Discussiones Mathematicae
Graph Theory 43 (2023) 965–978
https://doi.org/10.7151/dmgt.2487

ACYCLIC CHROMATIC INDEX OF IC-PLANAR GRAPHS

Wenyao Song

School of Mathematics and Statistics
Zaozhuang University, Zaozhuang 277160, China
e-mail: songwenyao@cumt.edu.cn

Lianying Miao

School of Mathematics
China University of Mining and Technology, Xuzhou 221116, China
e-mail: miaolianying@cumt.edu.cn

Yueying Zhao

School of Mathematics and Statistics
Xuzhou University of Technology, Xuzhou, 221018, China
e-mail: zlyue2006@126.com

and

Haiyang Zhu

Department of Flight Support Command
Air Force Logistics College, Xuzhou 221000, China
e-mail: tulunzhuhaiyang7@126.com

Abstract

Two distinct crossings are independent if the end-vertices of any two pairs of crossing edges are disjoint. If a graph G has a drawing in the plane such that every two crossings are independent, then we call G a plane graph with independent crossings or IC-planar graph for short. A proper edge coloring of a graph G is acyclic if there is no 2-colored cycle in G. The acyclic chromatic index of G is the least number of colors such that G has an acyclic edge coloring and denoted by $\chi'_a(G)$. In this paper, we prove that $\chi'_a(G) \leq \Delta(G)+17$, for any IC-planar graph G with maximum degree $\Delta(G)$.

Keywords: acyclic chromatic index, maximum degree, IC-planar graph.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote its vertex set, edge set, maximum degree and minimum degree, respectively. For a planar graph G, $F(G)$ denotes its face set, and $d(v)$ denotes the *degree* of a vertex v in G. The *length* or *degree* of a face f, denoted by $d(f)$, is the length of the boundary walk of f in G. We call $v \in V(G)$ a *k-vertex*, or a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively and call $f \in F(G)$ a *k-face*, or a *k^+-face*, or a *k^--face* if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. Any undefined notation follows that of Bondy and Murty [6].

A proper edge k-coloring of a graph G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that no pair of adjacent edges is colored with the same color. A proper edge coloring of a graph G is acyclic if there is no 2-colored cycle in G. The *acyclic chromatic index* of G is the least number of colors such that G has an acyclic edge coloring and denoted by $\chi'_a(G)$. Fiamčik [11] and later Alon *et al.* [3] proposed the following conjecture:

Conjecture 1. *For any graph G, $\chi'_a(G) \leq \Delta(G) + 2$.*

Alon *et al.* [2] proved that $\chi'_a(G) \leq 64\Delta(G)$ for any graph G. Molloy and Reed [18] improved this bound to $\chi'_a(G) \leq 16\Delta(G)$. Recently, Ndreca *et al.* [21] improved the upper bound to $\lceil 9.62(\Delta(G) - 1)\rceil$. With entropy compression method, Esperet and Parreau [10] further improved the result to $4\Delta(G) - 4$. The best known general bound is $\lceil 3.74(\Delta(G) - 1)\rceil$ due to Giotis *et al.* [13]. Alon *et al.* [3] proved that there is a constant c such that $\chi'_a(G) \leq \Delta(G) + 2$ for a graph G with girth at least $c\Delta(G)\log(\Delta(G))$. Něsetřil and Wormald [22] proved that $\chi'_a(G) \leq \Delta(G) + 1$ for a random $\Delta(G)$-regular graph G. The acyclic edge coloring of some special classes of graphs has been studied widely, including graphs with maximum degree 4 (Basavaraju and Chandran [5]), subcubic graphs (Basavaraju and Chandran [4]; Fiamčik [11]; Skulrattanakulchai [25]), sparse graphs (Muthu *et al.* [20]), series-parallel graphs (Hou *et al.* [15]; Wang and Shu [28]), outerplanar graphs (Hou *et al.* [16]; Muthu *et al.* [19]), planar graphs (Cohen *et al.* [8]; Dong and Xu [9]; Fiedorowicz *et al.* [12]; Shu and Wang [23, 24]; Wang *et al.* [29]; Yu *et al.* [30]).

For 1-planar graphs, where a graph is called 1-*planar* if it can be drawn in the plane such that each of its edgs is crossed by at most one other edge, Zhang *et al.* [31] proved that each 1-planar graph G with maximum degree $\Delta(G)$ is acyclically edge L-colorable where $L = \max\{2\Delta(G) - 2, \Delta(G) + 83\}$. Song and Miao [26] showed that each triangle-free 1-planar graph G admits an acyclic edge coloring with $\Delta(G) + 22$ colors. Chen [7] improved this bound to $\Delta(G) + 16$.

In this paper, we focus on IC-planar graphs which is a subclass of 1-planar graphs and prove the following theorem.

Theorem 2. *Let G be an* IC*-planar graph with maximum degree $\Delta(G)$. Then $\chi'_a(G) \leq \Delta(G) + 17$.*

Two distinct crossings are *independent* if the end-vertices of any two pairs of crossing edges are disjoint. If a graph G has a drawing in the plane in which every two crossings are independent, then we call G a *plane graph with independent crossings* or IC-*planar graph* for short throughout this paper. The definition of IC-planar graph was introduced by Alberson [1] in 2008. Confirming a conjecture of Alberson [1], Král and Stacho [17] proved that every IC-planar graph is 5-colorable.

We assume that every IC-planar graph G in this paper has been drawn in the plane with all its crossings independent and with the number of crossings minimum. We call such drawing an IC-*plane graph.* Let $C(G)$ be the set of all crossings and $E_0(G)$ be the non-crossed edges in G. The *associated plane graph* $G^\times$ of an IC-plane graph G is the plane graph such that $V(G^\times) = V(G) \cup C(G)$ and $E(G^\times) = E_0(G) \cup \{uw, vw | uv \in E(G)\backslash E_0(G)\}$, where w is the crossing on uv. Thus, all crossings of G become new real vertices of degree four in $G^\times$. For convenience, we call the new 4-vertices in $G^\times$ *false vertices.* A *false face* means a face f in $G^\times$ that is incident with one false vertex; otherwise, f is a *normal face.* We use $n_i(v)$ to denote the number of i-vertices which are adjacent to v. One can see that each non-false vertex in G is adjacent to at most one false vertex and incident with at most two false 3-faces in $G^\times$. Note that every IC-planar graph is also a 1-planar graph and hence all the properties of 1-planar graphs are also true for IC-planar graphs.

The rest of this paper is organized as follows. In Section 2, we give some definitions and lemmas on acyclic edge coloring. In Section 3, we give a structural lemma of IC-planar graphs which plays an important role in the proof of Theorem 2. In Section 4, we give the proof of Theorem 2.

2. Preliminary

In this paper, we use C to denote the color set under an acyclic edge coloring c. For $e \in E(G)$, the color α of C is said to be a *candidate* for e with respect to a partial acyclic edge coloring c of G if none of the adjacent edges of e is colored with α.

An (α, β)-*maximal bichromatic path* with respect to a partial coloring c in G is a maximal path consisting of edges that are colored with the colors α and β alternatingly. An (α, β, u, v)-*maximal bichromatic path* is an (α, β)-maximal bichromatic path which starts from the vertex u and ends at v. An (α, β, uv)-*critical path* for an edge uv is an (α, β, u, v)-maximal bichromatic path which starts at the vertex u with an edge colored α and ends at the vertex v with an edge colored α.

A graph G with $\Delta(G) \leq k$ is *k-deletion-minimal* if $\chi'_a(G) > k$ and $\chi'_a(H) \leq k$ for every proper subgraph H of G. Under an acyclic edge coloring c of G, we denote by $C(v)$ the set of colors which are assigned by c to the edges in $E(G)$ incident to v. We use $c(uv)$ to denote the color of edge uv with respect to the coloring c. Let $C_{uv} = C(v) - c(uv)$.

A *multiset* is a generalized set where each member can appear multiple times in the set. If an element x appears t times in the multiset S, then we say the *multiplicity* of x in S is t, denoted by $D_S(x)$. We use $\|S\| = \sum_{x\in S} D_S(x)$ to denote the cardinality of a finite multiset S. Let S and S' be two multisets. A multiset, denoted by $S \uplus S'$, is said to be the *union* of S and S' if the multiset $S \uplus S'$ has all the members of S and S' and $D_{S\uplus S'}(x) = D_S(x) + D_{S'}(x)$ for any member $x \in S \uplus S'$.

Lemma 3 [4]**.** *Given a pair of colors α and β of a proper coloring c of G, there can be at most one maximal (α, β)-bichromatic path containing a particular vertex v, with respect to c.*

Lemma 4 [27]**.** *If G is a k-deletion-minimal graph, then G is 2-connected.*

Lemma 5 [27]**.** *Let G be a k-deletion-minimal graph. If v is adjacent to a 2-vertex v_0 and $N_G(v_0) = \{w, v\}$, then v is adjacent to at least $k - d(w) + 1$ vertices of degree at least $k - d(v) + 2$. Moreover, if $k \geq \Delta(G) + 2$ and v is adjacent to precisely $k - \Delta(G) + 1$ vertices of degree at least $k - \Delta(G) + 2$, then v is adjacent to at most $d(v) + \Delta(G) - k - 3$ vertices of degree two and $d(v) \geq k - \Delta(G) + 4$.*

3. A Structural Lemma

Lemma 6. *If G is a connected* IC*-planar graph, then there exists a vertex v with d neighbors $v_1, \ldots, v_d$, where $d(v_1) \leq \cdots \leq d(v_d)$ such that one of the following holds:*

(A1) $d = 3$, $d(v_1) \leq 14$;

(A2) $d = 4$, $d(v_1) \leq 9$, $d(v_2) \leq 14$;

(A3) $d = 5$, $d(v_1) \leq 7$, $d(v_2) \leq 8$, $d(v_3) \leq 12$;

(A4) $d = 6$, $d(v_1) \leq 6$, $d(v_2) \leq 6$, $d(v_3) \leq 7$, $d(v_4) \leq 7$;

(A5) $d \leq 2$.

Before proving Lemma 6, we introduce a basic definition of the so-called *canonical triangulation* of an IC-planar graph which was given by Zhang *et al.* in [31]. The definition of canonical triangulation plays an important role in the proof of Lemma 6, so we include it.

A simple graph G is *triangulated* if every cycle of length at least 4 has an edge joining two nonadjacent vertices of the cycle. We say G_T is a *canonical triangulation* of a IC-planar graph G if G_T is obtained from G by the following operations.

Step 1. For each pair of edges ab, cd that cross each other at a point s, add edges ac, cb, bd and da "close to s", i.e., so that they form triangles asc, csb, bsd and dsa with empty interiors.

Step 2. Delete all multiple edges.

Step 3. If there are two edges that cross each other then delete one of them.

Step 4. Triangulate the planar graph obtained after the operation in Step 3 in any way.

Step 5. Add back the edges deleted in Step 3.

Note that the associated plane graph $G_T^\times$ of G_T is a special triangulation of $G^\times$ such that each crossing vertex remains to be of degree four, and each vertex v in $G_T^\times$ is incident with $d_{G_T^\times}(v)$ 3-faces. We use $cr(v)$ to denote the number of false vertices which are adjacent to v in $G_T^\times$. By the facts that G_T admits no multiple edge and the drawing of G_T minimizes the number of crossings, we have the following observations.

Observation 7. *For a canonical triangulation G_T of a simple* IC-*planar graph G, it holds.*

(1) *Any two false vertices are not adjacent in $G_T^\times$.*

(2) *If $d_{G_T^\times}(v) = 3$, then $cr(v) = 0$.*

(3) *If $d_{G_T^\times}(v) \geq 4$, then $cr(v) \leq 1$.*

***Proof of Lemma* 6.** The lemma is proved by contradiction. Let G be a simple IC-planar graph with a fixed embedding in the plane. Suppose G is a counterexample to the theorem. Note that if we can join two nonadjacent vertices in G by a new edge e so that $G + e$ is still an IC-planar graph, then $G + e$ shall also be a counterexample to the lemma. Without loss of generality, we always assume G is 2-connected and $G = G_T$ in the following, where G_T is a canonical triangulation of G that has been drawn on the plane properly. In other words, G is a canonical triangulation of itself.

Let v be a d-vertex and $v_1, v_2, \ldots, v_d$ be its d neighbors. A non-false 4-vertex v is *heavy* if $d(v_1) \geq 10$, *light* if $d(v_1) \leq 9$ and $d(v_2) \geq 15$. We use $n_{h4}(u)$ and $n_{l4}(u)$ to denote the number of heavy 4-vertices and light 4-vertices which are adjacent to u, respectively. A 5-vertex v is *heavy* if $d(v_1) \geq 8$, *light* if $d(v_1) \leq 7$ and $d(v_2) \geq 9$, *bad* if $d(v_1) \leq 7$, $d(v_2) \leq 8$ and $d(v_3) \geq 13$. We use $n_{h5}(u)$, $n_{l5}(u)$

and $n_{b5}(u)$ to denote the number of heavy 5-vertices, light 5-vertices and bad 5-vertices which are adjacent to u, respectively. A 6-vertex v is *heavy* if $d(v_2) \geq 7$, *light* if $d(v_2) \leq 6$ and $d(v_3) \geq 8$, *bad* if $d(v_2) \leq 6$, $d(v_3) \leq 7$ and $d(v_4) \geq 8$. We use $n_{h6}(u)$, $n_{l6}(u)$ and $n_{b6}(u)$ to denote the number of heavy 6-vertices, light 6-vertices and bad 6-vertices which are adjacent to u, respectively.

We apply the discharging method on the associated plane graph $G^\times$ of G and complete the proof by contradiction. Since $G^\times$ is a plane graph, we have

$$\sum_{v \in V(G^\times)} (d(v) - 4) + \sum_{f \in F(G^\times)} (d(f) - 4) = -8.$$

Now we define the initial charge function $ch(x)$ of $x \in V(G^\times) \cup F(G^\times)$. Let $ch(v) = d(v) - 4$ if $v \in V(G^\times)$ and $ch(f) = d(f) - 4$ if $f \in F(G^\times)$. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to redistribute the initial charge function $ch(x)$ to obtain a new charge function $ch'(x)$ on $V(G^\times) \cup F(G^\times)$ such that $ch'(x) \geq 0$ for all $x \in V(G^\times) \cup F(G^\times)$, then $0 \leq \sum_{x \in V(G^\times) \cup F(G^\times)} ch'(x) = \sum_{x \in V(G^\times) \cup F(G^\times)} ch(x) = -8$, a contradiction. Hence, we show the nonexistence of such a counterexample and complete the proof of Theorem 2.

For $v \in V(G^\times)$ and $f \in F(G^\times)$, we define the discharging rules as follows.

(R1) Let f be a false 3-face in $G^\times$. Then f receives $\frac{1}{2}$ from each incident real vertex.

(R2) Let f be a normal 3-face in $G^\times$. Then f receives $\frac{1}{3}$ from each incident vertex.

(R3) Each 7^+-vertex v sends $\frac{1}{15}$ to each adjacent heavy 6-vertex in G.

(R4) Each 8^+-vertex v sends $\frac{1}{5}$ to each adjacent heavy 5-vertex, $\frac{1}{9}$ to each adjacent light and bad 6-vertex in G.

(R5) Each 9^+-vertex v sends $\frac{1}{4}$ to each adjacent light 5-vertex in G.

(R6) Each 10^+-vertex v sends $\frac{5}{12}$ to each adjacent heavy 4-vertex in G.

(R7) Each 13^+-vertex v sends $\frac{1}{3}$ to each adjacent bad 5-vertex in G.

(R8) Each 15^+-vertex v sends $\frac{2}{3}$ to each adjacent 3-vertex, $\frac{5}{9}$ to each adjacent light 4-vertex in G.

Now we prove that $ch'(x) \geq 0$ for each $x \in V(G^\times) \cup F(G^\times)$.

Let f be a face of $G^\times$. Clearly, if $d(f) \geq 4$, then $ch'(f) = ch(f) = d(f) - 4 \geq 0$. Now we check the final charge of 3-faces in $G^\times$. If f is a false 3-face, then f receives $\frac{1}{2}$ from each real vertex incident with it by (R1). Thus f receives $\frac{1}{2} \times 2 = 1$. So we have $ch'(f) = ch(f) + 1 = 0$. If f is a normal 3-face, then

f receives $\frac{1}{3}$ from each real vertex incident with it by (R2). Thus f receives $\frac{1}{3} \times 3 = 1$. So we have $ch'(f) = ch(f) + 1 = 0$.

We next check the final charge of the vertex $v \in V(G^{\times})$. Since G has no $(A5)$, it follows $d(v) \geq 3$.

Suppose $d(v) = 3$. Since G has no $(A1)$, each neighbor of v is a 15^+-vertex. By (R8), v receives $\frac{2}{3} \times 3 = 2$ from its neighbors in G. Since v is not incident with any false 3-face by (2) of Observation 7, v sends at most 1 to the normal 3-faces by (R2). So we have $ch'(f) = ch(f) + 2 - 1 \geq 0$.

Suppose $d(v) = 4$. If v is false, then $ch'(v) = ch(v) \geq 0$. Otherwise, v is real in $G^{\times}$. By (3) of Observation 7, we get $cr(v) \leq 1$. Since $(A2)$ is forbidden in G, we consider the following two cases.

If v is heavy, then $ch'(v) \geq ch(v) + 4 \times \frac{5}{12} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = 0$ by (R1), (R2), (R6)–(R8).

If v is light, then $ch'(v) \geq ch(v) + 3 \times \frac{5}{9} - 2 \times \frac{1}{2} - 2 \times \frac{1}{3} = 0$ by (R1), (R2) and (R8).

Suppose $d(v) = 5$. By (3) of Observation 7, we have $cr(v) \leq 1$. Since $(A3)$ is forbidden in G, we consider the following three cases.

If v is heavy, then $ch'(v) \geq ch(v) + 5 \times \frac{1}{5} - 2 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by (R1), (R2), (R4)–(R8).

If v is light, then $ch'(v) \geq ch(v) + 4 \times \frac{1}{4} - 2 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by (R1), (R2) and (R5)–(R8).

If v is bad, then $ch'(v) \geq ch(v) + 3 \times \frac{1}{3} - 2 \times \frac{1}{2} - 3 \times \frac{1}{3} = 0$ by (R1), (R2), (R7) and (R8).

Suppose $d(v) = 6$. By (3) of Observation 7, we get $cr(v) \leq 1$. Since $(A4)$ is forbidden in G, we consider the following three cases.

If v is heavy, then $ch'(v) \geq ch(v) + 5 \times \frac{1}{15} - 2 \times \frac{1}{2} - 4 \times \frac{1}{3} = 0$ by (R1)–(R8).

If v is light, then $ch'(v) \geq ch(v) + 4 \times \frac{1}{9} - 2 \times \frac{1}{2} - 4 \times \frac{1}{3} > 0$ by (R1), (R2) and (R4)–(R8).

If v is bad, then $ch'(v) \geq ch(v) + 3 \times \frac{1}{9} - 2 \times \frac{1}{2} - 4 \times \frac{1}{3} = 0$ by (R1), (R2) and (R4)–(R8).

Suppose $d(v) = 7$. Since G has no $(A4)$, v is adjacent to at most four heavy 6-vertices. So we have $ch'(v) \geq ch(v) - 4 \times \frac{1}{15} - 2 \times \frac{1}{2} - 5 \times \frac{1}{3} > 0$ by (R1)–(R3) and (3) of Observation 7.

Suppose $d(v) = 8$. By (3) of Observation 7, we get $cr(v) \leq 1$. If $n_{h5}(v) = 0$, then we have $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 6 \times \frac{1}{3} - 8 \times \frac{1}{9} > 0$ by (R1), (R2) and (R4). Otherwise, $n_{h5}(v) \geq 1$. Since $(A3)$ is forbidden in G, $n_{h5}(v) \leq 4$. If $n_6(v) = 0$, then we have $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 6 \times \frac{1}{3} - \frac{n_{h5}(v)}{5} = \frac{5 - n_{h5}(v)}{5} > 0$ by (R1), (R2) and (R4). If $n_6(v) \geq 1$, then we have $2n_{h5}(v) + 1 + n_6(v) \leq d(v)$. Therefore, we have $ch'(v) \geq ch(v) - 2 \times \frac{1}{2} - 6 \times \frac{1}{3} - \frac{n_{h5}(v)}{5} - \frac{n_6(v)}{9} = \frac{10 + n_{h5}(v)}{45} > 0$ by (R1), (R2) and (R4).

Suppose $d(v) = 9$. According to the definitions of heavy 5-vertex and light

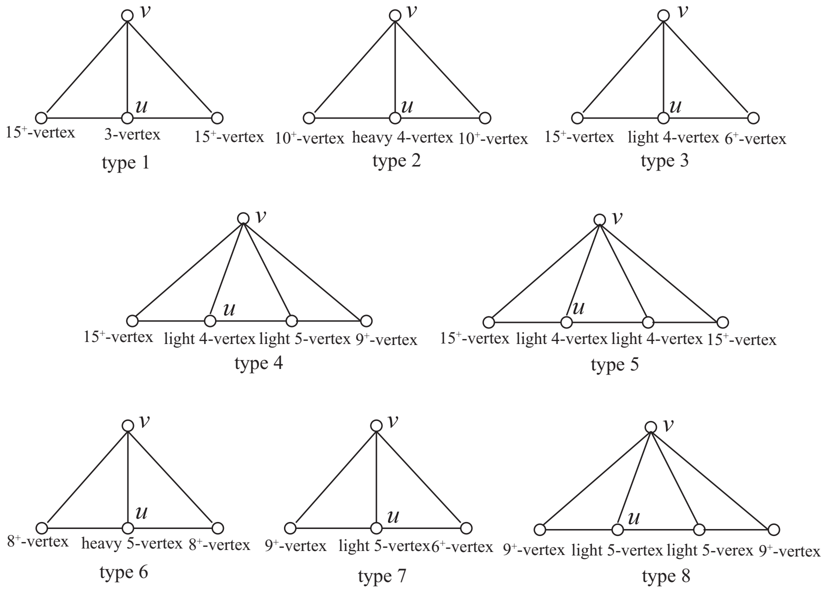

Figure 1. Eight different types of neighbors u of vertex v.

5-vertex, it follows that every two heavy 5-vertices u and v have two common 8^+-neighbors if $uv \in E(G^\times)$ and every two light 5-vertices w and v have at least one common 9^+-neighbor if $wv \in E(G^\times)$. Therefore, v is adjacent to at least $(n_{h5}(v)+\frac{1}{2}n_{l5}(v))$ 8^+-vertices (see type 4 and types 6–8 in Figure 1). Thus, we can easily obtain that $\frac{3}{2}n_{l5}(v)+2n_{h5}(v)+n_6(v) \le n_{l5}(v)+n_{h5}(v)+n_6(v)+n_{8^+}(v) \le 9$.

By (3) of Observation 7, we get $cr(v) \le 1$. So every 9-vertex sends at most $2 \times \frac{1}{2} + 7 \times \frac{1}{3} = \frac{10}{3}$ to incident 3-faces in $G^\times$ by (R1) and (R2). Next, we can calculate the largest possible value of the charges sent by v to adjacent 6^--vertices in G; we deduce that $\frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$ by (R3)–(R5). Let γ_9 be the largest possible value of the charges sent by v. We have $\gamma_9 = \frac{10}{3} + \frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$.

So $ch'(v) \ge ch(v) - \gamma_9 > 0$ since $\gamma_9 = \frac{1}{6}(\frac{3}{2}n_{l5}(v)+\frac{6}{5}n_{h5}(v)+\frac{2}{3}n_6(v))+\frac{10}{3} \le \frac{29}{6}$.

Suppose $d(v) = d$, $10 \le d \le 12$. According to the definition of heavy 4-vertex, it follows that every two heavy 4-vertices u and v have two common 10^+-neighbors if $uv \in E(G^\times)$. Therefore, v is adjacent to at least $(n_{h4}(v) + n_{h5}(v) + \frac{1}{2}n_{l5}(v))$ 8^+-vertices (see type 2, type 4 and types 6–8 in Figure 1). Thus, we can easily obtain that $2n_{h4}(v) + \frac{3}{2}n_{l5}(v) + 2n_{h5}(v) + n_6(v) \le n_{h4}(v) + n_{l5}(v) + n_{h5}(v) + n_6(v) + n_{8^+}(v) \le d$.

By (3) of Observation 7, we get $cr(v) \le 1$. So every d-vertex sends at most $1 + \frac{d-2}{3} = \frac{d+1}{3}$ to incident 3-faces in $G^\times$ by (R1) and (R2). Next, we can calculate the largest possible value of the charges sent by v to adjacent 6^--

vertices in G; we can deduce that $\frac{5}{12}n_{h4}(v) + \frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$ by (R3)–(R6). Let γ_d be the largest possible value of the charges sent by v. We have $\gamma_d = \frac{d+1}{3} + \frac{5}{12}n_{h4}(v) + \frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$.

So $ch'(v) \geq ch(v) - \gamma_d = \frac{11d-104}{24} > 0$ since $\gamma_d = \frac{5}{24}(2n_{h4}(v) + \frac{6}{5}n_{l5}(v) + \frac{24}{25}n_{h5}(v) + \frac{8}{15}n_6(v)) + \frac{d+1}{3} \leq \frac{13d+8}{24}$.

Suppose $d(v) = d$, $13 \leq d \leq 14$. Note that v may be adjacent to bad 5-vertices and is also adjacent at least $(n_{h4}(v)+n_{h5}(v)+\frac{1}{2}n_{l5}(v))$ 8^+-vertices. According to the above argument, it follows that $2n_{h4}(v)+\frac{3}{2}n_{l5}(v)+2n_{h5}(v)+n_{b5}(v)+n_6(v) \leq n_{h4}(v) + n_{l5}(v) + n_{h5}(v) + n_{b5}(v) + n_6(v) + n_{8^+}(v) \leq d$.

By (3) of Observation 7, we get $cr(v) \leq 1$. So every d-vertex sends at most $1 + \frac{d-2}{3} = \frac{d+1}{3}$ to incident 3-faces in $G^\times$ by (R1) and (R2). Next, we calculate the largest possible value of the charges sent by v to adjacent 6^--vertices in G; we deduce that $\frac{5}{12}n_{h4}(v) + \frac{1}{3}n_{b5}(v) + \frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$ by (R3)–(R5) and (R7). Let γ_d be the largest possible value of the charges sent by v. We have $\gamma_d = \frac{d+1}{3} + \frac{5}{12}n_{h4}(v) + \frac{1}{3}n_{b5}(v) + \frac{1}{4}n_{l5}(v) + \frac{1}{5}n_{h5}(v) + \frac{1}{9}n_6(v)$.

So $ch'(v) \geq ch(v) - \gamma_d = \frac{d-13}{3} \geq 0$ since $\gamma_d = \frac{1}{3}\big(\frac{5}{4}n_{h4}(v) + n_{b5}(v) + \frac{3}{4}n_{l5}(v) + \frac{3}{5}n_{h5}(v) + \frac{1}{3}n_6(v)\big) + \frac{d+1}{3} \leq \frac{2d+1}{3}$.

Suppose $d(v) = d \geq 15$. According to the definition of light 4-vertex, it follows that every two light 4-vertices u and v have at least one common 15^+-neighbors if $uv \in E(G^\times)$ and every two 3-vertices u and v have two common 15^+-neighbors if $wv \in E(G^\times)$. Therefore, v is adjacent to at least $(n_3(v) + n_{h4}(v) + \frac{1}{2}n_{l4}(v))$ 10^+-vertices (see types 1–8 in Figure 1). Thus, we can easily obtain that $2n_3(v)+2n_{h4}(v)+\frac{3}{2}n_{l4}(v)+n_5(v)+n_6(v) \leq n_3(v)+n_{h4}(v)+n_{l4}(v)+n_5(v) + n_6(v) + n_{10^+}(v) \leq d$.

By (3) of Observation 7, we get $cr(v) \leq 1$. So every d-vertex sends at most $1 + \frac{d-2}{3} = \frac{d+1}{3}$ to incident 3-faces in $G^\times$ by (R1) and (R2). Next, we calculate the largest possible value of the charges sent by v to adjacent 6^--vertices in G; we deduce that $\frac{2}{3}n_3(v) + \frac{5}{9}n_{l4}(v) + \frac{5}{12}n_{h4}(v) + \frac{1}{3}n_5(v) + \frac{1}{9}n_6(v)$ by (R3)–(R8). Let γ_d be the largest possible value of the charges sent by v. We have $\gamma_d = \frac{d+1}{3} + \frac{2}{3}n_3(v) + \frac{5}{9}n_{l4}(v) + \frac{5}{12}n_{h4}(v) + \frac{1}{3}n_5(v) + \frac{1}{9}n_6(v)$.

Note that $\gamma_d = \frac{1}{3}(2n_3(v) + \frac{5}{3}n_{l4}(v) + \frac{5}{4}n_{h4}(v) + n_5(v) + \frac{1}{3}n_6(v)) + \frac{d+1}{3} = \frac{1}{3}(2n_3(v)+\frac{3}{2}n_{l4}(v)+\frac{5}{4}n_{h4}(v)+n_5(v)+\frac{1}{3}n_6(v))+\frac{1}{18}n_{l4}(v)+\frac{d+1}{3} \leq \frac{2d+1}{3}+\frac{1}{18}n_{l4}(v)$. Since $(A2)$ is forbidden in G, we have $n_{l4}(v) \leq \lfloor\frac{2}{3}d\rfloor$. Therefore, $\gamma_d \leq \frac{19d+9}{27}$, we have $ch'(v) \geq ch(v) - \gamma_d = \frac{8d-117}{27} \geq 0$.

This completes the proof of Lemma 6. ■

4. PROOF OF THEOREM 2

Proof. Let G be a k-deletion-minimal graph with $k = \Delta(G) + 17$. Then G is 2-connected by Lemma 4. By Lemma 6, there exists a vertex $v \in V(G)$ with

d neighbors $v_1, \ldots, v_d$ such that v admits one of configurations (A1)–(A5). Let $H = G - vv_1$. By the minimality of G, H has an acyclic edge k-coloring c with the color set C=$\{1, 2, \ldots, k\}$. Moreover, we can choose the coloring c such that the value of $m = |C(v) \cap C(v_1)|$ is the minimum among all the acyclic edge colorings of H.

Suppose $m = 0$. Since $|C(v) \cup C(v_1)| \le 5 + \Delta(G) - 1 = \Delta(G) + 4 < k = |C|$, we have at least one available color for the edge vv_1 such that no bichromatic cycles are created. So we may assume that $m \ge 1$.

Case 1. $d(v) = 3$. The proof of this case is similar to that of Lemma 4 in [14], we omit it here.

Case 2. $d(v) = 4$. Let $N_G(v) = \{v_1, v_2, v_3, v_4\}$, where $d(v_1) \le d(v_2) \le d(v_3) \le d(v_4)$, $d(v_1) \le 9$ and $d(v_2) \le 14$. Without loss of generality, assume that $d(v_1) = 9, d(v_2) = 14$. Let x_i, $i = 1, \ldots, 8$, be the neighbors of v_1 other than v. Let $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4}$.

Suppose $m = 1$, without loss of generality, assume that $c(vv_4) = c(v_1x_1)$ =1. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash F| = \Delta(G) + 7$. Then there exists a color $\alpha \in C \backslash F$ such that no bichromatic cycle is created in G with edge vv_1 colored with α. Otherwise, for any color $\theta \in C \backslash F$, there exists a $(1, \theta, vv_1)$-critical path under c. We have $d(v_4) \ge |(C \backslash F) \cup c(vv_4)| = \Delta(G) + 7 + 1 = \Delta(G) + 8 > \Delta(G)$, a contradiction.

When $m \ge 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \ldots, m\}$. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash F| = \Delta(G) + 6 + m$. If there exists an color $\alpha \in C \backslash F$ such that we color the edge vv_1 with α and do not create any bichromatic cycle in G, then we get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\theta \in C \backslash F$, there exists an (i, θ, vv_1)-critical path for $i \in \{1, \ldots, m\}$ under c. Therefore, there is a color $\beta \in C \backslash F$ with $D_{S_v}(\beta) \le 1$ in S_v since $\|S_v\| = d(v_3) - 1 + d(v_4) - 1 + d(v_2) - 1 \le 2\Delta(G) + 11 < 2(\Delta(G) + 6 + m)$. Without loss of generality, assume that $\beta \in C(v_2)$. By Lemma 3, there is no $(1, \beta, vv_3)$-critical path through v_2 under c. We recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges incident with v and v_1 is smaller, a contradiction.

Case 3. $d(v) = 5$. Let $N_G(v) = \{v_1, v_2, v_3, v_4, v_5\}$, where $d(v_1) \le d(v_2) \le d(v_3) \le d(v_4) \le d(v_5)$, $d(v_1) \le 7, d(v_2) \le 8$ and $d(v_3) \le 12$. Without loss of generality, assume that $d(v_1) = 7, d(v_2) = 8, d(v_3) = 12$. Let x_i, $i = 1, \ldots, 6$, be the neighbors of v_1 other than v. Let $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4} \uplus C_{vv_5}$.

Suppose $m = 1$. Without loss of generality, assume that $c(vv_5) = c(v_1x_1) = 1$. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash F| = \Delta(G) + 8$. Then there is a color $\alpha \in C \backslash F$ such that no bichromatic cycle is created in G with edge vv_1 colored with α. Otherwise, for any color $\theta \in C \backslash F$, there is a $(1, \theta, vv_1)$-critical path under c. We have $d(v_5) \ge |(C \backslash F) \cup c(vv_5)| = \Delta(G) + 8 + 1 = \Delta(G) + 9 > \Delta(G)$, a contradiction.

When $m \geq 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \ldots, m\}$. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C \backslash F| = \Delta(G)+7+m$. If there exists a color $\alpha \in C\backslash F$ such that we color the edge vv_1 with α and do not create any bichromatic cycle in G, then we can get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\theta \in C\backslash F$, there exists a (i, θ, vv_1)-critical path for $i \in \{1, \ldots, m\}$ under c. Therefore, there is a color $\beta \in C\backslash F$ with $D_{S_v}(\beta) \leq 1$ in S_v since $\|S_v\| = d(v_5) - 1 + d(v_4) - 1 + d(v_3) - 1 + d(v_2) - 1 \leq 2\Delta(G) + 16 < 2(\Delta(G) + 7 + m)$. Without loss of generality, we assume that $\beta \in C(v_2)$. By Lemma 3, there is no $(1, \beta, vv_3)$-critical path through v_2 under c. So we recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges incident with v and v_1 is smaller, a contradiction.

Case 4. $d(v) = 6$. Let $N_G(v) = \{v_1, v_2, \ldots, v_6\}$, where $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_6)$, $d(v_1) \leq 6, d(v_2) \leq 6, d(v_3) \leq 7$ and $d(v_4) \leq 7$. Without loss of generality, assume that $d(v_1) = 6, d(v_2) = 6, d(v_3) = 7, d(v_4) = 7$. Let x_i, $i = 1, \ldots, 5$, be the neighbors of v_1 other than v. Let $S_v = C_{vv_2} \uplus C_{vv_3} \uplus C_{vv_4} \uplus C_{vv_5} \uplus C_{vv_6}$.

Suppose $m = 1$. Without loss of generality, assume that $c(vv_6) = c(v_1x_1) = 1$. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C\backslash F| = \Delta(G)+8$. Then there is a color $\alpha \in C\backslash F$, such that no bichromatic cycle is created in G with edge vv_1 colored with α. Otherwise, for any color $\theta \in C\backslash F$, there is a $(1, \theta, vv_1)$-critical path under c. We have $d(v_6) \geq |(C\backslash F) \cup c(vv_6)| = \Delta(G) + 8 + 1 = \Delta(G) + 9 > \Delta(G)$, a contradiction.

When $m \geq 2$, without loss of generality, assume that $c(vv_{i+1}) = c(v_1x_i) = i$, for $i \in \{1, \ldots, m\}$. Let $F = C(v) \cup C(v_1)$ in H. Therefore, $|C\backslash F| = \Delta(G)+7+m$. If there exists a color $\alpha \in C\backslash F$ such that we color the edge vv_1 with α and do not create any bichromatic cycle in G, then we can get an acyclic edge k-coloring of G, a contradiction. Otherwise, for any color $\theta \in C\backslash F$, there exists an (i, θ, vv_1)-critical path for $i \in \{1, \cdots, m\}$ under c. Therefore, there is a color $\beta \in C\backslash F$ with $D_{S_v}(\beta) \leq 1$ in S_v since $\|S_v\| = d(v_6)-1+d(v_5)-1+d(v_4)-1+d(v_3)-1+d(v_2)-1 \leq 2\Delta(G) + 15 < 2(\Delta(G) + 7 + m)$. Without loss of generality, we assume that $\beta \in C(v_2)$. By Lemma 3, there is no $(1, \beta, vv_3)$-critical path through v_2 under c. So we recolor the edge vv_3 with β to obtain an acyclic edge coloring c' of H, but the number of common colors on the edges incident with v and v_1 is smaller, a contradiction.

Now we consider the situation that there is no vertex v that belongs to configurations (A1), (A2), (A3) and (A4).

Case 5. G contains a 2-vertex. We delete all the 2-vertices in G to get a graph G'. By Lemma 5, if $d_{G'}(x) < d_G(x)$, then $d_{G'}(x) \geq 18$. So G' does not contain any 2-vertex and $d_{G'}(x) = d_G(x)$ if $3 \leq d_G(x) \leq 6$. Now we consider G'. By Lemma 6, there exists a vertex in G' such that at least one of (A1), (A2), (A3) and (A4) holds, say the vertex is v, then $3 \leq d_{G'}(v) \leq 6$ and $d_{G'}(v_1) \leq 14$. Thus, v_1 is adjacent to at least one 2-vertex in G since $d_{G'}(x) = d_G(x)$. This lead to a contradiction which $d_{G'}(v_1) \geq 18$ if v_1 is adjacent to a 2-vertex by Lemma 5.

This completes the proof of Theorem 2. ■

Acknowledgements

The authors would like to express their thanks to the referee for his valuable corrections and suggestions of the manuscript that greatly improve the format and correctness of it. This work was supported by National Natural Science Foundation of China (Grant No. 11771443, 12001481), Natural Science Foundation of Shandong Province, China (Grant No. ZR2019BA016, ZR2021MA103) and the foundation of Zaozhuang University, China (Grant No.YJG21066).

References

[1] M.O. Alberson, *Chromatic number, independent ratio, and crossing number*, Ars Math. Contemp. **1** (2008) 1–6.
https://doi.org/10.26493/1855-3974.10.2d0

[2] N. Alon, C. McDiarmid and B. Reed, *Acyclic coloring of graphs*, Random Structures Algorithms **2** (1991) 277–288.
https://doi.org/10.1002/rsa.3240020303

[3] N. Alon, B. Sudakov and A. Zaks, *Acyclic edge colorings of graphs*, J. Graph Theory **37** (2001) 157–167.
https://doi.org/10.1002/jgt.1010

[4] M. Basavaraju and L.S. Chandran, *Acyclic edge coloring of subcubic graphs*, Discrete Math. **308** (2008) 6650–6653.
https://doi.org/10.1016/j.disc.2007.12.036

[5] M. Basavaraju and L.S. Chandran, *Acyclic edge coloring of graphs with maximum degree* 4, J. Graph Theory **61** (2009) 192–209.
https://doi.org/10.1002/jgt.20376

[6] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York-Amsterdam-Oxford, 1982).

[7] J. Chen, T. Wang and H. Zhang, *Acyclic chromatic index of triangle-free* 1*-planar graphs*, Graphs Combin. **33** (2017) 859–868.
https://doi.org/10.1007/s00373-017-1809-0

[8] N. Cohen, F. Havet and T. Müller, *Acyclic edge-colouring of planar graphs. Extended abstract*, Electron. Notes Discrete Math. **34** (2009) 417–421.
https://doi.org/10.1016/j.endm.2009.07.069

[9] W. Dong and B. Xu, *Some results on acyclic edge coloring of plane graphs*, Inform. Process. Lett. **110** (2010) 887–892.
https://doi.org/10.1016/j.ipl.2010.07.019

[10] L. Esperet and A. Parreau, *Acyclic edge-coloring using entropy compression*, European J. Combin. **34** (2013) 1019–1027.
https://doi.org/10.1016/j.ejc.2013.02.007

[11] J. Fiamčik, *The acyclic chromatic class of a graph*, Math. Slovaca **28** (1978) 139–145, in Russian.

[12] A. Fiedorowicz, M. Hałuszczak and N. Narayanan, *About acyclic edge colourings of planar graphs*, Inform. Process. Lett. **108** (2008) 412–417.
https://doi.org/10.1016/j.ipl.2008.07.016

[13] I. Giotis, L. Kirousis, K.I. Psaromiligkos and D.M. Thilikos, *Acyclic edge coloring through the Lovász Local Lemma*, Theoret. Comput. Sci. **665** (2017) 40–50.
https://doi.org/10.1016/j.tcs.2016.12.011

[14] J.F. Hou, N. Roussel and J. Wu, *Acyclic chromatic index of planar graphs with triangles*, Inform. Process. Lett. **111** (2011) 836–840.
https://doi.org/10.1016/j.ipl.2011.05.023

[15] J. Hou, J. Wu, G. Liu and B. Liu, *Acyclic edge colorings of planar graphs and series-parallel graphs*, Sci. China Ser. A **52** (2009) 605–616.
https://doi.org/10.1007/s11425-008-0124-x

[16] J. Hou, J. Wu, G. Liu and B. Liu, *Acyclic edge chromatic number of outerplanar graphs*, J. Graph Theory **64** (2010) 22–36.
https://doi.org/10.1002/jgt.20436

[17] D. Král and L. Stacho, *Coloring plane graphs with independet crossings*, J. Graph Theory **64** (2010) 184–205.
https://doi.org/10.1002/jgt.20448

[18] M. Molloy and B. Reed, *Further algorithmic aspects of the local lemma*, in: Proc. 30th Annual ACM Symposium on Theory of Computing, (ACM, New York, 1998) 524–529.
https://doi.org/10.1145/276698.276866

[19] R. Muthu, N. Narayanan and C.R. Subramanian, *Acyclic edge colouring of outerplanar graphs*, Lecture Notes in Comput. Sci. **4508** (2007) 144–152.
https://doi.org/10.1007/978-3-540-72870-2_14

[20] R. Muthu, N. Narayanan and C.R. Subramanian, *Improved bounds on acyclic edge colouring*, Discrete Math. **307** (2007) 3063–3069.
https://doi.org/10.1016/j.disc.2007.03.006

[21] S. Ndreca, A. Procacci and B. Scoppola, *Improved bounds on coloring of graphs*, European J. Combin. **33** (2012) 592–609.
https://doi.org/10.1016/j.ejc.2011.12.002

[22] J. Něsetřil and N.C. Wormald, *The acyclic edge chromatic number of a random d-regular graph is* $d+1$, J. Graph Theory **49** (2005) 69–74.
https://doi.org/10.1002/jgt.20064

[23] Q. Shu and W. Wang, *Acyclic chromatic indices of planar graphs with girth at least five*, J. Comb. Optim. **23** (2012) 140–157.
https://doi.org/10.1007/s10878-010-9354-2

[24] Q. Shu, W. Wang and Y. Wang, *Acyclic chromatic indices of planar graphs with girth at least* 4, J. Graph Theory **73** (2013) 386–399.
https://doi.org/10.1002/jgt.21683

[25] S. Skulrattanakulchai, *Acyclic colorings of subcubic graphs*, Inform. Process. Lett. **92** (2004) 161–167.
https://doi.org/10.1016/j.ipl.2004.08.002

[26] W.Y. Song and L.Y. Miao, *Acyclic edge coloring of triangle-free* 1*-planar graphs*, Acta Mat. Sin. (Engl. Ser.) **31** (2015) 1563–1570.
https://doi.org/10.1007/s10114-015-4479-y

[27] T. Wang and Y. Zhang, *Acyclic edge coloring of graphs*, Discrete Appl. Math. **167** (2014) 290–303.
https://doi.org/10.1016/j.dam.2013.12.001

[28] W. Wang and Q. Shu, *Acyclic chromatic indices of K_4-minor free graphs*, Scientia Sin. Math. **41** (2011) 733–744.
https://doi.org/10.1360/012010-530

[29] W. Wang, Q. Shu, K. Wang and P. Wang, *Acyclic chromatic indices of planar graphs with large girth*, Discrete Appl. Math. **159** (2011) 1239–1253.
https://doi.org/10.1016/j.dam.2011.03.017

[30] D. Yu, J. Hou, G. Liu, B. Liu and L. Xu, *Acyclic edge coloring of planar graphs with large girth*, Theoret. Comput. Sci. **410** (2009) 5196-5200.
https://doi.org/10.1016/j.tcs.2009.08.015

[31] X. Zhang, G. Liu and J. Wu, *Structural properties of* 1*-planar graphs and an application to acyclic edge coloring*, Scientia Sin. Math. **40** (2010) 1025–1032.
https://doi.org/10.1360/012009-678

Received 8 September 2017
Revised 18 April 2020
Accepted 16 May 2021
Available online 21 February 2023