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# THE GENERALIZED 4-CONNECTIVITY OF BALANCED HYPERCUBES

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## Abstract

The balanced hypercube is a kind of highly symmetrical network and possesses many good properties. Generalized connectivity is a new measurement of interconnection networks' fault tolerance. The internally disjoint N-trees are edge-disjoint trees but with intersecting vertex set N. Let  $\kappa(N)$  be the maximum number of internally disjoint N-trees and the generalized k-connectivity of G be  $\kappa_k(G) = \min\{\kappa(N) \mid N \subset V(G) \text{ and } |N| = k\}$ . In this paper, we study the n-dimensional balanced hypercube  $BH_n$  and demonstrate that  $\kappa_4(BH_n) = 2n - 1$  for  $n \geq 1$ .

**Keywords:** interconnection network, balanced hypercube, generalized connectivity, fault tolerance.

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## 1. INTRODUCTION

The parallel and distributed system plays a significant role in social networks, cloud computing, Big Data, and so on. Interconnection network as the topological structure of parallel and distributed system has obtained widely studied and applied. An interconnection network (network briefly) is modeled by a graph, where the processors and communication links are corresponding to vertices and edges, respectively. The hypercube [16] is one of the best-known networks. Compared with the hypercube, the balanced hypercube not only keeps many good properties like the hypercube but also has other better properties than the hypercube, including the smaller diameter and that each vertex has a paired vertex which has the same neighborhood [19], so each processor has an alternative processor

when processes the same adjacent relationship tasks. Balanced hypercube's other properties have received extensive research [3, 4, 8, 13–15, 17–23].

Connectivity is a traditional way to measure a network's fault tolerance. The connectivity of G is  $\kappa(G) = \min\{|N| \mid N \subset V(G) \text{ and } G - N$  is disconnected or trivial}. For  $N \subset V(G)$ , the N-tree means that the tree connects each vertex of N. The n internally disjoint N-trees  $T_i$ s mean that  $T_i$ s are pairwise edge-disjoint but with intersecting vertex set N, where  $1 \leq i \leq n$ . The N-trees are important in information transportation in terms of parallel routing design for large-scale networks. The more applications of N-trees in computer communication networks are described in [5]. Let  $\kappa(N) = \max\{l \mid T_1, T_2, \ldots, T_l \text{ are internally disjoint } N$ -trees}. The generalized k-connectivity of G is  $\kappa_k(G) = \min\{\kappa(N) \mid N \subset V(G)$  and  $|N| = k\}$  [24]. Note that it is equal to connectivity of G when k is 2 [24]. Generalized connectivity [2] uses internally disjoint trees to connect more vertices, which is more important in the application of multi-party computation or communication [17]. So it is a generalization method to determine the fault tolerance of distributed networks.

It is NP-complete to compute  $\kappa_k(G)$  [6]. Just a few networks' generalized 4-connectivity were determined, including hypercube [10], hierarchical cubic networks [25], exchanged hypercubes [24], divide-and-swap cube [26], pancake graphs [27], (n, k)-star networks [9], crossed cubes [11], and folded hypercubes [12]. For the *n*-dimensional balanced hypercube  $BH_n$ , it was shown that  $\kappa_3(BH_n) = 2n - 1$  when  $n \ge 1$  [17]. In our paper,  $\kappa_4(BH_n) = 2n - 1$  is further obtained, where  $n \ge 1$ .

This paper includes four sections. The preliminaries and main results are in the next two sections, respectively, and the conclusion is in last section.

## 2. Preliminaries

In a graph G = (V(G), E(G)), if  $(u, v) \in E(G)$  is an edge, then u and v are each other's neighbors. The neighborhood of  $u \in V(G)$  is  $N_G(u) = \{v \mid (u, v) \in E(G), v \in V(G)\}$  and the degree of  $u \in V(G)$  is  $d_G(u) = |\{(u, v) \mid v \in V(G)\}|$ . Denote  $\delta(G)$  as the minimum of all  $d_G(u)$  for  $u \in V(G)$ . Denote  $P[x, y] = \langle x_0, x_1, x_2, \ldots, x_l \rangle$  as a path from x to y, where  $x_0 = x, x_l = y, x_i$ s  $(0 \le i \le l)$  are pairwise different, l is the path's length, and the path is l-path. If  $x_l = x_0$  and  $l \ge 3$ , P[x, y] becomes a cycle. For two distinct vertices a and c, the internally disjoint (a, c)-paths are vertex-disjoint paths except for the two common end vertices a and c. For a vertex a and a vertex set B such that  $a \notin B$ , the (a, B)-paths are vertex-disjoint paths connecting a and each vertex of B except for the only common end vertex a. For two vertex sets  $A = \{a_1, a_2, \ldots, a_k\}$  and  $B = \{b_1, b_2, \ldots, b_k\}$ , the paired (A, B)-paths are k vertex-disjoint paths  $P[a_i, b_i]$ s, where  $1 \leq i \leq k$ . The other terminology and notations not given here can be found in [1]. The  $BH_n$  has two methods to define. (Throughout this paper, among the labels of vertices of  $BH_n$ , the " $\pm$ " and "+" are by modulo 4 operation. We omit "(mod) 4" for simplicity.)

**Definition 1** [19].  $BH_n = (V(BH_n), E(BH_n))$ , where  $V(BH_n) = V_e \cup V_o$ , where  $V_e = \{(v_0, v_1, \dots, v_{n-1}) \mid v_i \in \{0, 1, 2, 3\}$  for  $1 \le i \le n-1, v_0 \in \{0, 2\}\}$  and  $V_o = \{(v_0, v_1, \dots, v_{n-1}) \mid v_i \in \{0, 1, 2, 3\}$  for  $1 \le i \le n-1, v_0 \in \{1, 3\}\}$ , and  $E(BH_n) = E_0 \cup E_i$ , where  $E_0 = \{((v_0, v_1, \dots, v_{n-1}), (v_0 \pm 1, v_1, \dots, v_{n-1}))\}$  and  $E_i = \{((v_0, v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{n-1}), (v_0 \pm 1, v_1, \dots, v_{i-1}, v_i + (-1)^{v_0}, v_{i+1}, \dots, v_{n-1})) \mid 1 \le i \le n-1\}.$ 

 $BH_n$  is a bipartite graph, and  $|V(BH_n)| = 2^{2n}$ . Let  $V_e$  be the set of white vertices and  $V_0$  the set of black vertices. Let  $E_0$  be the set of 0-dimensional edges and  $E_i$  the set of *i*-dimensional edges for  $1 \le i \le n-1$ .  $BH_1$  is a 4-cycle, two drawing methods of  $BH_2$  are depicted in Figure 1.



Figure 1. Two drawing methods of  $BH_2$ .

Another method to define  $BH_n$  is by a recursive definition.

**Definition 2** [19]. The recursive definition of  $BH_n$  is as follows.

(1)  $BH_1$  is a 4-cycle (0, 1, 2, 3, 0).

(2) For  $n \geq 2$ ,  $V(BH_n) = \bigcup_{i=0}^{3} V(BH_{n-1}^i)$ , where  $BH_{n-1}^i \cong BH_{n-1}$  for  $i \in \{0, 1, 2, 3\}$ . Every vertex  $(v_0, v_1, \ldots, v_{n-1}, i) \in V(BH_{n-1}^i)$   $(i \in \{0, 1, 2, 3\})$  has two extra neighbors:

(2.1)  $(v_0 \pm 1, v_1, \dots, v_{n-2}, i+1) \in V(BH_{n-1}^{i+1})$  if  $v_0$  is even.

(2.2)  $(v_0 \pm 1, v_1, \dots, v_{n-2}, i-1) \in V(BH_{n-1}^{i-1})$  if  $v_0$  is odd.

The  $BH_{n-1}^i$ s  $(0 \le i \le 3)$  are called sub-balanced hypercubes. In  $BH_n$ , two vertices with labels  $(v_0-1, v_1, \ldots, v_{n-1})$  and  $(v_0+1, v_1, \ldots, v_{n-1})$  are called paired vertices. By Definition 2, each vertex of  $BH_{n-1}^i$   $(0 \le i \le 3)$  has two neighbors in  $BH_{n-1}^{i+1}$  or  $BH_{n-1}^{i-1}$ , and these two neighbors are paired vertices. Two edges e = (r, s) and e' = (r', s') are called paired edges if r and r' (respectively, s and s') are paired vertices. Two cycles  $\langle r_1, r_2, \ldots, r_l \rangle$  and  $\langle r'_1, r'_2, \ldots, r'_l \rangle$  are called paired cycles if  $r_i$  and  $r'_i$  are paired vertices, where  $1 \le i \le l$ . Imaging two paired vertices as one vertex, and the four edges between the two paired vertices as one edge, we have the following graph  $\widetilde{BH}_n$  which is a contraction of  $BH_n$ .

**Definition 3.** Let  $\widetilde{BH_n} = (V(\widetilde{BH_n}), E(\widetilde{BH_n}))$  be a contraction of  $BH_n$ , where  $V(\widetilde{BH_n}) = \{V \mid V = \{v, v'\}, v \text{ and } v' \text{ are paired vertices of } BH_n\}$ , and  $E(\widetilde{BH_n}) = \{(U, V) \mid U = \{u, u'\}, V = \{v, v'\} \in V(\widetilde{BH_n}) \text{ such that } (u, v), (u, v'), (u', v), (u', v') \in E(BH_n)\}$ .  $\widetilde{BH_1}$  is an edge, denoted by (e, o). For  $n \ge 2$ , if  $V \in V(\widetilde{BH_n})$  is a white vertex, it is denoted by  $(e, v_1, v_2, \dots, v_{n-1})$ , otherwise it is denoted by  $(o, v_1, v_2, \dots, v_{n-1})$ , where  $e \in \{0, 2\}, o \in \{1, 3\}, \text{ and } v_i \in \{0, 1, 2, 3\}$  for  $1 \le i \le n-1$ .

The graphs of  $\widetilde{BH_2}$  and  $\widetilde{BH_3}$  are shown in Figure 2.



Figure 2.  $\widetilde{BH_2}$  and  $\widetilde{BH_3}$ .

**Lemma 4** [19].  $BH_n$  is 2*n*-regular and  $\kappa(BH_n) = 2n$ , where  $n \ge 1$ .

**Lemma 5** [17].  $\kappa_3(BH_n) = 2n - 1$ , where  $n \ge 1$ .

By Definition 3 and Lemma 4, we directly get the following lemma.

**Lemma 6.**  $\widetilde{BH_n}$  is *n*-regular, and  $\kappa(\widetilde{BH_n}) = n$ , where  $n \ge 1$ .

**Lemma 7** [19].  $BH_n$  is vertex-transitive, where  $n \ge 1$ .

**Lemma 8** [28].  $BH_n$  is edge-transitive, where  $n \ge 1$ .

**Lemma 9** [19]. In  $BH_n$ , any two paired vertices have the same neighborhood.

**Lemma 10** [3]. In  $BH_n$ , any edge (x, y) is included in 2n - 2 8-cycles  $C_8^j s$  such that  $C_8^j s$  are edge-disjoint except (x, y) and  $\left|E(C_8^j) \cap E(BH_{n-1}^i)\right| = 1$ , where  $1 \le j \le 2n - 2$  and  $i \in \{0, 1, 2, 3\}$ .

Since  $|N_{BH_{n-1}^i}(u)| = 2n - 2$  for  $u \in V(BH_{n-1}^i)$ , where  $i \in \{0, 1, 2, 3\}$ , by Lemmas 7, 8 and 10, we directly get the following lemma.

**Lemma 11.** In  $BH_n$ , any vertex u is contained in 2n-2 8-cycles  $C_8^j s$  such that  $C_8^j s$  are edge-disjoint and  $|E(C_8^j) \cap E(BH_{n-1}^i)| = 1$ , where  $1 \le j \le 2n-2$  and  $i \in \{0, 1, 2, 3\}$ .

**Lemma 12** [7]. If G includes (a, b) with  $d_G(a) = d_G(b) = \delta(G)$ , then  $\kappa_k(G) \leq \delta(G) - 1$ , where  $3 \leq k \leq |V(G)|$ .

**Lemma 13** [1]. If  $\kappa(G) = k$ , for  $a, b \in V(G)$ , then G includes k internally disjoint paths between a and b.

**Lemma 14** [1]. If  $\kappa(G) = k$ , for  $a \in V(G)$  and  $B \subset V(G) \setminus \{a\}$  with |B| = k, then G includes (a, B)-paths.

**Lemma 15** [1]. If  $\kappa(G) = k$ , for  $A \subset V(G)$ ,  $B \subset V(G)$  with |A| = |B| = k and  $A \cap B = \emptyset$ , then G includes paired (A, B)-paths.

## 3. Main Results

**Lemma 16.** Let  $P = \{p, p'\}$  and  $R = \{r, r'\}$  be any two vertices of  $\widetilde{BH_n}$  with  $(P, R) \notin E(\widetilde{BH_n})$ . Then any path connecting P and R of  $\widetilde{BH_n}$  is corresponding to two internally disjoint N-trees of  $BH_n$ , and two paired vertex-disjoint paths P[p, r] and P[p', r'], and P[p, r'] and P[p', r] of  $BH_n$ , where  $N = \{p, p', r, r'\}$  and  $n \geq 2$ .

**Proof.** Let  $\langle P, Q_1, Q_2, \ldots, Q_l, R \rangle$  be any path in  $\widehat{BH_n}$ , where  $Q_i = \{q_i, q'_i\}$  for  $1 \leq i \leq l$ . Then  $q_i$  and  $q'_i$  are paired vertices for  $1 \leq i \leq l$ . Let  $T_1 = \langle p, q_1, q_2, \ldots, q_l, r \rangle \cup (q_1, p') \cup (q_l, r')$  and  $T_2 = \langle p', q'_1, q'_2, \ldots, q'_l, r' \rangle \cup (p, q'_1) \cup (q'_l, r)$ . Then  $T_1$  and  $T_2$  are two internally disjoint N-trees of  $BH_n$ , where  $N = \{p, p', r, r'\}$ . Clearly,  $P[p, r] = \langle p, q_1, q_2, \ldots, q_l, r \rangle$  and  $P[p', r'] = \langle p', q'_1, q'_2, \ldots, q'_l, r' \rangle$  are two vertex-disjoint paths, and  $P[p, r'] = [p, q'_1, q'_2, \ldots, q'_l, r']$  and  $P[p', r] = \langle p', q_1, q_2, \ldots, q_l, r \rangle$  are two vertex-disjoint paths.

**Lemma 17.** In  $BH_n$ , any edge e = (r, s) and its paired edge e' = (r', s') are included in a 4-cycle  $\langle r, s, r', s', r \rangle$ , where  $n \ge 2$ .

**Proof.** By Lemma 8, we only need to consider e = (r, s), where  $r = (r_0, r_1, \ldots, r_{n-1})$ , and  $s = (r_0 + 1, r_1, \ldots, r_{n-1})$ . Let e' = (r', s'), where  $r' = (r_0 + 2, r_1, \ldots, r_{n-1})$  and  $s' = (r_0 + 3, r_1, \ldots, r_{n-1})$ . Then  $\langle r, s, r', s', r \rangle$  is a 4-cycle.

**Lemma 18.** In  $BH_n$  with  $n \geq 2$ , any edge e and its paired edge e' are included in two paired 8-cycles, denoted by  $R = \langle r_0, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_0 \rangle$  and  $R' = \langle r'_0, r'_1, r'_2, r'_3, r'_4, r'_5, r'_6, r'_7, r'_0 \rangle$ , respectively, where  $|E(R) \cap E(BH^i_{n-1})| = |E(R') \cap E(BH^i_{n-1})| = 1$  for  $i \in \{0, 1, 2, 3\}$ , and  $\langle r_j, r_{j+1}, r'_j, r'_{j+1}, r_j \rangle$  is a 4-cycle for  $0 \leq j \leq 7$  (The subscript "j + 1" is with operation modulo 8.).

**Proof.** By Lemma 10, e is included in an 8-cycle R satisfying that  $|E(R) \cap E(BH_{n-1}^i)| = 1$  for  $i \in \{0, 1, 2, 3\}$ . Let  $E(R) \cap E(BH_{n-1}^i) = e_i$  for  $i \in \{0, 1, 2, 3\}$ , where  $e_0 = (r_0, r_1), e_1 = (r_2, r_3), e_2 = (r_4, r_5), e_3 = (r_6, r_7)$ . By Lemma 17, each edge  $(r_j, r_{j+1})$  has a paired edge  $(r'_j, r'_{j+1})$ , and they are included in a 4-cycle  $\langle r_j, r_{j+1}, r'_j, r'_{j+1}, r_j \rangle$ , where  $j \in \{0, 2, 4, 6\}$ .  $r_k$  and  $r'_k$  are paired vertices, where  $0 \leq k \leq 7$ . By Lemma 9,  $r_k$  and  $r'_k$  have the same neighborhood. So  $\langle r_j, r_{j+1}, r'_j, r'_{j+1}, r_j \rangle$  is a 4-cycle, where  $j \in \{1, 3, 5, 7\}$ . Let  $R' = \langle r'_0, r'_1, r'_2, r'_3, r'_4, r'_5, r'_6, r'_7, r'_0 \rangle$ . Then  $e' \in E(R')$ . R and R' are paired 8-cycles, (see Figure 3). Hence, the lemma is true.

**Lemma 19.** In  $BH_n$  with  $n \ge 2$ , for any vertex a and vertex set B ( $a \notin B$ ) with |B| = 2n - 2 of some  $BH_{n-1}^i$  ( $i \in \{0, 1, 2, 3\}$ ), there exist (a, B)-paths in  $BH_{n-1}^i$ . Let  $d \in V(BH_{n-1}^i)$  be such that  $d \ne a$ . Then  $BH_{n-1}^i$  includes at least one edge (d, d') such that (d, d') is not in (a, B)-paths.

**Proof.** By Lemmas 4 and 14, there are (a, B)-paths in  $BH_{n-1}^i$ . By Definition 2, d has 2n-2 neighbors  $d_j$ s in  $BH_{n-1}^i$ , where  $1 \le j \le 2n-2$ . If all the  $(d, d_j)$ s are in (a, B)-paths, then the (a, B)-paths have two common vertices a and d, which is a contradiction. Hence,  $BH_{n-1}^i$  includes at least one edge (d, d') such that (d, d') is not in (a, B)-paths.



Figure 3. Two paired 8-cycles containing e and e', respectively.

**Lemma 20.** In  $BH_n$  with  $n \geq 2$ , for any two different vertices a and c (a and c are not paired vertices) of some  $BH_{n-1}^i$  ( $i \in \{0, 1, 2, 3\}$ ), there exist 2n - 2 internally disjoint (a, c)-paths  $P_k s$  ( $1 \leq k \leq 2n-2$ ) in  $BH_{n-1}^i$ . Let  $d \in V(BH_{n-1}^i)$  be such that  $d \notin \{a, c\}$ . Then  $BH_{n-1}^i$  includes at least one edge (d, d') such that (d, d') is not in  $\bigcup_{k=1}^{2n-2} P_k$ .

**Proof.** By Lemmas 4 and 13, there are 2n - 2 internally disjoint (a, c)-paths  $P_k$ s  $(1 \le k \le 2n - 2)$  in  $BH_{n-1}^i$ . By Definition 2, d has 2n - 2 neighbors  $d_j$ s in  $BH_{n-1}^i$ , where  $1 \le j \le 2n - 2$ . If all the  $(d, d_j)$ s are in  $\bigcup_{k=1}^{2n-2} P_k$ , then  $\bigcup_{k=1}^{2n-2} P_k$  have three common vertices a, c, and d, which is a contradiction. Hence,  $BH_{n-1}^i$  includes at least one edge (d, d') such that (d, d') is not in  $\bigcup_{k=1}^{2n-2} P_k$ .

**Lemma 21.** Let  $N \subset V(BH_n)$  be such that  $|N \cap V(BH_n)| = 4$  and N contains paired vertices. Then there are 2n - 1 internally disjoint N-trees in  $BH_n$ , where  $n \geq 2$ .

**Proof.** Denote  $N = \{p, q, r, s\}$ . We discuss two cases.

Case 1. Two vertices of N are paired vertices, say q and p are paired vertices. By Lemma 5,  $BH_n$  includes 2n-1 internally disjoint N'-trees  $T'_j$ 's  $(1 \le j \le 2n-1)$ , where  $N' = \{p, r, s\}$ . Let  $p_j$  be the neighbor of p in  $T'_j$  for  $1 \le j \le 2n-1$ . Then  $T_j = T'_j \cup (q, p_j)$  is N-tree and  $T_j$ 's are internally disjoint, where  $1 \le j \le 2n-1$ .

Case 2. Four vertices of N are two different paired vertices, say p, q are paired vertices and r, s are paired vertices.

Let  $P = \{p,q\}$  and  $R = \{r,s\}$ . By Definition 3, Lemmas 6 and 13,  $BH_n$  includes n internally disjoint paths connecting P and R. By Lemma 16,  $BH_n$  includes 2n internally disjoint N-trees.

**Lemma 22.**  $\kappa_4(BH_1) = 1$ .

**Proof.** By Lemmas 4 and 12,  $\kappa_4(BH_1) \leq 1$ . By Definition 2,  $BH_1$  is a 4-cycle, so  $BH_1$  includes a path including its four vertices. Hence, the lemma holds.

Lemma 23.  $\kappa_4(BH_2) = 3.$ 

**Proof.** The proof is in Appendix 1.

**Lemma 24.** Let  $N \subset V(BH_n)$  be such that  $|N \cap V(BH_n)| = 4$  and N does not contain paired vertices. If each sub-balanced hypercube has one vertex of N, then there are 2n - 1 internally disjoint N-trees in  $BH_n$ , where  $n \geq 3$ .

**Proof.** Without loss of generality, let  $N = \{p, q, r, s\}$  and  $p \in V(BH_{n-1}^0)$ ,  $q \in V(BH_{n-1}^3)$ ,  $r \in V(BH_{n-1}^2)$ ,  $s \in V(BH_{n-1}^1)$ . By Lemma 11, w is in an 8-cycle  $\langle w^7, w^0, w^1, w^2, w^3, w^4, w^5, w^6, w^7 \rangle$  for  $w \in \{p, q, r, s\}$ , where  $(w^7, w^0) \in E(BH_{n-1}^0)$ ,  $(w^1, w^2) \in E(BH_{n-1}^1)$ ,  $(w^3, w^4) \in E(BH_{n-1}^2)$ ,  $(w^5, w^6) \in E(BH_{n-1}^3)$ . Since  $|V(BH_{n-1}^i)| = 2^{2(n-1)}$ , we have  $2^{2(n-1)-1}$  black vertices and  $2^{2(n-1)-1}$  white vertices in  $BH_{n-1}^i$  for  $i \in \{0, 1, 2, 3\}$  and  $n \ge 3$ . Not considering the vertices of 8-cycles that contain p, q, r, s respectively in  $BH_{n-1}^i$ , since  $2^{2(n-1)-1} - 4 > 2n - 4$  for  $n \ge 3$ , by Lemma 11, we can pick another 2n - 4 vertex-disjoint 8-cycles  $\langle x_i^7, x_i^0, x_i^1, x_i^2, x_i^3, x_i^4, x_i^5, x_i^6, x_i^7 \rangle$  in  $BH_n$ , where  $(x_i^7, x_i^0) \in E(BH_{n-1}^0), (x_i^1, x_i^2) \in E(BH_{n-1}^1), (x_i^3, x_i^4) \in E(BH_{n-1}^2), (x_i^5, x_i^6) \in E(BH_{n-1}^3)$  for  $1 \le i \le 2n - 4$ . We deal with four cases.

 $\begin{array}{l} Case \ 1. \ \text{Each vertex of } \{p,q,r,s\} \ \text{is in different } C_8\text{s. Without loss of generality, let } p^7 = p, q^5 = q, r^3 = r \ \text{and } s^1 = s. \ \text{Let } X^0 = \left\{q^0,r^0,s^0,x_1^0,x_2^0,\ldots,x_{2n-5}^0\right\}, \\ X^1 = \left\{p^2,q^2,r^2,x_1^2,x_2^2,\ldots,x_{2n-5}^2\right\}, X^2 = \left\{p^4,q^4,s^4,x_1^4,x_2^4,\ldots,x_{2n-5}^4\right\} \ \text{and } X^3 = \left\{p^6,r^6,s^6,x_1^6,x_2^6,\ldots,x_{2n-5}^6\right\}. \ \text{By Lemmas } 4 \ \text{and } 14, \ BH_{n-1}^0 \ \text{includes } (p,X^0) - paths \ Q^0,R^0,S^0,X_1^0,X_2^0,\ldots,X_{2n-5}^0, BH_{n-1}^1 \ \text{includes } (s,X^1) - paths \ P^1,Q^1,R^1, \\ X_1^1,X_2^1,\ldots,X_{2n-5}^1,BH_{n-1}^2 \ \text{includes } (r,X^2) - paths \ P^2,Q^2,S^2,X_1^2,X_2^2,\ldots,X_{2n-5}^2, \\ \text{and } BH_{n-1}^3 \ \text{includes } (q,X^3) - paths \ P^3,R^3,S^3,X_1^3,X_2^3,\ldots,X_{2n-5}^3, \\ \text{where } Q^0 \ \text{connects } p \ \text{and } r^0, \ S^0 \ \text{connects } p \ \text{and } s^0, \ X_i^0 \ \text{connects } p \\ \text{and } x_i^0, \ P^1 \ \text{connects } s \ \text{and } p^2, \ Q^1 \ \text{connects } s \ \text{and } q^2, \ R^1 \ \text{connects } s \ \text{and } r^2, \ X_i^1 \\ \text{connects } s \ \text{and } x_i^2, \ P^2 \ \text{connects } r \ \text{and } p^4, \ Q^2 \ \text{connects } r \ \text{and } q^4, \ S^2 \ \text{connects } r \\ r \ \text{and } s^4, \ X_i^2 \ \text{connects } r \ \text{and } x_i^4, \ P^3 \ \text{connects } q \ \text{and } r^6, \ R^3 \ \text{connects } q \ \text{and } r^6, \\ S^3 \ \text{connects } q \ \text{and } x_i^3 \ \text{connects } q \ \text{and } x_i^6, \ \text{where } 1 \le i \le 2n-5. \ \text{Let } T_i = X_i^0 \cup X_i^1 \cup X_i^2 \cup X_i^3 \cup \left\langle x_i^4, x_i^5, x_i^6, x_i^7, x_i^0, x_i^1, x_i^2 \right\rangle \ \text{for } 1 \le i \le 2n-5, \ T_{2n-4} = S^0 \cup S^2 \cup S^3 \cup \left\langle s^4, s^5, s^6, s^7, s^0, s \right\rangle, \ T_{2n-3} = R^0 \cup R^1 \cup R^3 \cup \left\langle r^6, r^7, r^0, r^1, r^2, r \right\rangle, \end{array}$ 



Figure 4. The illustration of Case 1 in the proof of Lemma 24.

 $T_{2n-2} = Q^0 \cup Q^1 \cup Q^2 \cup \langle q^0, q^1, q^2, q^3, q^4, q \rangle$ , and  $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2 \rangle$ , (see Figure 4).

Case 2. Two vertices of  $\{p, q, r, s\}$  are in the same  $C_8$ .

Case 2.1. Two vertices of  $\{p, q, r, s\}$ , say q and r, are in two consecutive sub-balanced hypercubes.

Without loss of generality, let  $p^7 = p$ ,  $q^5 = q$ ,  $q^3 = r$  and  $s^1 = s$ , i.e., q and r are in the same 8-cycle  $C = \langle q^7, q^0, q^1, q^2, r, q^4, q, q^6, q^7 \rangle$ . By Lemma 18, C has a paired 8-cycle C'. Denote  $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7 \rangle$ , where  $(y^7, y^0) \in E(BH_{n-1}^0)$ ,  $(y^1, y^2) \in E(BH_{n-1}^1)$ ,  $(y^3, y^4) \in E(BH_{n-1}^2)$ ,  $(y^5, y^6) \in E(BH_{n-1}^3)$ , and  $\langle q^7, y^0, y^7, q^0, q^7 \rangle$ ,  $\langle q^0, y^1, y^0, q^1, q^0 \rangle$ ,  $\langle q^1, y^2, y^1, q^2, q^1 \rangle$ ,  $\langle q^2, y^3, y^2, r, q^2 \rangle$ ,  $\langle r, y^4, y^3, q^4, r \rangle$ ,  $\langle q^4, y^5, y^4, q, q^4 \rangle$ ,  $\langle q, y^6, y^5, q^6, q \rangle$ ,  $\langle q^6, y^7, y^6, q^7, q^6 \rangle$  are 4-cycles. In  $BH_{n-1}^0$ ,  $BH_{n-1}^1$  and  $BH_{n-1}^3$ , the discussions are similar to Case 1 except that we need to use  $y^k$ s instead of  $r^k$ s for  $k \in \{0, 1, 2, 5, 6, 7\}$  and use  $Y^j$ s instead of  $R^j$ s for  $j \in \{0, 1, 3\}$  ( $Y^3 = (q, y^6)$ ). In  $BH_{n-1}^2$ , let  $X^2 = \{p^4, y^4, s^4, x_1^4, x_2^4, \dots, x_{2n-5}^4\}$ . By Lemmas 4 and 14,  $BH_{n-1}^2$  includes  $(r, X^2)$ -paths  $P^2, Y^2, S^2, X_1^2, \dots, X_{2n-5}^2$ , where  $P^2$  connects r and  $p^4, Y^2 = (r, y^4), S^2$  connects r and  $s^4$ , and  $X_i^2$  connects r and  $x_i^4$  for  $1 \le i \le 2n - 5$ . Let  $T_i = X_i^0 \cup X_i^1 \cup X_i^2 \cup X_i^3 \cup \langle x_i^2, x_i^1, x_0^i, x_i^7, x_6^i, x_5^5, x_4^i \rangle$  for  $1 \le i \le 2n - 5, T_{2n-4} = S^0 \cup S^2 \cup S^3 \cup \langle s, s^0, s^7, s^6, s^5, s^4 \rangle, T_{2n-3} = Y^0 \cup Y^1 \cup \langle r, y^4, y^3, y^2, y^1, y^0, y^7, y^6, q \rangle$ ,  $T_{2n-2} = Q^0 \cup Q^1 \cup \langle r, y^2, q^1, q^0, q^7, q^6, y^5, y^4, q \rangle \cup (q^1, q^2)$ , and  $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2 \rangle$ , (see Figure 5(a)).



Figure 5. The illustration of Case 2.1 and Case 2.2 in the proof of Lemma 24.

Case 2.2. Two vertices of  $\{p, q, r, s\}$ , say q and s, are in two opposite subbalanced hypercubes.

Without loss of generality, let  $p^7 = p$ ,  $s^5 = q$ ,  $q^3 = r$  and  $s^1 = s$ , i.e., qand s are in the 8-cycle  $C = \langle s^7, s^0, s, s^2, s^3, s^4, q, s^6, s^7 \rangle$ . By Lemma 18, C has a paired 8-cycle C'. Denote  $C' = \langle y^7, y^0, y^1, y^2, y^3, y^4, y^5, y^6, y^7 \rangle$ , where  $(y^7, y^0) \in E(BH_{n-1}^0)$ ,  $(y^1, y^2) \in E(BH_{n-1}^1)$ ,  $(y^3, y^4) \in E(BH_{n-1}^2)$ ,  $(y^5, y^6) \in E(BH_{n-1}^3)$ , and  $\langle y^7, s^0, s^7, y^0, y^7 \rangle$ ,  $\langle y^0, s, s^0, y^1, y^0 \rangle$ ,  $\langle y^1, s^2, s, y^2, y^1 \rangle$ ,  $\langle y^2, s^3, s^2, y^3, y^2 \rangle$ ,  $\langle y^3, s^4, s^3, y^4, y^3 \rangle$ ,  $\langle q, s^4, y^5, y^4, q \rangle$ ,  $\langle q, y^6, y^5, s^6, q \rangle$ ,  $\langle s^7, y^6, y^7, s^6, s^7 \rangle$  are 4-cycles. In  $BH_{n-1}^0, BH_{n-1}^1, BH_{n-1}^2$ , the proofs of this case are similar to Case 2.1, except that  $Y^2$  is now a path connecting r and  $y^4$ . Let  $X^3 = \{p^6, q^6, y^6, x_1^6, x_2^6, \dots, x_{2n-5}^6\}$ . By Lemma 14,  $BH_{n-1}^3$  contains  $(q, X^3)$ -paths  $P^3, Q^3, Y^3, X_1^3, X_2^3, \dots, X_{2n-5}^3$ , where  $P^3$  connects q and  $p^6, Q^3$  connects q and  $q^6, Y^3 = (q, y^6), X_i^3$  connects q and  $x_i^6$  for  $1 \le i \le 2n - 5$ . Let  $T_i = X_i^0 \cup X_i^1 \cup X_i^2 \cup X_i^3 \cup \langle x_i^2, x_i^1, x_i^0, x_i^7, x_i^6, x_i^5, x_i^4 \rangle$ for  $1 \le i \le 2n - 5$ ,  $T_{2n-4} = S^0 \cup S^2 \cup \langle s, s^0, s^7, s^6, q, s^4 \rangle$ ,  $T_{2n-3} = Y^0 \cup Y^1 \cup$  $Y^2 \cup \langle y^4, y^3, y^2, y^1, y^0, y^7, y^6, q \rangle$ ,  $T_{2n-2} = Q^0 \cup Q^1 \cup Q^3 \cup \langle q^2, q^1, q^0, q^7, q^6, q^5, q^4, r \rangle$ ,  $T_{2n-1} = P^1 \cup P^2 \cup P^3 \cup \langle p, p^6, p^5, p^4, p^3, p^2 \rangle$ , (see Figure 5(b)).

 $\begin{array}{l} r,r^2\rangle, \left\langle q^4,y^3,y^4,r,q^4\right\rangle, \left\langle q,y^4,y^5,q^4,q\right\rangle, \left\langle q,y^6,y^5,q^6,q\right\rangle, \left\langle q^7,y^6,y^7,q^6,q^7\right\rangle \, \mathrm{are} \, 4- \mathrm{cycles}. \ \mathrm{Since} \, |V(BH_{n-1}^i)| = 2^{2(n-1)} \ \mathrm{with} \, 2^{2(n-1)-1} \ \mathrm{black} \ \mathrm{vertices} \ \mathrm{and} \, 2^{2(n-1)-1} \\ \mathrm{white} \ \mathrm{vertices} \ \mathrm{and} \, 2^{2(n-1)-1} > 2n-1 \ \mathrm{for} \ i \in \{0,1,2,3\} \ \mathrm{and} \ n \geq 3, \ \mathrm{by} \ \mathrm{Lemma} \\ 11, \ \mathrm{we} \ \mathrm{can} \ \mathrm{find} \ \mathrm{another} \ 8-\mathrm{cycle} \, \left\langle z^7,z^0,z^1,z^2,z^3,z^4,z^5,z^6,z^7\right\rangle \ \mathrm{in} \ BH_n, \ \mathrm{where} \\ (z^7,z^0) \in E(BH_{n-1}^0), \, (z^1,z^2) \in E(BH_{n-1}^1), \, (z^3,z^4) \in E(BH_{n-1}^2), \ \mathrm{and} \, (z^5,z^6) \in \\ E(BH_{n-1}^3). \ \mathrm{In} \ \mathrm{ech} \ BH_{n-1}^i \ \mathrm{sfor} \ i \in \{0,2,3\}, \ \mathrm{the} \ \mathrm{discussions} \ \mathrm{are} \ \mathrm{similar} \ \mathrm{to} \ \mathrm{Case} \\ 2.1 \ \mathrm{except} \ \mathrm{that} \ \mathrm{we} \ \mathrm{only} \ \mathrm{need} \ \mathrm{to} \ \mathrm{use} \, z^k \ \mathrm{sinstead} \ \mathrm{of} \ s^k \ \mathrm{sfor} \ k \in \{0,3,4,5,6,7\} \ \mathrm{and} \\ \mathrm{use} \ Z^j \ \mathrm{sinstead} \ \mathrm{of} \ S^j \ \mathrm{sfor} \ j \in \{0,2,3\}. \ \mathrm{In} \ BH_{n-1}^1, \ \mathrm{let} \ X^1 = \{p^2,y^2,z^2,x_1^2,x_2^2,\ldots, \\ x_{2n-5}^2\}. \ \mathrm{By} \ \mathrm{Lemmas} \ 4 \ \mathrm{and} \ 14, \ BH_{n-1}^1 \ \mathrm{includes} \ (s,X^1) \ \mathrm{paths} \ P^1,Y^1,Z^1,X_1^1,\ldots, \\ X_{2n-5}^1, \ \mathrm{where} \ P^1 \ \mathrm{connects} \ s \ \mathrm{and} \ p^2, \ Y^1 = (s,y^2), \ Z^1 \ \mathrm{connects} \ s \ \mathrm{and} \ z^2, \ X_1^i \\ \mathrm{connects} \ s \ \mathrm{and} \ x_i^2 \ \mathrm{for} \ 1 \ \leq i \ 22n-5. \ \mathrm{Let} \ T_i = X_i^0 \cup X_i^1 \cup X_i^2 \cup X_i^3 \cup \\ \left\langle x_i^2, x_i^1, x_i^0, x_i^7, x_6^6, x_5^5, x_i^4 \right\rangle \ \mathrm{for} \ 1 \ \leq i \ \leq 2n-5, \ T_{2n-4} = \ Z^0 \cup Z^1 \cup Z^2 \cup Z^3 \cup \\ \left\langle z^2, z^1, z^0, z^7, z^6, z^5, z^4 \right\rangle, \ T_{2n-3} = Y^0 \cup Y^3 \cup \left\langle s, y^0, y^7, y^6, y^5, y^4, r \right\rangle, \ T_{2n-2} = \ S^0 \cup \\ \left\langle s^0, s, y^2, r \right\rangle \cup \left\langle y^2, y^3, y^4, q \right\rangle, \ \mathrm{and} \ T_{2n-1} = \ P^1 \cup P^2 \cup P^3 \cup \left\langle p, p^6, p^5, p^4, p^3, p^2 \right\rangle, \ (\mathrm{see} \ \mathrm{Figure} \ 6(\mathrm{a})). \end{array} \right$ 



Figure 6. The illustration of Case 3 and Case 4 in the proof of Lemma 24.

 $\begin{array}{l} x_{2}^{2},\ldots,x_{2n-4}^{2}\}, \ X^{2} \ = \ \{q^{2},y^{4},x_{1}^{4},x_{2}^{4},\ldots,x_{2n-4}^{4}\} \ \text{and} \ X^{3} \ = \ \{p^{3},y^{6},x_{1}^{6},x_{2}^{6},\ldots,x_{2n-4}^{6}\}. \ \text{By Lemmas 4 and 14}, \ BH_{n-1}^{0} \ \text{includes} \ (p,X^{0})\text{-paths } S^{0},Y^{0},X_{1}^{0},X_{2}^{0},\ldots,X_{2n-4}^{0}, \ BH_{n-1}^{1} \ \text{includes} \ (s,X^{1})\text{-paths } R^{1},Y^{1},X_{1}^{1},X_{2}^{1},\ldots,X_{2n-4}^{1}, \ BH_{n-1}^{2} \ \text{includes} \ (s,X^{1})\text{-paths } R^{1},Y^{1},X_{1}^{1},X_{2}^{1},\ldots,X_{2n-4}^{1}, \ BH_{n-1}^{2} \ \text{includes} \ Q^{2},Y^{2},X_{1}^{2},X_{2}^{2},\ldots,X_{2n-4}^{2}, \ \text{and} \ BH_{n-1}^{3} \ \text{includes} \ (q,X^{3})\text{-paths} \ P^{3},Y^{3},X_{1}^{3},X_{2}^{3},\ldots,X_{2n-4}^{3}, \ \text{where} \ S^{0} \ = \ (p,s^{0}), \ Y^{0} \ = \ (p,y^{0}), \ X_{i}^{0} \ \text{connects} \ p \ \text{and} \ x_{i}^{0}, \ R^{1} \ = \ (s,r^{1}), \ Y^{1} \ = \ (s,y^{2}), \ X_{i}^{1} \ \text{connects} \ s \ \text{and} \ x_{i}^{2}, \ Q^{2} \ = \ (r,q^{2}), \ Y^{2} \ = \ (r,y^{4}), \ X_{i}^{2} \ \text{connects} \ r \ \text{and} \ x_{i}^{4}, \ P^{3} \ = \ (q,p^{3}), \ Y^{3} \ = \ (q,y^{6}), \ X_{i}^{3} \ \text{connects} \ q \ \text{and} \ x_{i}^{6}, \ \text{where} \ 1 \ \le i \ \le 2n-4. \ \text{Let} \ T_{i} \ = \ X_{i}^{0} \ \cup \ X_{i}^{1} \ \cup \ X_{i}^{2} \ \cup \ X_{i}^{3} \ \cup \ \langle x_{i}^{2}, x_{i}^{1}, x_{0}^{0}, x_{i}^{7}, x_{i}^{6}, x_{i}^{5}, x_{i}^{4} \ \text{for} \ 1 \ \le i \ \le 2n-4, \ T_{2n-3} \ \le \ (p,s^{0},y^{1},r^{1},r,q^{2},q) \ \cup \ (s^{0},s), \ T_{2n-2} \ = \ \langle q,y^{6},p,y^{0},s,y^{2},r \ \rangle, \ \text{and} \ T_{2n-1} \ = \ \langle p,p^{3},q,y^{4},y^{3},r^{1},s \ \lor \ (y^{4},r), \ (\text{see Figure 6(b)}). \end{array}$ 

**Lemma 25.** Let  $N \subset V(BH_n)$  be such that  $|N \cap V(BH_n)| = 4$  and N does not contain paired vertices. If there exist two sub-balanced hypercubes such that each has two vertices of N, then there are 2n - 1 internally disjoint N-trees in  $BH_n$ , where  $n \geq 3$ .

**Proof.** Denote  $N = \{p, q, r, s\}$ . Without loss of generality, let  $N \cap V(BH_{n-1}^0) = \{p, q\}$ . By symmetry of  $BH_{n-1}^1$  and  $BH_{n-1}^3$ , we only need to consider that both r and s are in  $BH_{n-1}^1$  or  $BH_{n-1}^2$ .

Let p and q be different color vertices, and r and s be different colors. Without loss of generality, let p and r be black vertices and q and s be white vertices. If pand q are the same color (since  $BH_n$  is a bipartite graph), we only need to consider that p and q are black vertices (see Figure 12–14), or p and q are different colors but r and s are the same color (see Figure 15–16), the proofs are similar. To save space, we only show the graphs in the Appendix 2.) By Lemmas 4 and 13,  $BH_{n-1}^0$  includes 2n - 2 internally disjoint paths  $P_i$ s connecting p and q, where  $1 \le i \le 2n - 2$ . By Definition 2, p has a neighbor  $p^3 \in V(BH_{n-1}^3)$ , and q has a neighbor  $q^1 \in V(BH_{n-1}^1)$ . By Lemma 19 and Definition 2,  $BH_n$  includes a path  $\tilde{Q} = \langle q, q^0, q^2, q^3, q^4 \rangle$ , where  $q^0 \in V(BH_{n-1}^1)$ ,  $\{q^2, q^3\} \subset V(BH_{n-1}^0)$  and  $q^4 \in V(BH_{n-1}^3)$ .

Case 1.  $\{r,s\} \subset V(BH_{n-1}^1)$ . By Lemmas 4 and 13,  $BH_{n-1}^1$  includes 2n-2 internally disjoint paths  $R_i$ s connecting r and s, where  $1 \leq i \leq 2n-2$ . (If  $(p,q) \in E(BH_{n-1}^0)$ , let  $P_{2n-3} = (p,q)$ . If  $(r,s) \in E(BH_{n-1}^1)$ , let  $R_{2n-2} = (r,s)$ .) By Definition 2, r has a neighbor  $r^7 \in V(BH_{n-1}^0)$ , and s has a neighbor  $s^2 \in V(BH_{n-1}^2)$ . By Lemma 19 and Definition 2,  $BH_n$  includes a path  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , where  $r^0 \in V(BH_{n-1}^0)$ ,  $\{r^1, r^2\} \subset V(BH_{n-1}^1)$ , and  $r^3 \in V(BH_{n-1}^2)$ . Since  $BH_{n-1}^0$  is connected,  $BH_{n-1}^0$  includes a path  $R^1$  connecting  $r^7$  and q. Let v be the first intersection vertex of  $R^1$  and  $\bigcup_{i=1}^{2n-2} P_i$ . Since  $BH_{n-1}^1$  is connected,  $BH_{n-1}^1$  includes a path  $v \in V(P_{2n-2})$  and  $u \in V(R_{2n-3})$ . Let  $R[r^7, v]$  be the sub-path of  $R^1$  and  $Q[q^1, u]$  be the sub-path of  $Q^1$ . Pick black vertex

 $x_j^0 \in V(P_j)$  and white vertex  $y_j^1 \in V(R_j)$  for  $1 \le j \le 2n - 4$ . By Definition 2,  $x_j^0$  has a neighbor  $x_j^3 \in V(BH_{n-1}^3)$  and  $y_j^1$  has a neighbor  $y_j^2 \in V(BH_{n-1}^2)$  for  $1 \le j \le 2n - 4$ .

By Lemma 17,  $BH_n$  includes a 4-cycle  $\langle b^3, b^2, c^3, c^2, b^3 \rangle$ , where  $\{b^2, c^2\} \subset V(BH_{n-1}^2)$  and  $\{b^3, c^3\} \subset V(BH_{n-1}^3)$ . Pick another 2n - 4  $(a_j^3, a_j^2)$ s, where  $a_j^3 \in V(BH_{n-1}^3)$  and  $a_j^2 \in V(BH_{n-1}^2)$  for  $1 \leq j \leq 2n - 4$ . Let  $X^3 = \{p^3, q^4, x_1^3, x_2^3, \ldots, x_{2n-4}^3\}$ ,  $A^3 = \{b^3, c^3, a_1^3, a_2^3, \ldots, a_{2n-4}^3\}$ ,  $Y^2 = \{s^2, r^3, y_1^2, y_2^2, \ldots, y_{2n-4}^2\}$ , and  $A^2 = \{b^2, c^2, a_1^2, a_2^2, \ldots, a_{2n-4}^2\}$ . By Lemmas 4 and 15,  $BH_{n-1}^3$  includes  $(X^3, A^3)$ -paths  $P, Q, Q_1, Q_2, \ldots, Q_{2n-4}$  and  $BH_{n-1}^2$  includes  $(A^2, Y^2)$ -paths  $R, S, S_1, S_2, \ldots, S_{2n-4}$ , where P connects  $p^3$  and  $b^3, Q$  connects  $q^4$  and  $c^3, Q_j$  connects  $x_j^3$  and  $a_j^3$ , R connects  $b^2$  and  $s^2, S$  connects  $c^2$  and  $r^3$ , and  $S_j$  connects  $a_j^2$  and  $y_j^2, 1 \leq j \leq 2n - 4$ . Let  $T_j = P_j \cup (x_j^0, x_j^3) \cup Q_j \cup (a_j^3, a_j^2) \cup S_j \cup (y_j^2, y_j^1) \cup R_j$  for  $1 \leq j \leq 2n - 4$ ,  $T_{2n-3} = P_{2n-3} \cup (q, q^1) \cup Q[q^1, u] \cup R_{2n-3}, T_{2n-2} = P_{2n-2} \cup R[v, r^7] \cup (r^7, r) \cup R_{2n-2}$ , and  $T_{2n-1} = P \cup Q \cup R \cup S \cup \tilde{Q} \cup (p, p^3) \cup \langle b^3, b^2, c^3, c^2 \rangle \cup \tilde{R} \cup (s, s^2)$ , (see Figure 7(a)).



Figure 7. The illustrations of Case 1 and Case 2 in the proof of Lemma 25.

Case 2.  $\{r,s\} \subset V(BH_{n-1}^2)$ . By Lemmas 4 and 13,  $BH_{n-1}^2$  includes 2n-2 internally disjoint paths  $Q_i$ s connecting r and s, where  $1 \leq i \leq 2n-2$ . (If  $(p,q) \in E(BH_{n-1}^0)$ , let  $P_{2n-2} = (p,q)$ . If  $(r,s) \in E(BH_{n-1}^2)$ , let  $Q_{2n-3} = (r,s)$ .) By Lemma 19 and Definition 2,  $BH_n$  includes a path  $\tilde{R} = \langle r, r^1, r^5, r^4, r^3 \rangle$ , an edge  $(r, r^2)$ , and an edge  $(s, s^3)$ , where  $\{r^1, r^2\} \subset V(BH_{n-1}^1), \{r^4, r^5\} \subset V(BH_{n-1}^2)$ , and  $\{r^3, s^3\} \subset V(BH_{n-1}^3)$ . Select white vertex  $x_j^0 \in V(P_j)$  (re-

spectively, black vertex  $y_k^2 \in V(Q_k)$ ), by Definition 2,  $x_j^0$  (respectively,  $y_k^2$ ) has a neighbor  $x_j^1 \in V(BH_{n-1}^1)$  (respectively,  $y_k^1 \in V(BH_{n-1}^1)$ ), where  $1 \leq j \leq 2n-3$  (respectively,  $1 \leq k \leq 2n-2$ ). Let  $(y_{2n-3}^2, y_{2n-3}^1) = (r, r^2)$ . Let  $X^1 = \{x_1^1, x_2^1, \dots, x_{2n-4}^1, x_{2n-3}^1, q^1\}$  and  $Y^1 = \{y_1^1, y_2^1, \dots, y_{2n-4}^1, r^2, y_{2n-2}^1\}$ . By Lemmas 4 and 15,  $BH_{n-1}^1$  includes paired  $(X^1, Y^1)$ -paths  $R_1, R_2, \dots, R_{2n-2}$ , where  $R_j$  connects  $x_j^1$  and  $y_{2n-2}^1$ . Since  $BH_{n-1}^3$  is connected,  $BH_{n-1}^3$  includes a tree  $T'_{2n-1}$  connecting  $q^4, p^3, r^3$  and  $s^3$ . Let  $T_i = P_j \cup (x_j^0, x_j^1) \cup R_j \cup (y_j^1, y_j^2) \cup Q_j$ for  $1 \leq j \leq 2n-4$ ,  $T_{2n-3} = P_{2n-3} \cup (x_{2n-3}^0, x_{2n-3}^1) \cup R_{2n-3} \cup (r^2, r) \cup Q_{2n-3}$ ,  $T_{2n-2} = P_{2n-2} \cup (q, q^1) \cup R_{2n-2} \cup (y_{2n-2}^1, y_{2n-2}^2) \cup Q_{2n-2}$ , and  $T_{2n-1} = T'_{2n-1} \cup \tilde{Q} \cup (p^3, p) \cup \tilde{R} \cup (s^3, s)$ , (see Figure 7(b)).

**Lemma 26.** Let  $N \subset V(BH_n)$  be such that  $|N \cap V(BH_n)| = 4$  and N does not contain paired vertices. If there exist three sub-balanced hypercubes having 2, 1, and 1 vertices of N, respectively, then there are 2n-1 internally disjoint N-trees in  $BH_n$ , where  $n \geq 3$ .

**Proof.** The proof is in Appendix 3.

**Lemma 27.** Let  $N \subset V(BH_n)$  be such that  $|N \cap V(BH_n)| = 4$  and N does not contain paired vertices. If there exist two sub-balanced hypercubes having 3 and 1 vertices of N, respectively, then there are 2n - 1 internally disjoint N-trees in  $BH_n$ , where  $n \geq 3$ .

**Proof.** Denote  $N = \{p, q, r, s\}$ . Without loss of generality, let  $\{p, q, r\} \subset V(BH_{n-1}^0)$ , p, r be black vertices, and q be white vertex. (If p, r are white vertices and q is black vertex, or p, q, r are the same color, the discussions are similar except that we need to use Lemma 19 and Definition 2 to find two paths connecting p or r are in  $BH_{n-1}^3$  and the other end vertices of the two paths connecting q are in  $BH_{n-1}^2$  for Case 1, and find two neighbors of p or r in  $BH_{n-1}^1$  and find two paths connecting with q and the other end vertices in  $BH_{n-1}^1$  for Case 2.) By Definition 2, p has two neighbors  $p^3, p^4$  in  $BH_{n-1}^3$ . The two neighbors  $r^3, r^4$  in  $BH_{n-1}^3$ , and q has two neighbors  $q^1, q^2$  in  $BH_{n-1}^1$ . By Lemmas 4 and 5,  $BH_{n-1}^0$  includes 2n - 3 internally disjoint N-trees  $T'_j$ s, where  $1 \le j \le 2n - 3$ . Since  $BH_n$  is symmetric, we deal with the following Case 1 and Case 2.

Case 1.  $s \in V(BH_{n-1}^1)$ . Pick one white vertex  $x_j^0 \in V(T'_j)$  for  $1 \le j \le 2n-3$ . By Definition 2,  $x_j^0$  has a neighbor  $x_j^1 \in V(BH_{n-1}^1)$ , where  $1 \le j \le 2n-3$ . Let  $X^1 = \{x_1^1, x_2^1, \ldots, x_{2n-3}^1\}$ . By Lemmas 4 and 14,  $BH_{n-1}^1$  includes  $(s, X^1)$ -paths  $R_j$ s, where  $R_j$  connects s and  $x_j^1$  for  $1 \le j \le 2n-3$ . Let  $T_j = T'_j \cup (x_j^0, x_j^1) \cup R_j$  for  $1 \le j \le 2n-3$ . By Lemma 19 and Definition 2,  $BH_n$  includes two paths  $\tilde{Q_1} = \langle q, q^1, q^3, q^5 \rangle$  and  $\tilde{Q_2} = \langle q, q^2, q^4, q^6 \rangle$ , where  $\{q^1, q^2, q^3, q^4\} \subset V(BH_{n-1}^1)$  and

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 $\begin{array}{l} \{q^5,q^6\} \subset V(BH_{n-1}^2). \text{ By Lemma 17 and Definition 2, } BH_n \text{ includes two vertex-disjoint 4-cycles } \left\langle c^3,c^2,d^3,d^2,c^3\right\rangle \text{ and } \left\langle a^3,a^2,b^3,b^2,a^3\right\rangle, \text{ where } \left\{a^3,b^3,c^3,d^3\right\} \subset V(BH_{n-1}^3) \text{ and } \left\{a^2,b^2,c^2,d^2\right\} \subset V(BH_{n-1}^2). \end{array}$ 

Case 1.1. s is white vertex. By Definition 2, s has two neighbors  $s^1$  and  $s^2$  in  $BH_{n-1}^2$ . By Lemma 15,  $BH_{n-1}^3$  includes vertex-disjoint paths P, Q, P', Q' connecting  $r^3, p^4, r^4, p^3$  and  $c^3, d^3, a^3, b^3$ , respectively, and  $BH_{n-1}^2$  includes vertex-disjoint paths R, S, R', S' connecting  $q^6, s^1, s^2, q^5$  and  $c^2, d^2, a^2, b^2$ , respectively. Let  $T_{2n-2} = P \cup Q \cup R \cup S \cup (p, p^4) \cup (r, r^3) \cup (s, s^1) \cup \tilde{Q}_2 \cup \langle c^3, c^2, d^3, d^2 \rangle$ , and  $T_{2n-1} = P' \cup Q' \cup R' \cup S' \cup (p, p^3) \cup (r, r^4) \cup (s, s^2) \cup \tilde{Q}_1 \cup \langle a^3, a^2, b^3, b^2 \rangle$ , (see Figure 8(a)).



Figure 8. The illustrations of Case 1.1 and Case 1.2 in the proof of Lemma 27.

Case 1.2. s is black vertex. The proof of this case is similar to Case 1.1 except that we need to use  $\tilde{S}_1 = \langle s, s^0, s^2, s^4, s^6 \rangle$  and  $\tilde{S}_2 = \langle s, s^1, s^3 \rangle$  to replace  $(s, s^1)$  and  $(s, s^2)$  of Case 1.1, respectively, where  $s^0 \in V(BH_{n-1}^0)$ ,  $s^2$  is a paired vertex of s in  $BH_{n-1}^1$ ,  $s^1$  is not in  $(s, X^1)$ -paths,  $s^4$  is in some path of  $(s, X^1)$ -paths,  $(s^2, s^1) \in E(BH_{n-1}^1)$ ,  $\{s^6, s^3\} \subset V(BH_{n-1}^2)$ . (Since  $|N_{BH_{n-1}^1}(s)| = 2n - 2$ , there exists one neighbor  $s^1$  not in  $(s, X^1)$ -paths.) Let  $T_{2n-2} = P \cup Q \cup R \cup S \cup (p, p^4) \cup (r, r^3) \cup \langle c^3, c^2, d^3, d^2 \rangle \cup \tilde{S}_1 \cup \tilde{Q}_2$  and  $T_{2n-1} = P' \cup Q' \cup R' \cup S' \cup (p, p^3) \cup (r, r^4) \cup \langle a^3, a^2, b^3, b^2 \rangle \cup \tilde{S}_2 \cup \tilde{Q}_1$ , (see Figure 8(b)).

Case 2.  $s \in V(BH_{n-1}^2)$ . Select 2n-3 edges  $(x_j^0, x_j^3)$ s, where  $x_j^0 \in V(T'_j)$  and

 $x_j^3 \in V(BH_{n-1}^3)$  for  $1 \leq j \leq 2n-3$ . Pick p and r's paired vertices and denote by  $\tilde{p}$  and  $\tilde{r}$ , respectively. Since p (respectively, r) has 2n-2 neighbors in  $BH_{n-1}^0$ , there exists one neighbor  $p^1$  (respectively,  $r^1$ ) in  $BH_{n-1}^0$  such that  $(p, p^1)$  (respectively,  $(r, r^1)$ ) is not in  $T_j$ s for  $1 \leq j \leq 2n-3$ . By Lemma 18,  $(p, p^1)$  and its paired edge  $(\tilde{p}, \tilde{p}^1)$  are in two paired cycles C and C', respectively. Let  $\langle p, p^1, p^2 \rangle$  (respectively,  $\langle \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle$ ) be part of C (respectively, C'), where  $\{p^2, \tilde{p}^2\} \subset V(BH_{n-1}^1)$ . The discussion for vertex r is similar, thus we have that  $\langle r, r^1, r^2 \rangle$  and  $\langle \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle$  are parts of two paired cycles, respectively, where  $\{r^2, \tilde{r}^2\} \subset V(BH_{n-1}^1)$ . Let  $\widetilde{Q} = \{q^1, q^2\}, \ \widetilde{P} = \{p^2, \tilde{p}^2\}, \ \widetilde{R} = \{r^2, \tilde{r}^2\}$ .

 $\begin{array}{l} Case \ 2.1. \ s \ \text{is black vertex. By Definition 2, $s$ has two neighbors $s^1$ and $s^2$ in $BH_{n-1}^1$. Let $\widetilde{S} = \{s^1, s^2\}$. By Definition 3 and Lemmas 6 and 14, $BH_{n-1}^1$ has $(\widetilde{S}, \{\widetilde{P}, \widetilde{Q}, \widetilde{R}\})$-paths. By Lemma 16, $BH_{n-1}^1$ includes vertex-disjoint paths $Q[q^1, s^1], Q[q^2, s^2], P[\widetilde{p}^2, s^1], P[p^2, s^2], R[r^2, s^1], R[\widetilde{r}^2, s^2]$. Select $2n-3$ neighbors of $s$ in $BH_{n-1}^2$ and denote by $s_j^2$s for $1 \le j \le 2n-3$. By Definition 2, let $s_j^3$ be a neighbor of $s_j^2$ in $BH_{n-1}^3$, where $1 \le j \le 2n-3$. Let $X^3 = \{x_1^3, x_2^3, \ldots, x_{2n-3}^3\}$ and $S^3 = \{s_1^3, s_2^3, \ldots, s_{2n-3}^3\}$. By Lemmas 4 and 15, there are paired $(X^3, S^3)$-paths $Q_j$s in $BH_{n-1}^3$, where $Q_j$ connects $x_j^3$ and $s_j^3$ for $1 \le j \le 2n-3$. Let $T_j = T_j' \cup (x_j^0, x_j^3) \cup Q_j \cup \langle s_j^3, s_j^2, s \rangle$ for $1 \le j \le 2n-3$, $T_{2n-2} = (q, q^1) \cup $Q[q^1, s^1] \cup \langle p, p^3, \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle \cup P[\tilde{p}^2, s^1] \cup \langle r, r^1, r^2 \rangle \cup R[r^2, s^1] \cup (s^1, s)$ and $T_{2n-1} = (q, q^2) \cup Q[q^2, s^2] \cup \langle p, p^1, p^2 \rangle \cup P[p^2, s^2] \cup \langle r, r^3, \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle \cup R[\tilde{r}^2, s^2] \cup (s^2, s)$, (see Figure 9(a)). \\ \end{array}$ 

Case 2.2. s is white vertex. The proof of this case is similar to Case 2.1 except that we need to use  $\tilde{S}_j = \langle s, s_j, s_j^2, s_j^3 \rangle$  to instead of  $\langle s, s_j^2, s_j^3 \rangle$  of Case 2.1 for  $1 \leq j \leq 2n-3$ , and use  $S_1 = \langle s, s^3, \tilde{s}, s_1, s^1 \rangle$  and  $S_2 = \langle s, \tilde{s}^3, \tilde{s}, \tilde{s}^2, \tilde{s}^1 \rangle$  to instead of  $(s, s^1)$  and  $(s, s^2)$  of Case 2.1, respectively, where  $\tilde{s}$  is the paired vertex of s,  $\tilde{s}^3$  and  $s^3$  are common neighbors of s and  $\tilde{s}$  in  $BH_{n-1}^3$ ,  $\tilde{s}^2 \notin \{s_1, s_2, \ldots, s_{2n-3}\}$ , and  $s^1$  and  $\tilde{s}^1$  are common neighbors of  $\tilde{s}^2$  and  $s_1$  and they are paired vertices. (Since  $\tilde{s}$  has 2n-2 neighbors in  $BH_{n-1}^2$ , there exists such vertex  $\tilde{s}^2$ . Without loss of generality, let  $s_1$  be the paired vertex of  $\tilde{s}^2$ .) So  $(\tilde{s}, s_j) \in E(BH_{n-1}^2)$  for  $1 \leq j \leq 2n-3$ . Let  $T_j = T'_j \cup (x_j^0, x_j^3) \cup Q_j \cup \tilde{S}_j$  for  $1 \leq j \leq 2n-3$ ,  $T_{2n-2} = (q, q^1) \cup Q[q^1, s^1] \cup \langle p, p^3, \tilde{p}, \tilde{p}^1, \tilde{p}^2 \rangle \cup P[\tilde{p}^2, s^1] \cup \langle r, r^1, r^2 \rangle \cup R[r^2, s^1] \cup S_1$  and  $T_{2n-1} = (q, q^2) \cup Q[q^2, \tilde{s}^1] \cup \langle p, p^1, p^2 \rangle \cup P[p^2, \tilde{s}^1] \cup \langle r, r^3, \tilde{r}, \tilde{r}^1, \tilde{r}^2 \rangle \cup R[\tilde{r}^2, \tilde{s}^1] \cup S_2$ , (see Figure 9(b)).

**Theorem 28.**  $\kappa_4(BH_n) = 2n - 1$ , where  $n \ge 1$ .

**Proof.** The proof is by induction hypothesis on n. By Lemmas 22 and 23, the theorem holds when  $n \leq 2$ . Assume that the theorem holds for  $m \leq n - 1$ . We prove that the theorem holds for  $m = n \geq 3$  as follows. For any  $N \subset V(BH_n)$  with |N| = 4, we denote  $N = \{p, q, r, s\}$ . By Lemmas 4 and 12,  $\kappa_4(BH_n) \leq 2n-1$ .

We need to show that  $BH_n$  includes 2n-1 internally disjoint *N*-trees. By Lemma 21,  $BH_n$  includes 2n-1 internally disjoint *N*-trees if *N* contains paired vertices. In the following, we consider that *N* contains no paired vertices for  $n \ge 3$ .



Figure 9. The illustrations of Case 2.1 and Case 2.2 in the proof of Lemma 27.

Case 1. All the vertices of N are in the same sub-balanced hypercube. By symmetry of  $BH_n$ , let  $N \subset V(BH_{n-1}^0)$ . By induction hypothesis,  $BH_{n-1}^0$  contains 2n-3 internally disjoint N-trees  $T_j$ s for  $1 \leq j \leq 2n-3$ . By Definition 1, each vertex of p, q, r, s has paired neighbors  $p^1, p^2, q^1, q^2, r^1, r^2, s^1, s^2$  in  $BH_{n-1}^1$  or  $BH_{n-1}^3$ , respectively. By Lemma 18, each paired neighbors of N is included in vertex-disjoint 8-cycles of  $BH_n$ . Let  $P[p^1, p']$  and  $P[p^2, p''], Q[q^1, q']$  and  $Q[q^2, q''], R[r^1, r']$  and  $R[r^2, r'']$ , and  $S[s^1, s']$  and  $S[s^2, s'']$  be sub-paths of the two disjoint 8-cycles, respectively, where  $N' = \{p', p'', q', q'', r', r'', s', s''\} \subset V(BH_{n-1}^2)$ . Select one vertex  $v \in V(BH_{n-1}^2)$  and  $v \notin N'$ . By Lemmas 4 and 14,  $BH_{n-1}^2$  includes (v, N')-paths  $P^1, P^2, Q^1, Q^2, R^1, R^2, S^1, S^2$ , where  $X^1$  connects x' and v and  $X^2$  connects x'' and v for X = P, Q, R, S and x = p, q, r, s, respectively. Let  $T_{2n-2} = P^1 \cup Q^1 \cup R^1 \cup S^1 \cup P[p^1, p'] \cup Q[q^1, q'] \cup R[r^1, r'] \cup S[s^1, s'] \cup (p, p^1) \cup (q, q^1) \cup (r, r^1) \cup (s, s^1)$  and  $T_{2n-1} = P^2 \cup Q^2 \cup R^2 \cup S^2 \cup P[p^2, p''] \cup Q[q^2, q''] \cup R[r^2, r''] \cup S[s^2, s''] \cup (p, p^2) \cup (q, q^2) \cup (r, r^2) \cup (s, s^2)$ .

Case 2. Each sub-balanced hypercube has one vertex of N.

Case 3. Two sub-balanced hypercubes have two vertices of N, respectively.

Case 4. Three sub-balanced hypercubes have 2, 1, and 1 vertices of N, respectively.

Case 5. Two sub-balanced hypercubes have 3 and 1 vertices of N, respectively.

By Lemmas 24, 25, 26, and 27,  $BH_n$  includes 2n-1 internally disjoint N-trees for the above Cases 2–5, respectively.

Hence, the theorem holds.

## 4. Conclusion

In [17],  $\kappa_3(BH_n) = 2n - 1$  is determined, in this paper, we further obtain that  $k_4(BH_n) = 2n - 1$ , where  $n \ge 1$ . Since it is NP-complete to compute  $\kappa_k(G)$  when G is general [6], the method of our paper can be a reference to determine the generalized 4-connectivity of other special networks.

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## Appendix

#### Appendix 1. The proof of Lemma 23.

**Proof.** For any vertex set  $N \subset V(BH_2)$  with |N| = 4, we denote  $N = \{p, q, s, t\}$ . By Lemmas 4 and 12,  $\kappa_4(BH_2) \leq 2 \times 2 - 1 = 3$ . We need to show that  $BH_2$ includes 3 internally disjoint N-trees. Note that any two black vertices (respectively, white vertices) of  $BH_1^i$  are paired vertices, where  $i \in \{0, 1, 2, 3\}$ . If  $3 \leq |N \cap V(BH_1^i)| \leq 4$  for some  $i \in \{0, 1, 2, 3\}$ , then N contains paired vertices. By Lemma 21,  $BH_2$  contains 3 internally disjoint N-trees if N contains paired vertices. Hence, we only need to consider the following two cases.

Case 1.  $|N \cap V(BH_1^i)| = 2$ , say  $N \cap V(BH_1^i) = \{p,q\}$ , and  $(p,q) \in E(BH_1^i)$ , where  $i \in \{0, 1, 2, 3\}$ . Without loss of generality, let i = 0. By Lemma 8, we only need to consider  $\{p,q\} = \{(0,0), (3,0)\}$ . By symmetry of  $BH_2$ , we only need to consider  $(s,t) \in E(BH_1^1), (s,t) \in E(BH_1^2)$ , and s,t are in two different sub-balanced hypercubes of  $BH_1^1 \cup BH_1^2 \cup BH_1^3$ . Since the two black vertices (respectively, white vertices) of  $BH_1^i$  for  $i \in \{0, 1, 2, 3\}$  are paired vertices, we only need to consider the distributions of s, t shown in Figure 10. The 3 internally disjoint N-trees with red, green, blue colors, respectively, are shown in Figure 10.

Case 2.  $|N \cap V(BH_1^i)| = 1$  for any  $i \in \{0, 1, 2, 3\}$ . Without loss of generality, let p, q, s, t be in  $BH_1^0, BH_1^3, BH_1^1, BH_1^2$ , respectively. By Lemma 7, let p = (0, 0). Since the two black vertices (respectively, white vertices) of  $BH_1^i$  for  $i \in \{0, 1, 2, 3\}$  are paired vertices, we only need to consider  $s \in \{(1, 1), (0, 1)\}, q \in \{(0, 3), (1, 3)\}, t \in \{(0, 2), (3, 2)\}$ . The 3 internally disjoint N-trees with red, green, blue colors, respectively, are shown in Figure 11.



Figure 10. The illustration of Case 1 in the proof of Lemma 23.



Figure 11. The illustration of Case 2 in the proof of Lemma 23.



Appendix 2. The graphs of other cases in the proof of Lemma 25.

Figure 12. p and q are black vertices of  $BH_{n-1}^0$ , and r and s are the same color of  $BH_{n-1}^1$  in the proof of Lemma 25.



Figure 13. p and q are black vertices of  $BH_{n-1}^0$ , and r and s are the same color of  $BH_{n-1}^2$  in the proof of Lemma 25.



Figure 14. p and q are black vertices of  $BH_{n-1}^0$ , and r and s are different colors of  $BH_{n-1}^1$  and  $BH_{n-1}^2$  in the proof of Lemma 25.



Figure 15. p and q are different colors of  $BH_{n-1}^0$ , and r and s are the same color of  $BH_{n-1}^1$  in the proof of Lemma 25.



Figure 16. p and q are different colors of  $BH_{n-1}^0$ , and r and s are the same color of  $BH_{n-1}^2$  in the proof of Lemma 25.

## Appendix 3. The proof of Lemma 26.

**Proof.** For any vertex set  $N \,\subset V(BH_2)$  with |N| = 4, we denote  $N = \{p, q, s, t\}$ . By symmetry of  $BH_n$ , let  $\{p, q\} \subset V(BH_{n-1}^0)$ . Without loss of generality, let p and q be different colors, say p is white vertex and q is black vertex. (If p and q are with the same color, by Lemma 19 and Definition 2,  $BH_{n-1}^0$  includes a path or an edge connecting p or q with the other end vertex in  $BH_{n-1}^3$  (respectively,  $BH_{n-1}^2$ ) for Case 1 (respectively, Case 2).) By Lemmas 4 and 13,  $BH_{n-1}^0$  includes 2n-2 internally disjoint paths  $P_j$ s connecting p and q, where  $1 \leq j \leq 2n-2$ . Without loss of generality, we only need to consider that r is black vertex and s is white vertex. (If r is white vertex and s is black vertex, or r and s are with the same colors, by Lemma 19 and Definition 2,  $BH_n$  includes a path or an edge connecting r or s such that the other end vertices are in  $BH_{n-1}^3$  (respectively,  $BH_{n-1}^2$ ) for Case 1 (respectively, Case 2).) By symmetry of  $BH_{n-1}^1$  and  $BH_{n-1}^3$ , we only need to consider that the other end vertices are in  $BH_{n-1}^3$  (respectively,  $BH_{n-1}^2$ ) for Case 1 (respectively, Case 2).) By symmetry of  $BH_{n-1}^1$  and  $BH_{n-1}^3$ , we only need to consider two cases.

Case 1. r and s are in  $BH_{n-1}^1$  and  $BH_{n-1}^2$ , respectively, say  $r \in V(BH_{n-1}^1)$ and  $s \in V(BH_{n-1}^2)$ . By Definition 2, we select one edge  $(x_j^0, x_j^1)$ , where  $x_j^0 \in V(P_j)$ , and  $x_j^1 \in V(BH_{n-1}^1)$  for  $1 \leq j \leq 2n-2$ . Let  $X^1 = \{x_1^1, x_2^1, \dots, x_{2n-2}^1\}$ . By Lemmas 4 and 14,  $BH_{n-1}^1$  includes  $(r, X^1)$ -paths  $Q_j$ s, where  $Q_j$  connects  $x_j^1$  and r for  $1 \leq j \leq 2n-2$ . Pick one white vertex  $y_j^1 \in Q_j$ , where  $1 \leq j \leq 2n-2$ . By Definition 2,  $y_j^1$  has a neighbor  $y_j^2 \in V(BH_{n-1}^2)$ , where  $1 \leq j \leq 2n-2$ . Let  $Y^2 = \{y_1^2, y_2^2, \dots, y_{2n-2}^2\}$ . By Lemmas 4 and 14,  $BH_{n-1}^2$  includes  $(s, Y^2)$ -paths  $R_j$ s, where  $R_j$  connects  $y_j^2$  and s for  $1 \le j \le 2n-2$ . Let  $T_j = P_j \cup (x_j^0, x_j^1) \cup Q_j \cup (y_j^1, y_j^2) \cup R_j$  for  $1 \le j \le 2n-2$ .

By Lemma 19 and Definition 2,  $BH_n$  includes three paths  $\tilde{P} = \langle p, p^1, p^0, p^6, p^3 \rangle$ and  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3, r^4, r^5 \rangle$ , where  $\{r^0, p^0, p^6\} \subset V(BH_{n-1}^0)$ ,  $\{p^1, r^1, r^2\} \subset V(BH_{n-1}^1)$ ,  $\{r^3, r^4\} \subset V(BH_{n-1}^2)$ , and  $\{p^3, r^5\} \subset V(BH_{n-1}^3)$ . By Definition 2, q (respectively, s) has a neighbor  $q^3$  (respectively,  $s^3$ ) in  $BH_{n-1}^3$ . Since  $BH_{n-1}^3$ is connected,  $BH_{n-1}^3$  includes a tree  $T'_{2n-1}$  connecting  $p^3, q^3, s^3$  and  $r^5$ . Let  $T_{2n-1} = T'_{2n-1} \cup \tilde{P} \cup (q^3, q) \cup (s^3, s) \cup \tilde{R}$ , (see Figure 17(a)).



Figure 17. The illustrations of Case l and Case 2 in the proof of Lemma 26.

Case 2. r and s are in  $BH_{n-1}^1$  and  $BH_{n-1}^3$ , respectively, say  $r \in V(BH_{n-1}^1)$ and  $s \in V(BH_{n-1}^3)$ .

Pick one white vertex  $x_j^0 \in V(P_j)$  and denote  $(x_j^0, y_j^0) \in E(P_j)$  for  $1 \leq j \leq 2n-2$ . By Definition 2,  $x_j^0$  (respectively,  $y_j^0$ ) has a neighbor  $x_j^1 \in V(BH_{n-1}^1)$  (respectively,  $y_j^3 \in V(BH_{n-1}^3)$ ), where  $1 \leq j \leq 2n-2$ . Let  $X^1 = \{x_1^1, x_2^1, \dots, x_{2n-2}^1\}$  and  $Y^3 = \{y_1^3, y_2^3, \dots, y_{2n-2}^3\}$ . By Lemmas 4 and 14,  $BH_{n-1}^1$  includes  $(r, X^1)$ -paths  $Q_j$ s, where  $Q_j$  connects  $x_j^1$  and r for  $1 \leq j \leq 2n-2$ . By Lemmas 4 and 14,  $BH_{n-1}^1$  includes  $(r, X^1)$ -paths  $Q_j$ s, where  $Q_j$  connects  $x_j^1$  and r for  $1 \leq j \leq 2n-2$ . By Lemmas 4 and 14,  $BH_{n-1}^3$  includes  $(s, Y^3)$ -paths  $R_j$ s, where  $R_j$  connects  $y_j^3$  and s for  $1 \leq j \leq 2n-2$ . Let  $T_j = P_j \cup Q_j \cup R_j \cup (x_j^0, x_j^1) \cup (y_j^0, y_j^3)$  for  $1 \leq j \leq 2n-2$ .

By Lemma 19 and Definition 2,  $BH_n$  includes four paths  $\tilde{P} = \langle p, p^1, p^2, p^3 \rangle$ ,  $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , and  $\tilde{S} = \langle s, s^0, s^1, s^2, s^3 \rangle$ , where  $\{p^1, p^2, p^3\}$ ,  $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , and  $\tilde{S} = \langle s, s^0, s^1, s^2, s^3 \rangle$ , where  $\{p^1, p^2, p^3\}$ ,  $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , and  $\tilde{S} = \langle s, s^0, s^1, s^2, s^3 \rangle$ , where  $\{p^1, p^2, q^3\}$ ,  $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , and  $\tilde{S} = \langle s, s^0, s^1, s^2, s^3 \rangle$ ,  $\tilde{Q} = \langle p, p^1, p^2, q^3 \rangle$ ,  $\tilde{Q} = \langle q, q^1, q^2, q^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ , and  $\tilde{S} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^1, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^2, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^2, r^2, r^2, r^2, r^3 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^2, r^2, r^2, r^2, r^2 \rangle$ ,  $\tilde{R} = \langle r, r^0, r^2, r^2, r^2, r^2, r^2 \rangle$ ,  $\tilde{R} = \langle r, r^2, r^2, r^2, r^2 \rangle$ ,  $\tilde{R} = \langle r, r^2, r^2, r^2 \rangle$ ,  $\tilde{R} = \langle r, r^2 \rangle$ ,  $\tilde{R} =$ 

 $\begin{array}{l} r^{1}, r^{2} \} \subset V(BH_{n-1}^{1}), \{p^{3}, q^{3}, r^{3}, s^{3}\} \subset V(BH_{n-1}^{2}), \text{ and } \{q^{1}, q^{2}, s^{1}, s^{2}\} \subset V(BH_{n-1}^{3}). \\ \text{Since } BH_{n-1}^{2} \text{ is connected}, BH_{n-1}^{2} \text{ includes a tree } T'_{2n-1} \text{ connecting } p^{3}, q^{3}, r^{3} \text{ and } s^{3}. \\ \text{Let } T_{2n-1} = T'_{2n-1} \cup \tilde{P} \cup \tilde{Q} \cup \tilde{S} \cup \tilde{R}, \text{ (see Figure 17(b)).} \end{array}$ 

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