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HIGH-ORDERED SPECTRAL CHARACTERIZATION OF UNICYCLIC GRAPHS

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Abstract

In this paper we will apply the tensor and its traces to investigate the spectral characterization of unicyclic graphs. Let G be a graph and G^m be the m-th power (hypergraph) of G. The spectrum of G is referring to its adjacency matrix, and the spectrum of G^m is referring to its adjacency tensor. The graph G is called determined by high-ordered spectra (DHS, for short) if, whenever H is a graph such that H^m is cospectral with G^m for all m, then H is isomorphic to G. In this paper we first give formulas for the traces of the power of unicyclic graphs, and then provide some high-ordered cospectral invariants of unicyclic graphs. We prove that a class of unicyclic graphs with cospectral mates is DHS, and give two examples of infinitely many pairs of cospectral unicyclic graphs but with different high-ordered spectra.

Keywords: unicyclic graph, graph isomorphism, cospectral graphs, power hypergraph, adjacency tensor, trace.

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1. Introduction

The graph isomorphism problem is one of few standard problems in computational complexity theory belonging to NP. In 1956, Günthard and Primas [25] raised the question of determining the graphs by the spectrum. In 1957, Collatz and Sinogowitz [10] presented a pair of non-isomorphic cospectral trees. In 1973, Schwenk [36] proved that almost every tree has a cospectral mate by constructing a non-isomorphic cospectral tree for each tree of sufficiently large order. In 1982, Godsil and McKay [24] invented a powerful method called GM-switching, which can produce lots of pairs of cospectral graphs. A graph G is said to be *determined by the spectrum* (DS, for short) if, whenever H is a graph cospectral with G, then H must be isomorphic to G. All the known DS graphs have very special structures, and the techniques involved in proving them to be DS cannot be applied to general graphs; see [12, 13].

Since a graph cannot be determined by its spectrum in general, we need more information to recognize a graph. Note that two graphs are isomorphic if and only if their complements are isomorphic. Wang and Xu [40, 41] and Wang [39] applied the spectra of a graph and also its complement to investigate whether the graph is determined by its generalized spectrum. A similar idea appears in a recent work by Chen, Sun and Bu [7], who applied the spectra of the powers of a graph to characterize whether the graph is determined by high-ordered spectra in the setting of tensor eigenvalues.

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For an integer $m \geq 2$, the *m-th power of G*, denoted by $G^m = (V^m, E^m)$, is defined to be the m-uniform hypergraph with vertex set $V^m = V \cup \{i_{e,1}, \ldots, i_{e,m-2} : e \in E\}$ and edge set $E^m = \{e \cup \{i_{e,1}, \ldots, i_{e,m-2}\} : e \in E\}$, where $i_{e,1}, \ldots, i_{e,m-2}$ are new vertices inserted to each edge $e \in E$. Note if $m = 2$, $G^m = G$ and this case is trivial. Observe that two graphs are isomorphic if and only if their m-th powers are isomorphic for each integer $m \geq 2$. Chen, Sun and Bu [7] introduced the following notions on the spectrum of power hypergraphs. They called the spectrum of the adjacency tensor of G^m the *m-ordered spectrum* of G, and two graphs are *m-ordered cospectral* if they have the same m-ordered spectra. A graph G is called *determined by high-ordered spectra* (DHS, for short) if, whenever H is a graph that are m-ordered cospectral with G for all $m \geq 2$, then H must be isomorphic to G.

Surely, a graph that is DS must be DHS, but the converse does not hold. For example, van Dam and Haemers [13] showed that not all Smith's graphs are DS. However, all Smith's graphs are DHS, proved by Chen, Sun and Bu [7]. They also showed that every tree and its cospectral mate in Schwenk's construction have different high-ordered spectra [7].

The traces or spectral moments play an important role in DS problems. Let

G be a graph and let $\mathcal{A}(G)$ be the adjacency matrix of G. The d-th *trace* (or d-th *spectral moment*) of G, denoted by $\mathrm{Tr}_d(G)$, is defined to be the trace of $\mathcal{A}(G)^d$, which is the sum of the d-th powers of all eigenvalues of $\mathcal{A}(G)$, and is also equal to the number of closed walks of length d in G starting from each vertex of G. It is known that two graphs G and H are cospectral if and only if $\mathrm{Tr}_d(G) = \mathrm{Tr}_d(H)$ for all d (or for $d = 1, 2, \ldots, |V(G)|$). We should note that here the spectrum of a uniform hypergraph is defined as the spectrum of the adjacency tensor of the hypergraph. By the traces generalized from matrices to tensors due to Morozov and Shakirov [34], we still have the above equalities for two cospectral uniform hypergraphs. A key problem is how to interpreter the structural information from the traces of hypergraphs.

In this paper, we will extend the work of Chen, Sun and Bu [7] from trees to unicyclic graphs, and characterize the unicylic graphs that are DHS. The paper is organized as follows. In Section 2 we introduce some preliminary knowledge about the spectra and traces of hypergraphs. In Section 3 we give formulas for the traces of the power of unicyclic graphs by means of the sub-structure of the graph. In the last section, we provide some high-ordered cospectral invariants for general graphs especially for unicyclic graphs, and prove that a class of unicyclic graphs with copectral mates is DHS. We give two examples of infinitely many pairs of cospectral unicyclic graphs but with different high-ordered spectra. Our work implies that high-ordered spectra of graphs can recognize more structural information than the usual spectra.

2. Preliminaries

2.1. Tensors and hypergraphs

Let $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ be a complex tensor of order m and dimension n. Given a vector $x \in \mathbb{C}^n$, $\mathcal{T}x^{m-1} \in \mathbb{C}^n$, which is defined as follows:

$$\left(\mathcal{T}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m \in [n]} t_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, i \in [n].$$

Let $\mathcal{I} = (i_{i_1 i_2 \cdots i_m})$ be the *identity tensor* of order m and dimension n, that is, $i_{i_1 i_2 \cdots i_m} = 1$ if $i_1 = i_2 = \cdots = i_m \in [n]$ and $i_{i_1 i_2 \cdots i_m} = 0$ otherwise. In 2005, Lim [28] and Qi [35] introduced the eigenvalues of tensors independently as follows.

Definition [28, 35]**.** Let $\mathcal{T}$ be an m-th order n-dimensional tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{m-1} = 0$, or equivalently $\mathcal{T}x^{m-1} = \lambda x^{[m-1]}$, has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then λ is called an *eigenvalue* of $\mathcal{T}$ and x is an *eigenvector* of $\mathcal{T}$ associated with λ, where $x^{[m-1]} = \left(x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}\right)$.

A *hypergraph* $\mathcal{H} = (V, E)$ consists of a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ denoted by $V(\mathcal{H})$ and an edge set $E = \{e_1, e_2, \ldots, e_k\}$ denoted by $E(\mathcal{H})$, where $e_i \subseteq V$ for $i \in [k]$. If $|e_i| = m$ for each $i \in [k]$ and $m \geq 2$, then $\mathcal{H}$ is called an *m-uniform* hypergraph. The *degree* $d_v(\mathcal{H})$ of a vertex v in $\mathcal{H}$ is the number of edges of $\mathcal{H}$ containing the vertex v. A vertex v of $\mathcal{H}$ is called a *cored vertex* if it has degree one. A *walk* W in $\mathcal{H}$ is a sequence of alternate vertices and edges: $v_0e_1v_1e_2 \cdots e_lv_l$, where $v_i \neq v_{i+1}$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, l-1$. If $v_0 = v_l$, then W is called a *circuit*, and is called a *cycle* if no vertices or edges are repeated except $v_0 = v_l$. The hypergraph $\mathcal{H}$ is said to be *connected* if every two vertices are connected by a walk. The hypergraph $\mathcal{H}$ is called *simple* if there exists no $i \neq j$ such that $e_i \subseteq e_j$, and is called *nontrivial* if it contains more than one vertex. Throughout of this paper, all hypergraphs are considered nontrivial, connected, simple and m-uniform unless stated somewhere.

In 2012, Cooper and Dutle [9] introduced the adjacency tensor of a uniform hypergraph, and applied the eigenvalues of the tensor to characterize the structural property of the hypergraph.

Definition [9]**.** Let $\mathcal{H}$ be an m-uniform hypergraph on n vertices $v_1, v_2, \ldots, v_n$. The *adjacency tensor* of $\mathcal{H}$ is defined as $\mathcal{A}(\mathcal{H}) = (a_{i_1 i_2 \cdots i_m})$, an m-th order n-dimensional tensor, where

$$a_{i_1 i_2 \cdots i_m} = \begin{cases} \frac{1}{(m-1)!}, & \text{if } \{v_{i_1}, \ldots, v_{i_m}\} \in E(H); \\ 0, & \text{else.} \end{cases}$$

The *spectrum* and *eigenvalues* of $\mathcal{H}$ are referring to those of $\mathcal{A}(\mathcal{H})$. If $m = 2$, then $\mathcal{A}(\mathcal{H})$ is exactly the adjacency matrix of the graph $\mathcal{H}$. Since the Perron-Frobenius theorem of nonnegative matrices was generalized to nonnegative tensors [4,21,42–44], the spectral hypergraph theory develops rapidly on many topics, such as the spectral radius [2,18,22,27,30,31,33], the eigenvariety [14,15,19], the spectral symmetry [16,17,37,46], and the eigenvalues of hypertrees [45].

Zhou *et al.* [46], and Cardoso *et al.* [3] investigated the relationship between the eigenvalues of G^m and those of the subgraphs of G (including G). Chen *et al.* [6] gave a more detailed statement as follows.

Lemma 1 [6]**.** *Let G^m be the m-th power of a graph G.*

(1) *If $m = 3$, λ is an eigenvalue of G^3 if and only if there is a signed induced subgraph of G with eigenvalue β such that $\beta^2 = \lambda^3$.*

(2) *If $m \geq 4$, λ is an eigenvalue of G^m if and only if there is a signed subgraph of G with eigenvalue β such that $\beta^2 = \lambda^m$.*

From Lemma 1, we find that G^m contains more spectral information than G as its eigenvalues are closely related to all signed (induced) subgraphs of G (including G itself). So, it is natural that a graph has more probability to be DHS than to be DS.

2.2. Traces

We will introduce some knowledge about the traces of hypergraphs. Let $\mathcal{H}$ be an m-uniform hypergraph on n vertices. The *d-th trace of* $\mathcal{H}$, denoted by $\mathrm{Tr}_d(\mathcal{H})$, is referring to the d-th trace of $\mathcal{A}(\mathcal{H})$. Morozov and Shakirov [34] introduced the traces of polynomial maps f given by homogeneous polynomials of arbitrary degrees. As a tensor $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ of order m and dimension n naturally induces polynomial maps, the d-th trace $\mathrm{Tr}_d(\mathcal{T})$ of $\mathcal{T}$ is expressed as follow:
(1)

$$\mathrm{Tr}_d(\mathcal{T}) = (m-1)^{n-1} \sum_{\substack{d_1+\cdots+d_n=d,\\ d_i\in\mathbb{N}, i\in[n]}} \prod_{i=1}^{n} \frac{1}{(d_i(m-1))!} \left(\sum_{y_i\in[n]^{m-1}} t_{iy_i} \frac{\partial}{\partial a_{iy_i}} \right)^{d_i} \mathrm{Tr}(A^{d(m-1)}),$$

where $t_{iy_i} = t_{ii_2\cdots i_m}$ and $\frac{\partial}{\partial a_{iy_i}} = \frac{\partial}{\partial a_{ii_2}} \cdots \frac{\partial}{\partial a_{ii_m}}$ if $y_i = (i_2, \ldots, i_m)$.

Cooper and Dulte [9] gave an expression for the co-degree coefficients of the characteristic polynomial of $\mathcal{A}(\mathcal{H})$ of $\mathcal{H}$ in terms of traces of $\mathcal{H}$. Shao, Qi and Hu [37] gave a graph interpretation for the d-th trace of a general tensor $\mathcal{T}$ of order m and dimension n, and proved that

$$\mathrm{Tr}_d(\mathcal{T}) = \sum_{i=1}^{N} \lambda_i^d,$$

where $\lambda_1, \ldots, \lambda_N$ are all eigenvalues of $\mathcal{T}$, and $N = n(m-1)^{n-1}$. So two tensors $\mathcal{T}_1$ and $\mathcal{T}_2$ both of order m and dimension n are cospectral if and only if

$$\mathrm{Tr}_d(\mathcal{T}_1) = \mathrm{Tr}_d(\mathcal{T}_2)$$

for all d or $d = 1, 2, \ldots, n(m-1)^{n-1}$. Clark and Cooper [8] expressed the trace as a weighted sum over a family of Veblen hypergraphs. Chen, Bu and Zhou [5] gave a formula for the spectral moments (equivalently, the traces) of a hypertree in terms of the number of sub-hypertrees.

Given an ordering of the vertices of $\mathcal{H}$, let

$$\mathcal{F}_d(\mathcal{H}) = \{(e_1(v_1), \ldots, e_d(v_d)) : e_i \in E(\mathcal{H}), v_1 \leq \cdots \leq v_d\},$$

be the set of d-tuples of ordered rooted edges, where $e_i(v_i)$ is an edge e_i with root $v_i \in e_i$ for $i \in [d]$. Define a rooted directed star $S_{e_i}(v_i) = (e_i, \{(v_i, u) : u \in e_i \backslash \{v_i\}\})$ for each $i \in [d]$, and multi-directed graph $R(F) = \bigcup_{i=1}^{d} S_{e_i}(v_i)$ associated with $F \in \mathcal{F}_d(\mathcal{H})$. Let

$$\mathcal{F}_d^{\epsilon}(\mathcal{H}) = \{F \in \mathcal{F}_d(\mathcal{H}) : R(F) \text{ is Eulerian}\}.$$

For an $F \in \mathcal{F}_d^{\epsilon}(\mathcal{H})$, denote $V(F) = V(R(F))$, $r_v(F)$ the number of edges in F with v as the root, and $d_v^+(F) = (m-1)r_v(F)$ (namely, the outdegree of v in

$R(F)$). Denote by $\tau(F) = \tau_u(R(F))$ the number of arborescences of $R(F)$ with root u (namely, a directed u-rooted spanning tree such that all vertices except u has a directed path from itself to u), which is equal to the principal minor the Laplacian matrix $L(R(F))$ of $R(F)$ by deleting the row and column indexed by u ([1, 38]). As $R(F)$ is Eulerian, $\tau_u(R(F))$ is independent of the choice of the root u so that the root u is omitted. Fan *et al.* [20] give an expression of the d-th trace of $\mathcal{H}$ as follows.

Lemma 2 [20]**.** *For an m-uniform hypergraph $\mathcal{H}$ on n vertices,*

$$\mathrm{Tr}_d(\mathcal{H}) = d(m-1)^n \sum_{F \in \mathcal{F}_d^\epsilon(\mathcal{H})} \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)}. \tag{2}$$

For each $F \in \mathcal{F}_d^\epsilon(\mathcal{H})$, we get a multi-hypergraph induced by the edges in F by omitting the roots, denoted by $\mathcal{V}_F$, which is an m-uniform and m-valent multi-hypergraph called *Veblen hypergraph.* On the other side, given a Veblen hypergraph H, a *rooting* of H is an ordering $F = (e_1(v_1), \ldots, e_t(v_t))$ of all edges of H, where v_i is the root of e_i for $i \in [t]$, and $v_1 \le \cdots \le v_t$ under the given order of the vertices of H. If $R(F)$ is Eulerian, then F is called an *Euler rooting* of H; in this case, H is called *Euler rooted* with each edge rooted as in F by omitting the order. Denote by $\mathcal{R}(H)$ the set of Euler rooting of H.

Denote by $\mathcal{V}_d(\mathcal{H})$ the set of Veblen hypergraphs with d edges associated with $\mathcal{H}$ as follows:

$$\mathcal{V}_d(\mathcal{H}) = \bigcup_{G \in \mathcal{C}(\mathcal{H})} \left\{ \mathcal{V}_F : F \in \mathcal{F}_d^\epsilon(\mathcal{H}), \underline{\mathcal{V}_F} = G \right\},$$

where $\mathcal{C}(\mathcal{H})$ denotes the set of representatives of the isomorphic classes of connected sub-hypergraphs of $\mathcal{H}$, and $\underline{H}$ the underlying hypergraph of a multi-hypergraph H obtained by removing duplicate edges of H. For each $H \in \mathcal{V}_d(\mathcal{H})$, denote

$$C_H = \sum_{F \in \mathcal{R}(H)} \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)},$$

and $N_{\mathcal{H}}(\underline{H})$ the number of sub-hypergraphs of $\mathcal{H}$ that is isomorphic to $\underline{H}$. By Lemma 2, we have

$$\mathrm{Tr}_d(\mathcal{H}) = d(m-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} \left(\sum_{F \in \mathcal{R}(H)} \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)} \right) N_{\mathcal{H}}(\underline{H}).$$

So we get another expression of $\mathrm{Tr}_d(\mathcal{H})$.

Corollary 3. *For an m-uniform hypergraph $\mathcal{H}$ on n vertices,*

$$\mathrm{Tr}_d(\mathcal{H}) = d(m-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H N_{\mathcal{H}}(\underline{H}). \tag{3}$$

If we use $\overline{\mathcal{V}}_d(\mathcal{H})$ denote the set of representatives of the isomorphism classes of the hypergraphs in $\mathcal{V}_d(\mathcal{H})$, then we have the following expression of $\mathrm{Tr}_d(\mathcal{H})$, which was proved by Clark and Cooper [8].

$$\begin{aligned}\mathrm{Tr}_d(\mathcal{H}) &= d(m-1)^n \sum_{H\in\overline{\mathcal{V}}_d(\mathcal{H})} C_H N_{\mathcal{H}}(\underline{H}) \frac{|\mathrm{Aut}(\underline{H})|}{|\mathrm{Aut}(H)|} \\ &= d(m-1)^n \sum_{H\in\overline{\mathcal{V}}_d(\mathcal{H})} C_H (\#H \subseteq \mathcal{H}),\end{aligned} \tag{4}$$

where

$$(\#H \subseteq \mathcal{H}) = N_{\mathcal{H}}(\underline{H}) \frac{|\mathrm{Aut}(\underline{H})|}{|\mathrm{Aut}(H)|},$$

and $\mathrm{Aut}(G)$ denotes the automorphism group of a hypergraph G.

2.3. Traces of hypertrees

A *hypertree* is a connected and acyclic hypergraph. Let $\mathcal{T}$ be an m-uniform hypertree. Then we have $|V(\mathcal{T})| = (m-1)|E(\mathcal{T})| + 1$. We need the following lemma to characterize the Veblen hypergraphs $H \in \mathcal{V}_d(\mathcal{T})$.

Lemma 4 [20]**.** *Let H be an m-uniform Veblen multi-hypergraph whose underlying hypergraph $\underline{H}$ is a hypertree. Then H is uniquely Euler rooted such that all vertices of each edge occur as roots of the edge in a same number of times, and hence every edge of H repeats in a multiple of m times.*

By Lemma 4, for $d \in \mathbb{Z}^+$, $\mathcal{V}_d(\mathcal{T}) \neq \emptyset$ if and only if $m \mid d$. So, in this subsection we always assume that *d is a positive multiple of m* when discussing $\mathrm{Tr}_d(\mathcal{T})$; otherwise, $\mathrm{Tr}_d(\mathcal{T}) = 0$ by Corollary 3. For each $H \in \mathcal{V}_d(\mathcal{T})$, let $\underline{H} = \hat{\mathcal{T}}$, a sub-hypertree of $\mathcal{T}$. Then H can be expressed as a weighted hypertree $\hat{\mathcal{T}}(\omega)$, where

$$\omega : E(\hat{\mathcal{T}}) \to \mathbb{Z}^+,$$

such that the multiplicity of an edge $e \in E(H)$ is $m\omega(e)$, and $\omega(\hat{\mathcal{T}}) = \sum_{e\in E(\hat{\mathcal{T}})} \omega(e) = d/m$, implying that $\hat{\mathcal{T}}$ contains at most d/m edges. So we have

$$\mathcal{V}_d(\mathcal{T}) = \bigcup_{\hat{\mathcal{T}}\in\mathcal{C}(\mathcal{T})} \{\hat{\mathcal{T}}(\omega) : \omega(\hat{\mathcal{T}}) = d/m\}.$$

For each $F \in \mathcal{R}(\hat{\mathcal{T}}(\omega))$, by a direct computation, we have

$$\tau(F) = \prod_{e\in E(\hat{\mathcal{T}})} \omega(e)^{m-1} m^{m-2} = m^{(m-2)|E(\hat{\mathcal{T}})|} \left(\prod_{e\in E(\hat{\mathcal{T}})} \omega(e)\right)^{m-1},$$

$$\prod_{v\in V(F)} d_v^+(F) = \prod_{v\in V(\hat{\mathcal{T}})} (m-1)d_v(\hat{\mathcal{T}}(\omega)) = (m-1)^{|V(\hat{\mathcal{T}})|} \prod_{v\in V(\hat{\mathcal{T}})} d_v(\hat{\mathcal{T}}(\omega)),$$

where $d_v(\hat{\mathcal{T}}(\omega)) = \sum_{e:v\in e}\omega(e)$, the weighted degree of the vertex v in $\hat{\mathcal{T}}(\omega)$.

By Lemma 4, for each $F \in \mathcal{R}(\hat{\mathcal{T}}(\omega))$, every vertex $v \in V(\hat{\mathcal{T}})$ occurs as a root in $d_v(\mathcal{T}(\omega))$ times of $d_v(\hat{\mathcal{T}})$ distinct edges e with multiplicity $\omega(e)$ respectively. So $\mathcal{R}(\hat{\mathcal{T}}(\omega))$ has $\prod_{v\in V(\hat{\mathcal{T}})} \frac{d_v(\hat{\mathcal{T}}(\omega))!}{r_v(\hat{\mathcal{T}}(\omega))}$ Euler rootings due to the ordering of the same roots in different edges, where $r_v(\hat{\mathcal{T}}(\omega) = \prod_{e:v\in e}\omega(e)!$. So, for a given $H = \hat{\mathcal{T}}(\omega) \in \mathcal{V}_d(\mathcal{T})$,

$$\begin{aligned} C_H &= \prod_{v\in V(\hat{\mathcal{T}})} \frac{d_v(\hat{\mathcal{T}}(\omega))!}{r_v(\hat{\mathcal{T}}(\omega))} \cdot \frac{\tau(F)}{\prod_{v\in V(F)} d_v^+(F)} \\ &= (m-1)^{-|V(\hat{\mathcal{T}})|} m^{(m-2)|E(\hat{\mathcal{T}})|} \left(\prod_{e\in E(\hat{\mathcal{T}})} \omega(e)\right)^{m-1} \prod_{v\in V(\hat{\mathcal{T}})} \frac{(d_v(\hat{\mathcal{T}}(\omega))-1)!}{r_v(\hat{\mathcal{T}}(\omega))}. \end{aligned} \tag{5}$$

Denote

$$c_{d,m}(\hat{\mathcal{T}}) = \sum_{\omega:\omega(\hat{\mathcal{T}})=d/m} \left(\prod_{e\in E(\hat{\mathcal{T}})} \omega(e)\right)^{m-1} \prod_{v\in V(\hat{\mathcal{T}})} \frac{(d_v(\hat{\mathcal{T}}(\omega))-1)!}{r_v(\hat{\mathcal{T}}(\omega))}.$$

Denote by $\mathbf{T}^m_{\leq t}$ (respectively, $\mathbf{T}^m_t$) the set of m-uniform hypertrees with at most t edges (respectively, exactly t edges) up to isomorphism. Then by Corollary 3, for $m \mid d$,

$$\begin{aligned} \mathrm{Tr}_d(\mathcal{T}) &= d(m-1)^{|V(\mathcal{T})|} \sum_{H\in\mathcal{V}_d(\mathcal{T})} C_H N_{\mathcal{T}}(\underline{H}) \\ &= \sum_{\hat{\mathcal{T}}(\omega):\hat{\mathcal{T}}\in\mathbf{T}^m_{\leq d/m},\omega(\hat{\mathcal{T}})=d/m} d(m-1)^{|V(\mathcal{T})|-|V(\hat{\mathcal{T}})|} m^{(m-2)|E(\hat{\mathcal{T}})|} \\ &\quad \cdot \left(\prod_{e\in E(\hat{\mathcal{T}})} \omega(e)\right)^{m-1} \prod_{v\in V(\hat{\mathcal{T}})} \frac{(d_v(\hat{\mathcal{T}}(\omega))-1)!}{r_v(\hat{\mathcal{T}}(\omega))} \cdot N_{\mathcal{T}}(\hat{\mathcal{T}}) \\ &= \sum_{\hat{\mathcal{T}}\in\mathbf{T}^m_{\leq d/m}} d(m-1)^{|V(\mathcal{T})|-|V(\hat{\mathcal{T}})|} m^{(m-2)|E(\hat{\mathcal{T}})|} \\ &\quad \cdot \sum_{\omega:\omega(\hat{\mathcal{T}})=d/m} \left(\prod_{e\in E(\hat{\mathcal{T}})} \omega(e)\right)^{m-1} \prod_{v\in V(\hat{\mathcal{T}})} \frac{(d_v(\hat{\mathcal{T}}(\omega))-1)!}{r_v(\hat{\mathcal{T}}(\omega))} \cdot N_{\mathcal{T}}(\hat{\mathcal{T}}) \end{aligned} \tag{6}$$

$$= \sum_{\hat{\mathcal{T}} \in \mathbf{T}^m_{\le d/m}} d(m-1)^{(m-1)(|E(\mathcal{T})|-|E(\hat{\mathcal{T}})|)} m^{(m-2)|E(\hat{\mathcal{T}})|} c_{d,m}(\hat{\mathcal{T}}) N_{\mathcal{T}}(\hat{\mathcal{T}})$$

$$= \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(\mathcal{T})|-k|)} m^{k(m-2)} \sum_{\hat{\mathcal{T}} \in \mathbf{T}^m_k} c_{d,m}(\hat{\mathcal{T}}) N_{\mathcal{T}}(\hat{\mathcal{T}}).$$

Now suppose that $\mathcal{T} = T^m$, the m-th power of a tree T. Then $H = \hat{\mathcal{T}}(\omega) = \hat{T}^m(\omega)$ for some sub-tree $\hat{T}$ of T. For each edge $e = \{u, v\} \in E(T)$, it corresponds to an edge $e^m = \{u, v, e_{i1}, \ldots, e_{i,m-2}\} \in E(T^m)$. The weighted hypertree $\hat{T}^m(\omega)$ induces a weighted tree $\hat{T}(\omega)$ with weight $\omega : E(\hat{T}) \to \mathbb{Z}^+$ such that $\omega(e) = \omega(e^m)$ and $\omega(\hat{T}) = \sum_{e \in E(\hat{T})} \omega(e) = d/m$. We have

$$\prod_{v \in V(\hat{T}^m)} \frac{(d_v(\hat{T}^m(\omega)) - 1)!}{r_v(\hat{T}^m(\omega))} = \prod_{v \in V(\hat{T})} \frac{(d_v(\hat{T}(\omega)) - 1)!}{r_v(\hat{T}(\omega))} \prod_{e \in E(\hat{T})} \left(\frac{(\omega(e) - 1)!}{\omega(e)!} \right)^{m-2}.$$

Hence

$$c_{d,m}(\hat{T}^m) = \sum_{\omega : \omega(\hat{T}) = d/m} \prod_{e \in E(\hat{T})} \omega(e) \prod_{v \in V(\hat{T})} \frac{(d_v(\hat{T}(\omega)) - 1)!}{r_v(\hat{T}(\omega))} =: \tilde{c}_{d/m}(\hat{T}). \tag{7}$$

So, for $m \mid d$,

$$\mathrm{Tr}_d(T^m) = \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(T)|-k)} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_T(\hat{T}). \tag{8}$$

We note that equations (6) and (8) with a slightly modification were proved by Chen, Bu and Zhou [5]. Here we show them in a different way also for the convenience in the following discussion.

3. Traces of Power of Unicyclic Graphs

In this section, we will give a decomposition formula for the traces of the power of unicyclic graphs. Denote by C_n a cycle on n vertices (as a graph). Let U^m be the m-th power of a unicyclic graph U which contains a cycle C_n, where $m \ge 3$. The following lemma is important for our discussion.

Lemma 5 [20]**.** *Let H be an m-uniform Veblen multi-hypergraph, and let e be an edge of $\underline{H}$ which contains a cored vertex. If H has an Euler rooting, then e repeats $k \cdot m$ times for some positive integer k, and all cored vertices in e occur as a root of e in k times.*

Let $H \in \mathcal{V}_d(U^m)$. As $m \geq 3$, each edges of H contains $m-2$ cored vertices. By Lemma 5, each edge of H occurs in a multiple of m times. So, for $d \in \mathbb{Z}^+$, $\mathcal{V}_d(U^m) \neq \emptyset$ if and only if $m \mid d$. We will assume $m \mid d$ in this section.

3.1. Traces of power of cycles

We first consider a special case, namely, $U = C_n$, where C_n has vertices $v_1, \ldots, v_n$ and edges $e_i = \{v_i, v_{i+1}\}$ for $i \in [n]$, $v_{n+1} = v_1$. We label the edges of C_n^m as $e_i^m = \{v_i, v_{i+1}, e_{i1}, \ldots, e_{i,m-2}\}$ for $i \in [n]$. Let

$$\mathcal{V}_d\big(C_n^m; [C_n^m]\big) = \big\{H \in \mathcal{V}_d(C_n^m) : \underline{H} = C_n^m\big\}.$$

Let $H \in \mathcal{V}_d(C_n^m; [C_n^m])$. By Lemma 5, H is a weighted cycle $C_n^m(\omega)$ with $\omega : E(C_n^m) \to \mathbb{Z}^+$ such that e_i^m repeats $m\omega(e_i^m)$ times for $i \in [n]$, where $\omega(C_n^m) = \sum_{i=1}^n \omega(e_i^m) = d/m$. So

$$\mathcal{V}_d\big(C_n^m; [C_n^m]\big) = \big\{C_n^m(\omega) : \omega(C_n^m) = d/m\big\}.$$

Let $H = C_n^m(\omega)$ and $F \in \mathcal{R}(C_n^m(\omega))$. It is known that each cored vertex of e_i^m occurs as a root of e_i^m in $\omega(e_i^m)$ times for $i \in [n]$ by Lemma 5. Let t_{ii} (respectively, $t_{i,i+1}$) be the times of v_i (respectively, v_{i+1}) as a root of e_i^m for $i \in [n]$, where the subscripts are taken modulo n. As e_i^m repeats $m\omega(e_i^m)$ times and each cored vertex of e_i^m occurs as a root of e_i^m in $\omega(e_i^m)$ times, we have

$$t_{ii} + t_{i,i+1} = 2\omega(e_i^m). \tag{9}$$

As $R(F)$ is Eulerian,

$$(t_{i-1,i} + t_{ii})(m-1) = m\omega(e_{i-1}^m) - t_{i-1,i} + m\omega(e_i^m) - t_{ii},$$

which implies that

$$t_{i-1,i} + t_{i,i} = \omega(e_{i-1}^m) + \omega(e_i^m). \tag{10}$$

Suppose that $i_0 \in [n]$ such that $\omega(e_{i_0}^m) = \min_{i \in [n]} \omega(e_i^m) = \omega_{\min}$. By equations (9) and (10), we have

$$t_{ii} = \omega(e_i^m) - \omega(e_{i_0}^m) + t_{i_0 i_0},\ t_{i,i+1} = \omega(e_i^m) + \omega(e_{i_0}^m) - t_{i_0 i_0}, t_{i_0 i_0} \in \big[0, 2\omega(e_{i_0}^m)\big],$$

or equivalently

$$t_{ii} = \omega(e_i^m) - \omega_{\min} + x,\ t_{i,i+1} = \omega(e_i^m) + \omega_{\min} - x, x \in \big[0, 2\omega_{\min}\big].$$

So

$$\mathcal{R}(C_n^m(\omega)) = \bigcup_{x=0}^{2\omega_{\min}} \mathcal{R}(C_n^m(\omega); x),$$

where $\mathcal{R}(C_n^m(\omega);x)$ consists of those rootings $F \in \mathcal{R}(C_n^m(\omega))$ such that each cored vertex of e_i^m acts as a root of e_i^m in $\omega(e_i^m)$ times, and for $i \in [n]$, v_i (respectively, v_{i+1}) acts as a root of e_i^m in $t_{ii} = \omega(e_i^m) - \omega_{\min} + x$ (respectively, $t_{i,i+1} = \omega(e_i^m) + \omega_{\min} - x$) times.

For each $F \in \mathcal{R}(C_n^m(\omega);x)$, we now calculate $\tau(F)$. Note that $\tau(F)$ is the principal minor of $L(R(F))$ by deleting the row and column both indexed by a specified vertex v. Here we take $v = v_1$ and write the $L(R(F))$ as follows, where $t_i = (t_{i-1,i} + t_{ii})(m-1)$, $\omega_i = \omega(e_i^m)$ for $i \in [n]$, $\mathbf{1}$ is an all-one's column vector of size $m-2$, $K = mI - J$ of size $m-2$, I is an identity matrix and J is an all-one's matrix.

$$L(R(F)) = \begin{bmatrix} t_1 & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & A & B \\ \mathbf{d} & C & D \end{bmatrix},$$

where

$$\mathbf{a} = (-t_{11}, 0, \ldots, 0, -t_{n1}),\ \mathbf{b} = \left(-t_{11}\mathbf{1}^\top, 0, \ldots, 0, -t_{n1}\mathbf{1}^\top\right),$$

$$\mathbf{c} = (-t_{12}, 0, \ldots, 0, -t_{nn})^\top,\ \mathbf{d} = \left(-\omega_1\mathbf{1}^\top, 0, \ldots, 0, -\omega_n\mathbf{1}^\top\right)^\top,$$

$$A = \begin{bmatrix} t_2 & -t_{22} & 0 & \cdots & 0 \\ -t_{23} & t_3 & -t_{33} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -t_{n-2,n-1} & t_{n-1} & -t_{n-1,n-1} \\ 0 & \cdots & 0 & -t_{n-1,n} & t_n \end{bmatrix},$$

$$B = \begin{bmatrix} -t_{12}\mathbf{1}^\top & -t_{22}\mathbf{1}^\top & 0 & \cdots & 0 \\ 0 & -t_{23}\mathbf{1}^\top & -t_{33}\mathbf{1}^\top & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -t_{n-2,n-1}\mathbf{1}^\top & -t_{n-1,n-1}\mathbf{1}^\top & 0 \\ 0 & \cdots & 0 & -t_{n-1,n}\mathbf{1}^\top & -t_{n,n}\mathbf{1}^\top \end{bmatrix},$$

$$C = \begin{bmatrix} -\omega_1\mathbf{1} & 0 & 0 & \cdots & 0 \\ -\omega_2\mathbf{1} & -\omega_2\mathbf{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\omega_{n-1}\mathbf{1} & -\omega_{n-1}\mathbf{1} \\ 0 & 0 & \cdots & 0 & -\omega_n\mathbf{1} \end{bmatrix},$$

$$D = \begin{bmatrix} \omega_1 K & O & O & \cdots & O \\ O & \omega_2 K & O & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & \cdots & O & \omega_{n-1}K & O \\ O & \cdots & O & O & \omega_n K \end{bmatrix}.$$

So

$$\tau(F) = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \det(A - BD^{-1}C).$$

We easily get

$$\det D = 2^n m^{n(m-3)} \left(\prod_{i=1}^{n} \omega_i\right)^{m-2}.$$

Note that $\mathbf{1}^\top K = 2\mathbf{1}^\top$ so that $\mathbf{1}^\top K^{-1} = \frac{1}{2}\mathbf{1}^\top$. We have

$$BD^{-1}C = \frac{m-2}{2} \begin{bmatrix} \frac{t_2}{m-1} & t_{22} & 0 & \cdots & 0 \\ t_{23} & \frac{t_3}{m-1} & t_{33} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & t_{n-2,n-1} & \frac{t_{n-1}}{m-1} & t_{n-1,n-1} \\ 0 & \cdots & 0 & t_{n-1,n} & \frac{t_n}{m-1} \end{bmatrix},$$

and hence

$$A - BD^{-1}C = \frac{m}{2} \begin{bmatrix} t_{12}+t_{22} & -t_{22} & 0 & \cdots & 0 \\ -t_{23} & t_{23}+t_{33} & -t_{33} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -t_{n-2,n-1} & t_{n-2,n-1}+t_{n-1,n-1} & -t_{n-1,n-1} \\ 0 & \cdots & 0 & -t_{n-1,n} & t_{n-1,n}+t_{nn} \end{bmatrix}.$$

We get

$$\begin{aligned} \det(A - BD^{-1}C) &= \left(\frac{m}{2}\right)^{n-1} \left(\prod_{i=2}^{n} t_{ii} + \sum_{l=1}^{n-2} \prod_{i=1}^{l} t_{i,i+1} \prod_{i=l+2}^{n} t_{ii} + \prod_{i=1}^{n-1} t_{i,i+1}\right) \\ &= \left(\frac{m}{2}\right)^{n-1} \sum_{l=0}^{n-1} \prod_{i=1}^{l} t_{i,i+1} \prod_{i=l+2}^{n} t_{ii}, \end{aligned}$$

and therefore

$$\tau(F) = 2m^{n(m-2)-1} \left(\prod_{i=1}^{n} \omega_i\right)^{m-2} \sum_{l=0}^{n-1} \prod_{i=1}^{l} t_{i,i+1} \prod_{i=l+2}^{n} t_{ii}.$$

By the diagonal entries of $L(R(F))$, we also get

$$\begin{aligned} \prod_{v \in V(F)} d_v^+(F) &= \prod_{i=1}^{n} t_i \prod_{i=1}^{n} (\omega_i(m-1))^{m-2} \\ &= (m-1)^{n(m-1)} \left(\prod_{i=1}^{n} \omega_i\right)^{m-2} \prod_{i=1}^{n} (t_{i-1,i} + t_{ii}). \end{aligned}$$

Note $\mathcal{R}(C_n^m(\omega);x)$ contains exactly $\prod_{i=1}^n \binom{t_{i-1,i}+t_{ii}}{t_{ii}}$ rootings F, all corresponding the same values of $\tau(F)$ and $\prod_{v\in V(F)} d_v^+(F)$. So we get C_H for $H = C_n^m(\omega)$ as follows:

$$\begin{aligned}
C_{C_n^m(\omega)} &= \sum_{F\in\mathcal{R}(C_n^m(\omega))} \frac{\tau(F)}{\prod_{v\in V(F)} d_v^+(F)} \\
&= \sum_{x=0}^{2\omega_{\min}} \sum_{F\in\mathcal{R}(C_n^m(\omega);x)} \frac{\tau(F)}{\prod_{v\in V(F)} d_v^+(F)} \\
&= \sum_{x=0}^{2\omega_{\min}} \prod_{i=1}^{n} \binom{t_{i-1,i}+t_{ii}}{t_{ii}} \cdot \frac{2m^{n(m-2)-1}\sum_{l=0}^{n-1}\prod_{i=1}^{l} t_{i,i+1}\prod_{i=l+2}^{n} t_{ii}}{(m-1)^{n(m-1)}\prod_{i=1}^{n}(t_{i-1,i}+t_{ii})} \qquad (11)\\
&= \frac{2m^{n(m-2)-1}}{(m-1)^{n(m-1)}\prod_{i=1}^{n}(\omega_{i-1}+\omega_i)} \\
&\quad \cdot \sum_{x=0}^{2\omega_{\min}} \prod_{i=1}^{n} \binom{\omega_{i-1}+\omega_i}{\omega_i-\omega_{\min}+x} \sum_{l=0}^{n-1}\prod_{i=1}^{l}(\omega_i+\omega_{\min}-x)\prod_{i=l+2}^{n}(\omega_i-\omega_{\min}+x).
\end{aligned}$$

Denote

$$\begin{aligned}
f_{C_n}(\omega) &= \frac{1}{\prod_{i=1}^{n}(\omega_{i-1}+\omega_i)} \\
&\quad \cdot \sum_{x=0}^{2\omega_{\min}} \prod_{i=1}^{n} \binom{\omega_{i-1}+\omega_i}{\omega_i-\omega_{\min}+x} \sum_{l=0}^{n-1}\prod_{i=1}^{l}(\omega_i+\omega_{\min}-x)\prod_{i=l+2}^{n}(\omega_i-\omega_{\min}+x)
\end{aligned}$$

and

$$\mathrm{Tr}_d(C_n^m;[C_n^m]) = d(m-1)^{|V(C_n^m)|}\sum_{H\in\mathcal{V}_d(C_n^m;[C_n^m])} C_H N_{C_n^m}(\underline{H}).$$

Lemma 6. *For $m \mid d$,*

$$\mathrm{Tr}_d(C_n^m;[C_n^m]) = \sum_{\omega:\omega(C_n^m)=d/m} 2dm^{n(m-2)-1} f_{C_n}(\omega). \qquad (12)$$

If $d/m = n$, then

$$\mathrm{Tr}_{nm}(C_n^m;[C_n^m]) = 2n(n+1)m^{n(m-2)}. \qquad (13)$$

Proof. By the previous discussion, each $H \in \mathcal{V}_d(C_n^m;[C_n^m])$ is a weighted cycle $C_n^m(\omega)$, and $\underline{H} = C_n^m$, implying that $N_{C_n^m}(\underline{H}) = 1$. By definition and equation (11),

$$\begin{aligned}\mathrm{Tr}_d(C_n^m;[C_n^m]) &= d(m-1)^{n(m-1)} \sum_{H\in\mathcal{V}_d(C_n^m;[C_n^m])} C_H N_{C_n^m}(\underline{H}) \\ &= \sum_{\omega:\omega(C_n^m)=d/m} d(m-1)^{n(m-1)} C_{C_n^m(\omega)} \\ &= \sum_{\omega:\omega(C_n^m)=d/m} d(m-1)^{n(m-1)} \frac{2m^{n(m-2)-1}}{(m-1)^{n(m-1)}} f_{C_n}(\omega) \\ &= \sum_{\omega:\omega(C_n^m)=d/m} 2dm^{n(m-2)-1} f_{C_n}(\omega).\end{aligned}$$

If $d/m=n$, then $\omega_i=1$ for $i\in[n]$, $f_{C_n}(\omega)=n+1$, and therefore

$$\mathrm{Tr}_{nm}(C_n^m;[C_n^m]) = 2n(n+1)m^{n(m-2)}.$$

■

3.2. Trace of power of unicyclic graphs

Let U^m be the m-th power of a unicyclic graph U which contains a cycle C_n. The labeling of the vertices and edges of C_n^m is the same as in Section 3.1. We have a decomposition:

$$\mathcal{V}_d(U^m) = \mathcal{V}_d(U^m;[\hat{C}_n^m]) \cup \mathcal{V}_d(U^m;[C_n^m]), \tag{14}$$

where $\mathcal{V}_d(U^m;[\hat{C}_n^m])$ (respectively, $\mathcal{V}_d(U^m;[C_n^m])$) denotes the subset of $\mathcal{V}_d(U^m)$ which consists of Veblen hypergraphs that contain not all edges of C_n^m (respectively, contain all edges of C_n^m). By Lemma 5, either of three sets in (14) is nonempty if and only if $m \mid d$. So we assume $m\mid d$ in the following discussion.

For each $H\in\mathcal{V}_d(U^m;[\hat{C}_n^m])$, $\underline{H}$ is a hypertree, and by Lemma 4, H is uniquely rooted and $H=\hat{T}^m(\omega)$ for some tree $\hat{T}$ contained in U with $\omega(\hat{T})=d/m$. So by Corollary 3, equations (5) and (7),

$$\begin{aligned}\mathrm{Tr}_d(U^m;[\hat{C}_n^m]) &= d(m-1)^{|V(U^m)|} \sum_{H\in\mathcal{V}_d(U^m;[\hat{C}_n^m])} C_H N_{U^m}(\underline{H}) \\ &= \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)} \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_U(\hat{T}).\end{aligned} \tag{15}$$

Suppose that U is obtained from C_n by attaching rooted trees $T_1,\ldots,T_n$ with their roots identified with $v_1,\ldots,v_n$ of the cycle respectively, where some trees T_i may be trivial containing only one vertex (namely v_i). Here we stress that each T_i is a rooted tree with root v_i for $i\in[n]$. Note an isomorphism between two rooted trees is one preserving the roots. If $\hat{T}$ is a rooted tree with root v, then

$N_{T_i}(\hat{T})$ denotes the number of subgraphs of T_i with root v_i that is isomorphic to $\hat{T}$ mapping v_i to v.

For each $H \in \mathcal{V}_d(U^m; [C_n^m])$, $H|_{T_i^m}$ (if nonempty) for $i \in [n]$ and $H|_{C_n^m}$ are all Vebelen hypergraphs. By Lemma 4, $H|_{T_i^m} = \hat{T}_i^m(\omega^i)$ for some subtree $\hat{T}_i$ of T_i containing the vertex v_i such that each edge e of $\hat{T}_i^m$ repeat $m\omega^i(e)$ times in $H|_{T_i^m}$. Similarly, by Lemma 5, $H|_{C_n^m} = C_n^m(\omega^0)$ such that each edge e of C_n^m repeat $m\omega^0(e)$ times in $H|_{C_n^m}$. Surely, the number of edges of $H|_{T_i^m}$ for $i \in [n]$ and the number of edges of $H|_{C_n^m}$ are multiples of m. We also note $\mathcal{V}_d(U^m; [C_n^m]) \neq \emptyset$ only if $d/m \geq n$, with equality only if $\mathcal{V}_d(U^m; [C_n^m]) = \{C_n^m(\omega^0)\}$ with $\omega^0(e) = 1$ for each edge $e \in E(C_n^m)$.

By the discussion before, we have

$$\mathcal{V}_d(U^m; [C_n^m]) = \bigcup_{\substack{d_0+\cdots+d_n=d, d_0/m \geq n \\ m|d_i, i \in \{0,1,\ldots,n\}}} \bigcup_{0 \leq s_i \leq d_i/m} \mathcal{V}_{d_0,d_1,\ldots,d_n}^{s_1,\ldots,s_n}(U^m), \tag{16}$$

where $\mathcal{V}_{d_0,d_1,\ldots,d_n}^{s_1,\ldots,s_n}(U^m)$ is the set of Veblen hypergraphs contains $d_0 > 0$ edges of C_n^m, and d_i edges of T_i^m among of which ms_i edges contains the vertex v_i for $i \in [n]$. Furthermore,

$$\begin{aligned}\mathcal{V}_{d_0,d_1,\ldots,d_n}^{s_1,\ldots,s_n}(U^m) = \cup \Bigg\{ & C_n^m(\omega^0) \cup \bigcup_{i=1}^{n} \hat{T}_i^m(\omega^i) : C_n^m(\omega^0) \in \mathcal{V}_{d_0}(C_n^m; [C_n^m]), \\ & \hat{T}_i^m(\omega^i) \in \mathcal{V}_{d_i;s_i}(T_i^m; [v_i]), i \in [n] \Bigg\},\end{aligned}$$

where $\mathcal{V}_{d_i;s_i}(T_i^m; [v_i]) = \{\hat{T}_i^m(\omega^i) \in \mathcal{V}_{d_i}(T_i^m) : d_{v_i}(\hat{T}_i^m(\omega^i)) = s_i, \hat{T}_i \subseteq T_i\}$, and $\hat{T}_i$ is a rooted tree with root v_i. Note that in the above decomposition, some d_i's are zeros, and $d_i = 0$ if and only if $s_i = 0$.

Denote $N = (m-1)|E(U)|$ the number of vertices of U^m, $N_i = (m-1)$ $|E(T_i)| + 1$ the number of vertices of T_i^m for $i \in [n]$, and $N_0 = (m-1)n$ the number of vertices of C_n^m. Then $N = N_0 + \sum_{i \in [n]} N_i - n$. Let $H = C_n^m(\omega^0) \cup \bigcup_{i \in I(s)} \hat{T}_i^m(\omega^i) \in \mathcal{V}_{d_0,d_1,\ldots,d_n}^{s_1,\ldots,s_n}(U^m)$, where $I(s) = \{i \in [n] : s_i > 0\}$. For each $F \in \mathcal{R}(H)$, $F_0 = F|_{C_n^m(\omega^0)}$ is an Euler rooting of $C_n^m(\omega^0)$, and $d_{v_i}^+(F_0) = (\omega^0(e_{i-1}^m) + \omega^0(e_i^m))(m-1) =: t_i(m-1)$ for $i \in [n]$; $F_i = F|_{\hat{T}_i^m(\omega^i)}$ is also an Euler rooting of $\hat{T}_i^m(\omega^i)$, and $d_{v_i}^+(F_i) = s_i(m-1)$ for $i \in I(s)$. We have $\tau(F) = \tau(F_0) \prod_{i \in I(s)} \tau(F_i)$ and $N_{U^m}(\underline{H}) = \prod_{i \in I(s)} N_{T_i}(\hat{T}_i)$. Denote

$$\mathrm{Tr}_{d_i;s_i}(T_i^m; [v_i]) = d(m-1)^{N_i} \sum_{H \in \mathcal{V}_{d_i;s_i}(T_i^m; [v_i])} C_H N_{T_i^m}(\underline{H}), i \in I(s).$$

By Lemma 6, we have

$$
\begin{aligned}
\mathrm{Tr}^{s_1,\dots,s_n}_{d_0,d_1,\dots,d_n}(U^m) &= d(m-1)^N \sum_{H\in\mathcal{V}^{s_1,\dots,s_n}_{d_0,d_1,\dots,d_n}} C_H N_{U^m}(\underline{H}) \\
&= d(m-1)^N \sum_{H\in\mathcal{V}^{s_1,\dots,s_n}_{d_0,d_1,\dots,d_n}} \left(\sum_{F\in\mathcal{R}(H)} \frac{\tau(F)}{\prod_{v\in V(F)} d_v^+(F)} \right) N_{U^m}(\underline{H}) \\
&= d(m-1)^N \sum_{C_n^m(\omega^0)\in\mathcal{V}_{d_0},\hat{T}_i^m(\omega^i)\in\mathcal{V}_{d_i;s_i},i\in I(s)} \prod_{i\in I(s)} \binom{s_i+t_i}{s_i} \\
&\quad \cdot \sum_{F\in\mathcal{R}(\cup_{i\in I(s)}\hat{T}_i^m(\omega^i))} \prod_{i\in I(s)} \frac{s_i t_i(m-1)}{s_i+t_i} \frac{\tau(F_i)}{\prod_{v\in V(F_i)} d_v^+(F_i)} N_{T_i}(\hat{T}_i) \\
&\quad \cdot \sum_{F_0\in\mathcal{R}(C_n^m(\omega_0))} \frac{\tau(F_0)}{\prod_{v\in V(F_0)} d_v^+(F_0)} = \frac{d(m-1)^{N-N_0-\sum_{i\in I(s)}(N_i-1)}}{d_0} \\
&\quad \cdot \prod_{i\in I(s)} \frac{s_i}{d_i} \sum_{T_i^m(\omega^i)\in\mathcal{V}_{d_i;s_i}} \sum_{F\in\mathcal{R}(\hat{T}_i^m(\omega^i))} \frac{d_i(m-1)^{N_i}\tau(F_i)}{\prod_{v\in V(F_i)} d_v^+(F_i)} N_{T_i}(\hat{T}_i) \\
&\quad \cdot \sum_{C_n^m(\omega_0)\in\mathcal{V}_{d_0}} \prod_{i\in I(s)} \frac{t_i}{s_i+t_i} \binom{s_i+t_i}{t_i} \sum_{F\in\mathcal{R}(C_n^m(\omega^0))} \frac{d_0(m-1)^{N_0}\tau(F_0)}{\prod_{v\in V(F_0)} d_v^+(F_0)} \\
&= \frac{d(m-1)^{N-N_0-\sum_{i\in I(s)}(N_i-1)}}{d_0} \prod_{i\in I(s)} \frac{s_i}{d_i} \mathrm{Tr}_{d_i;s_i}(T_i^m;[v_i]) \\
&\quad \cdot \sum_{\omega^0:\omega^0(C_n^m)=\frac{d_0}{m}} \prod_{i=1}^n \binom{\omega^0_{i-1}+\omega^0_i+s_i-1}{s_i} \cdot 2d_0 m^{n(m-2)-1} f_{C_n}(\omega_0),
\end{aligned} \tag{17}
$$

where $\mathcal{V}^{s_1,\dots,s_n}_{d_0,d_1,\dots,d_n} = \mathcal{V}^{s_1,\dots,s_n}_{d_0,d_1,\dots,d_n}(U^m)$, $\mathcal{V}_{d_0} = \mathcal{V}_{d_0}(C_n^m;[C_n^m])$, $\mathcal{V}_{d_i;s_i} = \mathcal{V}_{d_i;s_i}(T_i^m;[v_i])$.

By a similar discussion as in Section 2.3, we have a formula for $\mathrm{Tr}_{d;s}(T^m;[v])$. Denote by $T(k;[v])$ the set of representatives of the isomorphic classes of subtrees of T with root v and k edges, and for each $\hat{T}\in T(k;[v])$ denote $\tilde{c}_{\ell;s}(\hat{T})$ as follows:

$$
\tilde{c}_{\ell;s}(\hat{T}) = \sum_{\omega:\omega(\hat{T})=\ell,d_v(\hat{T}(\omega))=s} \prod_{e\in E(\hat{T})} \omega(e) \prod_{u\in V(\hat{T})} \frac{(d_u(\hat{T}(\omega))-1)!}{r_u(\hat{T}(\omega))}.
$$

We have

$$
\mathrm{Tr}_{d;s}(T^m;[v]) = \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(T)|-k)} m^{k(m-2)} \sum_{\hat{T}\in T(k;[v])} \tilde{c}_{d/m;s}(\hat{T}) N_T(\hat{T}). \tag{18}
$$

So

$$\prod_{i\in I}\mathrm{Tr}_{d_i;s_i}(T_i^m;[v_i]) = \prod_{i\in I} d_i \sum_{\substack{k_i\in[d_i/m]\\ i\in I}}(m-1)^{(m-1)\sum_{i\in I}(|E(T_i)|-k_i)}m^{(m-2)\sum_{i\in I}k_i} \cdot \prod_{\substack{\hat{T}_i\in T_i(k_i;[v_i])\\ i\in I}}\tilde{c}_{d_i/m;s_i}(\hat{T}_i)N_{T_i}(\hat{T}_i). \tag{19}$$

We now give a formula for the traces of the power of unicyclic graphs.

Theorem 7. *Let U be a unicyclic graph which is obtained from a cycle C_n by attaching rooted trees $T_1,\ldots,T_n$ with their roots identified with the vertices $v_1,\ldots,v_n$ of the cycle respectively. For each d with $m\mid d$,*

$$\begin{aligned}\mathrm{Tr}_d(U^m)=&\sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(U)|-k)-1}m^{k(m-2)}\sum_{\hat{T}\in\mathbf{T}_k}\tilde{c}_{d/m}(\hat{T})N_U(\hat{T})\\ &+2m^{n(m-2)-1}\sum_{\substack{\sum_{i=0}^n d_i=d,\frac{d_0}{m}\geq n\\ m|d_i,i\in\{0,1,\ldots,n\}}}\sum_{\substack{0\leq s_i\leq d_i/m\\ i\in[n]}} d(m-1)^{(m-1)(|E(U)|-n-\sum_{i\in I(s)}|E(T_i)|)}\prod_{i\in I(s)}s_i\\ &\cdot\sum_{\substack{k_i\in[d_i/m]\\ i\in I(s)}}(m-1)^{(m-1)\sum\limits_{i\in I(s)}(|E(T_i)|-k_i)}m^{(m-2)\sum\limits_{i\in I(s)}k_i}\prod_{\substack{\hat{T}_i\in T_i(k_i;[v_i])\\ i\in I(s)}}\tilde{c}_{d_i/m;s_i}(\hat{T}_i)N_{T_i}(\hat{T}_i)\\ &\cdot\sum_{\omega^0:\omega^0(C_n^m)=d_0/m}\prod_{i=1}^n\binom{\omega_{i-1}^0+\omega_i^0+s_i-1}{s_i}f_{C_n}(\omega_0).\end{aligned}\tag{20}$$

Proof. By the decomposition (16), equations (17) and (19), we have

$$\begin{aligned}\mathrm{Tr}_d(U^m;[C_n^m]) &= d(m-1)^{|V(U^m)|}\sum_{H\in\mathcal{V}_d(U^m;[C_n^m])}C_H N_{U^m}(\underline{H})\\ &=\sum_{\substack{\sum_{i=0}^n d_i=d,\frac{d_0}{m}\geq n\\ m|d_i,i\in\{0,1,\ldots,n\}}}\sum_{\substack{0\leq s_i\leq d_i/m\\ i\in[n]}}\mathrm{Tr}^{s_1,\ldots,s_n}_{d_0,d_1,\ldots,d_n}(U^m)\\ &=\sum_{\substack{\sum_{i=0}^n d_i=d,\frac{d_0}{m}\geq n\\ m|d_i,i\in\{0,1,\ldots,n\}}}\sum_{\substack{0\leq s_i\leq d_i/m\\ i\in[n]}}\frac{d(m-1)^{N-N_0-\sum\limits_{i\in I(s)}(N_i-1)}}{d_0}\prod_{i\in I(s)}\frac{s_i}{d_i}\mathrm{Tr}_{d_i;s_i}(T_i^m;[v_i])\\ &\cdot\sum_{\omega^0:\omega^0(C_n^m)=\frac{d_0}{m}}\prod_{i=1}^n\binom{\omega_{i-1}^0+\omega_i^0+s_i-1}{s_i}2d_0m^{n(m-2)-1}f_{C_n}(\omega_0)\\ &=2m^{n(m-2)-1}\sum_{\substack{\sum_{i=0}^n d_i=d,\frac{d_0}{m}\geq n\\ m|d_i,i\in\{0,1,\ldots,n\}}}\sum_{\substack{0\leq s_i\leq d_i/m\\ i\in[n]}}d(m-1)^{N-N_0-\sum_{i\in I(s)}(N_i-1)}\prod_{i\in I(s)}s_i\end{aligned}\tag{21}$$

$$\cdot \sum_{\substack{k_i \in [d_i/m] \\ i \in I(s)}} (m-1)^{(m-1)\sum\limits_{i \in I(s)} (|E(T_i)|-k_i)} m^{(m-2)\sum\limits_{i \in I(s)} k_i} \prod_{\substack{\hat{T}_i \in T_i(k_i;[v_i]) \\ i \in I(s)}} \tilde{c}_{d_i/m;s_i}(\hat{T}_i) N_{T_i}(\hat{T}_i)$$

$$\cdot \sum_{\omega^0 : \omega^0(C_n^m) = d_0/m} \prod_{i=1}^{n} \binom{\omega_{i-1}^0 + \omega_i^0 + s_i - 1}{s_i} f_{C_n}(\omega_0),$$

where $N = (m-1)|E(U)|$, $N_0 = (m-1)n$ and $N_i = (m-1)|E(T_i)| + 1$ are respectively the number of vertices of U^m, C_n^m and T_i^m for $i \in [n]$.

By the decomposition (14), we have

$$\mathrm{Tr}_d(U^m) = \mathrm{Tr}_d(U^m; [\hat{C}_n^m]) + \mathrm{Tr}_d(U^m; [C_n^m])$$

and get the desired result by combing equations (15) and (21). ■

Corollary 8. *Let U be the unicyclic graph as defined in Theorem 7. If $d/m = n$, then*

$$\begin{aligned} \mathrm{Tr}_d(U^m) = & \sum_{k=1}^{n} n(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)+1} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_n(\hat{T}) N_U(\hat{T}) \\ & + 2n(n+1)(m-1)^{(m-1)(|E(U)|-n)} m^{n(m-2)}. \end{aligned} \tag{22}$$

If $d/m < n$, then

$$\mathrm{Tr}_d(U^m) = \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_U(\hat{T}). \tag{23}$$

Proof. If $d/m = n$, then $d_0 = d = mn$, $d_i = s_i = 0$ for $i \in [n]$, and $I(s) = \emptyset$ in equation (20). Also in this case, $\omega_i^0 = 1$ for $i \in [n]$, and $f_{C_n}(\omega^0) = n+1$. So we have

$$\begin{aligned} \mathrm{Tr}_{nm}(U^m) = & \sum_{k=1}^{d/m} d(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_U(\hat{T}) \\ & + 2d(m-1)^{(m-1)(|E(U)|-n)} m^{n(m-2)-1} (n+1) \\ = & \sum_{k=1}^{n} n(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)+1} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_n(\hat{T}) N_U(\hat{T}) \\ & + 2n(n+1)(m-1)^{(m-1)(|E(U)|-n)} m^{n(m-2)}. \end{aligned}$$

If $d/m < n$, then the second summand in equation (20) does not appear. The result follows. ■

Recall that the *girth* of a graph G, denoted by $g(G)$, is the minimum length of the cycles of G. If G contains no cycles, then we define $g(G) = +\infty$. At the end of this section, we will consider some special traces of a general graph with girth g.

Theorem 9. *Let G be a graph with n vertices and p edges. Then for each $m \geq 3$ and each d with $m \mid d$ and $d/m < g(G)$,*

$$\mathrm{Tr}_d(G^m) = \sum_{k=1}^{d/m} d(m-1)^{n-1-p+(m-1)(p-k)} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_G(\hat{T}). \tag{24}$$

Proof. Observe that G^m has $N = n + p(m-2)$ vertices. As $m \geq 3$, each edge of G^m contains a cored vertex. By Lemma 5, each edge of $H \in \mathcal{V}_d(G^m)$ repeats in a multiple of m times, which implying that d is multiple of m if $\mathcal{V}_d(G^m) \neq \emptyset$. Also H is a weighted graph, namely $H = \underline{H}(\omega)$ such that each edge e of $\underline{H}$ repeats $m\omega(e)$ times in H, where $\omega : E(\underline{H}) \to \mathbb{Z}^+$. So, $\omega(\underline{H}) = \sum_{e \in E(\underline{H})} \omega(e) = d/m$, which implies that $\underline{H}$ has at most d/m edges.

Now assume that $m \mid d$ and $d/m < g(G)$. Then $\underline{H}$ contains no cycles, and hence $\underline{H} = \hat{T}^m$ for some tree $\hat{T} \subseteq G$. So

$$\mathcal{V}_d(G^m) = \{\hat{T}^m(\omega) : \hat{T} \subseteq G, \omega(\hat{T}^m) = d/m\}.$$

By Corollary 3, equation (7) and a similar discussion as in equation (6) by replacing $\mathcal{T}$ with G^m, $\hat{\mathcal{T}}$ with $\hat{T}^m$,

$$\begin{aligned}
\mathrm{Tr}_d(G^m) &= d(m-1)^N \sum_{H \in \mathcal{V}_d(G^m)} C_H N_{G^m}(\underline{H}) \\
&= \sum_{\hat{T}^m \in \mathbf{T}^m_{\leq d/m}} d(m-1)^{N-|V(\hat{T}^m|)} m^{(m-2)|E(\hat{T}^m)|} c_{d,m}(\hat{T}^m) N_{G^m}(\hat{T}^m) \\
&= \sum_{k=1}^{d/m} d(m-1)^{n-1-p+(m-1)(p-k)} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T}) N_G(\hat{T}).
\end{aligned}$$

■

We note that if G is a tree T or unicyclic graph U, then equation (24) becomes equations (8) or (23).

4. Spectral Characterization of Unicyclic Graphs

In this section, we give some high-ordered cospectral invariants of graphs, in particular for unicylic graphs. As applications, we prove that a class of unicylic graphs which are not DS but DHS. We give two examples of infinitely many pairs of cospectral unicyclic graphs with different high-ordered spectra. Denote by P_n a path on n vertices (as a graph).

4.1. General graphs

It is known that if G_1 is cospectral with G_2, then they have the same number of vertices and edges, respectively, namely

$$N_{G_1}(P_1) = N_{G_2}(P_1), N_{G_1}(P_2) = N_{G_2}(P_2).$$

We will prove that $N_G(P_3)$ is a high-ordered cospectral invariant of G.

Lemma 10. *If G_1 is high-ordered cospectral with G_2, then*

$$\sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_\ell(\hat{T})N_{G_1}(\hat{T}) = \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_\ell(\hat{T})N_{G_2}(\hat{T}) \tag{25}$$

for all ℓ, k with $1 \le k \le \ell < \min\{g(G_1), g(G_2)\}$.

Proof. As $\mathrm{Tr}_d(G_1^m) = \mathrm{Tr}_d(G_2^m)$, by Theorem 9, for $m|d$ and $\frac{d}{m} < \min\{g(G_1), g(G_2)\}$,

$$\sum_{k=1}^{d/m}(m-1)^{n-1-p+(m-1)(p-k)}m^{k(m-2)} \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_{d/m}(\hat{T})(N_{G_1}(\hat{T}) - N_{G_2}(\hat{T})) = 0,$$

where n, p are respectively the number of vertices and edges of G_1 or G_2. Let $d/m = \ell$, $g(m,k) = (m-1)^{n-1-p+(m-1)(p-k)}m^{k(m-2)}$, $y(k) = \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_\ell(\hat{T})(N_{U_2}(\hat{T}) - N_{U_1}(\hat{T}))$. Then we have

$$\sum_{k=1}^{\ell} g(m,k)y(k) = 0.$$

As $g(m,k) = g(m,k-1)\frac{m^{m-2}}{(m-1)^{m-1}} =: g(m,k-1)\alpha(m)$ for $k > 1$, $g(m,k) = g(m,1)\alpha(m)^{k-1}$. Now taking m as ℓ different integers $m_1, \ldots, m_\ell$ that are greater than 2, we have

$$\begin{bmatrix} 1 & \alpha(m_1) & \alpha(m_1)^2 & \cdots & \alpha(m_1)^{\ell-1} \\ 1 & \alpha(m_2) & \alpha(m_2)^2 & \cdots & \alpha(m_2)^{\ell-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha(m_\ell) & \alpha(m_2)^2 & \cdots & \alpha(m_\ell)^{\ell-1} \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(\ell) \end{bmatrix} = 0. \tag{26}$$

As the matrix in equation (26) is nonsingular, we have $y(k) = 0$ for $k \in [\ell]$. ∎

Remark 11. In Lemma 10, suppose $\mathbf{T}_k$ has exactly t_k trees $\hat{T}_1, \ldots, \hat{T}_{t_k}$. If we can take ℓ as t_k different integers $\ell_1, \ldots, \ell_{t_k}$ such that $\min\{\ell_i : i \in [t_k]\} \ge k$ and $\max\{\ell_i : i \in [t_k]\} < \min\{g(G_1), g(G_2)\}$, then by Lemma 10, we have

$$\begin{bmatrix} \tilde{c}_{\ell_1}(\hat{T}_1) & \tilde{c}_{\ell_1}(\hat{T}_2) & \cdots & \tilde{c}_{\ell_1}(\hat{T}_{t_k}) \\ \tilde{c}_{\ell_2}(\hat{T}_1) & \tilde{c}_{\ell_2}(\hat{T}_2) & \cdots & \tilde{c}_{\ell_2}(\hat{T}_{t_k}) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{c}_{\ell_{t_k}}(\hat{T}_1) & \tilde{c}_{\ell_{t_k}}(\hat{T}_2) & \cdots & \tilde{c}_{\ell_{t_k}}(\hat{T}_{t_k}) \end{bmatrix} \begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(t_k) \end{bmatrix} = 0, \tag{27}$$

where $z(i) = N_{G_1}(\hat{T}_i) - N_{G_2}(\hat{T}_i)$ for $i \in [t_k]$. If the matrix in equation (27), denoted by $\tilde{C}$, is nonsingular, then the equation in equation (27) has only zero solution, which implies that for all $\hat{T} \in \mathbf{T}_k$,

$$N_{G_1}(\hat{T}) = N_{G_2}(\hat{T}).$$

Obviously, $\mathbf{T}_2 = \{P_3\}$. By definition, $\tilde{c}_2(P_3) > 0$. As any graph has girth greater than 2, if G_1 is high-order cospectral with G_2, then

$$N_{G_1}(P_3) = N_{G_2}(P_3). \tag{28}$$

Chen, Sun and Bu [7] defined a parameter $c_{2\ell}(\hat{T})$ and a matrix $C = (c_{2\ell_i}(\hat{T}_j))$ of size t_k similar to $\tilde{C}$. By definition, we find that

$$\tilde{c}_\ell(\hat{T}) = \frac{c_{2\ell}(\hat{T})}{2\ell}, \tag{29}$$

which implies that

$$\tilde{C} = \operatorname{diag}\left\{\frac{1}{2\ell_1}, \ldots, \frac{1}{2\ell_{t_k}}\right\} C.$$

So $\tilde{C}$ and C have the same singularity.

Corollary 12. *If G_1 is high-ordered cospectral with G_2, then*

$$N_{G_1}(P_3) = N_{G_2}(P_3).$$

Corollary 13. *If G_1 is high-ordered cospectral with G_2, then*

$$N_{G_1}(P_k) = N_{G_2}(P_k),\ k \in \{1, 2, 3\},$$

and

$$N_{G_1}(C_3) = N_{G_2}(C_3),\ N_{G_1}(C_4) = N_{G_2}(C_4).$$

Proof. It is known for a graph G, $\mathrm{Tr}_3(G) = 6N_G(C_3)$ and $\mathrm{Tr}_4(G) = 2N_G(P_2) + 4N_G(P_3) + 8N_G(C_4)$ in [11]. The result follows by Corollary 12 and the above equalities. ■

Corollary 14. *Let G_1 and G_2 be two graphs with girth greater than or equal to g. If G_1 is high-order cospectral with G_2, then*

(1) *if $g = 5$, then $N_{G_1}(\hat{T}) = N_{G_2}(\hat{T})$ for all trees $\hat{T}$ with at most 3 edges;*

(2) *if $g = 7$, then $N_{G_1}(\hat{T}) = N_{G_2}(\hat{T})$ for all trees $\hat{T}$ with at most 4 edges;*

(3) *if $g = 11$, then $N_{G_1}(\hat{T}) = N_{G_2}(\hat{T})$ for all trees $\hat{T}$ with at most 5 edges.*

Proof. By Tables 1 and 2 in [7] and Remark 11, $\mathbf{T}_3$ has 2 trees, and the matrix in (27) is nonsingular by taking $\ell \in \{3, 4\}$. Similarly, $\mathbf{T}_4$ has 3 trees, and the matrix in (27) is nonsingular by taking $\ell \in \{4, 5, 6\}$; $\mathbf{T}_5$ has 6 trees, and the matrix in (27) is nonsingular by taking $\ell \in \{5, 6, \ldots, 10\}$. The result follows by Remark 11. ■

4.2. Unicylic graphs

We first prove that the girth is a high-ordered cospectral invariant of unicyclic graphs.

Lemma 15. *Let U_1 (respectively, U_2) be a unicyclic graph containing a cycle C_{n_1} (respectively, C_{n_2}). If U_1 is high-order cospectral with U_2, then $n_1 = n_2$.*

Proof. Since U_1 is high-order cospectral with U_2, we have $\mathrm{Tr}_d(U_1^m) = \mathrm{Tr}_d(U_2^m)$ for all d and $m \geq 3$. Suppose that $n_1 < n_2$. Now taking $d = n_1 m$, by Corollary 8 we have

$$\sum_{k=1}^{n_1} n_1(m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)+1} \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_{n_1}(\hat{T})(N_{U_2}(\hat{T}) - N_{U_1}(\hat{T}))$$
$$= 2n_1(n_1+1)(m-1)^{(m-1)(|E(U)|-n_1)} m^{n_1(m-2)}.$$

By a simple calculation, we have

$$\sum_{k=1}^{n_1}(m-1)^{(m-1)(n_1-k)-1} m^{(k-n_1)(m-2)+1} \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_{n_1}(\hat{T})(N_{U_2}(\hat{T}) - N_{U_1}(\hat{T})) = 2(n_1+1).$$

Let $h(m,k) = (m-1)^{(m-1)(n_1-k)-1} m^{(k-n_1)(m-2)+1}$, $y(k) = \sum_{\hat{T}\in\mathbf{T}_k} \tilde{c}_{n_1}(\hat{T})(N_{U_2}(\hat{T})$ $-N_{U_1}(\hat{T}))$. Then $h(m,k) = h(m,1)\alpha(m)^{k-1}$, and

$$\sum_{k=1}^{n_1} h(m,1)\alpha(m)^{k-1} y(k) = 2(n_1+1),$$

where $\alpha(m) = \frac{m^{m-2}}{(m-1)^{m-1}}$. Taking m as n_1 different integers $m_1, \ldots, m_{n_1}$ that are greater than 2, we have

(30)
$$\begin{bmatrix} h(m_1,1) & h(m_1,1)\alpha(m_1) & \cdots & h(m_1,1)\alpha(m_1)^{n_1-1} \\ h(m_2,1) & h(m_2,1)\alpha(m_2) & \cdots & h(m_2,1)\alpha(m_2)^{n_1-1} \\ \vdots & \vdots & \vdots & \vdots \\ h(m_{n_1},1) & h(m_{n_1},1)\alpha(m_{n_1}) & \cdots & h(m_{n_1},1)\alpha(m_{n_1})^{n_1-1} \end{bmatrix} \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(n_1) \end{bmatrix} = \begin{bmatrix} 2(n_1+1) \\ 2(n_1+1) \\ \vdots \\ 2(n_1+1) \end{bmatrix}.$$

It is easily found the matrix $H = (h(m_i,1)\alpha(m_i)^{j-1})$ in equation (30) is nonsingular, and by Cramer's rule

$$y(1) = (\det H)^{-1} \det M, \tag{31}$$

where

$$M = \begin{bmatrix} 2(n_1+1) & h(m_1,1)\alpha(m_1) & \cdots & h(m_1,1)\alpha(m_1)^{n_1-1} \\ 2(n_1+1) & h(m_2,1)\alpha(m_2) & \cdots & h(m_2,1)\alpha(m_2)^{n_1-1} \\ \vdots & \vdots & \cdots & \vdots \\ 2(n_1+1) & h(m_{n_1},1)\alpha(m_{n_1}) & \cdots & h(m_{n_1},1)\alpha(m_{n_1})^{n_1-1} \end{bmatrix},$$

which is obtained from H by replacing the first column by $(2(n_1+1), \ldots, 2(n_1+1))^\top$.

We will prove that there exists distinct integers $m_1, m_2, \ldots, m_{n_1}$ such that $\det M \neq 0$. First given any $n_1 - 1$ distinct integers $m_1, m_2, \ldots, m_{n_1-1}$, the matrix $M(n_1, 1)$ is nonsingular so that the equation

$$(2(n_1+1), \ldots, 2(n_1+1))^\top = M(n_1, 1)x$$

has a unique solution $x = (x_1, \ldots, x_{n_1-1})^\top$, where $M(n_1, 1)$ is obtained from M by deleting the last row and the first column. Let ℓ be the smallest index such that $x_\ell \neq 0$. Then $\ell \leq n_1 - 2$; otherwise we have

$$\begin{aligned}&\big(2(n_1+1), \ldots, 2(n_1+1)\big)^\top \\ &= \big(h(m_1, 1)\alpha(m_1)^{n_1-1}, \ldots, h(m_{n_1-1}, 1)\alpha(m_{n_1-1})^{n_1-1}\big)^\top \cdot x_{n_1-1},\end{aligned}$$

a contradiction. Now consider the following equation in the variable m:

$$2(n_1+1) = h(m, 1)\alpha(m)x_1 + \cdots + h(m, 1)\alpha(m)^{n_1-1}x_{n_1-1}. \tag{32}$$

Let

$$\begin{aligned} f(m) &= h(m, 1)\alpha(m)x_1 + \cdots + h(m, 1)\alpha(m)^{n_1-1}x_{n_1-1} \\ &= h(m, 2)x_1 + \cdots + h(m, n_1)x_{n_1-1}. \end{aligned}$$

Note that $h(m, k) \sim e^{-(n_1-k)}(m-1)^{n_1-k}$ when $m \to +\infty$. As $x_\ell \neq 0$ for some smallest $\ell \leq n_1 - 2$,

$$f(m) \sim x_\ell h(m, \ell+1) \sim x_\ell e^{-(n_1-\ell-1)}(m-1)^{n_1-\ell-1}.$$

When $m \to +\infty$, then $|f(m)| \to +\infty$. So, there exists a positive integer $m = m_0$ such that equation (32) does not hold. Surely, m_0 is not equal to any of $m_1, \ldots, m_{n_1-1}$ by the definition of x. Taking $m_{n_1} = m_0$, $\det M \neq 0$ as the first column cannot be a linear combination of other columns.

By equation (31), we get $y(1) \neq 0$. However, by the definition of $y(k)$, $y(1) = 0$ as $N_{U_2}(\hat{T}) = N_{U_1}(\hat{T})$ when $\hat{T} = P_2$. So we get the desired result, namely $n_1 = n_2$. ∎

Lemma 16. *Let U_1 (respectively, U_2) be a unicyclic graph containing a cycle C_n. If U_1 is high-order cospectral with U_2, then*

$$\sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_\ell(\hat{T}) N_{U_1}(\hat{T}) = \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_\ell(\hat{T}) N_{U_2}(\hat{T})$$

for all ℓ, k with $1 \leq k \leq \ell \leq n$.

Proof. As $\mathrm{Tr}_d(U_1^m) = \mathrm{Tr}_d(U_2^m)$, by Corollary 8, for $m \mid d$ and $d/m \le n$, we have

$$\sum_{k=1}^{d/m} (m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)} \sum_{\hat{T}\in \mathbf{T}_k} \tilde{c}_{d/m}(\hat{T})(N_{U_2}(\hat{T}) - N_{U_1}(\hat{T})) = 0.$$

Let $d/m = \ell$, $g(m,k) = (m-1)^{(m-1)(|E(U)|-k)-1} m^{k(m-2)}$, $y(k) = \sum_{\hat{T}\in \mathbf{T}_k} \tilde{c}_\ell(\hat{T})$ $(N_{U_2}(\hat{T}) - N_{U_1}(\hat{T}))$. Then we have

$$\sum_{k=1}^{\ell} g(m,k)y(k) = 0.$$

By a similar discussion as in Lemma 10, we get the desired result. ∎

Remark 17. In Lemma 16, suppose $\mathbf{T}_k$ has exactly t_k trees $\hat{T}_1, \dots, \hat{T}_{t_k}$. If we can take ℓ as t_k different integers $\ell_1, \dots, \ell_{t_k}$ such that $\min\{\ell_i : i \in [t_k]\} \ge k$ and $\max\{\ell_i : i \in [t_k]\} \le n$, then by Lemma 16, we have the matrix equation in equation (27): $\tilde{C}z = 0$. If the matrix $\tilde{C}$ is nonsingular, then z has only zero solution, which implies that for all $\hat{T} \in \mathbf{T}_k$,

$$N_{U_1}(\hat{T}) = N_{U_2}(\hat{T}).$$

Corollary 18. *Let U_1 and U_2 be unicyclic graphs both with girth g. If U_1 is high-order cospectral with U_2, then*

(1) *if $g \ge 4$, then $N_{U_1}(\hat{T}) = N_{U_2}(\hat{T})$ for all trees $\hat{T}$ with at most 3 edges;*

(2) *if $g \ge 6$, then $N_{U_1}(\hat{T}) = N_{U_2}(\hat{T})$ for all trees $\hat{T}$ with at most 4 edges;*

(3) *if $g \ge 10$, then $N_{U_1}(\hat{T}) = N_{U_2}(\hat{T})$ for all trees $\hat{T}$ with at most 5 edges.*

Proof. The proof is similar to that of Corollary 14 by noting that the number ℓ in Lemma 16 is allowed to equal the girth g. We omit the details. ∎

We further investigate the high-order cospectral pairs of unicyclic graphs with girth at most 4.

Corollary 19. *Let U_1 and U_2 be high-order cospectral unicyclic graphs with girth g.*

(1) *If $g = 3$, then*

$$N_{U_1}(P_4) + 2N_{U_1}(S_4) = N_{U_2}(P_4) + 2N_{U_2}(S_4).$$

(2) *If $g = 4$, then*

$$N_{U_1}(P_5) + 2N_{U_1}(Q_5) + 6N_{U_1}(S_5) = N_{U_2}(P_5) + 2N_{U_2}(Q_5) + 6N_{U_2}(S_5).$$

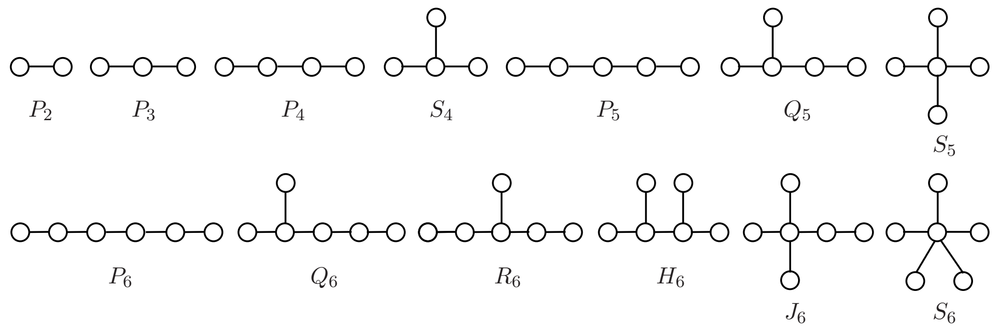


Figure 1. The trees with at most 5 edges.

Proof. By Lemma 16, we have

$$\sum_{\hat{T}\in\mathbf{T}_g} \tilde{c}_g(\hat{T})N_{U_1}(\hat{T}) = \sum_{\hat{T}\in\mathbf{T}_g} \tilde{c}_g(\hat{T})N_{U_2}(\hat{T}). \tag{33}$$

If $g = 3$, then $\mathbf{T}_3 = \{P_4, S_4\}$ and $\tilde{c}_3(P_4) = 1$, $\tilde{c}_3(S_4) = 2$; and if $g = 4$, then $\mathbf{T}_4 = \{P_5, Q_5, S_5\}$ and $\tilde{c}_4(P_5) = 1$, $\tilde{c}_4(Q_5) = 2$ and $\tilde{c}_4(S_5) = 6$, where the graphs S_4, Q_5, S_5 are listed in Figure 1. The result follows by substituting the above values into equation (33). ■

4.3. A class of DHS unicylic graphs

By Lemma 15, the girth is a high-ordered cospectral invariant of unicyclic graphs. However, it does not hold for the usual cospectral invariant of unicyclic graphs. Let $H(n; q, n_1, n_2)$ denotes the unicyclic graph on n vertices which is obtained from a cycle C_q by attaching two paths P_{n_1+1} and P_{n_2+1} at the same vertex of the cycle C_q, and $n_1, n_2 \geq 1$. Liu *et al.* [32] proved the following cospectral mates: $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, $H(n; 2a+6, a, a+2)$ and $\Lambda(a, a, 2a+2)$, $H(n; 2b, b, b)$ and $\Theta(b-2, 2b-3, b-1)$, where $\Lambda(a, a, 2a+2)$ is a unicyclic graph with girth 6 and a being positive even number, and $\Theta(b-2, 2b-3, b-1)$ is a bicyclic graph with b being even number greater than 2; see Theorem 3.4, Lemma 5.8 and Lemma 5.11 in [32]. We now further investigate the high-ordered spectral property of $H(n; q, n_1, n_2)$.

Theorem 20. *The unicyclic graph $H(n; q, n_1, n_2)$ is DHS when $q \geq 5$.*

Proof. Let $H = H(n; q, n_1, n_2)$. Suppose that G is a graph high-ordered cospectral with H, which has n_1 vertices of degree 1, n_2 vertices of degree 2 and n_3 vertices of degree greater than or equal to 3. By the relation $N_H(P_i) = N_G(P_i)$

for $i \in [3]$, and noting that $N_H(P_3) = n + 3$, we have

$$n_1 + n_2 + n_3 = n, n_1 + 2n_2 + \sum_{v:d_v \geq 3} d_v = 2n, n_2 + \sum_{v:d_v \geq 3} \binom{d_v}{2} = n + 3. \tag{34}$$

By the 3rd and 2nd equalities,

$$n + 3 \geq n_2 + \sum_{v:d_v \geq 3} d_v = 2n - n_1 - n_2,$$

which implies that $n_3 \leq 3$. The result will be arrived by the following discussion.

Case 1. $n_3 = 3$. Then for each v with $d_v \geq 3$, $d_v = 3$. So we get $n_1 = 3, n_2 = n - 6, n_3 = 3$. By Corollary 13, $N_H(C_k) = N_G(C_k)$ for $k = 3, 4$. As H contains no C_3 or C_4, the girth of G is at least 5. Observe that $N_G(S_4) = 3$ and $N_H(S_4) = 4$. However, by Corollary 14, $N_H(S_4) = N_G(S_4)$; a contradiction. So this case cannot happen.

Case 2. $n_3 = 2$. Then G contains exactly two vertices, say u, v, both with degree greater than or equal to 3. By the 1st and 2nd equalities in equation (34), we have $d_u + d_v = n + 2 - n_2$. If both d_u and d_v are greater than or equal to 4, then by the 3rd equality in equation (34),

$$n + 3 \geq n_2 + \frac{3}{2}(d_u + d_v) = n_2 + \frac{3}{2}(n + 2 - n_2) = \frac{3}{2}n - \frac{1}{2}n_2 + 3,$$

which implies that $n \leq n_2$; a contradiction. So there exists a vertex among u, v, say u with $d_u = 3$. Also by equation (34),

$$d_v = n - n_2 - 1, \binom{d_v}{2} = n - n_2.$$

If letting $x = n - n_2$, by the above equalities, we have

$$x^2 - 5x + 2 = 0,$$

which implies that x is an irrational number; a contradiction. So this case also cannot happen.

Case 3. $n_3 = 1$. Then G have only one vertex say v with degree greater than or equal to 3. By equation (34), we have

$$d_v = n - n_2 + 1, \binom{d_v}{2} = n - n_2 + 3.$$

By a simple computation, we have $n_1 = 2, n_2 = n - 3, n_3 = 1$ and $d_v = 4$.

Let G_0 be a component of G containing the vertex v. Then G_0 must contain a cycle; otherwise G_0 is a tree with at least 4 vertices of degree 1; a contradiction to $n_1 = 2$. As G_0 contains exactly one vertex greater than 2, namely the vertex v with degree 4, G_0 is of form $H(n; q', n_1', n_2')$.

We assert that G has no other components other than G_0. Otherwise, let G_1 be another component of G. Then G_1 is a cycle with the largest eigenvalue $\lambda_1(G_1) = 2$ as it only contains vertices with degree 2. We also note $\lambda_1(G_0) > 2$ as G_0 contains the cycle C_q as a proper subgraph. So the second eigenvalue $\lambda_2(G)$ of G holds that

$$\lambda_2(G) \geq \min\{\lambda_1(G_0), \lambda_1(G_1)) \geq 2.$$

Let $\tilde{v}$ be the vertex of H with degree 4. Note that $H - \tilde{v}$ consists of paths, and the largest eigenvalue of a path is less than 2. By the interlacing theorem,

$$\lambda_2(H) \leq \lambda_1(H - \tilde{v}) < 2,$$

which yields a contradiction as $\lambda_2(G) = \lambda_2(H)$. So G is of form $H(n; q', n_1', n_2')$. By Theorem 3.4 of [32], except for $H(12; 6, 1, 5)$ and $H(12; 8, 2, 2)$, no two non-isomorphic graphs of form $H(n'; q', n_1', n_2')$ are cospectral. As two high-ordered cospectral unicyclic graphs has the same girth by Lemma 15, G is isomorphic to $H(n; q, n_1, n_2)$.

Case 4. $n_3 = 0$. Then G is a union of paths and/or cycles. This is impossible as $\lambda_1(G) \leq 2 < \lambda_1(H)$. ■

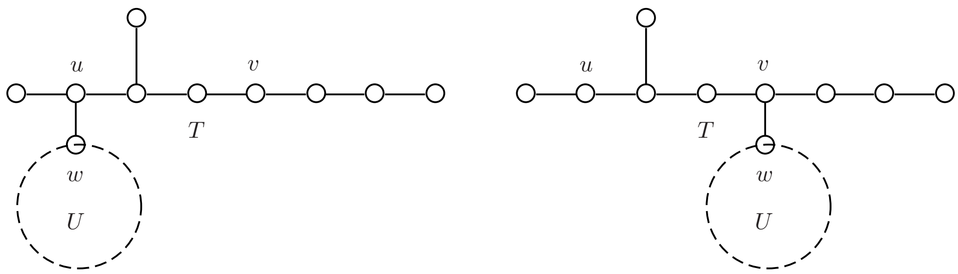
4.4. Cospectral unicyclic graph mates with different high-ordered spectra

There are two methods to construct cospectral graphs. The first is to use coalescence introduced by Schwenk [36] (or see [12]). Let G be a graph with root u and H be a graph with root v. The *coalescence* of G and H is obtained by identifying the root u of G with the root v of H, denoted by $G(u) \odot H(v)$.

Lemma 21 [12,36]**.** *Let G be a graph with root u and H be a graph with root v. If G is cospectral with H and $G - u$ is cospectral with $H - v$, then for any graph Γ with root w, $G(u) \odot \Gamma(w)$ is cospectral with $H(v) \odot \Gamma(w)$.*

The second method is to use rooted product introduced by Godsil and McKay [23] (or see [29]). Let G be a graph and let Γ be a rooted graph. The *root product* $G(\Gamma)$ is the graph obtained by identifying each vertex of G with the root of a copy of Γ.

Lemma 22 [29]**.** *If G is cospectral with a graph H, then for any rooted graph Γ, $G(\Gamma)$ is cospectral with $H(\Gamma)$.*



(a) $T(u) \ominus U(w)$ (b) $T(v) \ominus U(w)$

Figure 2. The graphs $T(u) \ominus U(w)$ and $T(v) \ominus U(w)$.

We now construct infinitely many pairs of cospectral unicyclic graphs with different high-order spectrum. Let T be the tree in Figure 2 (the graph with solid edges) with two specified vertices u and v. It was shown that $T-u$ and $T-v$ have the same spectrum [36]. Let U be any unicyclic graph with root w. Denote by $T(u) \ominus U(w)$ (respectively, $T(v) \ominus U(w)$) the graph obtained by adding an edge between u and w (respectively, between v and w); see Figure 2. By Lemma 21, $T(u) \ominus U(w)$ is cospectral with $T(v) \ominus U(w)$. We now prove that $T(u) \ominus U(w)$ is not high-ordered cospectral with $T(v) \ominus U(w)$.

Corollary 23. *$T(u) \ominus U(w)$ is not high-ordered cospectral with $T(v) \ominus U(w)$.*

Proof. Let $G_{uw} = T(u) \ominus U(w)$ and $G_{vw} = T(v) \ominus U(w)$, and let C_n be the cycle contained in U. We will compare $\mathrm{Tr}_d(G_{uw}^m)$ and $\mathrm{Tr}_d(G_{vw}^m)$ with $d = 5m$.

Let $H \in \mathcal{V}_d(G_{uw}^m, [C_n^m])$. By Lemma 5, H is a weighted hypergraph $\underline{H}(\omega)$ which contains C_n^m such that each edge e of $\underline{H}$ repeats in $m\omega(e)$ times. So $\omega(\underline{H}) = 5m/5 = 5$, which implies that $\underline{H}$ has at most 5 edges. As C_n^m has at least 3 edges, $\underline{H}$ has at most 2 edges outside U^m. Let $\tilde{G}_{uw}$ be the subgraph of G_{uw} induced by the vertices of U, u and its neighbors. Then $\underline{H}$ is contained in $\tilde{G}_{uw}^m$. So $\mathcal{V}_d(G_{uw}^m, [C_n^m]) = \mathcal{V}_d(\tilde{G}_{uw}^m, [C_n^m])$. Similarly, if letting $\tilde{G}_{vw}$ be the subgraph of G_{vw} induced by the vertices of U, v and its neighbors, then for each $H \in \mathcal{V}_d(G_{vw}^m, [C_n^m])$, $\underline{H}$ is contained in $\tilde{G}_{vw}^m$, and hence $\mathcal{V}_d(G_{vw}^m, [C_n^m]) = \mathcal{V}_d(\tilde{G}_{vw}^m, [C_n^m])$. Note that $\tilde{G}_{uw}$ is isomorphic to $\tilde{G}_{vw}$. So $\mathcal{V}_d(G_{uw}^m, [C_n^m])$ is equal to $\mathcal{V}_d(G_{vw}^m, [C_n^m])$ under isomorphism.

By definition,

$$\mathrm{Tr}_d(G_{uw}^m; [C_n^m]) = d(m-1)^{|V(G_{uw}^m)|} \sum_{H \in \mathcal{V}_d(G_{uw}^m, [C_n^m])} C_H N_{G_{uw}^m}(\underline{H})$$

$$
\begin{aligned}
&= d(m-1)^{|V(G^m_{uw})|} \sum_{H \in \mathcal{V}_d(\tilde{G}^m_{uw}, [C^m_n])} C_H N_{\tilde{G}^m_{uw}}(\underline{H}) \\
&= d(m-1)^{|V(G^m_{vw})|} \sum_{H \in \mathcal{V}_d(\tilde{G}^m_{vw}, [C^m_n])} C_H N_{\tilde{G}^m_{vw}}(\underline{H}) \\
&= d(m-1)^{|V(G^m_{vw})|} \sum_{H \in \mathcal{V}_d(G^m_{vw}, [C^m_n])} C_H N_{G^m_{vw}}(\underline{H}) \\
&= \mathrm{Tr}_d(G^m_{vw}; [C^m_n]).
\end{aligned}
$$

By equations (14) and (15), we have

$$
\begin{aligned}
&\mathrm{Tr}_d(G^m_{uw}) - \mathrm{Tr}_d(G^m_{vw}) \\
&= \mathrm{Tr}_d(G^m_{uw}; [\hat{C}^m_n]) - \mathrm{Tr}_d(G^m_{vw}; [\hat{C}^m_n]) \\
&= \sum_{k=1}^{5} d(m-1)^{(m-1)(p-k)-1} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_5(\hat{T})(N_{G_{uw}}(\hat{T}) - N_{G_{vw}}(\hat{T})),
\end{aligned}
$$

where p denotes the number of edges of G_{uw} or G_{vw}. We find that for each tree $\hat{T} \in \mathbf{T}_{\leq 5} \setminus \{P_5, Q_5, P_6, Q_6, H_6\}$, $N_{G_{uw}}(\hat{T}) = N_{G_{vw}}(\hat{T})$, where Q_5, Q_6, H_6 are listed in Figure 1. So

$$
\begin{aligned}
\mathrm{Tr}_d(G^m_{uw}) - \mathrm{Tr}_d(G^m_{vw}) = dg(m,4)\big[&\tilde{c}_5(P_5)(N_{G_{uw}}(P_5) - N_{G_{vw}}(P_5)) \\
&+ \tilde{c}_5(Q_5)(N_{G_{uw}}(Q_5) - N_{G_{vw}}(Q_5)\big] \\
&+ dg(m,5)\big[\tilde{c}_5(P_6)(N_{G_{uw}}(P_6) - N_{G_{vw}}(P_6)) \\
&+ \tilde{c}_5(Q_6)(N_{G_{uw}}(Q_6) - N_{G_{vw}}(Q_6)) \\
&+ \tilde{c}_5(H_6)(N_{G_{uw}}(H_6) - N_{G_{vw}}(H_6))\big],
\end{aligned}
$$

where $g(m,k) = (m-1)^{(m-1)(p-k)-1} m^{k(m-2)}$. By a direct computation or referring Tables 1 and 2 in [7] with the relation (29), we have

$$
\tilde{c}_5(P_5) = 6, \tilde{c}_5(Q_5) = 14, \tilde{c}_5(P_6) = 1, \tilde{c}_5(Q_6) = 2, \tilde{c}_5(H_6) = 4.
$$

We also have

$$
\begin{gathered}
N_{G_{uw}}(P_5) - N_{G_{vw}}(P_5) = -2, N_{G_{uw}}(Q_5) - N_{G_{vw}}(Q_5) = 1, \\
N_{G_{uw}}(P_6) - N_{G_{vw}}(P_6) = -2d_{w(U)}, N_{G_{uw}}(Q_6) - N_{G_{vw}}(Q_6) = d_{w(U)} - 3, \\
N_{G_{uw}}(H_6) - N_{G_{vw}}(H_6) = 1.
\end{gathered}
$$

So,

$$
\begin{aligned}
\mathrm{Tr}_d(G^m_{uw}) - \mathrm{Tr}_d(G^m_{vw}) &= 2dg(m,4) - 2dg(m,5) \\
&= 2d(m-1)^{(m-1)(p-5)-1} m^{4(m-2)} \left((m-1)^{m-1} - m^{m-2}\right).
\end{aligned}
$$

Clearly, there exists $m = m_0$ (e.g. $m_0 = 3$) such that $\mathrm{Tr}_d(G_{uw}^m) - \mathrm{Tr}_d(G_{vw}^m) \neq 0$, which implies that $G_{uw}^{m_0}$ is not cospectral with $G_{vw}^{m_0}$. The result follows. ∎

Finally we note that the smallest cospectral pair of graphs are $G_1 = K_{1,4}$ (also S_5 in Figure 1) and $G_2 = C_4 + K_1$. Consider $\mathrm{Tr}_d(G_1^m)$ and $\mathrm{Tr}_d(G_2^m)$ with $d/m = 2$. By equation (8),

$$\mathrm{Tr}_d(G_1^m) = \sum_{k=1}^{2} d(m-1)^{(m-1)(4-k)} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_2(\hat{T}) N_{G_1}(\hat{T}).$$

Noting that $V(G_2^m) = V(C_4^m) \cup V(K_1)$, by equation (23) we have

$$\mathrm{Tr}_d(G_2^m) = \sum_{k=1}^{2} d(m-1)^{(m-1)(4-k)} m^{k(m-2)} \sum_{\hat{T} \in \mathbf{T}_k} \tilde{c}_2(\hat{T}) N_{G_2}(\hat{T}).$$

As $N_{G_1}(P_2) = N_{G_2}(P_2) = 4$ and $N_{G_1}(P_3) = 6 \neq N_{G_2}(P_3) = 4$, we have $\mathrm{Tr}_d(G_1^m) \neq \mathrm{Tr}_d(G_2^m)$. So $K_{1,4}$ and $C_4 + K_1$ are not high-order cospectral pair.

We consider the rooted products $G_1(P_n)$ and $G_2(P_n)$, where P_n has one of its pendent vertices as the root and $n \geq 2$. By Lemma 22, $G_1(P_n)$ and $G_2(P_n)$ are cospectral. As $N_{G_1(P_n)}(P_3) - N_{G_2(P_n)}(P_3) = 2 \neq 0$, by a similar discussion we have $\mathrm{Tr}_d((G_1(P_n))^m) \neq \mathrm{Tr}_d((G_2(P_n))^m)$ for $d/m = 2$. So $G_1(P_n)$ and $G_2(P_n)$ are also not high-order cospectral pair.

Corollary 24. *Let $G_1 = K_{1,4}$ and $G_2 = C_4 + K_1$, and let P_n be a path on n vertices with one of its pendent vertices as root. Then $G_1(P_n)$ and $G_2(P_n)$ are not high-order cospectral pair.*

5. Conclusion

In this paper, we investigate the spectral characterization of unicyclic graphs by the high-ordered spectra of unicyclic graphs. It is seen that the high-ordered spectra can recognize more structural information than the usual spectra, which implies that high-ordered spectra may have potential value on the graph isomorphism problem. We also find there are two questions for further study on high-ordered spectral characterization of graphs.

(1) Recall that two graphs G_1 and G_2 are high-ordered cospectral if the spectrum of G_1^m is the same as that of G_2^m for all $m \geq 2$. In fact, from the discussion in Section 4 (e.g. Lemmas 10 and 15), we only need finitely many m's such that G_1 and G_2 are m-ordered cospectral. So, could we give an upper bound for the m on the definition of high-ordered cospectral? If it does, we will save times on comparing the high-ordered spectra of two graphs.

In addition, it is known that G_1^m is cospectral with G_2^m if and only if $\mathrm{Tr}_d(G_1^m) = \mathrm{Tr}_d(G_2^m)$ for all d or $d = 1, 2, \ldots, n(m-1)^{n-1}$, where n is the number of vertices of G_1^m or G_2^m. If $m \geq 3$, then $\mathrm{Tr}_d(G^m) \neq 0$ only if $m \mid d$. As seen in Lemmas 10 and 16, Corollaries 14, 18, 19, 23 and 24, we care more about d/m than d. This also can be found from the definition $\tilde{c}_{d/m}(\hat{T})$ in equation (7).

(2) It is harder to compute the spectra or the characteristic polynomials of uniform hypergraphs than graphs, as it is closely related to the computation of resultants. Though Chen *et al.* [6] gave a method to get the eigenvalues of the power G^m from the eigenvalues of signed subgraphs of G (see Lemma 1), we still do not know the multiplicities of the eigenvalues of G^m. So, how to compute the spectra of the power hypergraphs is a key question. Fortunately, we can use traces instead of spectra to recognize high-ordered cospectral graphs, and can compute the traces of hypergraphs with simple structure by Corollary 3.

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References

[1] T. van Aardenne-Ehrenfest and N.G. de Bruijn, *Circuits and trees in oriented linear graphs*, in: Classic Papers in Combinatorics, I. Gessel and G.C. Rota (Ed(s)) (Boston, Birkhäuser Boston, 1987) 149–163.

[2] S. Bai and L. Lu, *A bound on the spectral radius of hypergraphs with e edges*, Linear Algebra Appl. **549** (2018) 203–218.
https://doi.org/10.1016/j.laa.2018.03.030

[3] K. Cardoso, C. Hoppen and V. Trevisan, *The spectrum of a class of uniform hypergraphs*, Linear Algebra Appl. **590** (2020) 243–257.
https://doi.org/10.1016/j.laa.2019.12.042

[4] K.C. Chang, K. Pearson and T. Zhang, *Perron-Frobenius theorem for nonnegative tensors*, Commun. Math. Sci. **6** (2008) 507–520.
https://doi.org/10.4310/CMS.2008.v6.n2.a12

[5] L. Chen, C. Bu and J. Zhou, *Spectral moments of hypertrees and their applications*, Linear Multilinear Algebra (2021), in press.
https://doi.org/10.1080/03081087.2021.1953431

[6] L. Chen, E.R. van Dam and C. Bu, *All eigenvalues of the power hypergraph and signed subgraphs of a graph* (2022, a manuscript).
arXiv:2209.03709

[7] L. Chen, L. Sun and C. Bu, *High-ordered spectral characterizations of graphs* (2021, a manuscript).
arXiv:2111.03877

[8] G.J. Clark and J.N. Cooper, *A Harary-Sachs theorem for hypergraphs*, J. Combin. Theory Ser. B **149** (2021) 1–15.
https://doi.org/10.1016/j.jctb.2021.01.002

[9] J.N. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl. **436** (2012) 3268–3292.
https://doi.org/10.1016/j.laa.2011.11.018

[10] L. von Collatz and U. Sinogowitz, *Spektren endlicher Grafen*, Abh. Math. Semin. Univ. Hambg. **21** (1957) 63–77.
https://doi.org/10.1007/BF02941924

[11] D. Cvetković and P. Rowlinson, *Spectra of unicyclic graphs*, Graphs Combin. **3** (1987) 7–23.
https://doi.org/10.1007/BF01788525

[12] E.R. van Dam and W.H. Haemers, *Which graphs are determined by their spectrum*?, Linear Algebra Appl. **373** (2003) 241–272.
https://doi.org/10.1016/S0024-3795(03)00483-X

[13] E.R. van Dam and W.H. Haemers, *Developments on spectral characterizations of graphs*, Discrete Math. **309** (2009) 576–586.
https://doi.org/10.1016/j.disc.2008.08.019

[14] Y.-Z. Fan, Y.-H. Bao and T. Huang, *Eigenvariety of nonnegative symmetric weakly irreducible tensors associated with spectral radius and its application to hypergraphs*, Linear Algebra Appl. **564** (2019) 72–94.
https://doi.org/10.1016/j.laa.2018.11.027

[15] Y.-Z. Fan, T. Huang and Y.-H. Bao, *The dimension of eigenvariety of nonnegative tensors associated with spectral radius*, Proc. Amer. Math. Soc. **150** (2022) 2287–2299.
https://doi.org/10.1090/proc/15781

[16] Y.-Z. Fan, T. Huang, Y.-H. Bao, C. L. Zhuan-Sun and Y.-P. Li, *The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs*, Trans. Amer. Math. Soc. **372** (2019) 2213–2233.
https://doi.org/10.1090/tran/7741

[17] Y.-Z. Fan, M. Li and Y. Wang, *The cyclic index of adjacency tensor of generalized power hypergraphs*, Discrete Math. **344(5)** (2021) 112329.
https://doi.org/10.1016/j.disc.2021.112329

[18] Y.-Z. Fan, Y.-Y. Tan, X.-X. Peng and A.-H. Liu, *Maximizing spectral radii of uniform hypergraphs with few edges*, Discuss. Math. Graph Theory **36** (2016) 845–856.
https://doi.org/10.7151/dmgt.1906

[19] Y.-Z. Fan, M.-Y. Tian and M. Li, *The stabilizing index and cyclic index of the coalescence and Cartesian product of uniform hypergraphs*, J. Combin. Theory Ser. A **185** (2022) 105537.
https://doi.org/10.1016/j.jcta.2021.105537

[20] Y.-Z. Fan, Y. Yang, C.-M. She, J. Zheng, Y.-M. Song and H.-X. Yang, *The trace and Estrada index of uniform hypergraphs with cut vertices*, Linear Algebra Appl. **660** (2023) 89–117.
https://doi.org/10.1016/j.laa.2022.12.006

[21] S. Friedland, S. Gaubert and L. Han, *Perron-Frobenius theorem for nonnegative multilinear forms and extensions*, Linear Algebra Appl. **438** (2013) 738–749.
https://doi.org/10.1016/j.laa.2011.02.042

[22] G. Gao, A. Chang and Y. Hou, *Spectral radius on linear r-graphs without expanded K_{r+1}*, SIAM J. Discrete. Math. **36** (2022) 1000–1011.
https://doi.org/10.1137/21M1404740

[23] C.D. Godsil and B.D. McKay, *A new graph product and its spectrum*, Bull. Aust. Math. Soc. **18** (1978) 21–28.
https://doi.org/10.1017/S0004972700007760

[24] C.D. Godsil and B.D. McKay, *Constructing cospectral graphs*, Aequationes Math. **25** (1982) 257–268.
https://doi.org/10.1007/BF02189621

[25] Hs.H. Günthard and H. Primas, *Zusammenhang von Graphentheorie und MO-Theorie von Molekeln mit Systemen konjugierter Bindungen*, Helv. Chim. Acta **39** (1956) 1645–1653.
https://doi.org/10.1002/hlca.19560390623

[26] S. Hu, L. Qi and J. Shao, *Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues*, Linear Algebra Appl. **439** (2013) 2980–2998.
https://doi.org/10.1016/j.laa.2013.08.028

[27] P. Keevash, J. Lenz and D. Mubayi, *Spectral extremal problems for hypergraphs*, SIAM J. Discrete Math. **28** (2014) 1838–1854.
https://doi.org/10.1137/130929370

[28] L.-H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, in: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adapative Processing (CAMSAP'05) (New Jersey, 2005) 129–132.

[29] M. Lepović, *Some results on starlike trees and sunlike graphs*, J. Appl. Math. Comput. **11** (2003) 109–122.
https://doi.org/10.1007/BF02935725

[30] H. Li, J.-Y. Shao and L. Qi, *The extremal spectral radii of k-uniform supertrees*, J. Comb. Optim. **32** (2016) 741–764.
https://doi.org/10.1007/s10878-015-9896-4

[31] L. Liu, L. Kang and E. Shan, *On the irregularity of uniform hypergraphs*, European J. Comb. **71** (2018) 22–32.
https://doi.org/10.1016/j.ejc.2018.02.034

[32] X. Liu, S. Wang, Y. Zhang and X. Yong, *On the spectral characterization of some unicyclic graphs*, Discrete Math. **311** (2011) 2317–2336.
https://doi.org/10.1016/j.disc.2011.05.034

[33] L. Lu and S. Man, *Connected hypergraphs with small spectral radius*, Linear Algebra Appl. **509** (2016) 206–227.
https://doi.org/10.1016/j.laa.2016.07.013

[34] A. Morozov and Sh. Shakirov, *Analogue of the identity Log Det = Trace Log for resultants*, J. Geom. Phys. **61** (2011) 708–726.
https://doi.org/10.1016/j.geomphys.2010.12.001

[35] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005) 1302–1324.
https://doi.org/10.1016/j.jsc.2005.05.007

[36] A.J. Schwenk, *Almost all trees are cospectral*, in: New Directions in the Theory of Graphs, F. Harary (Ed(s)) (New York: Academic Press, 1973) 275–307.

[37] J.-Y. Shao, L. Qi and S. Hu, *Some new trace formulas of tensors with applications in spectral hypergraph theory*, Linear Multilinear Algebra **63** (2015) 971–992.
https://doi.org/10.1080/03081087.2014.910208

[38] W.T. Tutte and C.A.B. Smith, *On unicursal paths in a network of degree* 4, Amer. Math. Monthly **48** (1941) 233–237.
https://doi.org/10.1080/00029890.1941.11991103

[39] W. Wang, *A simple arithmetic criterion for graphs being determined by their generalized spectra*, J. Combin. Theory Ser. B **122** (2017) 438–451.
https://doi.org/10.1016/j.jctb.2016.07.004

[40] W. Wang and C.-X. Xu, *A sufficient condition for a family of graphs being determined by their generalized spectra*, European J. Combin. **27** (2006) 826–840.
https://doi.org/10.1016/j.ejc.2005.05.004

[41] W. Wang and C.-X. Xu, *An excluding algorithm for testing whether a family of graphs are determined by their generalized spectra*, Linear Algebra Appl. **418** (2006) 62–74.
https://doi.org/10.1016/j.laa.2006.01.016

[42] Y. Yang and Q. Yang, *Further results for Perron-Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl. **31** (2010) 2517–2530.
https://doi.org/10.1137/090778766

[43] Q. Yang and Y. Yang, *Further results for Perron-Frobenius theorem for nonnegative tensors* II, SIAM J. Matrix Anal. Appl. **32** (2011) 1236–1250.
https://doi.org/10.1137/100813671

[44] Y. Yang and Q. Yang, *On some properties of nonnegative weakly irreducible tensors* (2011, a manuscript).
arXiv:1111.0713v2

[45] W. Zhang, L. Kang, E. Shan and Y. Bai, *The spectra of uniform hypertrees*, Linear Algebra Appl. **533** (2017) 84–94.
https://doi.org/10.1016/j.laa.2017.07.018

[46] J. Zhou, L. Sun, W. Wang and C. Bu, *Some spectral properties of uniform hypergraphs*, Electron. J. Combin. **21(4)** (2014) #P4.24.
https://doi.org/10.37236/4430

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