Figure 1.

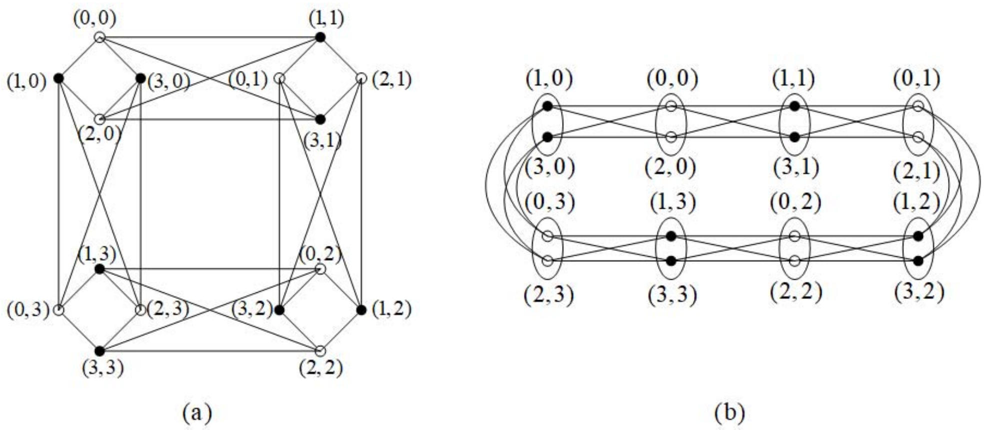

Figure 1. Two drawing methods of BH_2.

Another method to define BH_n is by a recursive definition.

Definition 2 [19]**.** The recursive definition of BH_n is as follows.

(1) BH_1 is a 4-cycle $\langle 0,1,2,3,0 \rangle$.

(2) For $n \geq 2$, $V(BH_n) = \bigcup_{i=0}^{3} V(BH_{n-1}^i)$, where $BH_{n-1}^i \cong BH_{n-1}$ for $i \in \{0,1,2,3\}$. Every vertex $(v_0, v_1, \ldots, v_{n-1}, i) \in V(BH_{n-1}^i)$ $(i \in \{0,1,2,3\})$ has two extra neighbors:

(2.1) $(v_0 \pm 1, v_1, \ldots, v_{n-2}, i+1) \in V(BH_{n-1}^{i+1})$ if v_0 is even.

(2.2) $(v_0 \pm 1, v_1, \ldots, v_{n-2}, i-1) \in V(BH_{n-1}^{i-1})$ if v_0 is odd.

The BH_{n-1}^is $(0 \le i \le 3)$ are called sub-balanced hypercubes. In BH_n, two vertices with labels $(v_0-1, v_1, \ldots, v_{n-1})$ and $(v_0+1, v_1, \ldots, v_{n-1})$ are called paired vertices. By Definition 2, each vertex of BH_{n-1}^i $(0 \le i \le 3)$ has two neighbors in BH_{n-1}^{i+1} or BH_{n-1}^{i-1}, and these two neighbors are paired vertices. Two edges $e = (r, s)$ and $e' = (r', s')$ are called paired edges if r and r' (respectively, s and s') are paired vertices. Two cycles $\langle r_1, r_2, \ldots, r_l \rangle$ and $\langle r'_1, r'_2, \ldots, r'_l \rangle$ are called paired cycles if r_i and r'_i are paired vertices, where $1 \le i \le l$. Imaging two paired vertices as one vertex, and the four edges between the two paired vertices as one edge, we have the following graph $\widetilde{BH_n}$ which is a contraction of BH_n.

Definition 3. Let $\widetilde{BH_n} = (V(\widetilde{BH_n}), E(\widetilde{BH_n}))$ be a contraction of BH_n, where $V(\widetilde{BH_n}) = \{V \mid V = \{v, v'\}, v \text{ and } v' \text{ are paired vertices of } BH_n\}$, and $E(\widetilde{BH_n}) = \{(U, V) \mid U = \{u, u'\}, V = \{v, v'\} \in V(\widetilde{BH_n}) \text{ such that } (u, v), (u, v'), (u', v), (u', v') \in E(BH_n)\}$. $\widetilde{BH_1}$ is an edge, denoted by (e, o). For $n \ge 2$, if $V \in V(\widetilde{BH_n})$ is a white vertex, it is denoted by $(e, v_1, v_2, \ldots, v_{n-1})$, otherwise it is denoted by $(o, v_1, v_2, \ldots, v_{n-1})$, where $e \in \{0, 2\}$, $o \in \{1, 3\}$, and $v_i \in \{0, 1, 2, 3\}$ for $1 \le i \le n-1$.

The graphs of $\widetilde{BH_2}$ and $\widetilde{BH_3}$ are shown in Figure 2.

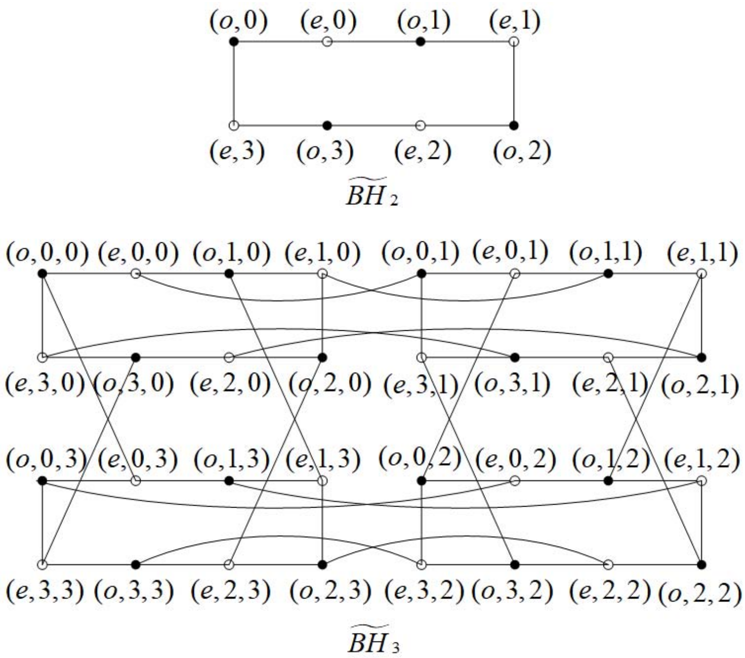

Figure 2. $\widetilde{BH_2}$ and $\widetilde{BH_3}$.

Lemma 4 [19]**.** *BH_n is $2n$-regular and $\kappa(BH_n) = 2n$, where $n \ge 1$.*

Lemma 5 [17]**.** *$\kappa_3(BH_n) = 2n - 1$, where $n \geq 1$.*

By Definition 3 and Lemma 4, we directly get the following lemma.

Lemma 6. *$\widetilde{BH_n}$ is n-regular, and $\kappa(\widetilde{BH_n}) = n$, where $n \geq 1$.*

Lemma 7 [19]**.** *BH_n is vertex-transitive, where $n \geq 1$.*

Lemma 8 [28]**.** *BH_n is edge-transitive, where $n \geq 1$.*

Lemma 9 [19]**.** *In BH_n, any two paired vertices have the same neighborhood.*

Lemma 10 [3]**.** *In BH_n, any edge (x, y) is included in $2n - 2$ 8-cycles C_8^js such that C_8^js are edge-disjoint except (x, y) and $\left|E(C_8^j) \cap E(BH_{n-1}^i)\right| = 1$, where $1 \leq j \leq 2n - 2$ and $i \in \{0, 1, 2, 3\}$.*

Since $\left|N_{BH_{n-1}^i}(u)\right| = 2n - 2$ for $u \in V(BH_{n-1}^i)$, where $i \in \{0, 1, 2, 3\}$, by Lemmas 7, 8 and 10, we directly get the following lemma.

Lemma 11. *In BH_n, any vertex u is contained in $2n - 2$ 8-cycles C_8^js such that C_8^js are edge-disjoint and $\left|E(C_8^j) \cap E(BH_{n-1}^i)\right| = 1$, where $1 \leq j \leq 2n - 2$ and $i \in \{0, 1, 2, 3\}$.*

Lemma 12 [7]**.** *If G includes (a, b) with $d_G(a) = d_G(b) = \delta(G)$, then $\kappa_k(G) \leq \delta(G) - 1$, where $3 \leq k \leq |V(G)|$.*

Lemma 13 [1]**.** *If $\kappa(G) = k$, for $a, b \in V(G)$, then G includes k internally disjoint paths between a and b.*

Lemma 14 [1]**.** *If $\kappa(G) = k$, for $a \in V(G)$ and $B \subset V(G) \setminus \{a\}$ with $|B| = k$, then G includes (a, B)-paths.*

Lemma 15 [1]**.** *If $\kappa(G) = k$, for $A \subset V(G)$, $B \subset V(G)$ with $|A| = |B| = k$ and $A \cap B = \emptyset$, then G includes paired (A, B)-paths.*

3. Main Results

Lemma 16. *Let $P = \{p, p'\}$ and $R = \{r, r'\}$ be any two vertices of $\widetilde{BH_n}$ with $(P, R) \notin E(\widetilde{BH_n})$. Then any path connecting P and R of $\widetilde{BH_n}$ is corresponding to two internally disjoint N-trees of BH_n, and two paired vertex-disjoint paths $P[p, r]$ and $P[p', r']$, and $P[p, r']$ and $P[p', r]$ of BH_n, where $N = \{p, p', r, r'\}$ and $n \geq 2$.*

Proof. Let $\langle P, Q_1, Q_2, \ldots, Q_l, R\rangle$ be any path in $\widetilde{BH_n}$, where $Q_i = \{q_i, q_i'\}$ for $1 \le i \le l$. Then q_i and q_i' are paired vertices for $1 \le i \le l$. Let $T_1 = \langle p, q_1, q_2, \ldots, q_l, r\rangle \cup (q_1, p') \cup (q_l, r')$ and $T_2 = \langle p', q_1', q_2', \ldots, q_l', r'\rangle \cup (p, q_1') \cup (q_l', r)$. Then T_1 and T_2 are two internally disjoint N-trees of BH_n, where $N = \{p, p', r, r'\}$. Clearly, $P[p, r] = \langle p, q_1, q_2, \ldots, q_l, r\rangle$ and $P[p', r'] = \langle p', q_1', q_2', \ldots, q_l', r'\rangle$ are two vertex-disjoint paths, and $P[p, r'] = [p, q_1', q_2', \ldots, q_l', r']$ and $P[p', r] = \langle p', q_1, q_2, \ldots, q_l, r\rangle$ are two vertex-disjoint paths. ■

Lemma 17. *In BH_n, any edge $e = (r, s)$ and its paired edge $e' = (r', s')$ are included in a 4-cycle $\langle r, s, r', s', r\rangle$, where $n \ge 2$.*

Proof. By Lemma 8, we only need to consider $e = (r, s)$, where $r = (r_0, r_1, \ldots, r_{n-1})$, and $s = (r_0 + 1, r_1, \ldots, r_{n-1})$. Let $e' = (r', s')$, where $r' = (r_0 + 2, r_1, \ldots, r_{n-1})$ and $s' = (r_0 + 3, r_1, \ldots, r_{n-1})$. Then $\langle r, s, r', s', r\rangle$ is a 4-cycle. ■

Lemma 18. *In BH_n with $n \ge 2$, any edge e and its paired edge e' are included in two paired 8-cycles, denoted by $R = \langle r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_0\rangle$ and $R' = \langle r_0', r_1', r_2', r_3', r_4', r_5', r_6', r_7', r_0'\rangle$, respectively, where $|E(R) \cap E(BH_{n-1}^i)| = |E(R') \cap E(BH_{n-1}^i)| = 1$ for $i \in \{0, 1, 2, 3\}$, and $\langle r_j, r_{j+1}, r_j', r_{j+1}', r_j\rangle$ is a 4-cycle for $0 \le j \le 7$ (The subscript "$j + 1$" is with operation modulo 8.).*

Proof. By Lemma 10, e is included in an 8-cycle R satisfying that $|E(R) \cap E(BH_{n-1}^i)| = 1$ for $i \in \{0, 1, 2, 3\}$. Let $E(R) \cap E(BH_{n-1}^i) = e_i$ for $i \in \{0, 1, 2, 3\}$, where $e_0 = (r_0, r_1), e_1 = (r_2, r_3), e_2 = (r_4, r_5), e_3 = (r_6, r_7)$. By Lemma 17, each edge (r_j, r_{j+1}) has a paired edge (r_j', r_{j+1}'), and they are included in a 4-cycle $\langle r_j, r_{j+1}, r_j', r_{j+1}', r_j\rangle$, where $j \in \{0, 2, 4, 6\}$. r_k and r_k' are paired vertices, where $0 \le k \le 7$. By Lemma 9, r_k and r_k' have the same neighborhood. So $\langle r_j, r_{j+1}, r_j', r_{j+1}', r_j\rangle$ is a 4-cycle, where $j \in \{1, 3, 5, 7\}$. Let $R' = \langle r_0', r_1', r_2', r_3', r_4', r_5', r_6', r_7', r_0'\rangle$. Then $e' \in E(R')$. R and R' are paired 8-cycles, (see Figure 3). Hence, the lemma is true. ■

Lemma 19. *In BH_n with $n \ge 2$, for any vertex a and vertex set B ($a \notin B$) with $|B| = 2n - 2$ of some BH_{n-1}^i ($i \in \{0, 1, 2, 3\}$), there exist (a, B)-paths in BH_{n-1}^i. Let $d \in V(BH_{n-1}^i)$ be such that $d \ne a$. Then BH_{n-1}^i includes at least one edge (d, d') such that (d, d') is not in (a, B)-paths.*

Proof. By Lemmas 4 and 14, there are (a, B)-paths in BH_{n-1}^i. By Definition 2, d has $2n - 2$ neighbors d_js in BH_{n-1}^i, where $1 \le j \le 2n - 2$. If all the (d, d_j)s are in (a, B)-paths, then the (a, B)-paths have two common vertices a and d, which is a contradiction. Hence, BH_{n-1}^i includes at least one edge (d, d') such that (d, d') is not in (a, B)-paths. ■

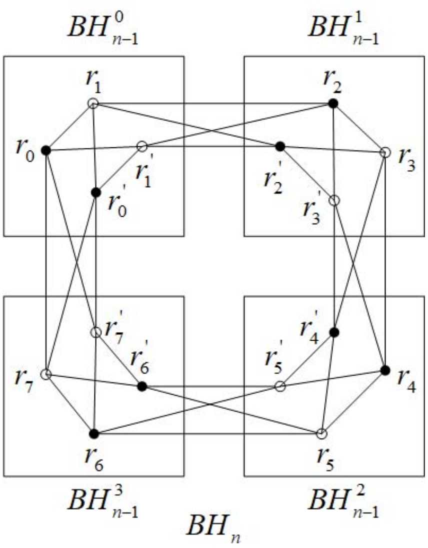

Figure 3. Two paired 8-cycles containing e and e', respectively.

Lemma 20. *In BH_n with $n \geq 2$, for any two different vertices a and c (a and c are not paired vertices) of some BH^i_{n-1} ($i \in \{0,1,2,3\}$), there exist $2n-2$ internally disjoint (a,c)-paths P_ks ($1 \leq k \leq 2n-2$) in BH^i_{n-1}. Let $d \in V(BH^i_{n-1})$ be such that $d \notin \{a,c\}$. Then BH^i_{n-1} includes at least one edge (d,d') such that (d,d') is not in $\bigcup_{k=1}^{2n-2} P_k$.*

Proof. By Lemmas 4 and 13, there are $2n-2$ internally disjoint (a,c)-paths P_ks ($1 \leq k \leq 2n-2$) in BH^i_{n-1}. By Definition 2, d has $2n-2$ neighbors d_js in BH^i_{n-1}, where $1 \leq j \leq 2n-2$. If all the (d,d_j)s are in $\bigcup_{k=1}^{2n-2} P_k$, then $\bigcup_{k=1}^{2n-2} P_k$ have three common vertices a, c, and d, which is a contradiction. Hence, BH^i_{n-1} includes at least one edge (d,d') such that (d,d') is not in $\bigcup_{k=1}^{2n-2} P_k$. ■

Lemma 21. *Let $N \subset V(BH_n)$ be such that $|N \cap V(BH_n)| = 4$ and N contains paired vertices. Then there are $2n-1$ internally disjoint N-trees in BH_n, where $n \geq 2$.*

Proof. Denote $N = \{p,q,r,s\}$. We discuss two cases.

Case 1. Two vertices of N are paired vertices, say q and p are paired vertices. By Lemma 5, BH_n includes $2n-1$ internally disjoint N'-trees T'_js ($1 \leq j \leq 2n-1$), where $N' = \{p,r,s\}$. Let p_j be the neighbor of p in T'_j for $1 \leq j \leq 2n-1$. Then $T_j = T'_j \cup (q,p_j)$ is N-tree and T_js are internally disjoint, where $1 \leq j \leq 2n-1$.

Case 2. Four vertices of N are two different paired vertices, say p,q are paired vertices and r,s are paired vertices.

Let $P = \{p, q\}$ and $R = \{r, s\}$. By Definition 3, Lemmas 6 and 13, $\widetilde{BH_n}$ includes n internally disjoint paths connecting P and R. By Lemma 16, BH_n includes $2n$ internally disjoint N-trees. ■

Lemma 22. $\kappa_4(BH_1) = 1$.

Proof. By Lemmas 4 and 12, $\kappa_4(BH_1) \leq 1$. By Definition 2, BH_1 is a 4-cycle, so BH_1 includes a path including its four vertices. Hence, the lemma holds. ■

Lemma 23. $\kappa_4(BH_2) = 3$.

Proof. The proof is in Appendix 1. ■

Lemma 24. *Let $N \subset V(BH_n)$ be such that $|N \cap V(BH_n)| = 4$ and N does not contain paired vertices. If each sub-balanced hypercube has one vertex of N, then there are $2n-1$ internally disjoint N-trees in BH_n, where $n \geq 3$.*

Proof. Without loss of generality, let $N = \{p, q, r, s\}$ and $p \in V(BH^0_{n-1})$, $q \in V(BH^3_{n-1})$, $r \in V(BH^2_{n-1})$, $s \in V(BH^1_{n-1})$. By Lemma 11, w is in an 8-cycle $\langle w^7, w^0, w^1, w^2, w^3, w^4, w^5, w^6, w^7\rangle$ for $w \in \{p, q, r, s\}$, where $(w^7, w^0) \in E(BH^0_{n-1})$, $(w^1, w^2) \in E(BH^1_{n-1})$, $(w^3, w^4) \in E(BH^2_{n-1})$, $(w^5, w^6) \in E(BH^3_{n-1})$. Since $|V(BH^i_{n-1})| = 2^{2(n-1)}$, we have $2^{2(n-1)-1}$ black vertices and $2^{2(n-1)-1}$ white vertices in BH^i_{n-1} for $i \in \{0, 1, 2, 3\}$ and $n \geq 3$. Not considering the vertices of 8-cycles that contain p, q, r, s respectively in BH^i_{n-1}, since $2^{2(n-1)-1} - 4 > 2n - 4$ for $n \geq 3$, by Lemma 11, we can pick another $2n-4$ vertex-disjoint 8-cycles $\langle x^7_i, x^0_i, x^1_i, x^2_i, x^3_i, x^4_i, x^5_i, x^6_i, x^7_i\rangle$ in BH_n, where $(x^7_i, x^0_i) \in E(BH^0_{n-1}), (x^1_i, x^2_i) \in E(BH^1_{n-1}), (x^3_i, x^4_i) \in E(BH^2_{n-1}), (x^5_i, x^6_i) \in E(BH^3_{n-1})$ for $1 \leq i \leq 2n-4$. We deal with four cases.

Case 1. Each vertex of $\{p, q, r, s\}$ is in different C_8s. Without loss of generality, let $p^7 = p$, $q^5 = q$, $r^3 = r$ and $s^1 = s$. Let $X^0 = \{q^0, r^0, s^0, x^0_1, x^0_2, \ldots, x^0_{2n-5}\}$, $X^1 = \{p^2, q^2, r^2, x^2_1, x^2_2, \ldots, x^2_{2n-5}\}$, $X^2 = \{p^4, q^4, s^4, x^4_1, x^4_2, \ldots, x^4_{2n-5}\}$ and $X^3 = \{p^6, r^6, s^6, x^6_1, x^6_2, \ldots, x^6_{2n-5}\}$. By Lemmas 4 and 14, BH^0_{n-1} includes (p, X^0)-paths $Q^0, R^0, S^0, X^0_1, X^0_2, \ldots, X^0_{2n-5}$, BH^1_{n-1} includes (s, X^1)-paths $P^1, Q^1, R^1, X^1_1, X^1_2, \ldots, X^1_{2n-5}$, BH^2_{n-1} includes (r, X^2)-paths $P^2, Q^2, S^2, X^2_1, X^2_2, \ldots, X^2_{2n-5}$, and BH^3_{n-1} includes (q, X^3)-paths $P^3, R^3, S^3, X^3_1, X^3_2, \ldots, X^3_{2n-5}$, where Q^0 connects p and q^0, R^0 connects p and r^0, S^0 connects p and s^0, X^0_i connects p and x^0_i, P^1 connects s and p^2, Q^1 connects s and q^2, R^1 connects s and r^2, X^1_i connects s and x^2_i, P^2 connects r and p^4, Q^2 connects r and q^4, S^2 connects r and s^4, X^2_i connects r and x^4_i, P^3 connects q and p^6, R^3 connects q and r^6, S^3 connects q and s^6, and X^3_i connects q and x^6_i, where $1 \leq i \leq 2n-5$. Let $T_i = X^0_i \cup X^1_i \cup X^2_i \cup X^3_i \cup \langle x^4_i, x^5_i, x^6_i, x^7_i, x^0_i, x^1_i, x^2_i\rangle$ for $1 \leq i \leq 2n-5$, $T_{2n-4} = S^0 \cup S^2 \cup S^3 \cup \langle s^4, s^5, s^6, s^7, s^0, s\rangle$, $T_{2n-3} = R^0 \cup R^1 \cup R^3 \cup \langle r^6, r^7, r^0, r^1, r^2, r\rangle$,

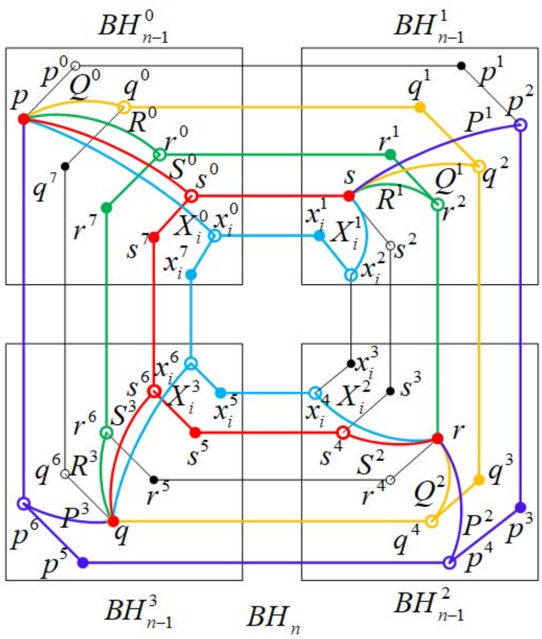

Figure 4. The illustration of Case 1 in the proof of Lemma 24.

$T_{2n-2} = Q^0 \cup Q^1 \cup Q^2 \cup \langle q^0, q^1, q^2, q^3, q^4, q \rangle$, and $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2 \rangle$, (see Figure 4).

Case 2. Two vertices of $\{p, q, r, s\}$ are in the same C_8.

Case 2.1. Two vertices of $\{p, q, r, s\}$, say q and r, are in two consecutive sub-balanced hypercubes.

Without loss of generality, let $p^7 = p$, $q^5 = q$, $q^3 = r$ and $s^1 = s$, i.e., q and r are in the same 8-cycle $C = \langle q^7, q^0, q^1, q^2, r, q^4, q, q^6, q^7 \rangle$. By Lemma 18, C has a paired 8-cycle C'. Denote $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7 \rangle$, where $(y^7, y^0) \in E(BH^0_{n-1})$, $(y^1, y^2) \in E(BH^1_{n-1})$, $(y^3, y^4) \in E(BH^2_{n-1})$, $(y^5, y^6) \in E(BH^3_{n-1})$, and $\langle q^7, y^0, y^7, q^0, q^7 \rangle$, $\langle q^0, y^1, y^0, q^1, q^0 \rangle$, $\langle q^1, y^2, y^1, q^2, q^1 \rangle$, $\langle q^2, y^3, y^2, r, q^2 \rangle$, $\langle r, y^4, y^3, q^4, r \rangle$, $\langle q^4, y^5, y^4, q, q^4 \rangle$, $\langle q, y^6, y^5, q^6, q \rangle$, $\langle q^6, y^7, y^6, q^7, q^6 \rangle$ are 4-cycles. In BH^0_{n-1}, BH^1_{n-1} and BH^3_{n-1}, the discussions are similar to Case 1 except that we need to use y^ks instead of r^ks for $k \in \{0, 1, 2, 5, 6, 7\}$ and use Y^js instead of R^js for $j \in \{0, 1, 3\}$ ($Y^3 = (q, y^6)$). In BH^2_{n-1}, let $X^2 = \{p^4, y^4, s^4, x^4_1, x^4_2, \ldots, x^4_{2n-5}\}$. By Lemmas 4 and 14, BH^2_{n-1} includes (r, X^2)-paths $P^2, Y^2, S^2, X^2_1, \ldots, X^2_{2n-5}$, where P^2 connects r and p^4, $Y^2 = (r, y^4)$, S^2 connects r and s^4, and X^2_i connects r and x^4_i for $1 \le i \le 2n-5$. Let $T_i = X^0_i \cup X^1_i \cup X^2_i \cup X^3_i \cup \langle x^2_i, x^1_i, x^0_i, x^7_i, x^6_i, x^5_i, x^4_i \rangle$ for $1 \le i \le 2n-5$, $T_{2n-4} = S^0 \cup S^2 \cup S^3 \cup \langle s, s^0, s^7, s^6, s^5, s^4 \rangle$, $T_{2n-3} = Y^0 \cup Y^1 \cup \langle r, y^4, y^3, y^2, y^1, y^0, y^7, y^6, q \rangle$, $T_{2n-2} = Q^0 \cup Q^1 \cup \langle r, y^2, q^1, q^0, q^7, q^6, y^5, y^4, q \rangle \cup (q^1, q^2)$, and $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2 \rangle$, (see Figure 5(a)).

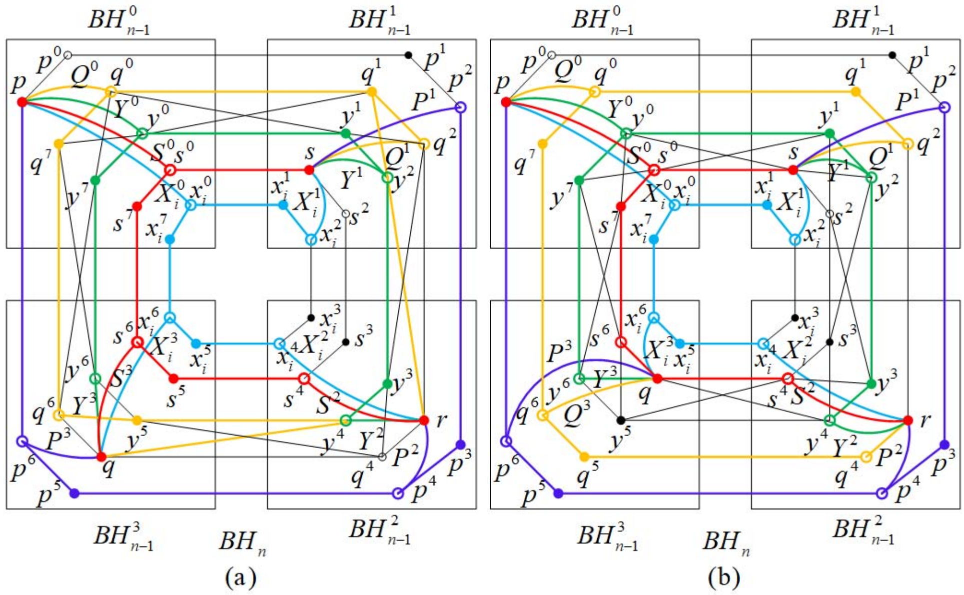

Figure 5. The illustration of Case 2.1 and Case 2.2 in the proof of Lemma 24.

Case 2.2. Two vertices of $\{p, q, r, s\}$, say q and s, are in two opposite sub-balanced hypercubes.

Without loss of generality, let $p^7 = p$, $s^5 = q$, $q^3 = r$ and $s^1 = s$, i.e., q and s are in the 8-cycle $C = \langle s^7, s^0, s, s^2, s^3, s^4, q, s^6, s^7\rangle$. By Lemma 18, C has a paired 8-cycle C'. Denote $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7\rangle$, where $(y^7, y^0) \in E(BH^0_{n-1})$, $(y^1, y^2) \in E(BH^1_{n-1})$, $(y^3, y^4) \in E(BH^2_{n-1})$, $(y^5, y^6) \in E(BH^3_{n-1})$, and $\langle y^7, s^0, s^7, y^0, y^7\rangle$, $\langle y^0, s, s^0, y^1, y^0\rangle$, $\langle y^1, s^2, s, y^2, y^1\rangle$, $\langle y^2, s^3, s^2, y^3, y^2\rangle$, $\langle y^3, s^4, s^3, y^4, y^3\rangle$, $\langle q, s^4, y^5, y^4, q\rangle$, $\langle q, y^6, y^5, s^6, q\rangle$, $\langle s^7, y^6, y^7, s^6, s^7\rangle$ are 4-cycles. In $BH^0_{n-1}, BH^1_{n-1}, BH^2_{n-1}$, the proofs of this case are similar to Case 2.1, except that Y^2 is now a path connecting r and y^4. Let $X^3 = \{p^6, q^6, y^6, x^6_1, x^6_2, \ldots, x^6_{2n-5}\}$. By Lemma 14, BH^3_{n-1} contains (q, X^3)-paths $P^3, Q^3, Y^3, X^3_1, X^3_2, \ldots, X^3_{2n-5}$, where P^3 connects q and p^6, Q^3 connects q and q^6, $Y^3 = (q, y^6)$, X^3_i connects q and x^6_i for $1 \le i \le 2n-5$. Let $T_i = X^0_i \cup X^1_i \cup X^2_i \cup X^3_i \cup \langle x^2_i, x^1_i, x^0_i, x^7_i, x^6_i, x^5_i, x^4_i\rangle$ for $1 \le i \le 2n-5$, $T_{2n-4} = S^0 \cup S^2 \cup \langle s, s^0, s^7, s^6, q, s^4\rangle$, $T_{2n-3} = Y^0 \cup Y^1 \cup Y^2 \cup \langle y^4, y^3, y^2, y^1, y^0, y^7, y^6, q\rangle$, $T_{2n-2} = Q^0 \cup Q^1 \cup Q^3 \cup \langle q^2, q^1, q^0, q^7, q^6, q^5, q^4, r\rangle$, $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2\rangle$, (see Figure 5(b)).

Case 3. Three vertices of $\{p, q, r, s\}$ are in the same C_8. Without loss of generality, let q, r, s be in the same 8-cycle C, i.e., $C = \langle q^7, s^0, s, r^2, r, q^4, q, q^6, q^7\rangle$. By Lemma 18, C has a paired 8-cycle $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7\rangle$, where $(y^7, y^0) \in E(BH^0_{n-1})$, $(y^1, y^2) \in E(BH^1_{n-1})$, $(y^3, y^4) \in E(BH^2_{n-1})$, $(y^5, y^6) \in E(BH^3_{n-1})$, and $\langle q^7, y^0, y^7, s^0, q^7\rangle$, $\langle s^0, y^1, y^0, s, s^0\rangle$, $\langle s, y^2, y^1, r^2, s\rangle$, $\langle r^2, y^3, y^2,$

$r, r^2\rangle$, $\langle q^4, y^3, y^4, r, q^4\rangle$, $\langle q, y^4, y^5, q^4, q\rangle$, $\langle q, y^6, y^5, q^6, q\rangle$, $\langle q^7, y^6, y^7, q^6, q^7\rangle$ are 4-cycles. Since $|V(BH^i_{n-1})| = 2^{2(n-1)}$ with $2^{2(n-1)-1}$ black vertices and $2^{2(n-1)-1}$ white vertices and $2^{2(n-1)-1} > 2n-1$ for $i \in \{0,1,2,3\}$ and $n \geq 3$, by Lemma 11, we can find another 8-cycle $\langle z^7, z^0, z^1, z^2, z^3, z^4, z^5, z^6, z^7\rangle$ in BH_n, where $(z^7, z^0) \in E(BH^0_{n-1})$, $(z^1, z^2) \in E(BH^1_{n-1})$, $(z^3, z^4) \in E(BH^2_{n-1})$, and $(z^5, z^6) \in E(BH^3_{n-1})$. In each BH^i_{n-1}s for $i \in \{0,2,3\}$, the discussions are similar to Case 2.1 except that we only need to use z^ks instead of s^ks for $k \in \{0,3,4,5,6,7\}$ and use Z^js instead of S^js for $j \in \{0,2,3\}$. In BH^1_{n-1}, let $X^1 = \{p^2, y^2, z^2, x^2_1, x^2_2, \ldots, x^2_{2n-5}\}$. By Lemmas 4 and 14, BH^1_{n-1} includes (s, X^1)-paths $P^1, Y^1, Z^1, X^1_1, \ldots, X^1_{2n-5}$, where P^1 connects s and p^2, $Y^1 = (s, y^2)$, Z^1 connects s and z^2, X^1_i connects s and x^2_i for $1 \leq i \leq 2n-5$. Let $T_i = X^0_i \cup X^1_i \cup X^2_i \cup X^3_i \cup \langle x^2_i, x^1_i, x^0_i, x^7_i, x^6_i, x^5_i, x^4_i\rangle$ for $1 \leq i \leq 2n-5$, $T_{2n-4} = Z^0 \cup Z^1 \cup Z^2 \cup Z^3 \cup \langle z^2, z^1, z^0, z^7, z^6, z^5, z^4\rangle$, $T_{2n-3} = Y^0 \cup Y^3 \cup \langle s, y^0, y^7, y^6, y^5, y^4, r\rangle$, $T_{2n-2} = S^0 \cup \langle s^0, s, y^2, r\rangle \cup \langle y^2, y^3, y^4, q\rangle$, and $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2\rangle$, (see Figure 6(a)).

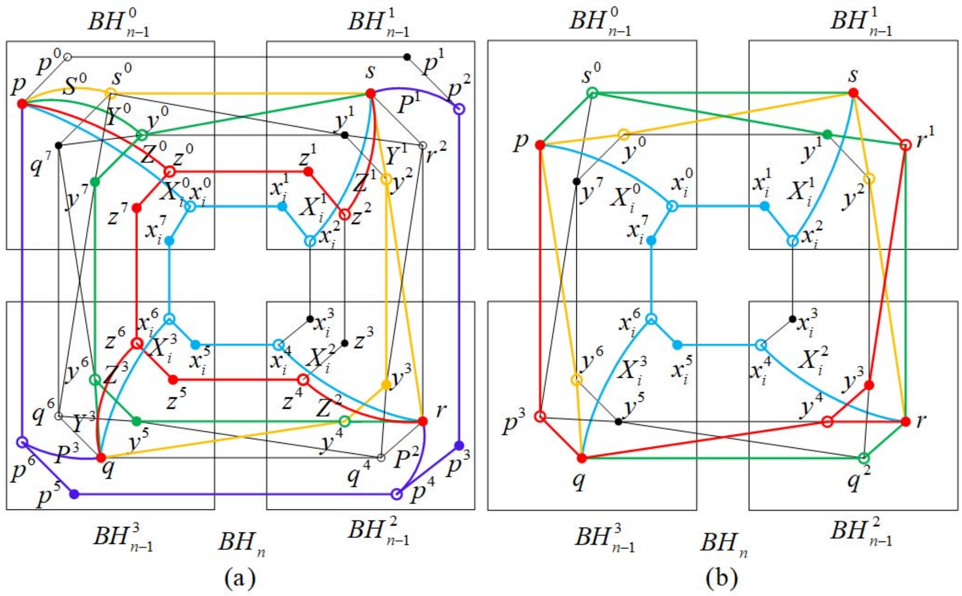

Figure 6. The illustration of Case 3 and Case 4 in the proof of Lemma 24.

Case 4. Four vertices of $\{p, q, r, s\}$ are in the same C_8. Denote $C = \langle p, s^0, s, r^1, r, q^2, q, p^3, p\rangle$. By Lemma 18, C has a paired cycle C'. Denote $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7\rangle$, where $(y^7, y^0) \in E(BH^0_{n-1})$, $(y^1, y^2) \in E(BH^1_{n-1})$, $(y^3, y^4) \in E(BH^2_{n-1})$, $(y^5, y^6) \in E(BH^3_{n-1})$, and $\langle p, y^0, y^7, s^0, p\rangle$, $\langle s^0, y^1, y^0, s, s^0\rangle$, $\langle s, y^2, y^1, r^1, s\rangle$, $\langle r^1, y^3, y^2, r, r^1\rangle$, $\langle r, y^4, y^3, q^2, r\rangle$, $\langle q, y^4, y^5, q^2, q\rangle$, $\langle p^3, y^5, y^6, q, p^3\rangle$, $\langle p, y^6, y^7, p^3, p\rangle$ are 4-cycles. Let $X^0 = \{s^0, y^0, x^0_1, x^0_2 \ldots, x^0_{2n-4}\}$, $X^1 = \{r^1, y^2, x^2_1,$

$x^2_2, \ldots, x^2_{2n-4}\}$, $X^2 = \{q^2, y^4, x^4_1, x^4_2, \ldots, x^4_{2n-4}\}$ and $X^3 = \{p^3, y^6, x^6_1, x^6_2, \ldots, x^6_{2n-4}\}$. By Lemmas 4 and 14, BH^0_{n-1} includes (p, X^0)-paths $S^0, Y^0, X^0_1, X^0_2, \ldots, X^0_{2n-4}$, BH^1_{n-1} includes (s, X^1)-paths $R^1, Y^1, X^1_1, X^1_2, \ldots, X^1_{2n-4}$, BH^2_{n-1} includes $Q^2, Y^2, X^2_1, X^2_2, \ldots, X^2_{2n-4}$, and BH^3_{n-1} includes (q, X^3)-paths $P^3, Y^3, X^3_1, X^3_2, \ldots, X^3_{2n-4}$, where $S^0 = (p, s^0)$, $Y^0 = (p, y^0)$, X^0_i connects p and x^0_i, $R^1 = (s, r^1)$, $Y^1 = (s, y^2)$, X^1_i connects s and x^2_i, $Q^2 = (r, q^2)$, $Y^2 = (r, y^4)$, X^2_i connects r and x^4_i, $P^3 = (q, p^3)$, $Y^3 = (q, y^6)$, X^3_i connects q and x^6_i, where $1 \leq i \leq 2n-4$. Let $T_i = X^0_i \cup X^1_i \cup X^2_i \cup X^3_i \cup \langle x^2_i, x^1_i, x^0_i, x^7_i, x^6_i, x^5_i, x^4_i \rangle$ for $1 \leq i \leq 2n-4$, $T_{2n-3} = \langle p, s^0, y^1, r^1, r, q^2, q \rangle \cup (s^0, s)$, $T_{2n-2} = \langle q, y^6, p, y^0, s, y^2, r \rangle$, and $T_{2n-1} = \langle p, p^3, q, y^4, y^3, r^1, s \rangle \cup (y^4, r)$, (see Figure 6(b)). ∎

Lemma 25. *Let $N \subset V(BH_n)$ be such that $|N \cap V(BH_n)| = 4$ and N does not contain paired vertices. If there exist two sub-balanced hypercubes such that each has two vertices of N, then there are $2n-1$ internally disjoint N-trees in BH_n, where $n \geq 3$.*

Proof. Denote $N = \{p, q, r, s\}$. Without loss of generality, let $N \cap V(BH^0_{n-1}) = \{p, q\}$. By symmetry of BH^1_{n-1} and BH^3_{n-1}, we only need to consider that both r and s are in BH^1_{n-1} or BH^2_{n-1}.

Let p and q be different color vertices, and r and s be different colors. Without loss of generality, let p and r be black vertices and q and s be white vertices. If p and q are the same color (since BH_n is a bipartite graph), we only need to consider that p and q are black vertices (see Figure 12–14), or p and q are different colors but r and s are the same color (see Figure 15–16), the proofs are similar. To save space, we only show the graphs in the Appendix 2.) By Lemmas 4 and 13, BH^0_{n-1} includes $2n-2$ internally disjoint paths P_is connecting p and q, where $1 \leq i \leq 2n-2$. By Definition 2, p has a neighbor $p^3 \in V(BH^3_{n-1})$, and q has a neighbor $q^1 \in V(BH^1_{n-1})$. By Lemma 19 and Definition 2, BH_n includes a path $\tilde{Q} = \langle q, q^0, q^2, q^3, q^4 \rangle$, where $q^0 \in V(BH^1_{n-1})$, $\{q^2, q^3\} \subset V(BH^0_{n-1})$ and $q^4 \in V(BH^3_{n-1})$.

Case 1. $\{r, s\} \subset V(BH^1_{n-1})$. By Lemmas 4 and 13, BH^1_{n-1} includes $2n-2$ internally disjoint paths R_is connecting r and s, where $1 \leq i \leq 2n-2$. (If $(p, q) \in E(BH^0_{n-1})$, let $P_{2n-3} = (p, q)$. If $(r, s) \in E(BH^1_{n-1})$, let $R_{2n-2} = (r, s)$.) By Definition 2, r has a neighbor $r^7 \in V(BH^0_{n-1})$, and s has a neighbor $s^2 \in V(BH^2_{n-1})$. By Lemma 19 and Definition 2, BH_n includes a path $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$, where $r^0 \in V(BH^0_{n-1})$, $\{r^1, r^2\} \subset V(BH^1_{n-1})$, and $r^3 \in V(BH^2_{n-1})$. Since BH^0_{n-1} is connected, BH^0_{n-1} includes a path R^1 connecting r^7 and q. Let v be the first intersection vertex of R^1 and $\bigcup_{i=1}^{2n-2} P_i$. Since BH^1_{n-1} is connected, BH^1_{n-1} includes a path Q^1 connecting q^1 and r. Let u be the first intersection vertex of Q^1 and $\bigcup_{i=1}^{2n-2} R_i$. Suppose that $v \in V(P_{2n-2})$ and $u \in V(R_{2n-3})$. Let $R[r^7, v]$ be the sub-path of R^1 and $Q[q^1, u]$ be the sub-path of Q^1. Pick black vertex

$x_j^0 \in V(P_j)$ and white vertex $y_j^1 \in V(R_j)$ for $1 \leq j \leq 2n-4$. By Definition 2, x_j^0 has a neighbor $x_j^3 \in V(BH_{n-1}^3)$ and y_j^1 has a neighbor $y_j^2 \in V(BH_{n-1}^2)$ for $1 \leq j \leq 2n-4$.

By Lemma 17, BH_n includes a 4-cycle $\langle b^3, b^2, c^3, c^2, b^3 \rangle$, where $\{b^2, c^2\} \subset V(BH_{n-1}^2)$ and $\{b^3, c^3\} \subset V(BH_{n-1}^3)$. Pick another $2n-4$ (a_j^3, a_j^2)s, where $a_j^3 \in V(BH_{n-1}^3)$ and $a_j^2 \in V(BH_{n-1}^2)$ for $1 \leq j \leq 2n-4$. Let $X^3 = \{p^3, q^4, x_1^3, x_2^3, \ldots, x_{2n-4}^3\}$, $A^3 = \{b^3, c^3, a_1^3, a_2^3, \ldots, a_{2n-4}^3\}$, $Y^2 = \{s^2, r^3, y_1^2, y_2^2, \ldots, y_{2n-4}^2\}$, and $A^2 = \{b^2, c^2, a_1^2, a_2^2, \ldots, a_{2n-4}^2\}$. By Lemmas 4 and 15, BH_{n-1}^3 includes (X^3, A^3)-paths $P, Q, Q_1, Q_2, \ldots, Q_{2n-4}$ and BH_{n-1}^2 includes (A^2, Y^2)-paths $R, S, S_1, S_2, \ldots, S_{2n-4}$, where P connects p^3 and b^3, Q connects q^4 and c^3, Q_j connects x_j^3 and a_j^3, R connects b^2 and s^2, S connects c^2 and r^3, and S_j connects a_j^2 and y_j^2, $1 \leq j \leq 2n-4$. Let $T_j = P_j \cup (x_j^0, x_j^3) \cup Q_j \cup (a_j^3, a_j^2) \cup S_j \cup (y_j^2, y_j^1) \cup R_j$ for $1 \leq j \leq 2n-4$, $T_{2n-3} = P_{2n-3} \cup (q, q^1) \cup Q[q^1, u] \cup R_{2n-3}$, $T_{2n-2} = P_{2n-2} \cup R[v, r^7] \cup (r^7, r) \cup R_{2n-2}$, and $T_{2n-1} = P \cup Q \cup R \cup S \cup \tilde{Q} \cup (p, p^3) \cup \langle b^3, b^2, c^3, c^2 \rangle \cup \tilde{R} \cup (s, s^2)$, (see Figure 7(a)).

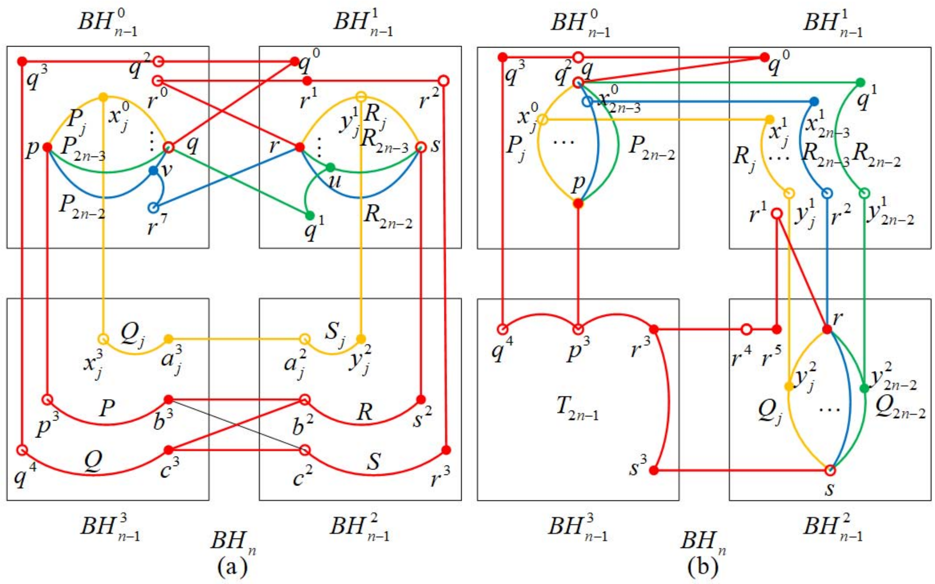

Figure 7. The illustrations of Case 1 and Case 2 in the proof of Lemma 25.

Case 2. $\{r, s\} \subset V(BH_{n-1}^2)$. By Lemmas 4 and 13, BH_{n-1}^2 includes $2n-2$ internally disjoint paths Q_is connecting r and s, where $1 \leq i \leq 2n-2$. (If $(p, q) \in E(BH_{n-1}^0)$, let $P_{2n-2} = (p, q)$. If $(r, s) \in E(BH_{n-1}^2)$, let $Q_{2n-3} = (r, s)$.) By Lemma 19 and Definition 2, BH_n includes a path $\tilde{R} = \langle r, r^1, r^5, r^4, r^3 \rangle$, an edge (r, r^2), and an edge (s, s^3), where $\{r^1, r^2\} \subset V(BH_{n-1}^1)$, $\{r^4, r^5\} \subset V(BH_{n-1}^2)$, and $\{r^3, s^3\} \subset V(BH_{n-1}^3)$. Select white vertex $x_j^0 \in V(P_j)$ (respectively, black vertex $y_k^2 \in V(Q_k)$), by Definition 2, x_j^0 (respectively, y_k^2) has a neighbor $x_j^1 \in V(BH_{n-1}^1)$ (respectively, $y_k^1 \in V(BH_{n-1}^1)$), where $1 \le j \le 2n-3$ (respectively, $1 \le k \le 2n-2$). Let $(y_{2n-3}^2, y_{2n-3}^1) = (r, r^2)$. Let $X^1 = \{x_1^1, x_2^1, \ldots, x_{2n-4}^1, x_{2n-3}^1, q^1\}$ and $Y^1 = \{y_1^1, y_2^1, \ldots, y_{2n-4}^1, r^2, y_{2n-2}^1\}$. By Lemmas 4 and 15, BH_{n-1}^1 includes paired (X^1, Y^1)-paths $R_1, R_2, \ldots, R_{2n-2}$, where R_j connects x_j^1 and y_j^1 for $1 \le j \le 2n-4$, R_{2n-3} connects x_{2n-3}^1 and r^2, and R_{2n-2} connects q^1 and y_{2n-2}^1. Since BH_{n-1}^3 is connected, BH_{n-1}^3 includes a tree T'_{2n-1} connecting q^4, p^3, r^3 and s^3. Let $T_i = P_j \cup (x_j^0, x_j^1) \cup R_j \cup (y_j^1, y_j^2) \cup Q_j$ for $1 \le j \le 2n-4$, $T_{2n-3} = P_{2n-3} \cup (x_{2n-3}^0, x_{2n-3}^1) \cup R_{2n-3} \cup (r^2, r) \cup Q_{2n-3}$, $T_{2n-2} = P_{2n-2} \cup (q, q^1) \cup R_{2n-2} \cup (y_{2n-2}^1, y_{2n-2}^2) \cup Q_{2n-2}$, and $T_{2n-1} = T'_{2n-1} \cup \tilde{Q} \cup (p^3, p) \cup \tilde{R} \cup (s^3, s)$, (see Figure 7(b)). ■

Lemma 26. *Let $N \subset V(BH_n)$ be such that $|N \cap V(BH_n)| = 4$ and N does not contain paired vertices. If there exist three sub-balanced hypercubes having 2, 1, and 1 vertices of N, respectively, then there are $2n-1$ internally disjoint N-trees in BH_n, where $n \ge 3$.*

Proof. The proof is in Appendix 3. ■

Lemma 27. *Let $N \subset V(BH_n)$ be such that $|N \cap V(BH_n)| = 4$ and N does not contain paired vertices. If there exist two sub-balanced hypercubes having 3 and 1 vertices of N, respectively, then there are $2n-1$ internally disjoint N-trees in BH_n, where $n \ge 3$.*

Proof. Denote $N = \{p, q, r, s\}$. Without loss of generality, let $\{p, q, r\} \subset V(BH_{n-1}^0)$, p, r be black vertices, and q be white vertex. (If p, r are white vertices and q is black vertex, or p, q, r are the same color, the discussions are similar except that we need to use Lemma 19 and Definition 2 to find two paths connecting p or q or r such that the other end vertices of the two paths connecting p or r are in BH_{n-1}^3 and the other end vertices of the two paths connecting q are in BH_{n-1}^2 for Case 1, and find two neighbors of p or r in BH_{n-1}^1 and find two paths connecting with q and the other end vertices in BH_{n-1}^1 for Case 2.) By Definition 2, p has two neighbors p^3, p^4 in BH_{n-1}^3, r has two neighbors r^3, r^4 in BH_{n-1}^3, and q has two neighbors q^1, q^2 in BH_{n-1}^1. By Lemmas 4 and 5, BH_{n-1}^0 includes $2n-3$ internally disjoint N-trees T'_js, where $1 \le j \le 2n-3$. Since BH_n is symmetric, we deal with the following Case 1 and Case 2.

Case 1. $s \in V(BH_{n-1}^1)$. Pick one white vertex $x_j^0 \in V(T'_j)$ for $1 \le j \le 2n-3$. By Definition 2, x_j^0 has a neighbor $x_j^1 \in V(BH_{n-1}^1)$, where $1 \le j \le 2n-3$. Let $X^1 = \{x_1^1, x_2^1, \ldots, x_{2n-3}^1\}$. By Lemmas 4 and 14, BH_{n-1}^1 includes (s, X^1)-paths R_js, where R_j connects s and x_j^1 for $1 \le j \le 2n-3$. Let $T_j = T'_j \cup (x_j^0, x_j^1) \cup R_j$ for $1 \le j \le 2n-3$. By Lemma 19 and Definition 2, BH_n includes two paths $\tilde{Q}_1 = \langle q, q^1, q^3, q^5 \rangle$ and $\tilde{Q}_2 = \langle q, q^2, q^4, q^6 \rangle$, where $\{q^1, q^2, q^3, q^4\} \subset V(BH_{n-1}^1)$ and

$\{q^5, q^6\} \subset V(BH^2_{n-1})$. By Lemma 17 and Definition 2, BH_n includes two vertex-disjoint 4-cycles $\langle c^3, c^2, d^3, d^2, c^3\rangle$ and $\langle a^3, a^2, b^3, b^2, a^3\rangle$, where $\{a^3, b^3, c^3, d^3\} \subset V(BH^3_{n-1})$ and $\{a^2, b^2, c^2, d^2\} \subset V(BH^2_{n-1})$.

Case 1.1. s is white vertex. By Definition 2, s has two neighbors s^1 and s^2 in BH^2_{n-1}. By Lemma 15, BH^3_{n-1} includes vertex-disjoint paths P, Q, P', Q' connecting r^3, p^4, r^4, p^3 and c^3, d^3, a^3, b^3, respectively, and BH^2_{n-1} includes vertex-disjoint paths R, S, R', S' connecting q^6, s^1, s^2, q^5 and c^2, d^2, a^2, b^2, respectively. Let $T_{2n-2} = P \cup Q \cup R \cup S \cup (p, p^4) \cup (r, r^3) \cup (s, s^1) \cup \tilde{Q}_2 \cup \langle c^3, c^2, d^3, d^2\rangle$, and $T_{2n-1} = P' \cup Q' \cup R' \cup S' \cup (p, p^3) \cup (r, r^4) \cup (s, s^2) \cup \tilde{Q}_1 \cup \langle a^3, a^2, b^3, b^2\rangle$, (see Figure 8(a)).

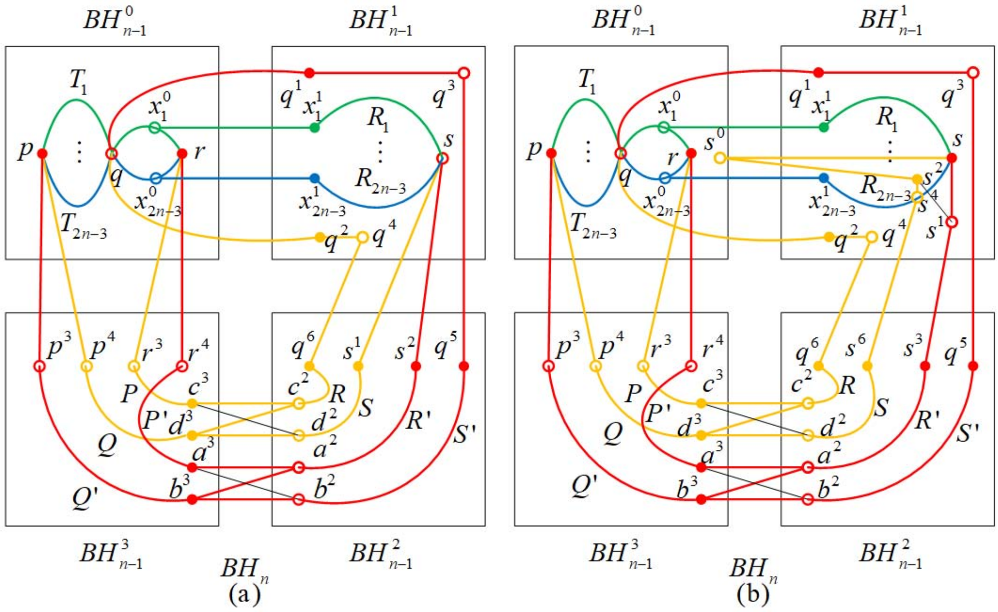

Figure 8. The illustrations of Case 1.1 and Case 1.2 in the proof of Lemma 27.

Case 1.2. s is black vertex. The proof of this case is similar to Case 1.1 except that we need to use $\tilde{S}_1 = \langle s, s^0, s^2, s^4, s^6\rangle$ and $\tilde{S}_2 = \langle s, s^1, s^3\rangle$ to replace (s, s^1) and (s, s^2) of Case 1.1, respectively, where $s^0 \in V(BH^0_{n-1})$, s^2 is a paired vertex of s in BH^1_{n-1}, s^1 is not in (s, X^1)-paths, s^4 is in some path of (s, X^1)-paths, $(s^2, s^1) \in E(BH^1_{n-1})$, $\{s^6, s^3\} \subset V(BH^2_{n-1})$. (Since $|N_{BH^1_{n-1}}(s)| = 2n - 2$, there exists one neighbor s^1 not in (s, X^1)-paths.) Let $T_{2n-2} = P \cup Q \cup R \cup S \cup (p, p^4) \cup (r, r^3) \cup \langle c^3, c^2, d^3, d^2\rangle \cup \tilde{S}_1 \cup \tilde{Q}_2$ and $T_{2n-1} = P' \cup Q' \cup R' \cup S' \cup (p, p^3) \cup (r, r^4) \cup \langle a^3, a^2, b^3, b^2\rangle \cup \tilde{S}_2 \cup \tilde{Q}_1$, (see Figure 8(b)).

Case 2. $s \in V(BH^2_{n-1})$. Select $2n - 3$ edges (x^0_j, x^3_j)s, where $x^0_j \in V(T'_j)$ and

$x_j^3 \in V(BH_{n-1}^3)$ for $1 \le j \le 2n-3$. Pick p and r's paired vertices and denote by $\tilde{p}$ and $\tilde{r}$, respectively. Since p (respectively, r) has $2n-2$ neighbors in BH_{n-1}^0, there exists one neighbor p^1 (respectively, r^1) in BH_{n-1}^0 such that (p, p^1) (respectively, (r, r^1)) is not in T_js for $1 \le j \le 2n-3$. By Lemma 18, (p, p^1) and its paired edge $(\tilde{p}, \tilde{p}^1)$ are in two paired cycles C and C', respectively. Let $\langle p, p^1, p^2 \rangle$ (respectively, $\langle \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle$) be part of C (respectively, C'), where $\{p^2, \tilde{p}^2\} \subset V(BH_{n-1}^1)$. The discussion for vertex r is similar, thus we have that $\langle r, r^1, r^2 \rangle$ and $\langle \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle$ are parts of two paired cycles, respectively, where $\{r^2, \tilde{r}^2\} \subset V(BH_{n-1}^1)$. Let $\widetilde{Q} = \{q^1, q^2\}$, $\widetilde{P} = \{p^2, \tilde{p}^2\}$, $\widetilde{R} = \{r^2, \tilde{r}^2\}$.

Case 2.1. s is black vertex. By Definition 2, s has two neighbors s^1 and s^2 in BH_{n-1}^1. Let $\widetilde{S} = \{s^1, s^2\}$. By Definition 3 and Lemmas 6 and 14, $\widetilde{BH_{n-1}^1}$ has $(\widetilde{S}, \{\widetilde{P}, \widetilde{Q}, \widetilde{R}\})$-paths. By Lemma 16, BH_{n-1}^1 includes vertex-disjoint paths $Q[q^1, s^1]$, $Q[q^2, s^2]$, $P[\tilde{p}^2, s^1]$, $P[p^2, s^2]$, $R[r^2, s^1]$, $R[\tilde{r}^2, s^2]$. Select $2n-3$ neighbors of s in BH_{n-1}^2 and denote by s_j^2s for $1 \le j \le 2n-3$. By Definition 2, let s_j^3 be a neighbor of s_j^2 in BH_{n-1}^3, where $1 \le j \le 2n-3$. Let $X^3 = \{x_1^3, x_2^3, \ldots, x_{2n-3}^3\}$ and $S^3 = \{s_1^3, s_2^3, \ldots, s_{2n-3}^3\}$. By Lemmas 4 and 15, there are paired (X^3, S^3)-paths Q_js in BH_{n-1}^3, where Q_j connects x_j^3 and s_j^3 for $1 \le j \le 2n-3$. Let $T_j = T_j' \cup (x_j^0, x_j^3) \cup Q_j \cup \langle s_j^3, s_j^2, s \rangle$ for $1 \le j \le 2n-3$, $T_{2n-2} = (q, q^1) \cup Q[q^1, s^1] \cup \langle p, p^3, \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle \cup P[\tilde{p}^2, s^1] \cup \langle r, r^1, r^2 \rangle \cup R[r^2, s^1] \cup (s^1, s)$ and $T_{2n-1} = (q, q^2) \cup Q[q^2, s^2] \cup \langle p, p^1, p^2 \rangle \cup P[p^2, s^2] \cup \langle r, r^3, \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle \cup R[\tilde{r}^2, s^2] \cup (s^2, s)$, (see Figure 9(a)).

Case 2.2. s is white vertex. The proof of this case is similar to Case 2.1 except that we need to use $\tilde{S}_j = \langle s, s_j, s_j^2, s_j^3 \rangle$ to instead of $\langle s, s_j^2, s_j^3 \rangle$ of Case 2.1 for $1 \le j \le 2n-3$, and use $S_1 = \langle s, s^3, \tilde{s}, s_1, s^1 \rangle$ and $S_2 = \langle s, \tilde{s}^3, \tilde{s}, \tilde{s}^2, \tilde{s}^1 \rangle$ to instead of (s, s^1) and (s, s^2) of Case 2.1, respectively, where $\tilde{s}$ is the paired vertex of s, $\tilde{s}^3$ and s^3 are common neighbors of s and $\tilde{s}$ in BH_{n-1}^3, $\tilde{s}^2 \notin \{s_1, s_2, \ldots, s_{2n-3}\}$, and s^1 and $\tilde{s}^1$ are common neighbors of $\tilde{s}^2$ and s_1 and they are paired vertices. (Since $\tilde{s}$ has $2n-2$ neighbors in BH_{n-1}^2, there exists such vertex $\tilde{s}^2$. Without loss of generality, let s_1 be the paired vertex of $\tilde{s}^2$.) So $(\tilde{s}, s_j) \in E(BH_{n-1}^2)$ for $1 \le j \le 2n-3$. Let $T_j = T_j' \cup (x_j^0, x_j^3) \cup Q_j \cup \tilde{S}_j$ for $1 \le j \le 2n-3$, $T_{2n-2} = (q, q^1) \cup Q[q^1, s^1] \cup \langle p, p^3, \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle \cup P[\tilde{p}^2, s^1] \cup \langle r, r^1, r^2 \rangle \cup R[r^2, s^1] \cup S_1$ and $T_{2n-1} = (q, q^2) \cup Q[q^2, \tilde{s}^1] \cup \langle p, p^1, p^2 \rangle \cup P[p^2, \tilde{s}^1] \cup \langle r, r^3, \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle \cup R[\tilde{r}^2, \tilde{s}^1] \cup S_2$, (see Figure 9(b)). ■

Theorem 28. $\kappa_4(BH_n) = 2n-1$, *where* $n \ge 1$.

Proof. The proof is by induction hypothesis on n. By Lemmas 22 and 23, the theorem holds when $n \le 2$. Assume that the theorem holds for $m \le n-1$. We prove that the theorem holds for $m = n \ge 3$ as follows. For any $N \subset V(BH_n)$ with $|N| = 4$, we denote $N = \{p, q, r, s\}$. By Lemmas 4 and 12, $\kappa_4(BH_n) \le 2n-1$.

We need to show that BH_n includes $2n-1$ internally disjoint N-trees. By Lemma 21, BH_n includes $2n-1$ internally disjoint N-trees if N contains paired vertices. In the following, we consider that N contains no paired vertices for $n \geq 3$.

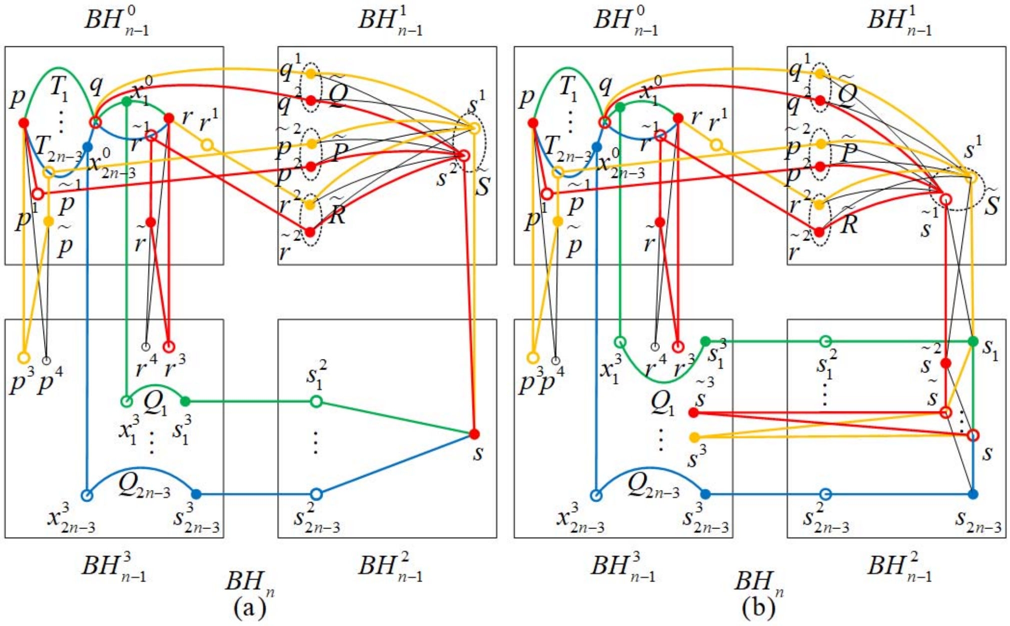

Figure 9. The illustrations of Case 2.1 and Case 2.2 in the proof of Lemma 27.

Case 1. All the vertices of N are in the same sub-balanced hypercube. By symmetry of BH_n, let $N \subset V(BH^0_{n-1})$. By induction hypothesis, BH^0_{n-1} contains $2n-3$ internally disjoint N-trees T_js for $1 \leq j \leq 2n-3$. By Definition 1, each vertex of p, q, r, s has paired neighbors p^1, p^2, q^1, q^2, r^1, r^2, s^1, s^2 in BH^1_{n-1} or BH^3_{n-1}, respectively. By Lemma 18, each paired neighbors of N is included in vertex-disjoint 8-cycles of BH_n. Let $P[p^1, p']$ and $P[p^2, p'']$, $Q[q^1, q']$ and $Q[q^2, q'']$, $R[r^1, r']$ and $R[r^2, r'']$, and $S[s^1, s']$ and $S[s^2, s'']$ be sub-paths of the two disjoint 8-cycles, respectively, where $N' = \{p', p'', q', q'', r', r'', s', s''\} \subset V(BH^2_{n-1})$. Select one vertex $v \in V(BH^2_{n-1})$ and $v \notin N'$. By Lemmas 4 and 14, BH^2_{n-1} includes (v, N')-paths $P^1, P^2, Q^1, Q^2, R^1, R^2, S^1, S^2$, where X^1 connects x' and v and X^2 connects x'' and v for $X = P, Q, R, S$ and $x = p, q, r, s$, respectively. Let $T_{2n-2} = P^1 \cup Q^1 \cup R^1 \cup S^1 \cup P[p^1, p'] \cup Q[q^1, q'] \cup R[r^1, r'] \cup S[s^1, s'] \cup (p, p^1) \cup (q, q^1) \cup (r, r^1) \cup (s, s^1)$ and $T_{2n-1} = P^2 \cup Q^2 \cup R^2 \cup S^2 \cup P[p^2, p''] \cup Q[q^2, q''] \cup R[r^2, r''] \cup S[s^2, s''] \cup (p, p^2) \cup (q, q^2) \cup (r, r^2) \cup (s, s^2)$.

Case 2. Each sub-balanced hypercube has one vertex of N.

Case 3. Two sub-balanced hypercubes have two vertices of N, respectively.

Case 4. Three sub-balanced hypercubes have 2, 1, and 1 vertices of N, respectively.

Case 5. Two sub-balanced hypercubes have 3 and 1 vertices of N, respectively.

By Lemmas 24, 25, 26, and 27, BH_n includes $2n-1$ internally disjoint N-trees for the above Cases 2–5, respectively.

Hence, the theorem holds. ■

4. Conclusion

In [17], $\kappa_3(BH_n) = 2n-1$ is determined, in this paper, we further obtain that $k_4(BH_n) = 2n-1$, where $n \geq 1$. Since it is NP-complete to compute $\kappa_k(G)$ when G is general [6], the method of our paper can be a reference to determine the generalized 4-connectivity of other special networks.

Acknowledgments

This work was supported by China Scholarship Council (CSC NO. 202006785015), and was completed during the period of the author visiting Nanyang Technological University with financial support under this grant. This work was also supported by Guangdong Basic and Applied Basic Research Foundation (2023A1515011049).

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, in: Grad. Texts in Math. **244** (Springer, New York, 2008).

[2] G. Chartrand, S.F. Kapoor, L. Lesniak and D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. **2** (1984) 1–6.

[3] D. Cheng, R.-X. Hao and Y.-Q. Feng, *Two node-disjoint paths in balanced hypercubes*, Appl. Math. Comput. **242** (2014) 127–142. https://doi.org/10.1016/j.amc.2014.05.037

[4] D. Cheng, *The h-restricted connectivity of balanced hypercubes*, Discrete Appl. Math. **305** (2021) 133–141. https://doi.org/10.1016/j.dam.2021.08.036

[5] D. Du and X. Hu, Steiner Tree Problem in Computer Communication Networks (World Scientific, Singapore, 2008). https://doi.org/10.1142/6729

[6] S. Li, X. Li and W. Zhou, *Sharp bounds for the generalized connectivity $\kappa_3(G)$*, Discrete Math. **310** (2010) 2147–2163. https://doi.org/10.1016/j.disc.2010.04.011

[7] S. Li, Some Topics on Generalized Connectivity of Graphs, Ph.D. Thesis (Nankai University, 2012).

[8] L. Li, X. Zhang, Q. Zhu and Y. Bai, *The* 3*-extra conditional diagnosability of balanced hypercubes under MM* model*, Discrete Appl. Math. **309** (2022) 310–316. https://doi.org/10.1016/j.dam.2021.04.004

[9] C. Li, S. Lin and S. Li, *The* 4*-set tree connectivity of* (n, k)*-star networks*, Theoret. Comput. Sci. **844** (2020) 81–86. https://doi.org/10.1016/j.tcs.2020.08.004

[10] S. Lin and Q. Zhang, *The generalized* 4*-connectivity of hypercubes*, Discrete Appl. Math. **220** (2017) 60–67. https://doi.org/10.1016/j.dam.2016.12.003

[11] H. Liu and D. Cheng, *The generalized* 3*-connectivity and* 4*-connectivity of crossed cube*, Discuss. Math. Graph Theory (2022), in press. https://doi.org/10.7151/dmgt.2474

[12] H. Liu and D. Cheng, *The generalized* 4*-connectivity of folded hypercube*, Int. J. Comput. Math. Comput. Syst. Theory **7(4)** (2022) 235–245. https://doi.org/10.1080/23799927.2022.2123405

[13] H. Liu and D. Cheng, *Structure fault tolerance of balanced hypercubes*, Theoret. Comput. Sci. **845** (2020) 198–207. https://doi.org/10.1016/j.tcs.2020.09.015

[14] H. Lü and H. Zhang, *Hyper-Hamiltonian laceability of balanced hypercubes*, J. Supercomput. **68** (2014) 302–314. https://doi.org/10.1007/s11227-013-1040-6

[15] H. Lü and T. Wu, *Edge-disjoint Hamiltonian cycles of balanced hypercubes*, Inform. Process. Lett. **144** (2019) 25–30. https://doi.org/10.1016/j.ipl.2018.12.004

[16] Y. Saad and M.H. Schultz, *Topological properties of hypercubes*, IEEE Trans. Comput. **37** (1988) 867–872. https://doi.org/10.1109/12.2234

[17] C. Wei, R.-X. Hao and J.-M. Chang, *The reliability analysis based on the generalized connectivity in balanced hypercubes*, Discrete Appl. Math. **292** (2021) 19–32. https://doi.org/10.1016/j.dam.2020.12.011

[18] C. Wei, R.-X. Hao and J.-M. Chang, *Two-disjoint-cycle-cover bipancyclicity of balanced hypercubes*, Appl. Math. Comput. **381** (2020) 125305. https://doi.org/10.1016/j.amc.2020.125305

[19] J. Wu and K. Huang, *The balanced hypercube: a cube-based system for fault-tolerant applications*, IEEE Trans. Comput. **46** (1997) 484–490. https://doi.org/10.1109/12.588063

[20] M. Xu, X.-D. Hu and J.-M. Xu, *Edge-pancyclicity and Hamiltonian laceability of the balanced hypercubes*, Appl. Math. Comput. **189** (2007) 1393–1401. https://doi.org/10.1016/j.amc.2006.12.036

[21] M. Xu and Y. Wei, *The h-edge tolerable diagnosability of balanced hypercubes*, Theoret. Comput. Sci. **795** (2019) 540–546.
https://doi.org/10.1016/j.tcs.2019.08.007

[22] M.-C. Yang, *Conditional diagnosability of balanced hypercubes under the PMC model*, Inform. Sci. **222** (2013) 754–760.
https://doi.org/10.1016/j.ins.2012.08.014

[23] Y. Yang and L. Zhang, *Fault-tolerant-prescribed hamiltonian laceability of balanced hypercubes*, Inform. Process. Lett. **145** (2019) 11–15.
https://doi.org/10.1016/j.ipl.2019.01.002

[24] S.-L. Zhao and R.-X. Hao, *The generalized* 4*-connectivity of exchanged hypercubes*, Appl. Math. Comput. **347** (2019) 342–353.
https://doi.org/10.1016/j.amc.2018.11.023

[25] S.-L. Zhao, R.-X. Hao and J. Wu, *The generalized* 4*-connectivity of hierarchical cubic networks*, Discrete Appl. Math. **289** (2021) 194–206.
https://doi.org/10.1016/j.dam.2020.09.026

[26] S-L. Zhao and J.-M. Chang, *Reliability assessment of the divide-and-swap cube in terms of generalized connectivity*, Theoret. Comput. Sci. **943** (2023) 1–15.
https://doi.org/10.1016/j.tcs.2022.12.005

[27] S.-L. Zhao, J.-M. Chang and H.-Z. Li, *The generalized* 4*-connectivity of pancake graphs*, Discrete Appl. Math. **327** (2023) 77–86.
https://doi.org/10.1016/j.dam.2022.11.020

[28] J.-X. Zhou, Z.-L. Wu, S.-C. Yang and K.-W. Yuan, *Symmetric property and reliability of balanced hypercube*, IEEE Trans. Comput. **64** (2015) 876–881.
https://doi.org/10.1109/TC.2014.2304391

APPENDIX

Appendix 1. The proof of Lemma 23.

Proof. For any vertex set $N \subset V(BH_2)$ with $|N| = 4$, we denote $N = \{p, q, s, t\}$. By Lemmas 4 and 12, $\kappa_4(BH_2) \leq 2 \times 2 - 1 = 3$. We need to show that BH_2 includes 3 internally disjoint N-trees. Note that any two black vertices (respectively, white vertices) of BH_1^i are paired vertices, where $i \in \{0, 1, 2, 3\}$. If $3 \leq |N \cap V(BH_1^i)| \leq 4$ for some $i \in \{0, 1, 2, 3\}$, then N contains paired vertices. By Lemma 21, BH_2 contains 3 internally disjoint N-trees if N contains paired vertices. Hence, we only need to consider the following two cases.

Case 1. $|N \cap V(BH_1^i)| = 2$, say $N \cap V(BH_1^i) = \{p, q\}$, and $(p, q) \in E(BH_1^i)$, where $i \in \{0, 1, 2, 3\}$. Without loss of generality, let $i = 0$. By Lemma 8, we only need to consider $\{p, q\} = \{(0, 0), (3, 0)\}$. By symmetry of BH_2, we only need to consider $(s, t) \in E(BH_1^1)$, $(s, t) \in E(BH_1^2)$, and s, t are in two different sub-balanced hypercubes of $BH_1^1 \cup BH_1^2 \cup BH_1^3$. Since the two black vertices (respectively, white vertices) of BH_1^i for $i \in \{0, 1, 2, 3\}$ are paired vertices, we only need to consider the distributions of s, t shown in Figure 10. The 3 internally disjoint N-trees with red, green, blue colors, respectively, are shown in Figure 10.

Case 2. $|N \cap V(BH_1^i)| = 1$ for any $i \in \{0, 1, 2, 3\}$. Without loss of generality, let p, q, s, t be in $BH_1^0, BH_1^3, BH_1^1, BH_1^2$, respectively. By Lemma 7, let $p = (0, 0)$. Since the two black vertices (respectively, white vertices) of BH_1^i for $i \in \{0, 1, 2, 3\}$ are paired vertices, we only need to consider $s \in \{(1, 1), (0, 1)\}$, $q \in \{(0, 3), (1, 3)\}$, $t \in \{(0, 2), (3, 2)\}$. The 3 internally disjoint N-trees with red, green, blue colors, respectively, are shown in Figure 11. ■

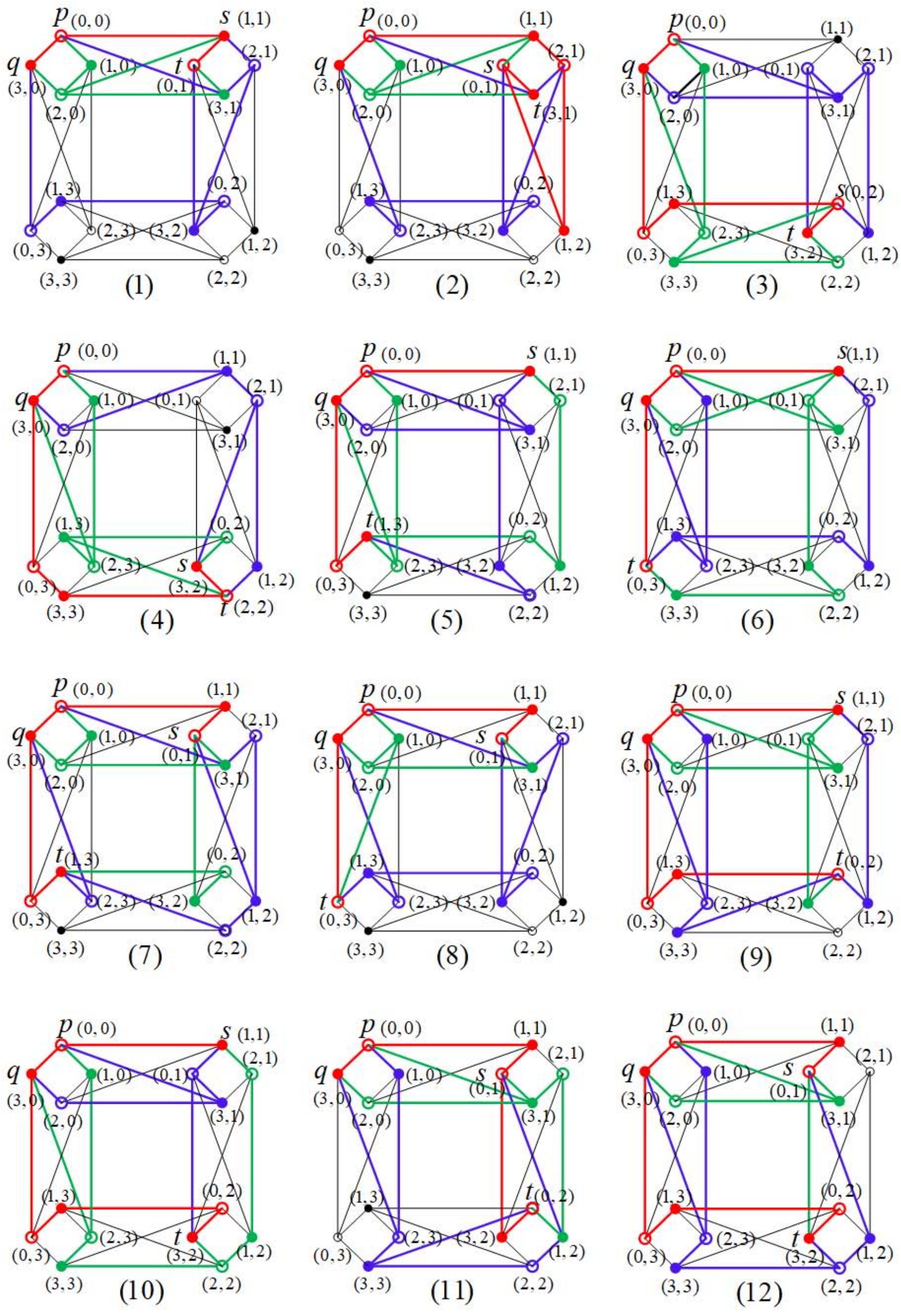

Figure 10. The illustration of Case 1 in the proof of Lemma 23.

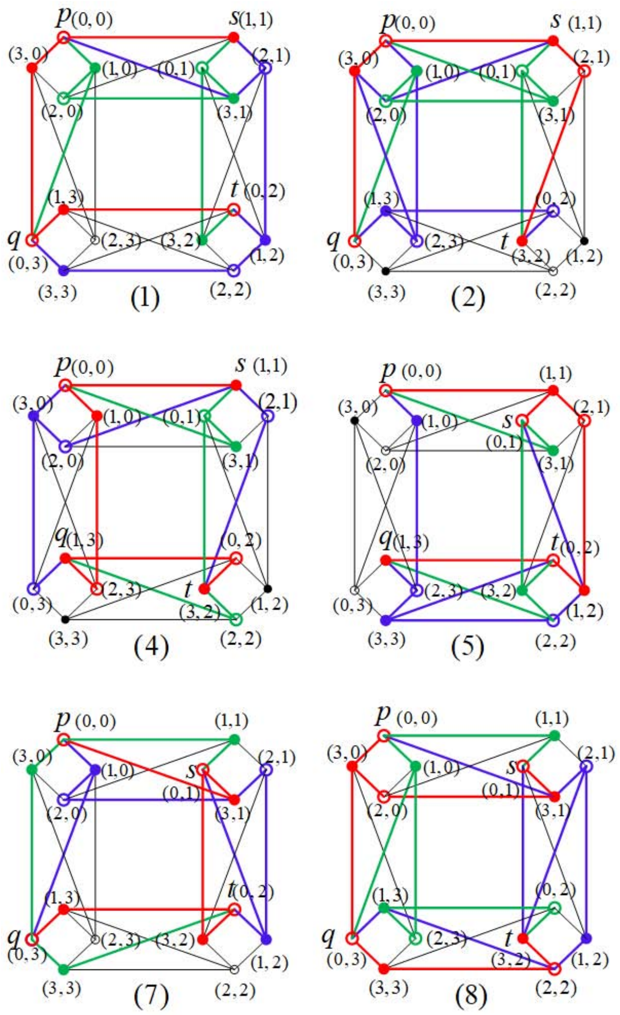

Figure 11. The illustration of Case 2 in the proof of Lemma 23.

Appendix 2. The graphs of other cases in the proof of Lemma 25.

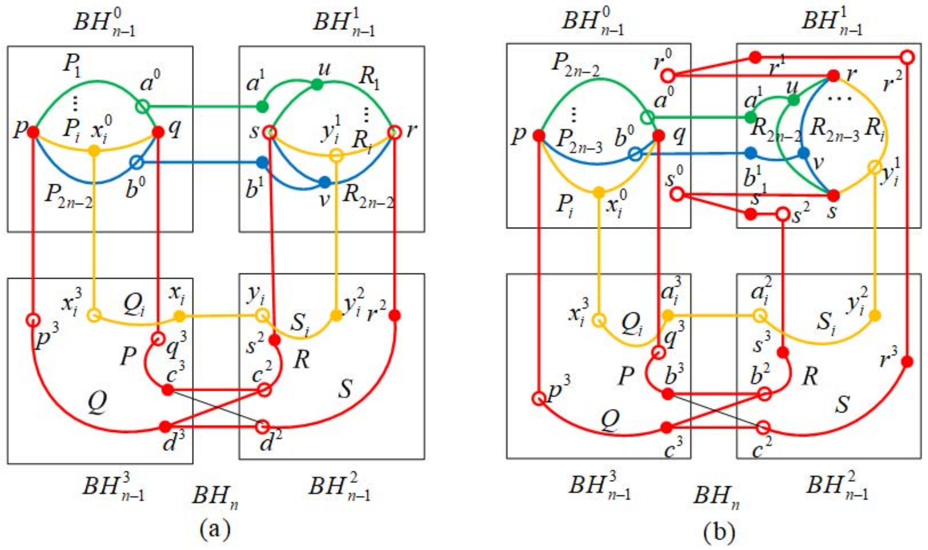

Figure 12. p and q are black vertices of BH^0_{n-1}, and r and s are the same color of BH^1_{n-1} in the proof of Lemma 25.

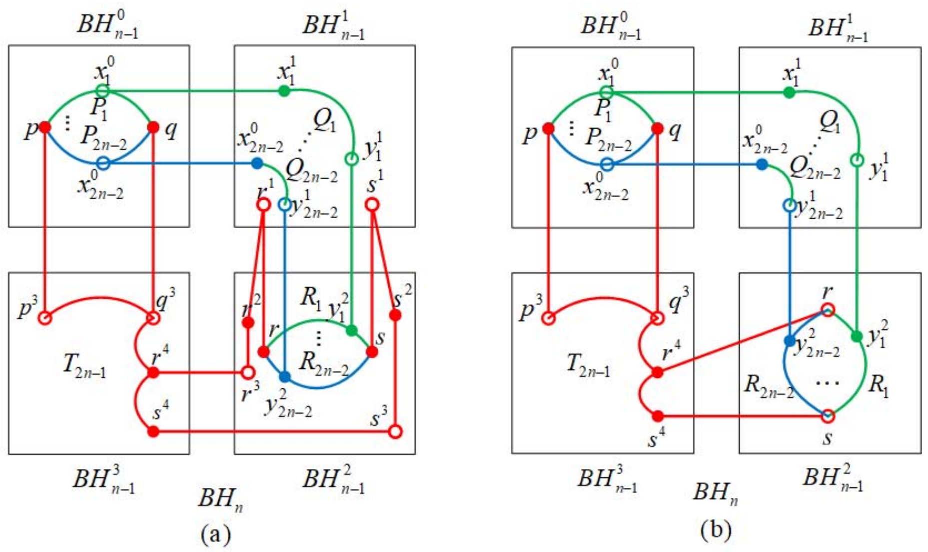

Figure 13. p and q are black vertices of BH^0_{n-1}, and r and s are the same color of BH^2_{n-1} in the proof of Lemma 25.

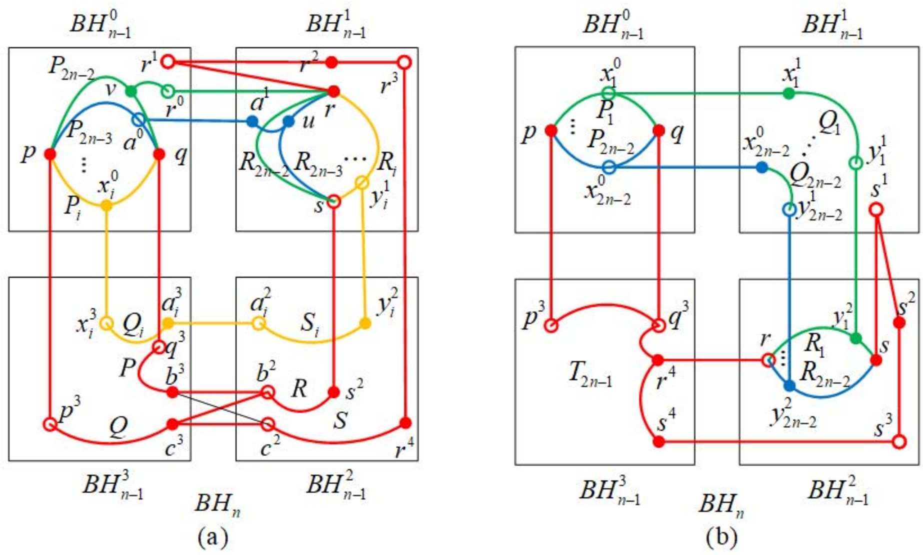

Figure 14. p and q are black vertices of BH^0_{n-1}, and r and s are different colors of BH^1_{n-1} and BH^2_{n-1} in the proof of Lemma 25.

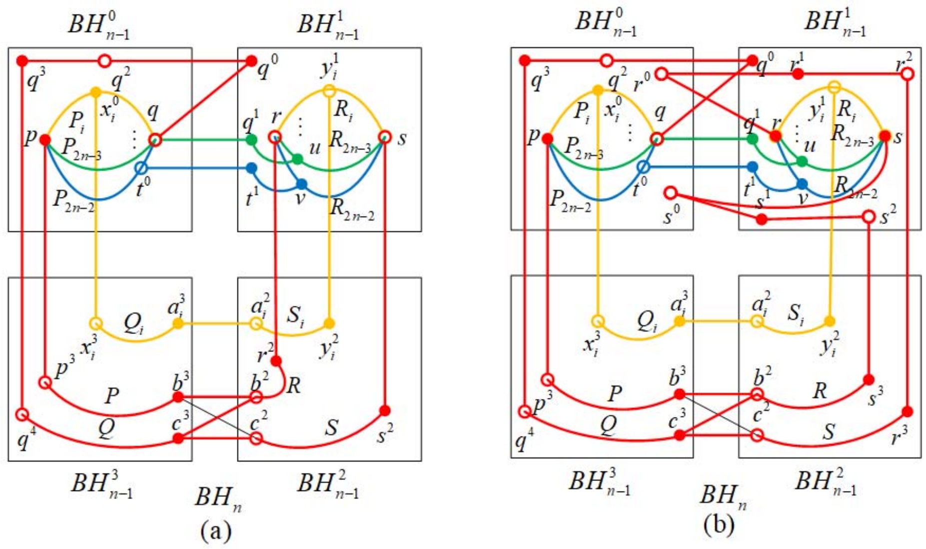

Figure 15. p and q are different colors of BH^0_{n-1}, and r and s are the same color of BH^1_{n-1} in the proof of Lemma 25.

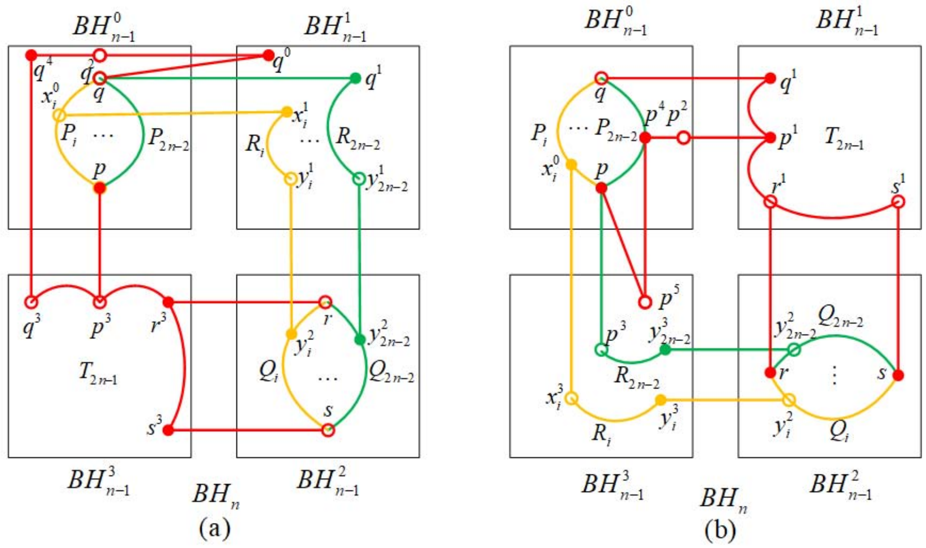

Figure 16. p and q are different colors of BH^0_{n-1}, and r and s are the same color of BH^2_{n-1} in the proof of Lemma 25.

Appendix 3. The proof of Lemma 26.

Proof. For any vertex set $N \subset V(BH_2)$ with $|N| = 4$, we denote $N = \{p, q, s, t\}$. By symmetry of BH_n, let $\{p, q\} \subset V(BH^0_{n-1})$. Without loss of generality, let p and q be different colors, say p is white vertex and q is black vertex. (If p and q are with the same color, by Lemma 19 and Definition 2, BH^0_{n-1} includes a path or an edge connecting p or q with the other end vertex in BH^3_{n-1} (respectively, BH^2_{n-1}) for Case 1 (respectively, Case 2).) By Lemmas 4 and 13, BH^0_{n-1} includes $2n-2$ internally disjoint paths P_js connecting p and q, where $1 \le j \le 2n-2$. Without loss of generality, we only need to consider that r is black vertex and s is white vertex. (If r is white vertex and s is black vertex, or r and s are with the same colors, by Lemma 19 and Definition 2, BH_n includes a path or an edge connecting r or s such that the other end vertices are in BH^3_{n-1} (respectively, BH^2_{n-1}) for Case 1 (respectively, Case 2).) By symmetry of BH^1_{n-1} and BH^3_{n-1}, we only need to consider two cases.

Case 1. r and s are in BH^1_{n-1} and BH^2_{n-1}, respectively, say $r \in V(BH^1_{n-1})$ and $s \in V(BH^2_{n-1})$. By Definition 2, we select one edge $\left(x^0_j, x^1_j\right)$, where $x^0_j \in V(P_j)$, and $x^1_j \in V(BH^1_{n-1})$ for $1 \le j \le 2n-2$. Let $X^1 = \left\{x^1_1, x^1_2, \ldots, x^1_{2n-2}\right\}$. By Lemmas 4 and 14, BH^1_{n-1} includes (r, X^1)-paths Q_js, where Q_j connects x^1_j and r for $1 \le j \le 2n-2$. Pick one white vertex $y^1_j \in Q_j$, where $1 \le j \le 2n-2$. By Definition 2, y^1_j has a neighbor $y^2_j \in V(BH^2_{n-1})$, where $1 \le j \le 2n-2$. Let $Y^2 = \left\{y^2_1, y^2_2, \ldots, y^2_{2n-2}\right\}$. By Lemmas 4 and 14, BH^2_{n-1} includes (s, Y^2)-paths

R_js, where R_j connects y_j^2 and s for $1 \le j \le 2n-2$. Let $T_j = P_j \cup (x_j^0, x_j^1) \cup Q_j \cup (y_j^1, y_j^2) \cup R_j$ for $1 \le j \le 2n-2$.

By Lemma 19 and Definition 2, BH_n includes three paths $\tilde{P} = \langle p, p^1, p^0, p^6, p^3 \rangle$ and $\tilde{R} = \langle r, r^0, r^1, r^2, r^3, r^4, r^5 \rangle$, where $\{r^0, p^0, p^6\} \subset V(BH_{n-1}^0)$, $\{p^1, r^1, r^2\} \subset V(BH_{n-1}^1)$, $\{r^3, r^4\} \subset V(BH_{n-1}^2)$, and $\{p^3, r^5\} \subset V(BH_{n-1}^3)$. By Definition 2, q (respectively, s) has a neighbor q^3 (respectively, s^3) in BH_{n-1}^3. Since BH_{n-1}^3 is connected, BH_{n-1}^3 includes a tree T'_{2n-1} connecting p^3, q^3, s^3 and r^5. Let $T_{2n-1} = T'_{2n-1} \cup \tilde{P} \cup (q^3, q) \cup (s^3, s) \cup \tilde{R}$, (see Figure 17(a)).

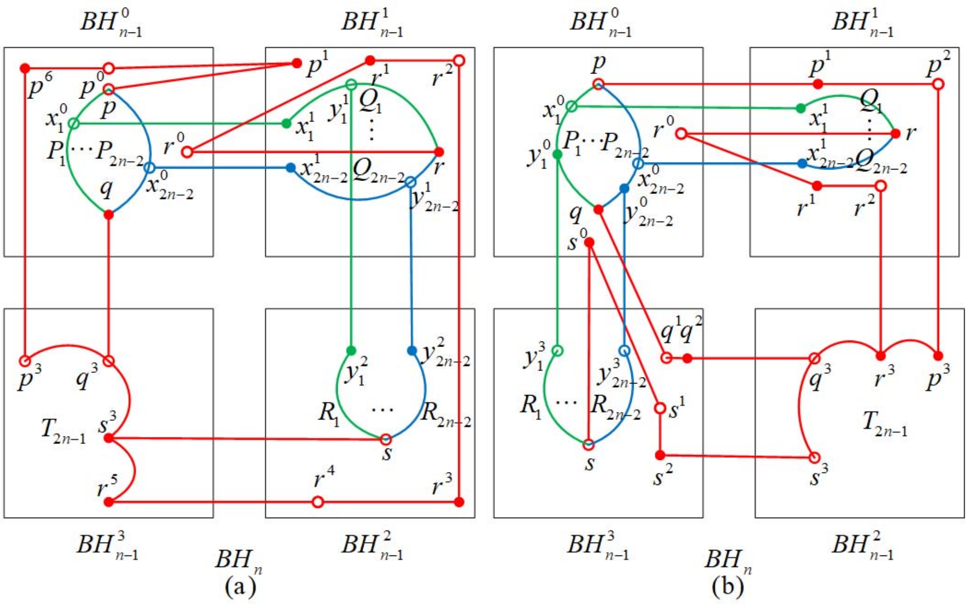

Figure 17. The illustrations of Case 1 and Case 2 in the proof of Lemma 26.

Case 2. r and s are in BH_{n-1}^1 and BH_{n-1}^3, respectively, say $r \in V(BH_{n-1}^1)$ and $s \in V(BH_{n-1}^3)$.

Pick one white vertex $x_j^0 \in V(P_j)$ and denote $(x_j^0, y_j^0) \in E(P_j)$ for $1 \le j \le 2n-2$. By Definition 2, x_j^0 (respectively, y_j^0) has a neighbor $x_j^1 \in V(BH_{n-1}^1)$ (respectively, $y_j^3 \in V(BH_{n-1}^3)$), where $1 \le j \le 2n-2$. Let $X^1 = \{x_1^1, x_2^1, \ldots, x_{2n-2}^1\}$ and $Y^3 = \{y_1^3, y_2^3, \ldots, y_{2n-2}^3\}$. By Lemmas 4 and 14, BH_{n-1}^1 includes (r, X^1)-paths Q_js, where Q_j connects x_j^1 and r for $1 \le j \le 2n-2$. By Lemmas 4 and 14, BH_{n-1}^3 includes (s, Y^3)-paths R_js, where R_j connects y_j^3 and s for $1 \le j \le 2n-2$. Let $T_j = P_j \cup Q_j \cup R_j \cup (x_j^0, x_j^1) \cup (y_j^0, y_j^3)$ for $1 \le j \le 2n-2$.

By Lemma 19 and Definition 2, BH_n includes four paths $\tilde{P} = \langle p, p^1, p^2, p^3 \rangle$, $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$, $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$, and $\tilde{S} = \langle s, s^0, s^1, s^2, s^3 \rangle$, where $\{p^1, p^2,$

$r^1, r^2\} \subset V(BH^1_{n-1})$, $\{p^3, q^3, r^3, s^3\} \subset V(BH^2_{n-1})$, and $\{q^1, q^2, s^1, s^2\} \subset V(BH^3_{n-1})$. Since BH^2_{n-1} is connected, BH^2_{n-1} includes a tree T'_{2n-1} connecting p^3, q^3, r^3 and s^3. Let $T_{2n-1} = T'_{2n-1} \cup \tilde{P} \cup \tilde{Q} \cup \tilde{S} \cup \tilde{R}$, (see Figure 17(b)). ■

Received 11 November 2022
Revised 6 February 2023
Accepted 7 February 2023
Available online 10 March 2023