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# ALL TIGHT DESCRIPTIONS OF FACES IN PLANE TRIANGULATIONS WITH MINIMUM DEGREE 4

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#### Abstract

It follows from the classical theorem by Lebesgue (1940) on the structure of minor faces in 3-polytopes that every plane triangulation with minimum degree at least 4 has a 3-face for which the set of degrees of its vertices is majorized by one of the following sequences:  $(4, 4, \infty)$ , (4, 5, 19), (4, 6, 11), (4, 7, 9), (5, 5, 9), (5, 6, 7).

In 1999, Jendrol' gave the following description of faces:  $(4, 4, \infty)$ , (4, 5, 13), (4, 6, 17), (4, 7, 8), (5, 5, 7), (5, 6, 6). Also, Jendrol' (1999) conjectured that there is a face of one of the types:  $(4, 4, \infty)$ , (4, 5, 10), (4, 6, 15), (4, 7, 7), (5, 5, 7), (5, 6, 6). In 2002, Lebesgue's description was strengthened by Borodin to  $(4, 4, \infty)$ , (4, 5, 17), (4, 6, 11), (4, 7, 8), (5, 5, 8), (5, 6, 6).

In 2014, we obtained the following tight description, which, in particular, disproves the above mentioned conjecture by Jendrol':  $(4, 4, \infty)$ , (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6). Recently, we obtained another tight description:  $(4, 4, \infty)$ , (4, 6, 10), (4, 7, 7), (5, 5, 8), (5, 6, 7).

The purpose of this paper is to give an exhausting list of tight descriptions of faces in plane triangulations with minimum degree at least 4, which turns out to consist of 32 items.

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### 1. INTRODUCTION

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [30], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs.

The degree d(x) of a vertex or face x in a 3-polytope P is the number of incident edges. A *k*-vertex and *k*-face is one of degree k, a  $k^+$ -vertex has degree at least k, and so on. A face f is minor if  $d(f) \leq 5$ . A face f with the set of incident vertices  $\{v_1, v_2, \ldots\}$  in P is of type  $(k_1, k_2, \ldots)$  or a  $(k_1, k_2, \ldots)$ -face if  $d(v_i) \leq k_i$  for each i.

By  $\mathbf{P}_{\delta}$  denote the class of 3-polytopes with minimum degree  $\delta$ ; in particular,  $\mathbf{P}_{3}$  is the set of all 3-polytopes. A plane triangulation is a 3-polytope with all faces of degree 3.

By  $\Delta$  and  $\delta$  denote the maximum and minimum vertex degree of P, respectively. The weight w(f) of a face f in P is the degree-sum of its boundary vertices, and w(P), or simply w, denotes the minimum weight of minor faces in P. The height h(f) of a face f in P is the maximum of the degrees of the vertices incident with f, and h(P) is the minimum of h(f) over the minor faces f in P.

We now recall some results on the structure of minor faces in 3-polytopes, beginning with the fundamental theorem of Lebesgue [26] from 1940.

**Theorem 1** (Lebesgue [26]). Every 3-polytope has a minor face of one of the following types:

 $(3, 6, \infty), (3, 7, 41), (3, 8, 23), (3, 9, 17), (3, 10, 14), (3, 11, 13), (4, 4, \infty), (4, 5, 19), (4, 6, 11), (4, 7, 9), (5, 5, 9), (5, 6, 7), (3, 3, 3, 3, \infty), (3, 3, 4, 11), (3, 3, 5, 7), (3, 4, 4, 5), (3, 3, 3, 3, 5).$ 

The classical Theorem 1, along with other ideas in Lebesgue [26], has a lot of applications to plane graph coloring problems (the first example of such applications and recent surveys can be found in [7, 13, 23, 28]).

Some parameters of Lebesgue's Theorem were improved for several narrow classes of plane graphs. In 1963, Kotzig [24] proved that every plane triangulation in  $\mathbf{P}_5$  satisfies  $w \leq 18$  and conjectured that  $w \leq 17$  holds. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

**Theorem 2** [2]. Every 3-polytope with  $\delta = 5$  has a (5, 5, 7)-face or a (5, 6, 6)-face, where all parameters are tight.

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Theorem 2 also confirmed a conjecture of Grübaum [20] from 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [29]).

We note that if  $(4, 4, \infty)$ - or  $(3, 3, 3, \infty)$ -faces, called *pyramidal faces*, are allowed in a 3-polytope, then it can happen that all its faces are of unbounded height (and weight).

For plane triangulations without 4-vertices, Kotzig [25] proved  $w \leq 39$ , and Borodin [3], confirming Kotzig's conjecture in [25], proved  $w \leq 29$ , which is best possible due to the dual of the twice-truncated dodecahedron. Borodin [5] further shows that each triangulated 3-polytope without  $(4, 4, \infty)$ -faces satisfies  $w \leq 29$ , and that for triangulations without 4-vertices adjacent to each other there holds a sharp bound  $w \leq 37$ .

For  $\mathbf{P_5}$ , Theorem 1 yields  $w \leq \max\{51, \Delta + 9\}$ . In 1996, Horňák and Jendrol' [21] strengthened this as follows: if there are no pyramidal faces, then  $w \leq 47$ . Borodin and Woodall [19] proved that forbidding  $(3, 3, 3, \infty)$ -faces implies  $w \leq \max\{29, \Delta + 8\}$ .

More generally, Horňák and Jendrol' [21] proved the following description.

**Theorem 3** (Horňák and Jendrol' [21]). Every 3-polytope has a minor face of one of the following types:

 $(3,5,39), (3,6,23), (3,7,18), (3,8,15), (3,9,14), (3,10,13), (4,4,\infty), (4,5,19), (4,6,11), (4,7,9), (5,5,9), (5,6,7), (3,3,3,\infty), (3,3,7,11), (3,4,5,7), (3,5,5,6), (3,3,3,3,5).$ 

Recently, the upper bound  $h \leq 10$  for quadrangulated 3-polytopes without  $(3,3,3,\infty)$ -faces, (Avgustinovich–Borodin [1]) was improved in Borodin– Ivanova [10] to  $h \leq 8$ , which is sharp.

In 2002, Borodin [6] strengthened Lebesgue's Theorem 1 as follows (the entries marked by an asterisk are proved in [6] to be best possible).

**Theorem 4** [6]. Every 3-polytope has a minor face of one of the following types:

 $(3, 6, \infty^*), (3, 8^*, 22), (3, 9^*, 15), (3, 10^*, 13), (3, 11^*, 12), (4, 4, \infty^*), (4, 5^*, 17), (4, 6^*, 11), (4, 7^*, 8), (5, 5^*, 8), (5, 6, 6^*), (3, 3, 3, \infty^*), (3, 3, 4^*, 11), (3, 3, 5^*, 7), (3, 4, 4, 5^*), (3, 3, 3, 3, 5^*).$ 

In 2013–2014, precise descriptions of the structure of faces were obtained for 3-polytopes in  $\mathbf{P}_4$  and for triangulated 3-polytopes.

**Theorem 5** [8]. Every 3-polytope in  $\mathbf{P}_4$  has a 3-face of one of the following types:  $(4, 4, \infty)$ , (4, 5, 14), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6), where all parameters are tight.

**Theorem 6** [16]. Every plane triangulation has a face of one of the following types: (3, 4, 31), (3, 5, 21), (3, 6, 20), (3, 7, 13), (3, 8, 14), (3, 9, 12), (3, 10, 12),  $(4, 4, \infty)$ , (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 6, 6), (5, 5, 7), where all parameters are tight.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in recent surveys [13,23], and also in [6,9,11,12,18,22,27].

Over the last decades, a lot of coloring problems have been posed, and some of them solved, in terms of forbidding these or those sets of cycle-lengths; for a survey see Borodin [7].

In 2017, Borodin, Ivanova, and Vasil'eva [17], suggested to apply a similar approach to finding tight versions of Theorems 1 and 3 by forbidding vertices from certain gradually narrowing degree-sets and obtained a tight description assuming the absence of vertices of degree 4 to 7.

Recently, we proved [14] that every 3-polytope without vertices of degree from 5 to 7 has a face of one of the types:  $(4, 4, \infty)$ , (3, 8, 14), (3, 9, 13), (3, 10, 12),  $(3, 3, 3, \infty)$ , (3, 3, 4, 11), (3, 4, 4, 4), (3, 3, 3, 3, 4), which description is tight.

Furthermore, we recently obtained [15] the following tight description of faces for triangulations without 3-vertices, which does not follow from the general Theorem 6.

**Theorem 7** [15]. Every plane triangulation in  $\mathbf{P}_4$  has a face of one of the following types:  $(4, 4, \infty)$ , (4, 6, 10), (5, 5, 8), (4, 7, 7), (5, 6, 7), where all parameters are best possible.

The purpose of this paper is to completely describe the set of tight descriptions of faces for triangulations in  $\mathbf{P}_4$ . (As usual, by a *tight description* of these or those fragments in a certain class of planar graphs we mean one in which no parameter can be lowered and no term dropped.)

**Theorem 8.** There exist precisely the following 32 tight descriptions of faces in plane triangulations without 3-vertices:

 $\begin{array}{l} (td1): \{(5,7,\infty)\}, \\ (td2): \{(4,7,\infty), (5,6,7)\}, \\ (td3): \{(4,7,\infty), (5,5,7), (5,6,6)\}, \\ (td4): \{(5,6,\infty), (4,7,7)\}, \\ (td5): \{(4,6,\infty), (5,7,7)\}, \\ (td6): \{(4,6,\infty), (4,7,7), (5,6,7)\}, \\ (td7): \{(4,6,\infty), (4,7,7), (5,5,7), (5,6,6)\}, \\ (td8): \{(5,5,\infty), (4,7,10), (5,6,6)\}, \\ (td9): \{(5,5,\infty), (4,6,10), (4,7,7), (5,6,6)\}, \\ (td10): \{(4,5,\infty), (4,7,10), (5,6,7)\}, \end{array}$ 

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 $(td11): \{(4,5,\infty), (4,7,10), (5,5,7), (5,6,6)\},\$ (td12): { $(4, 5, \infty), (4, 6, 10), (5, 7, 7)$ },  $(td13): \{(4,5,\infty), (4,6,10), (4,7,7), (5,6,7)\},\$  $(td14): \{(4,5,\infty), (4,6,10), (4,7,7), (5,5,7), (5,6,6)\},\$  $(td15): \{(4, 4, \infty), (5, 7, 10)\},\$ (td16): { $(4, 4, \infty), (5, 6, 10), (4, 7, 7)$ },  $(td17): \{(4, 4, \infty), (4, 7, 11), (5, 6, 7)\},\$  $(td18): \{(4, 4, \infty), (4, 7, 11), (5, 5, 7), (5, 6, 6)\},\$  $(td19): \{(4, 4, \infty), (4, 6, 11), (5, 7, 7)\},\$ (td20): { $(4, 4, \infty), (4, 6, 11), (4, 7, 7), (5, 6, 7)$ }, (td21): { $(4, 4, \infty), (4, 6, 11), (4, 7, 7), (5, 5, 7), (5, 6, 6)$ },  $(td22): \{(4,4,\infty), (4,5,11), (4,7,10), (5,6,7)\},\$ (td23): { $(4, 4, \infty), (4, 5, 11), (4, 7, 10), (5, 5, 7), (5, 6, 6)$ }, (td24): { $(4, 4, \infty), (4, 5, 11), (4, 6, 10), (5, 7, 7)$ },  $(td25): \{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 6, 7)\},\$ (td26): { $(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)$ }, (td27): { $(4, 4, \infty), (4, 7, 10), (5, 6, 8)$ }, (td28): { $(4, 4, \infty), (4, 7, 10), (5, 5, 8), (5, 6, 7)$ }, (td29): { $(4, 4, \infty), (4, 6, 10), (5, 7, 8)$ }, (td30): { $(4, 4, \infty), (4, 6, 10), (5, 6, 8), (4, 7, 7)$ },  $(td31): \{(4, 4, \infty), (4, 6, 10), (5, 5, 8), (5, 7, 7)\},\$  $(td32): \{(4,4,\infty), (4,6,10), (5,5,8), (4,7,7), (5,6,7)\}.$ 

### 2. Proving Theorem 8

We first define 3-polytopes  $H_1-H_7$  important for the proof. By  $H_1$  we denote the double *n*-pyramid, which has only (4, 4, n)-faces, and thus confirms the presence of  $\infty$  in each of descriptions (td1)-(td32).

Figure 1 [9] shows how to transform the (3, 3, 3, 3, 5) Archimedean solid, in which every vertex is incident with four 3-faces and a 5-face, into a 3-polytope which has only faces of types (4, 5, 11) and (5, 5, 8). This construction confirms that each of (td1)-(td32) must contain a triplet either  $(4^+, 5^+, 11^+)$  or  $(5^+, 5^+, 8^+)$ .

In Figure 2 we transform the (3, 3, 3, 3, 5) Archimedean solid into a 3-polytope which has only faces of types (4, 5, 11) and (5, 6, 7), which confirms the necessity or  $(4^+, 5^+, 11^+)$  or  $(5^+, 6^+, 7^+)$  in each of (td1)-(td32). This construction first appeared in [15].

Figure 3 shows well-known constructions  $H_4$ – $H_7$ , arising mainly from some Platonic and Archimedean solids, while  $H_6$  is obtained by gluing two halves along the outside cycle. The origin of these constructions is already difficult to



Figure 1.  $H_2$  has only (4, 5, 11)- and (5, 5, 8)-faces [9].



Figure 2.  $H_3$  has only (4, 5, 11)- and (5, 6, 7)-faces [15].

trace; at least we know that they were used in [4,8]. It is not hard to see that each of  $H_4$ ,  $H_5$ ,  $H_6$ ,  $H_7$  proves that the corresponding triplet from  $\{(4^+, 6^+, 10^+), (4^+, 7^+, 7^+), (5^+, 5^+, 7^+), (5^+, 6^+, 6^+)\}$  must appear in each of descriptions (td1)-(td32).

The rest of proving Theorem 8 falls into three lemmas, which consistently show that

- (i) each of the sets  $(td1), \ldots, (td32)$  is indeed a description,
- (ii) each of these descriptions is tight, and
- (iii) there are no tight descriptions different from  $(td1), \ldots, (td32)$ .

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Figure 3. Graphs  $H_4$ ,  $H_5$ ,  $H_6$ , and  $H_7$ .

**Lemma 9.** Each of the sets  $(td1), \ldots, (td32)$  is a description of faces in triangulations from  $\mathbf{P}_4$ .

**Proof.** For (td26) and (td32) this follows from the above mentioned results in [9] and [15], respectively.

Now note that, by definition, each (i, j, k)-path is also an (i', j', k')-path if  $i' \ge i, j' \ge j$ , and  $k' \ge k$ .

Therefore, for each of the sets of triplets in  $\{(td1), \ldots, (td25)\}$  it suffices (but maybe is a bit laborous) to check that all its triplets together cover all triplets in (td26). For example, the only triplet  $(5, 7, \infty)$  in (td1) covers each of the triplets in (td26).

In other words, as soon as any triangulation  $T_4$  in  $\mathbf{P}_4$  has a face of at least one of the types from the set  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ since (td1) is an already proven description, it follows that  $T_4$  has a  $(5, 7, \infty)$ -face, which proves that (td1) also describes  $T_4$ , as stated.

Similarly, for each of the sets  $(td27), \ldots, (td31)$  it is also easy to check that its triplets together cover all triplets in (td32). For example, the triplet (5, 6, 8)in (td30) covers the triplets (5, 5, 8) and (5, 6, 7) in (td32), while the other three triplets in (td30) are the same as in (td32). The rest of checking is left to the reader.

### **Lemma 10.** Each of the descriptions $(td1), \ldots, (td32)$ is tight.

**Proof.** The check is based on the properties of the graphs  $H_1-H_7$ . Namely, as mentioned above, each of  $(td1), \ldots, (td32)$  must contain

- (1) a triplet  $(4^+, 4^+, \infty)$  due to  $H_1$ ,
- (2) either  $(4^+, 5^+, 11^+)$  or  $(5^+, 5^+, 8^+)$  due to  $H_2$ ,
- (3) either  $(4^+, 5^+, 11^+)$  or  $(5^+, 6^+, 7^+)$  due to  $H_3$ ,
- (4)  $(4^+, 6^+, 10^+)$  due to  $H_4$ ,
- (5)  $(4^+, 7^+, 7^+)$  due to  $H_5$ ,
- (6)  $(5^+, 5^+, 7^+)$  due to  $H_6$ , and
- (7)  $(5^+, 6^+, 6^+)$  due to  $H_7$ .

As examples, let us consider several descriptions and make sure that none of their parameters can be lowered. Obviously,  $\infty$  cannot be replaced by any finite number due to  $H_1$ .

In (td1):  $\{(5,7,\infty)\}$ , we cannot decrease 5 by 4 due to  $H_6, H_7$ , and also cannot replace 7 by 6 due to  $H_5$ , which means that (td1) is tight.

Next consider (td8):  $\{(5,5,\infty), (4,7,10), (5,6,6)\}$ . Here,  $(5,5,\infty)$  cannot be replaced by  $(4,5,\infty)$  due to  $H_6$ . Furthermore, 7 cannot be lowered due to  $H_5$ , while 10 cannot be decreased due to  $H_4$ . As for (5,6,6), it cannot be replaced by (4,6,6) or (5,5,6) because of  $H_6$ .

We consider only one example more; namely, that of (td30):  $\{(4, 4, \infty), (4, 6, 10), (5, 6, 8), (4, 7, 7)\}$ . Here, the first entry of 6 cannot be lowered because of  $H_4$ , while the second 6 is responsible for covering  $H_7$ . Furthermore, 10 cannot be decreased since otherwise  $H_4$  does not obey such a stronger set of triplets. Now an attempt to diminish 8 is prevented by  $H_2$  since the set  $\{(4, 4, \infty), (4, 6, 10), (5, 6, 7), (4, 7, 7)\}$  has neither  $(4^+, 5^+, 11^+)$ - nor  $(5^+, 5^+, 8^+)$ -faces.

The rest of checking the tightness of descriptions in  $\{(td1), \ldots, (td31)\} \setminus (td26)$  is similar, and it is left to the reader.

An alternative approach to proving Lemma 10 can be as follows. We already know that  $\infty$  must appear in each of  $(td1), \ldots, (td32)$ .

Next look at 11, which appears only in  $(td17), \ldots, (td26)$ . An attempt to decrease 11 in any of them is invalid due to  $H_2$ , which has only (4, 5, 11)-faces and (5, 5, 8)-faces, neither of which is covered by a thus strengthened set of triplets.

Now note that 10 appears in descriptions  $(td8), \ldots, (td16), (td22), \ldots, (td26)$ , and it is necessary in each of them due to  $H_4$ , which has only (4, 6, 10)-faces.

Considering in a similar (but somewhere more involved) fashion every entry of 8, 7, 6, and 5 into each of  $(td1), \ldots, (td32)$ , we make sure that each entry is

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also best possible due to the above mentioned properties of appropriate graphs from  $\{H_2, \ldots, H_7\}$ .

**Lemma 11.** There are no tight descriptions of faces in triangulations  $\mathbf{T}_4$  from  $\mathbf{P}_4$  other than  $(td1), \ldots, (td32)$ .

**Proof.** Suppose  $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$ , where  $x_i \leq y_i \leq z_i$  whenever  $1 \leq i \leq k$ , is a tight description of faces in triangulations from **P**<sub>4</sub>. By definition, this means that

(1) every  $T_4 \in \mathbf{T}_4$  has a  $(x_i, y_i, z_i)$ -face for at least one *i* with  $1 \leq i \leq k$ , and

(2) if we delete any term  $(x_i, y_i, z_i)$  from D or decrease any parameter in D by one without changing the other 3k - 1 parameters, then the new description is not satisfied by at least one  $T_4 \in \mathbf{T}_4$ .

Note that, due to its tightness, the description D cannot have triplets (X, Y, Z) and (X', Y', Z') such that  $X \leq X', Y \leq Y'$ , and  $Z \leq Z'$ , otherwise  $D' = D \setminus \{(X, Y, Z)\}$  is equivalent to D but shorter. Also, all parameters in D should be at least 4 since we deal with  $\mathbf{T}_4$ .

Note that D must contain a term with  $z_i = \infty$  to be able to describe  $H_1$ . Therefore, we can assume that  $z_1 = \infty$ .

If D has a term  $(x_1, y_1, z_1) = (5^+, 7^+, \infty)$ , then D must coincide with the description (td1), which is tight by Lemma 10, so  $D = \{(5, 7, \infty)\}$ . Therefore, our case analysis splits into Cases 1–6, and we everywhere tacitly refer to Lemma 10.

Case 1.  $(x_1, y_1, z_1) = (4, 7^+, \infty)$ . Due to  $H_7$ , there should be a term  $(x_2, y_2, z_2) = (5^+, 6^+, 6^+)$  in D.

Now if  $(x_2, y_2, z_2) = (5^+, 6^+, 7^+)$ , then *D* coincides with the tight description  $\{(4, 7, \infty), (5, 6, 7)\}$ , that is with (td2). Otherwise, *D* must have triplets  $(x_2, y_2, z_2) = (5, 5, 7^+)$  to be able to cover  $H_7$  and  $(x_3, y_3, z_2) = (5^+, 6, 6)$  to cover  $H_6$ , and hence *D* coincides with (td3):  $\{(4, 7, \infty), (5, 5, 7), (5, 6, 6)\}$ .

Case 2.  $(x_1, y_1, z_1) = (5^+, 6, \infty)$ . Due to  $H_5$ , there should be a term  $(x_2, y_2, z_2) = (4^+, 7^+, 7^+)$  in D, which means that  $D = \{(5, 6, \infty), (4, 7, 7)\}$ , that is coincides with (td4).

Case 3.  $(x_1, y_1, z_1) = (4, 6, \infty)$ . Due to  $H_5$ , there should be a term  $(x_2, y_2, z_2) = (4^+, 7^+, 7^+)$  in D.

If actually  $(x_2, y_2, z_2) = (5^+, 7^+, 7^+)$ , then *D* coincides with (td5): { $(4, 6, \infty)$ , (5, 7, 7)}. Otherwise,  $(x_2, y_2, z_2) = (4, 7^+, 7^+)$ , so in view of  $H_6, H_7$  we have either  $(x_3, y_3, z_3) = (5^+, 6^+, 7^+)$ , in which case *D* is (td6): { $(4, 6, \infty), (4, 7, 7), (5, 6, 7)$ }, or else  $(x_3, y_3, z_3) = (5, 5, 7^+)$ , which term covers  $H_6$ , and there should be also a term  $(x_4, y_4, z_4) = (5^+, 6, 6)$  to cover  $H_7$ , so that *D* coincides with (td7): { $(4, 6, \infty), (4, 7, 7), (5, 5, 7), (5, 6, 6)$ }.

Case 4.  $(x_1, y_1, z_1) = (5, 5, \infty)$ . Since the first term of *D* does not cover  $H_4$ , we can assume that  $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$ .

Note that  $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$  is impossible due to the existence of a stronger description (td15):  $\{(4, 4, \infty), (5, 7, 10)\}$ . If  $(x_2, y_2, z_2) = (4, 7^+, 10^+)$ , then *D* is (td8):  $\{(5, 5, \infty), (4, 7, 10), (5, 6, 6)\}$  in view of necessity to cover  $H_7$  as well.

If  $(x_2, y_2, z_2) = (5^+, 6, 10^+)$ , then D must have  $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$  in order to satisfy  $H_5$ , but this contradicts (td16):  $\{(4, 4, \infty), (5, 6, 10), (4, 7, 7)\}$ .

Therefore, it remains to assume that  $(x_2, y_2, z_2) = (4, 6, 10^+)$ , in which case there should be also a triplet, say  $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$ , that covers  $H_5$ . If in fact  $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$ , then D contradicts to (td12):  $\{(4, 5, \infty), (4, 6, 10), (5, 7, 7)\}$ . Otherwise,  $(x_3, y_3, z_3) = (4, 7^+, 7^+)$ , which means that D should have  $(x_4, y_4, z_4) = (5^+, 6, 6^+)$  to be able to cover  $H_7$ , and hence Dcoincides with (td9):  $\{(5, 5, \infty), (4, 6, 10), (4, 7, 7), (5, 6, 6)\}$ .

Case 5.  $(x_1, y_1, z_1) = (4, 5, \infty)$ . Note that the first term of *D* does not cover  $H_4$ , so we can assume that  $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$ .

Subcase 5.1.  $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$ . This is impossible due to a stronger description (td15):  $\{(4, 4, \infty), (5, 7, 10)\}$ .

Subcase 5.2.  $(x_2, y_2, z_2) = (4, 7^+, 10^+)$ . Now  $H_5$  is already covered, so there should be a term to cover  $H_6$ . If  $(x_3, y_3, z_3) = (5^+, 6, 7^+)$ , then it covers both  $H_6$  and  $H_7$ , which means that D is actually (td10):  $\{(4, 5, \infty), (4, 7, 10), (5, 6, 7\}$ .

Otherwise,  $(x_3, y_3, z_3) = (5, 5, 7^+)$ , and then D must have  $(x_4, y_4, z_4) = (5^+, 6^+, 6^+)$  to cover  $H_7$ . Therefore,  $D = \{(4, 5, \infty), (4, 7, 10), (5, 5, 7), (5, 6, 6)\}$ , which is (td11).

Subcase 5.3.  $(x_2, y_2, z_2) = (5^+, 6, 10^+)$ . Now  $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$  in order for D to cover  $H_5$ , but this is impossible due to (td16): {(4, 4,  $\infty$ ), (5, 6, 10), (4, 7, 7)}.

Subcase 5.4.  $(x_2, y_2, z_2) = (4, 6, 10^+)$ . Now  $H_5$  is not covered by the second term, so D must have  $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$  to cover  $H_5$ .

If actually  $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$ , then  $D = \{(4, 5, \infty), (4, 6, 10), (5, 7, 7)\}$ , which is (td12). Further suppose  $(x_3, y_3, z_3) = (4, 7^+, 7^+)$ . Now since  $H_6$  and  $H_7$  should be covered, and this can be done either together or separately, we conclude that D is either (td13):  $\{(4, 5, \infty), (4, 6, 10), (4, 7, 7), (5, 6, 7)\}$  or (td14):  $\{(4, 5, \infty), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ , respectively.

Case 6.  $(x_1, y_1, z_1) = (4, 4, \infty)$ . Due to  $H_4$ , our D has a term  $(4^+, 6^+, 10^+)$ . Furthermore, D either has or does not have a term  $(4, 5, 11^+)$  to cover  $H_2, H_3$ . Therefore, our analysis splits in Subcases 6.1–6.5.

Subcase 6.1. If  $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$ , then *D* coincides with (td15):  $\{(4, 4, \infty), (5, 7, 10)\}$ . If  $(x_2, y_2, z_2) = (5^+, 6, 10^+)$ , then *D* should have  $(x_3, y_3, z_3)$ 

 $= (4, 7^+, 7^+)$  in order to be able to cover  $H_5$ , and hence D is (td16): { $(4, 4, \infty)$ , (5, 6, 10), (4, 7, 7)}.

Subcase 6.2. If  $(x_2, y_2, z_2) = (4, 7^+, 11^+)$ , then  $H_2, H_3$  are already covered, so it remains for D to cover  $H_6$  and  $H_7$  by other triplets, which can be done either by  $(x_3, y_3, z_3) = (5^+, 6^+, 7^+)$  alone, in which case D coincides with (td17):  $\{(4, 4, \infty), (4, 7, 11), (5, 6, 7)\}$ , or by  $(x_3, y_3, z_3) = (5^+, 5, 7^+)$  and  $(x_4, y_4, z_5) = (5^+, 6, 6)$ , and then D is (td18):  $\{(4, 4, \infty), (4, 7, 11), (5, 5, 7), (5, 6, 6)\}$ .

Subcase 6.3. Next suppose  $(x_2, y_2, z_2) = (4, 6, 11^+)$ . Now *D* must cover  $H_5, H_6$  and  $H_7$  by other triplets. This can be done by  $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$  alone, in which case *D* coincides with (td19): { $(4, 4, \infty), (4, 6, 11), (5, 7, 7)$ }.

Otherwise,  $H_5$  can be covered only by  $(x_3, y_3, z_3) = (4, 7^+, 7^+)$ . Then there are two ways for D to cover  $H_6$  and  $H_7$ . The first is to have  $(x_4, y_4, z_4) = (5^+, 6^+, 7^+)$ , in which case D is (td20):  $\{(4, 4, \infty), (4, 6, 11), (4, 7, 7), (5, 6, 7)\}$ . The second way is to cover  $H_6$  and  $H_7$  separately by having  $(x_4, y_4, z_4) = (5, 5, 7^+)$ and  $(x_5, y_5, z_5) = (5^+, 6^+, 6^+)$ , which results in  $D = \{(4, 4, \infty), (4, 6, 11), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ , that is in (td21).

Subcase 6.4. Now suppose  $(x_2, y_2, z_2) = (4, 5, 11^+)$ . Then *D* must cover all of  $H_4, \ldots, H_7$  by other triplets. To cover  $H_4$  alone, *D* must have a term, say the third, so we can assume that  $(x_3, y_3, z_3) = (4^+, 6^+, 10)$ . Now our proof splits even more.

Subcase 6.4.1.  $(x_3, y_3, z_3) = (5^+, 7^+, 10)$ . Here, *D* contradicts with a stronger description (td15):  $\{(4, 4, \infty), (5, 7, 10)\}$ .

Subcase 6.4.2.  $(x_3, y_3, z_3) = (5^+, 6, 10)$ . Now to cover  $H_5$ , our D must have a term  $(x_4, y_4, z_4) = (4^+, 7^+, 7^+)$ , and we arrive at a contradiction with (td16):  $\{(4, 4, \infty), (5, 6, 10), (4, 7, 7)\}$ .

Subcase 6.4.3.  $(x_3, y_3, z_3) = (4, 7^+, 10)$ . Here, it remains for D to cover  $H_6$  and  $H_7$ . As we know from above discussion, this happens by one or two other triplets, which results either in (td22):  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 6, 7)\}$  or (td23):  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ , respectively.

Subcase 6.4.4.  $(x_3, y_3, z_3) = (4, 6, 10)$ . Now to cover  $H_5$ , there should exist a term  $(x_4, y_4, z_4) = (4^+, 7^+, 7^+)$  in *D*. If actually  $(x_4, y_4, z_4) = (5^+, 7^+, 7^+)$ , then *D* coincides with (td24):  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (5, 7, 7)\}$ . Otherwise, we have  $(x_4, y_4, z_4) = (4, 7^+, 7^+)$ , and now the necessity to cover only  $H_6$ and  $H_7$  leads by means of our previous remark to (td25):  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 6, 7)\}$  or (td26):  $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ . Subcase 6.5. *D* has no term  $(4, 5, 11^+)$ . Here, *D* covers  $H_4$  by the second term, so  $(x_2, y_2, z_2) = (4, 6^+, 10^+)$  in view of Subcase 6.1, where  $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$  was settled.

Now to cover  $H_2$  and  $H_3$ , our D must have one or more triplets  $(5^+, 5^+, 8^+)$  and  $(5^+, 6^+, 7^+)$ . Note that these triplets automatically cover also  $H_6$  and  $H_7$ .

Subcase 6.5.1.  $(x_2, y_2, z_2) = (4, 7^+, 10^+)$ . If  $(x_3, y_3, z_3) = (5^+, 6^+, 8^+)$  then *D* is (td27): { $(4, 4, \infty)$ , (4, 7, 10), (5, 6, 8)}. Otherwise,  $(x_3, y_3, z_3) = (5, 5, 8^+)$ , so *D* needs another term  $(5^+, 6^+, 7^+)$  for satisfying  $H_3$ , and hence *D* reduces to (td28): { $(4, 4, \infty), (4, 7, 10), (5, 5, 8), (5, 6, 7)$ }.

Subcase 6.5.2.  $(x_2, y_2, z_2) = (4, 6, 10)$ . Note that now D needs a term  $(4^+, 7^+, 7^+)$  to satisfy  $H_5$ . If  $(x_3, y_3, z_3) = (5^+, 7^+, 8^+)$ , then D is (td29): {(4, 4,  $\infty$ ), (4, 6, 10), (5, 7, 8)}. If  $(x_3, y_3, z_3) = (5^+, 6, 8^+)$ , then D is (td30): {(4, 4,  $\infty$ ), (4, 6, 10), (5, 6, 8), (4, 7, 7)}.

Otherwise, we have  $(x_3, y_3, z_3) = (5, 5, 8^+)$ . Here, D needs another term  $(x_4, y_4, z_4) = (5^+, 6^+, 7)$  for  $H_3$ . If in fact  $(x_4, y_4, z_4) = (5^+, 7, 7)$ , then  $H_5$  is also covered, which means that D reduces to (td31):  $\{(4, 4, \infty), (4, 6, 10), (5, 5, 8), (5, 7, 7)\}$ . It remains to assume that  $(x_4, y_4, z_4) = (4, 7, 7)$ , which implies the necessity of  $(x_5, y_5, z_5) = (5^+, 6, 7)$ , and hence we have  $D = \{(4, 4, \infty), (4, 6, 10), (5, 6, 8), (4, 7, 7)\}$ , that is (td32).

This completes the proof of Theorem 8.

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