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ALL TIGHT DESCRIPTIONS OF FACES IN PLANE TRIANGULATIONS WITH MINIMUM DEGREE 4

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Abstract

It follows from the classical theorem by Lebesgue (1940) on the structure of minor faces in 3-polytopes that every plane triangulation with minimum degree at least 4 has a 3-face for which the set of degrees of its vertices is majorized by one of the following sequences: $(4,4,\infty)$, $(4,5,19)$, $(4,6,11)$, $(4,7,9)$, $(5,5,9)$, $(5,6,7)$.

In 1999, Jendrol' gave the following description of faces: $(4,4,\infty)$, $(4,5, 13)$, $(4,6,17)$, $(4,7,8)$, $(5,5,7)$, $(5,6,6)$. Also, Jendrol' (1999) conjectured that there is a face of one of the types: $(4,4,\infty)$, $(4,5,10)$, $(4,6,15)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$. In 2002, Lebesgue's description was strengthened by Borodin to $(4,4,\infty)$, $(4,5,17)$, $(4,6,11)$, $(4,7,8)$, $(5,5,8)$, $(5,6,6)$.

In 2014, we obtained the following tight description, which, in particular, disproves the above mentioned conjecture by Jendrol': $(4,4,\infty)$, $(4,5,11)$, $(4,6,10)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$. Recently, we obtained another tight description: $(4,4,\infty)$, $(4,6,10)$, $(4,7,7)$, $(5,5,8)$, $(5,6,7)$.

The purpose of this paper is to give an exhausting list of tight descriptions of faces in plane triangulations with minimum degree at least 4, which turns out to consist of 32 items.

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1. Introduction

By a 3-polytope we mean a finite convex 3-dimensional polytope. As proved by Steinitz [30], the 3-polytopes are in 1-1 correspondence with the 3-connected planar graphs.

The *degree* $d(x)$ of a vertex or face x in a 3-polytope P is the number of incident edges. A *k-vertex* and *k-face* is one of degree k, a *k^+-vertex* has degree at least k, and so on. A face f is *minor* if $d(f) \le 5$. A face f with the set of incident vertices $\{v_1, v_2, \dots\}$ in P is of *type* $(k_1, k_2, \dots)$ or a *$(k_1, k_2, \dots)$-face* if $d(v_i) \le k_i$ for each i.

By $\mathbf{P}_\delta$ denote the class of 3-polytopes with minimum degree δ; in particular, $\mathbf{P_3}$ is the set of all 3-polytopes. A plane triangulation is a 3-polytope with all faces of degree 3.

By Δ and δ denote the maximum and minimum vertex degree of P, respectively. The *weight* $w(f)$ of a face f in P is the degree-sum of its boundary vertices, and $w(P)$, or simply w, denotes the minimum weight of minor faces in P. The *height* $h(f)$ of a face f in P is the maximum of the degrees of the vertices incident with f, and $h(P)$ is the minimum of $h(f)$ over the minor faces f in P.

We now recall some results on the structure of minor faces in 3-polytopes, beginning with the fundamental theorem of Lebesgue [26] from 1940.

Theorem 1 (Lebesgue [26]). *Every 3-polytope has a minor face of one of the following types:*

$$\begin{gathered}(3,6,\infty),\ (3,7,41),\ (3,8,23),\ (3,9,17),\ (3,10,14),\ (3,11,13),\\ (4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),\\ (3,3,3,\infty),\ (3,3,4,11),\ (3,3,5,7),\ (3,4,4,5),\ (3,3,3,3,5).\end{gathered}$$

The classical Theorem 1, along with other ideas in Lebesgue [26], has a lot of applications to plane graph coloring problems (the first example of such applications and recent surveys can be found in [7, 13, 23, 28]).

Some parameters of Lebesgue's Theorem were improved for several narrow classes of plane graphs. In 1963, Kotzig [24] proved that every plane triangulation in $\mathbf{P_5}$ satisfies $w \le 18$ and conjectured that $w \le 17$ holds. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 [2]. *Every 3-polytope with $\delta = 5$ has a $(5,5,7)$-face or a $(5,6,6)$-face, where all parameters are tight.*

Theorem 2 also confirmed a conjecture of Grübaum [20] from 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [29]).

We note that if $(4,4,\infty)$- or $(3,3,3,\infty)$-faces, called *pyramidal faces*, are allowed in a 3-polytope, then it can happen that all its faces are of unbounded height (and weight).

For plane triangulations without 4-vertices, Kotzig [25] proved $w \leq 39$, and Borodin [3], confirming Kotzig's conjecture in [25], proved $w \leq 29$, which is best possible due to the dual of the twice-truncated dodecahedron. Borodin [5] further shows that each triangulated 3-polytope without $(4,4,\infty)$-faces satisfies $w \leq 29$, and that for triangulations without 4-vertices adjacent to each other there holds a sharp bound $w \leq 37$.

For $\mathbf{P_5}$, Theorem 1 yields $w \leq \max\{51, \Delta + 9\}$. In 1996, Horňák and Jendrol' [21] strengthened this as follows: if there are no pyramidal faces, then $w \leq 47$. Borodin and Woodall [19] proved that forbidding $(3,3,3,\infty)$-faces implies $w \leq \max\{29, \Delta + 8\}$.

More generally, Horňák and Jendrol' [21] proved the following description.

Theorem 3 (Horňák and Jendrol' [21])**.** *Every 3-polytope has a minor face of one of the following types:*

$$(3,5,39),\ (3,6,23)\ (3,7,18),\ (3,8,15),\ (3,9,14),\ (3,10,13),$$
$$(4,4,\infty),\ (4,5,19),\ (4,6,11),\ (4,7,9),\ (5,5,9),\ (5,6,7),$$
$$(3,3,3,\infty),\ (3,3,7,11),\ (3,4,5,7),\ (3,5,5,6),\ (3,3,3,3,5).$$

Recently, the upper bound $h \leq 10$ for quadrangulated 3-polytopes without $(3,3,3,\infty)$-faces, (Avgustinovich–Borodin [1]) was improved in Borodin–Ivanova [10] to $h \leq 8$, which is sharp.

In 2002, Borodin [6] strengthened Lebesgue's Theorem 1 as follows (the entries marked by an asterisk are proved in [6] to be best possible).

Theorem 4 [6]**.** *Every 3-polytope has a minor face of one of the following types:*

$$(3,6,\infty^*),\ (3,8^*,22),\ (3,9^*,15),\ (3,10^*,13),\ (3,11^*,12),$$
$$(4,4,\infty^*),\ (4,5^*,17),\ (4,6^*,11),\ (4,7^*,8),\ (5,5^*,8),\ (5,6,6^*),$$
$$(3,3,3,\infty^*),\ (3,3,4^*,11),\ (3,3,5^*,7),\ (3,4,4,5^*),\ (3,3,3,3,5^*).$$

In 2013–2014, precise descriptions of the structure of faces were obtained for 3-polytopes in $\mathbf{P_4}$ and for triangulated 3-polytopes.

Theorem 5 [8]**.** *Every 3-polytope in $\mathbf{P_4}$ has a 3-face of one of the following types:* $(4,4,\infty)$, $(4,5,14)$, $(4,6,10)$, $(4,7,7)$, $(5,5,7)$, $(5,6,6)$, *where all parameters are tight.*

Theorem 6 [16]. *Every plane triangulation has a face of one of the following types:* $(3,4,31)$, $(3,5,21)$, $(3,6,20)$, $(3,7,13)$, $(3,8,14)$, $(3,9,12)$, $(3,10,12)$, $(4,4,\infty)$, $(4,5,11)$, $(4,6,10)$, $(4,7,7)$, $(5,6,6)$, $(5,5,7)$, *where all parameters are tight.*

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in recent surveys [13, 23], and also in [6, 9, 11, 12, 18, 22, 27].

Over the last decades, a lot of coloring problems have been posed, and some of them solved, in terms of forbidding these or those sets of cycle-lengths; for a survey see Borodin [7].

In 2017, Borodin, Ivanova, and Vasil'eva [17], suggested to apply a similar approach to finding tight versions of Theorems 1 and 3 by forbidding vertices from certain gradually narrowing degree-sets and obtained a tight description assuming the absence of vertices of degree 4 to 7.

Recently, we proved [14] that every 3-polytope without vertices of degree from 5 to 7 has a face of one of the types: $(4,4,\infty)$, $(3,8,14)$, $(3,9,13)$, $(3,10,12)$, $(3,3,3,\infty)$, $(3,3,4,11)$, $(3,4,4,4)$, $(3,3,3,3,4)$, which description is tight.

Furthermore, we recently obtained [15] the following tight description of faces for triangulations without 3-vertices, which does not follow from the general Theorem 6.

Theorem 7 [15]. *Every plane triangulation in* $\mathbf{P_4}$ *has a face of one of the following types:* $(4,4,\infty)$, $(4,6,10)$, $(5,5,8)$, $(4,7,7)$, $(5,6,7)$, *where all parameters are best possible.*

The purpose of this paper is to completely describe the set of tight descriptions of faces for triangulations in $\mathbf{P_4}$. (As usual, by a *tight description* of these or those fragments in a certain class of planar graphs we mean one in which no parameter can be lowered and no term dropped.)

Theorem 8. *There exist precisely the following* 32 *tight descriptions of faces in plane triangulations without* 3*-vertices:*

$(td1)$: $\{(5,7,\infty)\}$,
$(td2)$: $\{(4,7,\infty),(5,6,7)\}$,
$(td3)$: $\{(4,7,\infty),(5,5,7),(5,6,6)\}$,
$(td4)$: $\{(5,6,\infty),(4,7,7)\}$,
$(td5)$: $\{(4,6,\infty),(5,7,7)\}$,
$(td6)$: $\{(4,6,\infty),(4,7,7),(5,6,7)\}$,
$(td7)$: $\{(4,6,\infty),(4,7,7),(5,5,7),(5,6,6)\}$,
$(td8)$: $\{(5,5,\infty),(4,7,10),(5,6,6\}$,
$(td9)$: $\{(5,5,\infty),(4,6,10),(4,7,7),(5,6,6)\}$,
$(td10)$: $\{(4,5,\infty),(4,7,10),(5,6,7\}$,

$(td11)$: $\{(4,5,\infty),(4,7,10),(5,5,7),(5,6,6)\}$,
$(td12)$: $\{(4,5,\infty),(4,6,10),(5,7,7)\}$,
$(td13)$: $\{(4,5,\infty),(4,6,10),(4,7,7),(5,6,7)\}$,
$(td14)$: $\{(4,5,\infty),(4,6,10),(4,7,7),(5,5,7),(5,6,6)\}$,
$(td15)$: $\{(4,4,\infty),(5,7,10)\}$,
$(td16)$: $\{(4,4,\infty),(5,6,10),(4,7,7)\}$,
$(td17)$: $\{(4,4,\infty),(4,7,11),(5,6,7)\}$,
$(td18)$: $\{(4,4,\infty),(4,7,11),(5,5,7),(5,6,6)\}$,
$(td19)$: $\{(4,4,\infty),(4,6,11),(5,7,7)\}$,
$(td20)$: $\{(4,4,\infty),(4,6,11),(4,7,7),(5,6,7)\}$,
$(td21)$: $\{(4,4,\infty),(4,6,11),(4,7,7),(5,5,7),(5,6,6)\}$,
$(td22)$: $\{(4,4,\infty),(4,5,11),(4,7,10),(5,6,7)\}$,
$(td23)$: $\{(4,4,\infty),(4,5,11),(4,7,10),(5,5,7),(5,6,6)\}$,
$(td24)$: $\{(4,4,\infty),(4,5,11),(4,6,10),(5,7,7)\}$,
$(td25)$: $\{(4,4,\infty),(4,5,11),(4,6,10),(4,7,7),(5,6,7)\}$,
$(td26)$: $\{(4,4,\infty),(4,5,11),(4,6,10),(4,7,7),(5,5,7),(5,6,6)\}$,
$(td27)$: $\{(4,4,\infty),(4,7,10),(5,6,8)\}$,
$(td28)$: $\{(4,4,\infty),(4,7,10),(5,5,8),(5,6,7)\}$,
$(td29)$: $\{(4,4,\infty),(4,6,10),(5,7,8)\}$,
$(td30)$: $\{(4,4,\infty),(4,6,10),(5,6,8),(4,7,7)\}$,
$(td31)$: $\{(4,4,\infty),(4,6,10),(5,5,8),(5,7,7)\}$,
$(td32)$: $\{(4,4,\infty),(4,6,10),(5,5,8),(4,7,7),(5,6,7)\}$.

2. Proving Theorem 8

We first define 3-polytopes H_1–H_7 important for the proof. By H_1 we denote the double n-pyramid, which has only $(4,4,n)$-faces, and thus confirms the presence of ∞ in each of descriptions $(td1)$–$(td32)$.

Figure 1 [9] shows how to transform the $(3,3,3,3,5)$ Archimedean solid, in which every vertex is incident with four 3-faces and a 5-face, into a 3-polytope which has only faces of types $(4,5,11)$ and $(5,5,8)$. This construction confirms that each of $(td1)$–$(td32)$ must contain a triplet either $(4^+,5^+,11^+)$ or $(5^+,5^+,8^+)$.

In Figure 2 we transform the $(3,3,3,3,5)$ Archimedean solid into a 3-polytope which has only faces of types $(4,5,11)$ and $(5,6,7)$, which confirms the necessity or $(4^+,5^+,11^+)$ or $(5^+,6^+,7^+)$ in each of $(td1)$–$(td32)$. This construction first appeared in [15].

Figure 3 shows well-known constructions H_4–H_7, arising mainly from some Platonic and Archimedean solids, while H_6 is obtained by gluing two halves along the outside cycle. The origin of these constructions is already difficult to

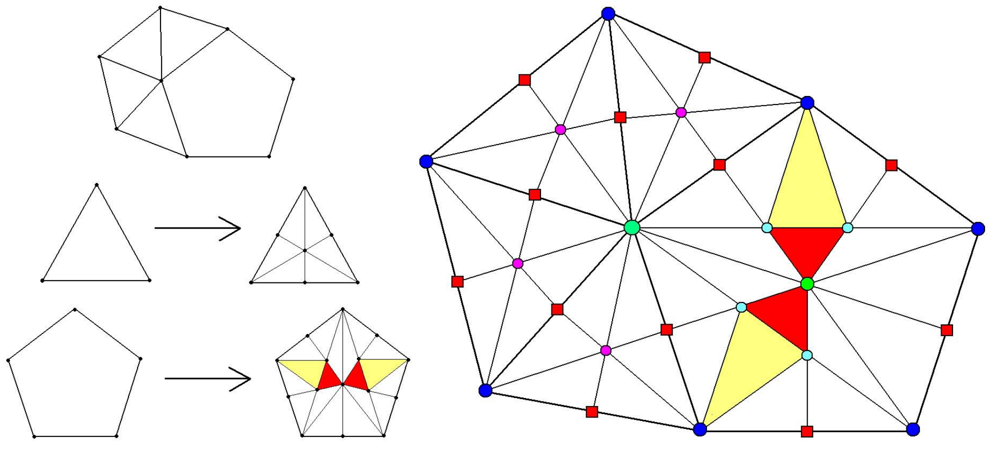

Figure 1. H_2 has only $(4,5,11)$- and $(5,5,8)$-faces [9].

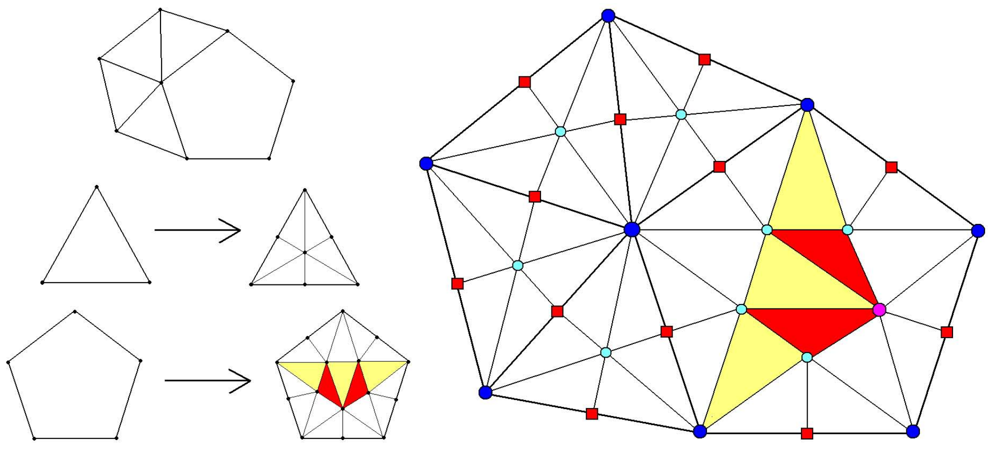

Figure 2. H_3 has only $(4,5,11)$- and $(5,6,7)$-faces [15].

trace; at least we know that they were used in [4, 8]. It is not hard to see that each of H_4, H_5, H_6, H_7 proves that the corresponding triplet from $\{(4^+,6^+,10^+), (4^+,7^+,7^+), (5^+,5^+,7^+), (5^+,6^+,6^+)\}$ must appear in each of descriptions $(td1)$–$(td32)$.

The rest of proving Theorem 8 falls into three lemmas, which consistently show that

(i) each of the sets $(td1),\dots,(td32)$ is indeed a description,

(ii) each of these descriptions is tight, and

(iii) there are no tight descriptions different from $(td1),\dots,(td32)$.

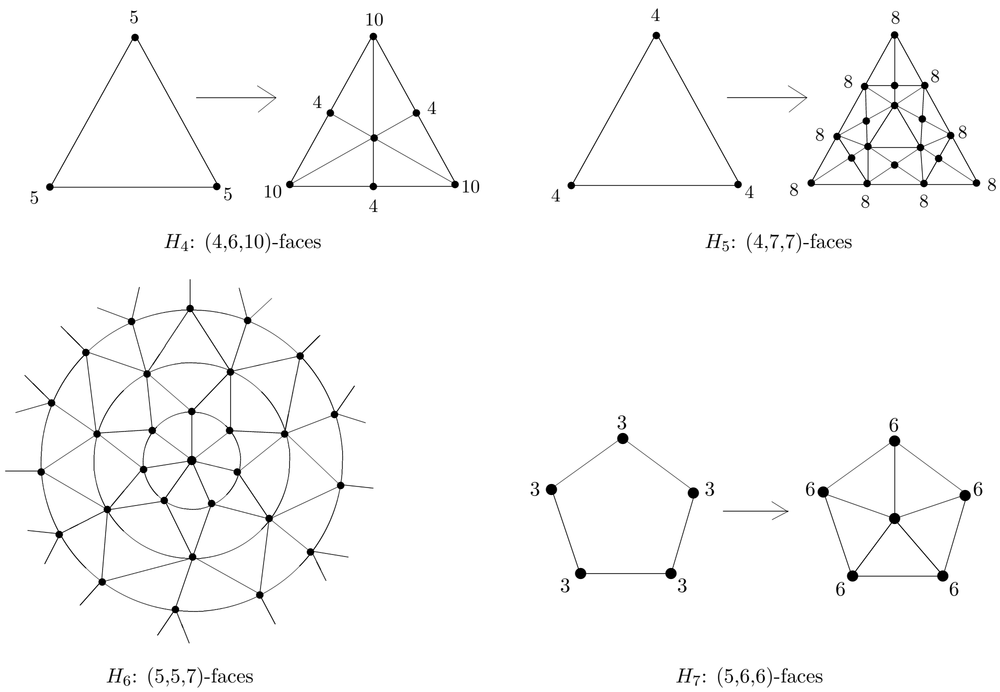

Figure 3. Graphs H_4, H_5, H_6, and H_7.

Lemma 9. *Each of the sets* $(td1), \ldots, (td32)$ *is a description of faces in triangulations from* $\mathbf{P_4}$.

Proof. For $(td26)$ and $(td32)$ this follows from the above mentioned results in [9] and [15], respectively.

Now note that, by definition, each (i, j, k)-path is also an (i', j', k')-path if $i' \geq i$, $j' \geq j$, and $k' \geq k$.

Therefore, for each of the sets of triplets in $\{(td1), \ldots, (td25)\}$ it suffices (but maybe is a bit laborous) to check that all its triplets together cover all triplets in $(td26)$. For example, the only triplet $(5, 7, \infty)$ in $(td1)$ covers each of the triplets in $(td26)$.

In other words, as soon as any triangulation T_4 in $\mathbf{P_4}$ has a face of at least one of the types from the set $\{(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$ since $(td1)$ is an already proven description, it follows that T_4 has a $(5, 7, \infty)$-face, which proves that $(td1)$ also describes T_4, as stated.

Similarly, for each of the sets $(td27), \ldots, (td31)$ it is also easy to check that its triplets together cover all triplets in $(td32)$. For example, the triplet $(5, 6, 8)$ in $(td30)$ covers the triplets $(5, 5, 8)$ and $(5, 6, 7)$ in $(td32)$, while the other three triplets in $(td30)$ are the same as in $(td32)$.

The rest of checking is left to the reader. ∎

Lemma 10. *Each of the descriptions* $(td1),\dots,(td32)$ *is tight.*

Proof. The check is based on the properties of the graphs H_1–H_7. Namely, as mentioned above, each of $(td1),\dots,(td32)$ must contain

(1) a triplet $(4^+,4^+,\infty)$ due to H_1,

(2) either $(4^+,5^+,11^+)$ or $(5^+,5^+,8^+)$ due to H_2,

(3) either $(4^+,5^+,11^+)$ or $(5^+,6^+,7^+)$ due to H_3,

(4) $(4^+,6^+,10^+)$ due to H_4,

(5) $(4^+,7^+,7^+)$ due to H_5,

(6) $(5^+,5^+,7^+)$ due to H_6, and

(7) $(5^+,6^+,6^+)$ due to H_7.

As examples, let us consider several descriptions and make sure that none of their parameters can be lowered. Obviously, ∞ cannot be replaced by any finite number due to H_1.

In (td1): $\{(5,7,\infty)\}$, we cannot decrease 5 by 4 due to H_6,H_7, and also cannot replace 7 by 6 due to H_5, which means that (td1) is tight.

Next consider (td8): $\{(5,5,\infty),(4,7,10),(5,6,6)\}$. Here, $(5,5,\infty)$ cannot be replaced by $(4,5,\infty)$ due to H_6. Furthermore, 7 cannot be lowered due to H_5, while 10 cannot be decreased due to H_4. As for $(5,6,6)$, it cannot be replaced by $(4,6,6)$ or $(5,5,6)$ because of H_6.

We consider only one example more; namely, that of (td30): $\{(4,4,\infty)$, $(4,6,10)$, $(5,6,8)$, $(4,7,7)\}$. Here, the first entry of 6 cannot be lowered because of H_4, while the second 6 is responsible for covering H_7. Furthermore, 10 cannot be decreased since otherwise H_4 does not obey such a stronger set of triplets. Now an attempt to diminish 8 is prevented by H_2 since the set $\{(4,4,\infty),(4,6,10),(5,6,7),(4,7,7)\}$ has neither $(4^+,5^+,11^+)$- nor $(5^+,5^+,8^+)$-faces.

The rest of checking the tightness of descriptions in $\{(td1),\dots,(td31)\}\setminus(td26)$ is similar, and it is left to the reader.

An alternative approach to proving Lemma 10 can be as follows. We already know that ∞ must appear in each of $(td1),\dots,(td32)$.

Next look at 11, which appears only in $(td17),\dots,(td26)$. An attempt to decrease 11 in any of them is invalid due to H_2, which has only $(4,5,11)$-faces and $(5,5,8)$-faces, neither of which is covered by a thus strengthened set of triplets.

Now note that 10 appears in descriptions $(td8),\dots,(td16)$, $(td22),\dots,(td26)$, and it is necessary in each of them due to H_4, which has only $(4,6,10)$-faces.

Considering in a similar (but somewhere more involved) fashion every entry of 8, 7, 6, and 5 into each of $(td1),\dots,(td32)$, we make sure that each entry is also best possible due to the above mentioned properties of appropriate graphs from $\{H_2, \ldots, H_7\}$. ■

Lemma 11. *There are no tight descriptions of faces in triangulations* $\mathbf{T_4}$ *from* $\mathbf{P_4}$ *other than* $(td1), \ldots, (td32)$.

Proof. Suppose $D = \{(x_1, y_1, z_1), \ldots (x_k, y_k, z_k)\}$, where $x_i \leq y_i \leq z_i$ whenever $1 \leq i \leq k$, is a tight description of faces in triangulations from $\mathbf{P_4}$. By definition, this means that

(1) every $T_4 \in \mathbf{T_4}$ has a (x_i, y_i, z_i)-face for at least one i with $1 \leq i \leq k$, and

(2) if we delete any term (x_i, y_i, z_i) from D or decrease any parameter in D by one without changing the other $3k - 1$ parameters, then the new description is not satisfied by at least one $T_4 \in \mathbf{T_4}$.

Note that, due to its tightness, the description D cannot have triplets (X, Y, Z) and (X', Y', Z') such that $X \leq X'$, $Y \leq Y'$, and $Z \leq Z'$, otherwise $D' = D \setminus \{(X, Y, Z)\}$ is equivalent to D but shorter. Also, all parameters in D should be at least 4 since we deal with $\mathbf{T_4}$.

Note that D must contain a term with $z_i = \infty$ to be able to describe H_1. Therefore, we can assume that $z_1 = \infty$.

If D has a term $(x_1, y_1, z_1) = (5^+, 7^+, \infty)$, then D must coincide with the description (td1), which is tight by Lemma 10, so $D = \{(5, 7, \infty)\}$. Therefore, our case analysis splits into Cases 1–6, and we everywhere tacitly refer to Lemma 10.

Case 1. $(x_1, y_1, z_1) = (4, 7^+, \infty)$. Due to H_7, there should be a term $(x_2, y_2, z_2) = (5^+, 6^+, 6^+)$ in D.

Now if $(x_2, y_2, z_2) = (5^+, 6^+, 7^+)$, then D coincides with the tight description $\{(4, 7, \infty), (5, 6, 7)\}$, that is with (td2). Otherwise, D must have triplets $(x_2, y_2, z_2) = (5, 5, 7^+)$ to be able to cover H_7 and $(x_3, y_3, z_2) = (5^+, 6, 6)$ to cover H_6, and hence D coincides with (td3): $\{(4, 7, \infty), (5, 5, 7), (5, 6, 6)\}$.

Case 2. $(x_1, y_1, z_1) = (5^+, 6, \infty)$. Due to H_5, there should be a term $(x_2, y_2, z_2) = (4^+, 7^+, 7^+)$ in D, which means that $D = \{(5, 6, \infty), (4, 7, 7)\}$, that is coincides with (td4).

Case 3. $(x_1, y_1, z_1) = (4, 6, \infty)$. Due to H_5, there should be a term $(x_2, y_2, z_2) = (4^+, 7^+, 7^+)$ in D.

If actually $(x_2, y_2, z_2) = (5^+, 7^+, 7^+)$, then D coincides with (td5): $\{(4, 6, \infty), (5, 7, 7)\}$. Otherwise, $(x_2, y_2, z_2) = (4, 7^+, 7^+)$, so in view of H_6, H_7 we have either $(x_3, y_3, z_3) = (5^+, 6^+, 7^+)$, in which case D is (td6): $\{(4, 6, \infty), (4, 7, 7), (5, 6, 7)\}$, or else $(x_3, y_3, z_3) = (5, 5, 7^+)$, which term covers H_6, and there should be also a term $(x_4, y_4, z_4) = (5^+, 6, 6)$ to cover H_7, so that D coincides with (td7): $\{(4, 6, \infty), (4, 7, 7), (5, 5, 7), (5, 6, 6)\}$.

Case 4. $(x_1, y_1, z_1) = (5, 5, \infty)$. Since the first term of D does not cover H_4, we can assume that $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$.

Note that $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$ is impossible due to the existence of a stronger description (td15): $\{(4, 4, \infty), (5, 7, 10)\}$. If $(x_2, y_2, z_2) = (4, 7^+, 10^+)$, then D is (td8): $\{(5, 5, \infty), (4, 7, 10), (5, 6, 6)\}$ in view of necessity to cover H_7 as well.

If $(x_2, y_2, z_2) = (5^+, 6, 10^+)$, then D must have $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$ in order to satisfy H_5, but this contradicts $(td16)$: $\{(4, 4, \infty), (5, 6, 10), (4, 7, 7)\}$.

Therefore, it remains to assume that $(x_2, y_2, z_2) = (4, 6, 10^+)$, in which case there should be also a triplet, say $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$, that covers H_5. If in fact $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$, then D contradicts to (td12): $\{(4, 5, \infty), (4, 6, 10), (5, 7, 7)\}$. Otherwise, $(x_3, y_3, z_3) = (4, 7^+, 7^+)$, which means that D should have $(x_4, y_4, z_4) = (5^+, 6, 6^+)$ to be able to cover H_7, and hence D coincides with (td9): $\{(5, 5, \infty), (4, 6, 10), (4, 7, 7), (5, 6, 6)\}$.

Case 5. $(x_1, y_1, z_1) = (4, 5, \infty)$. Note that the first term of D does not cover H_4, so we can assume that $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$.

Subcase 5.1. $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$. This is impossible due to a stronger description (td15): $\{(4, 4, \infty), (5, 7, 10)\}$.

Subcase 5.2. $(x_2, y_2, z_2) = (4, 7^+, 10^+)$. Now H_5 is already covered, so there should be a term to cover H_6. If $(x_3, y_3, z_3) = (5^+, 6, 7^+)$, then it covers both H_6 and H_7, which means that D is actually (td10): $\{(4, 5, \infty)$, $(4, 7, 10)$, $(5, 6, 7\}$.

Otherwise, $(x_3, y_3, z_3) = (5, 5, 7^+)$, and then D must have $(x_4, y_4, z_4) = (5^+, 6^+, 6^+)$ to cover H_7. Therefore, $D = \{(4, 5, \infty), (4, 7, 10), (5, 5, 7), (5, 6, 6)\}$, which is (td11).

Subcase 5.3. $(x_2, y_2, z_2) = (5^+, 6, 10^+)$. Now $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$ in order for D to cover H_5, but this is impossible due to (td16): $\{(4, 4, \infty)$, $(5, 6, 10)$, $(4, 7, 7)\}$.

Subcase 5.4. $(x_2, y_2, z_2) = (4, 6, 10^+)$. Now H_5 is not covered by the second term, so D must have $(x_3, y_3, z_3) = (4^+, 7^+, 7^+)$ to cover H_5.

If actually $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$, then $D = \{(4, 5, \infty), (4, 6, 10), (5, 7, 7)\}$, which is (td12). Further suppose $(x_3, y_3, z_3) = (4, 7^+, 7^+)$. Now since H_6 and H_7 should be covered, and this can be done either together or separately, we conclude that D is either (td13): $\{(4, 5, \infty)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 6, 7)\}$ or (td14): $\{(4, 5, \infty)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)\}$, respectively.

Case 6. $(x_1, y_1, z_1) = (4, 4, \infty)$. Due to H_4, our D has a term $(4^+, 6^+, 10^+)$. Furthermore, D either has or does not have a term $(4, 5, 11^+)$ to cover H_2, H_3. Therefore, our analysis splits in Subcases 6.1–6.5.

Subcase 6.1. If $(x_2, y_2, z_2) = (5^+, 7^+, 10^+)$, then D coincides with (td15): $\{(4, 4, \infty), (5, 7, 10)\}$. If $(x_2, y_2, z_2) = (5^+, 6, 10^+)$, then D should have (x_3, y_3, z_3)

$= (4, 7^+, 7^+)$ in order to be able to cover H_5, and hence D is (td16): $\{(4, 4, \infty), (5, 6, 10), (4, 7, 7)\}$.

Subcase 6.2. If $(x_2, y_2, z_2) = (4, 7^+, 11^+)$, then H_2, H_3 are already covered, so it remains for D to cover H_6 and H_7 by other triplets, which can be done either by $(x_3, y_3, z_3) = (5^+, 6^+, 7^+)$ alone, in which case D coincides with (td17): $\{(4, 4, \infty), (4, 7, 11), (5, 6, 7)\}$, or by $(x_3, y_3, z_3) = (5^+, 5, 7^+)$ and $(x_4, y_4, z_5) = (5^+, 6, 6)$, and then D is (td18): $\{(4, 4, \infty)$, $(4, 7, 11)$, $(5, 5, 7)$, $(5, 6, 6)\}$.

Subcase 6.3. Next suppose $(x_2, y_2, z_2) = (4, 6, 11^+)$. Now D must cover H_5, H_6 and H_7 by other triplets. This can be done by $(x_3, y_3, z_3) = (5^+, 7^+, 7^+)$ alone, in which case D coincides with (td19): $\{(4, 4, \infty)$, $(4, 6, 11)$, $(5, 7, 7)\}$.

Otherwise, H_5 can be covered only by $(x_3, y_3, z_3) = (4, 7^+, 7^+)$. Then there are two ways for D to cover H_6 and H_7. The first is to have $(x_4, y_4, z_4) = (5^+, 6^+, 7^+)$, in which case D is (td20): $\{(4, 4, \infty), (4, 6, 11), (4, 7, 7), (5, 6, 7)\}$. The second way is to cover H_6 and H_7 separately by having $(x_4, y_4, z_4) = (5, 5, 7^+)$ and $(x_5, y_5, z_5) = (5^+, 6^+, 6^+)$, which results in $D = \{(4, 4, \infty)$, $(4, 6, 11)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)\}$, that is in (td21).

Subcase 6.4. Now suppose $(x_2, y_2, z_2) = (4, 5, 11^+)$. Then D must cover all of $H_4, \ldots, H_7$ by other triplets. To cover H_4 alone, D must have a term, say the third, so we can assume that $(x_3, y_3, z_3) = (4^+, 6^+, 10)$. Now our proof splits even more.

Subcase 6.4.1. $(x_3, y_3, z_3) = (5^+, 7^+, 10)$. Here, D contradicts with a stronger description (td15): $\{(4, 4, \infty), (5, 7, 10)\}$.

Subcase 6.4.2. $(x_3, y_3, z_3) = (5^+, 6, 10)$. Now to cover H_5, our D must have a term $(x_4, y_4, z_4) = (4^+, 7^+, 7^+)$, and we arrive at a contradiction with (td16): $\{(4, 4, \infty), (5, 6, 10), (4, 7, 7)\}$.

Subcase 6.4.3. $(x_3, y_3, z_3) = (4, 7^+, 10)$. Here, it remains for D to cover H_6 and H_7. As we know from above discussion, this happens by one or two other triplets, which results either in (td22): $\{(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 6, 7)\}$ or (td23): $\{(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)\}$, respectively.

Subcase 6.4.4. $(x_3, y_3, z_3) = (4, 6, 10)$. Now to cover H_5, there should exist a term $(x_4, y_4, z_4) = (4^+, 7^+, 7^+)$ in D. If actually $(x_4, y_4, z_4) = (5^+, 7^+, 7^+)$, then D coincides with (td24): $\{(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(5, 7, 7)\}$. Otherwise, we have $(x_4, y_4, z_4) = (4, 7^+, 7^+)$, and now the necessity to cover only H_6 and H_7 leads by means of our previous remark to (td25): $\{(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 6, 7)\}$ or (td26): $\{(4, 4, \infty)$, $(4, 5, 11)$, $(4, 6, 10)$, $(4, 7, 7)$, $(5, 5, 7)$, $(5, 6, 6)\}$.

Subcase 6.5. D has no term $(4, 5, 11^+)$. Here, D covers H_4 by the second term, so $(x_2, y_2, z_2) = (4, 6^+, 10^+)$ in view of Subcase 6.1, where $(x_2, y_2, z_2) = (4^+, 6^+, 10^+)$ was settled.

Now to cover H_2 and H_3, our D must have one or more triplets $(5^+, 5^+, 8^+)$ and $(5^+, 6^+, 7^+)$. Note that these triplets automatically cover also H_6 and H_7.

Subcase 6.5.1. $(x_2, y_2, z_2) = (4, 7^+, 10^+)$. If $(x_3, y_3, z_3) = (5^+, 6^+, 8^+)$ then D is (td27): $\{(4, 4, \infty), (4, 7, 10), (5, 6, 8)\}$. Otherwise, $(x_3, y_3, z_3) = (5, 5, 8^+)$, so D needs another term $(5^+, 6^+, 7^+)$ for satisfying H_3, and hence D reduces to (td28): $\{(4, 4, \infty), (4, 7, 10), (5, 5, 8), (5, 6, 7)\}$.

Subcase 6.5.2. $(x_2, y_2, z_2) = (4, 6, 10)$. Note that now D needs a term $(4^+, 7^+, 7^+)$ to satisfy H_5. If $(x_3, y_3, z_3) = (5^+, 7^+, 8^+)$, then D is (td29): $\{(4, 4, \infty), (4, 6, 10), (5, 7, 8)\}$. If $(x_3, y_3, z_3) = (5^+, 6, 8^+)$, then D is (td30): $\{(4, 4, \infty), (4, 6, 10), (5, 6, 8), (4, 7, 7)\}$.

Otherwise, we have $(x_3, y_3, z_3) = (5, 5, 8^+)$. Here, D needs another term $(x_4, y_4, z_4) = (5^+, 6^+, 7)$ for H_3. If in fact $(x_4, y_4, z_4) = (5^+, 7, 7)$, then H_5 is also covered, which means that D reduces to (td31): $\{(4, 4, \infty), (4, 6, 10), (5, 5, 8), (5, 7, 7)\}$. It remains to assume that $(x_4, y_4, z_4) = (4, 7, 7)$, which implies the necessity of $(x_5, y_5, z_5) = (5^+, 6, 7)$, and hence we have $D = \{(4, 4, \infty), (4, 6, 10), (5, 6, 8), (4, 7, 7)\}$, that is (td32). ∎

This completes the proof of Theorem 8.

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