

GRAPH CLASSES EQUIVALENT TO 12-REPRESENTABLE GRAPHS

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Abstract

Jones *et al.* (2015) introduced the notion of u -representable graphs, where u is a word over $\{1, 2\}$ different from $22 \cdots 2$, as a generalization of word-representable graphs. Kitaev (2016) showed that if u is of length at least 3, then every graph is u -representable. This indicates that there are only two nontrivial classes in the theory of u -representable graphs: 11-representable graphs, which correspond to word-representable graphs, and 12-representable graphs. This study deals with 12-representable graphs.

Jones *et al.* (2015) provided a characterization of 12-representable trees in terms of forbidden induced subgraphs. Chen and Kitaev (2022) presented a forbidden induced subgraph characterization of a subclass of 12-representable grid graphs.

This paper shows that a bipartite graph is 12-representable if and only if it is an interval containment bigraph. The equivalence gives us a forbidden induced subgraph characterization of 12-representable bipartite graphs since the list of minimal forbidden induced subgraphs is known for interval containment bigraphs. We then have a forbidden induced subgraph characterization for grid graphs, which solves an open problem of Chen and Kitaev (2022). The study also shows that a graph is 12-representable if and only if it is the complement of a simple-triangle graph. This equivalence indicates that a necessary condition for 12-representability presented by Jones *et al.* (2015) is also sufficient. Finally, we show from these equivalences that 12-representability can be determined in $O(n^2)$ time for bipartite graphs and in $O(n(\bar{m} + n))$ time for arbitrary graphs, where n and \bar{m} are the number of vertices and edges of the complement of the given graph.

Keywords: 12-representable graphs, forbidden induced subgraphs, interval containment bigraphs, simple-triangle graphs, vertex ordering characterization.

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1. INTRODUCTION

The notion of u -representable graphs, where u is a word over $\{1, 2\}$ different from $22 \cdots 2$, was introduced by Jones *et al.* [11] as a generalization of a well-studied class of word-representable graphs [12, 14]. In this context, word-representable graphs correspond to 11-representable graphs.

Jones *et al.* [11] showed that any graph is 1^k -representable for every $k \geq 3$, where 1^k denotes k concatenated copies of 1. Extending this result, Kitaev [13] showed that for every $u \in \{1, 2\}^*$ of length at least 3, any graph is u -representable. Therefore, only two graph classes are nontrivial in the theory of u -representable graphs: 11-representable graphs and 12-representable graphs. This paper focuses on 12-representable graphs. Note that the class of 21-representable graphs is equivalent to that of 12-representable graphs, as shown in the next section.

The class of 12-representable graphs is a proper subclass of comparability graphs and a proper superclass of both co-interval graphs and permutation graphs [11]. The class of 12-representable graphs is not equivalent to that of 11-representable graphs since word-representable graphs (i.e., 11-representable graphs) generalize comparability graphs [12, 14]. It is also known that any cycle of length at least 5 is not 12-representable [11]. This implies that 12-representable graphs are weakly chordal since the graph $\overline{C_n}$ (the complement of the cycle of length n) is not a comparability graph for any $n \geq 5$, see, e.g., [7] and [8, Corollary 2.11].

Jones *et al.* [11] showed that a tree is 12-representable if and only if it is a *double caterpillar*, a tree in which every vertex is within distance 2 from a central path. It is easy to see that a tree is a double caterpillar if and only if it contains no T_3 as a subtree (see, e.g., [17, Lemma 18]), where T_3 is the tree in Figure 6(a). They also initiated the study of the 12-representability of grid graphs. They provided some 12-representable grid graphs and asked whether such graphs could be characterized. We use the term *grid graph* in this paper to mean an induced subgraph of a rectangular grid graph.

Chen and Kitaev [2] answered this question. They called a grid graph a *square grid graph* if every edge belongs to a cycle of length 4, and showed that a square grid graph is 12-representable if and only if it contains no X and no cycle of length $2n$ for $n \geq 4$ as an induced subgraph, where X is the graph in Figure 6(f). They also provided a conjecture for characterizing 12-representable *line grid graph*, grid graphs that are not square grid graphs [2, Conjecture 3.6]. We will deal with this conjecture in Remark 8.

Meanwhile, Jones *et al.* [11] gave a necessary condition for the 12-representability of a graph in terms of graph labelings (see Theorem 1). Chen and Kitaev [2] showed that the necessary condition is also sufficient for square grid graphs. Whether the condition is sufficient for arbitrary graphs was left as an open question [2].

This study shows that a bipartite graph is 12-representable if and only if it is an interval containment bigraph [10]. We also demonstrate that a graph is 12-representable if and only if it is the complement of a simple-triangle graph [3]. These equivalences provide some structural results on 12-representable graphs. In particular, we obtain a forbidden induced subgraph characterization of 12-representable bipartite graphs and then also for grid graphs. Moreover, we obtain from a characterization of simple-triangle graphs [19] that the necessary condition of Jones *et al.* [11] mentioned above is in fact also sufficient.

2. PRELIMINARIES

This section presents some definitions, notations, and results used in this paper.

All graphs in this paper are finite, simple, and undirected. We write xy for the edge joining two vertices x and y . For a graph G , we write $V(G)$ and $E(G)$ for the vertex set and the edge set of G , respectively. We usually denote the number of vertices and edges by n and m , respectively. The *complement* of a graph G is the graph \overline{G} such that $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for any two distinct vertices $x, y \in V(\overline{G})$.

2.1. Words and 12-representable graphs

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$ and $[n]^*$ be the set of all words over $[n]$. For a word $w \in [n]^*$, let $A(w)$ denote the set of letters occurring in w . For a subset $B \subseteq A(w)$, let w_B be a word obtained from w by removing all the letters of $A(w) \setminus B$. For a word $w \in [n]^*$, the *reduced form* of w , denoted by $\text{red}(w)$, is the word obtained from w by replacing each occurrence of the i th smallest letter with i . Let $u = u_1u_2 \cdots u_k$ with $\text{red}(u) = u$. A word $w = w_1w_2 \cdots w_m$ of $[n]^*$ has a *u -match* if there is an index i such that $\text{red}(w_iw_{i+1} \cdots w_{i+k-1}) = u$, that is, up to reduction, u occurs consecutively in w .

A *labeled graph* of a graph G is obtained from G by assigning an integer (label) to each vertex. This paper assumes that all labels are distinct and from $[n]$, where n denotes the number of vertices of the graph. Given a word $u \in [2]^*$ such that $\text{red}(u) = u$ (i.e., u is different from $22 \cdots 2$), a labeled graph G is *u -representable* if there is a word $w \in [n]^*$ such that $A(w) = [n]$ and for any $x, y \in V(G)$, $xy \in E(G)$ if and only if $w_{\{x,y\}}$ has no u -matches. In this case, we say that the word w *u -represents* the graph G and w is a *u -representant* of G . An unlabeled graph H is *u -representable* if there is a labeling of H such that the resulting labeled graph H' is u -representable.

By definition, the class of u -representable graphs is hereditary (i.e., closed under taking induced subgraphs). Note also that the class of u -representable graphs is equivalent to that of u^r -representable graphs, where u^r denotes the

reverse of u , since if a word w is a u -representant of a graph G , then its reverse w^r is a u^r -representant of G and vice versa. Thus, as noted in the introduction, the classes of 12-representable and 21-representable graphs are equivalent.

2.2. Necessary condition

Given a labeled graph G , the *reduced form* of G , denoted by $\text{red}(G)$, is the labeled graph obtained from G by relabeling so that the i th smallest label is replaced by i . For a graph G , a graph H is an *induced subgraph* if $V(H) \subseteq V(G)$ and $xy \in E(H) \iff xy \in E(G)$ for any $x, y \in V(H)$. We will use the following necessary condition to determine 12-representable graphs.

Theorem 1 [11]. *Let G be a labeled graph. If G has an induced subgraph H such that $\text{red}(H)$ is equal to one of I_3 , J_4 , or Q_4 in Figure 1, then G is not 12-representable.*

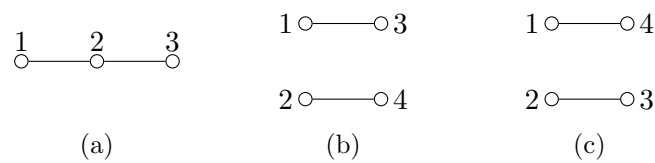


Figure 1. Forbidden labeled graphs I_3 (a), J_4 (b), and Q_4 (c).

We now define the notion of F -free labeling.

Definition 2. Let F be a set of labeled graphs. (Recall that we assume all labels are distinct and from $[n]$, where n is the number of vertices of the graph.) A graph labeling is F -free if it contains no induced subgraphs in F in the reduced form.

Note that $\{I_3, J_4, Q_4\}$ -free labeling is said to be *good* by Chen and Kitaev [2]. They showed that the existence of a good labeling for a square grid graph implies that the graph is 12-representable.

2.3. Interval containment bigraphs

A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets X and Y . Such a partition (X, Y) is called a *bipartition* of G . A bipartite graph G with bipartition (X, Y) is an *interval containment bigraph* [10] if there is an interval I_v for each vertex $v \in V(G)$ such that for any $x \in X$ and $y \in Y$, $xy \in E(G)$ if and only if I_x contains I_y . The set $\{I_v : v \in V(G)\}$ is called a *model* or *representation* of G . See Figures 2(a) and 2(b) for example.

The class of interval containment bigraphs is equivalent to some classes of graphs, such as bipartite graphs whose complements are circular-arc graphs [5]

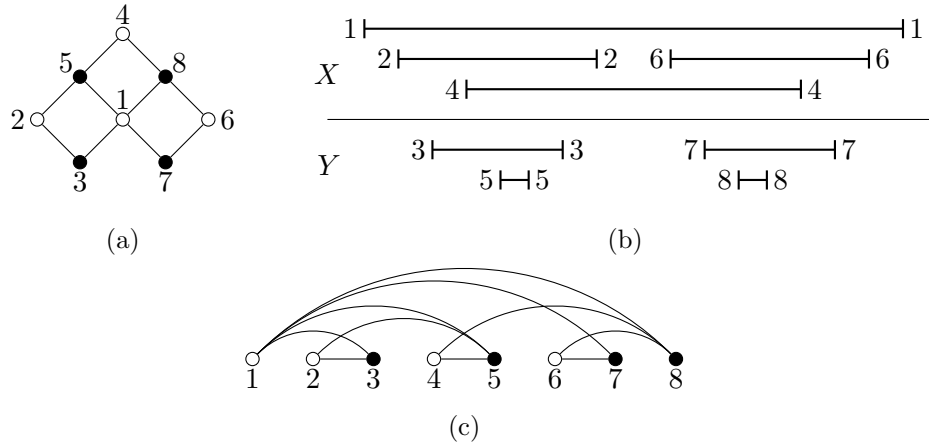


Figure 2. (a) An interval containment bigraph G_1 . (b) A model of G_1 . (c) An ordering of the vertices of G_1 . White and black vertices are in X and Y , respectively. The vertices are labeled based on the left endpoints of the intervals. As shown in Example 6, the word $w = 3578.53284761.1246$ is a 12-representant of G_1 (the dots are not part of the word, they are only included as delimiters of the word parts as constructed in the proof of Theorem 5).

and two-directional orthogonal ray graphs [17]. We will use this equivalence in Section 3. The other equivalent classes can be found in [16, 22]. Among those, we choose the model of interval containment bigraphs because of the simplicity of the construction of 12-representants.

Many results have been obtained for these classes, including a forbidden induced subgraph characterization [5, 17, 23] and polynomial-time recognition algorithms [17]. The class of interval containment bigraphs is a proper subclass of chordal bipartite graphs and a superclass of bipartite permutation graphs [17].

2.4. Simple-triangle graphs

Let L_1 and L_2 be two horizontal lines in the plane with L_1 above L_2 . A point on L_1 and an interval on L_2 define a triangle between L_1 and L_2 . A graph is a *simple-triangle graph* if there is a triangle T_v for each vertex $v \in V(G)$ such that for any $x, y \in V(G)$, $xy \in E(G)$ if and only if T_x intersects T_y . The set $\{T_v : v \in V(G)\}$ is called a *model* or *representation* of G . See Figures 3(a) and 3(b) for example.

Simple-triangle graphs were introduced in [3] as a generalization of both interval graphs and permutation graphs and have been studied under *PI graphs* [1, 18], where *PI* stands for *Point-Interval*. The recognition of simple-triangle graphs has been a longstanding open problem [18, Open Problem 13.3], and some polynomial-time recognition algorithms have been presented recently [15, 20, 21]. The class

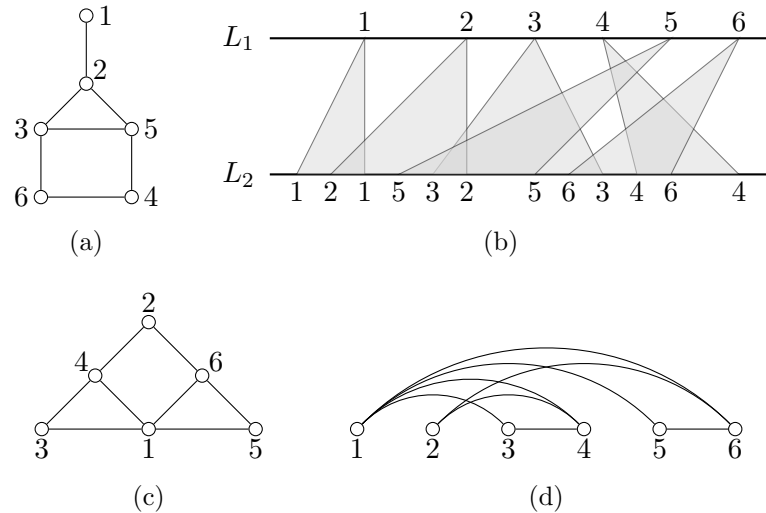


Figure 3. (a) A simple-triangle graph G_2 . (b) A model of G_2 . (c) The complement $\overline{G_2}$ of G_2 . (d) An ordering of the vertices of $\overline{G_2}$. The vertices are labeled based on the points on L_1 . As shown in Example 12, the word $w = 464365235121$ is a 12-representant of $\overline{G_2}$.

of simple-triangle graphs is known to be a proper subclass of trapezoid graphs [3]. It is also known that a simple-triangle graph is a cocomparability graph and alternately orientable [19].

2.5. Vertex ordering characterizations

Recall that the necessary condition for 12-representability (i.e., Theorem 1) is stated in terms of graph labelings. A labeling of a graph G can be viewed as ordering the vertices of G such that $x \prec y$ in the ordering if the label of x is smaller than that of y . Then, the graphs I_3 , J_4 , and Q_4 in Figure 1 correspond to ordered graphs in 4(a), 4(b), and 4(c), respectively. We will use characterizations of interval containment bigraphs and simple-triangle graphs defined in terms of forbidden ordered induced subgraphs. We will refer to such an ordered graph as a *pattern*.

An example of forbidden pattern characterization is as follows. A graph G is a *comparability graph* if each edge can be oriented so that if $x \rightarrow y$ and $y \rightarrow z$ then $x \rightarrow z$ for any $x, y, z \in V(G)$. It is known that a graph G is a comparability graph if and only if there is a vertex ordering σ of G such that for any $x, y, z \in V(G)$ with $x \prec y \prec z$ in σ , if $xy \in E(G)$ and $yz \in E(G)$ then $xz \in E(G)$. In other words, a graph is a comparability graph if and only if it has a vertex ordering which does not contain the pattern in Figure 4(a) as an induced pattern. Other examples can be found in [1, Section 7.4] and [6].

Theorem 3 [19]. *A graph G is a simple-triangle graph if and only if the complement \overline{G} of G has a vertex ordering which does not contain any pattern in Figure 4 as an induced pattern. Moreover, for any such ordering σ , there is a model of G such that σ coincides with the ordering of the points on L_1 . (Recall that the triangle in the model is defined by a point on L_1 and an interval on L_2 .) Such a model of G can be obtained in $O(n^2)$ time if σ is given.*

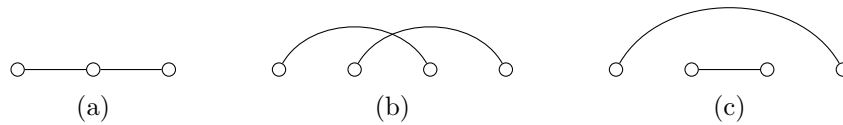


Figure 4. Forbidden patterns of complements of simple-triangle graphs.

Figures 3(c) and 3(d) show the complement $\overline{G_2}$ of G_2 and its vertex ordering, which does not contain any pattern in Figure 4.

Notice that an $\{I_3, J_4, Q_4\}$ -free labeling of a graph can be viewed as a vertex ordering which does not contain any pattern in Figure 4. Thus, we have from Theorems 1 and 3 that any 12-representable graph is the complement of a simple-triangle graph.

For interval containment bigraphs, the following characterization is known.

Theorem 4 [10]. *A bipartite graph G with bipartition (X, Y) is an interval containment bigraph if and only if G has a vertex ordering which does not contain any pattern in Figure 5 as an induced pattern.*

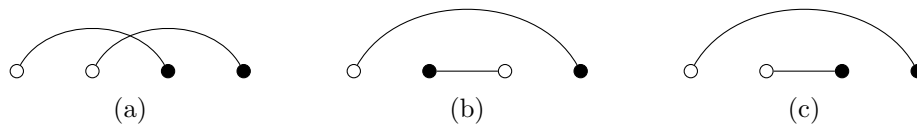


Figure 5. Forbidden patterns of interval containment bigraphs. White and black vertices are in X and Y , respectively, or the other way around.

Figure 2(c) shows a vertex ordering of G_1 , which does not contain any pattern in Figure 5. Notice that a $\{J_4, Q_4\}$ -free labeling of a bipartite graph can be viewed as a vertex ordering which does not contain any pattern in Figure 5. Thus, we have from Theorems 1 and 4 that any 12-representable bipartite graph is an interval containment bigraph.

3. INTERVAL CONTAINMENT BIGRAPHS

This section shows the equivalence of 12-representable bipartite graphs and interval containment bigraphs and its consequences.

Theorem 5. *Let G be a bipartite graph. The following statements are equivalent:*

- (i) G is 12-representable;
- (ii) there is a $\{J_4, Q_4\}$ -free labeling of G ;
- (iii) G is an interval containment bigraph.

Proof. The implications (i) \implies (ii) and (ii) \implies (iii) follow from Theorems 1 and 4, respectively. To prove (iii) \implies (i), we construct a labeling and a 12-representant of an interval containment bigraph. See Example 6 for an instance of construction.

Let G be an interval containment bigraph with bipartition (X, Y) such that there is an interval I_v for each $v \in V(G)$ and $xy \in E(G) \iff I_x \supseteq I_y$ for any $x \in X$ and $y \in Y$. As stated in [9], it is possible to choose intervals so that all endpoints are distinct. Thus, without loss of generality, we can assume that all endpoints are distinct. Let ℓ_v and r_v denote the left and right endpoint of the interval I_v , respectively. We assign a label i to a vertex $v \in V(G)$ if ℓ_v is the i th point among all left endpoints from left to right.

Let π_r be a permutation of $[n]$ such that the i th letter of π_r is the label of a vertex v if r_v is the i th point among all right endpoints from left to right. Let π_x and π_y be arbitrary permutations of the labels of vertices of X and Y , respectively.

We claim that $w = \pi_y \pi_r \pi_x$ is a 12-representant of G . Let u and v be two vertices of G with labels i and j , respectively. Without loss of generality, we assume $i < j$, that is, $\ell_u < \ell_v$. If $u, v \in X$ then $w_{\{i,j\}}$ has a 12-match since both π_r and π_x contain i and j . Similarly, if $u, v \in Y$ then $w_{\{i,j\}}$ has a 12-match since both π_y and π_r contain i and j . Suppose $u \in X$ and $v \in Y$. If $r_u > r_v$ then $w_{\{i,j\}} = jjii$, and if $r_u < r_v$ then $w_{\{i,j\}} = jiji$. Thus, $w_{\{i,j\}}$ has no 12-match if and only if I_u contains I_v . If $u \in Y$ and $v \in X$, then $w_{\{i,j\}}$ has a 12-match since π_y is to the left of π_x , which is consistent with the fact that I_v does not contain I_u . ■

Example 6. The graph G_1 in Figure 2(a) is an interval containment bigraph. The vertices are labeled based on the left endpoints of the intervals in Figure 2(b). By reading the labels of the right endpoints from left to right, we obtain the permutation $\pi_r = 53284761$. Let $\pi_x = 1246$ and $\pi_y = 3578$. It is straightforward to check that the word $w = 3578.53284761.1246$ is a 12-representant of G_1 (the dots are not part of the word, they are only included as delimiters of the word parts as constructed in the proof of Theorem 5).

Recall that the class of interval containment bigraphs coincides with the class of bipartite graphs whose complements are circular-arc graphs [5] and the class of two-directional orthogonal ray graphs [17]. As stated in [5, 17], Trotter and Moore [23] provide the list of minimal forbidden induced subgraphs for bipartite

graphs whose complements are circular-arc graphs. Therefore, Theorem 5 provides a forbidden induced subgraph characterization of 12-representable bipartite graphs. See [17] for figures of the forbidden subgraphs.

From the list of forbidden induced subgraphs for 12-representable bipartite graphs, we also have a characterization of 12-representable grid graphs.

Corollary 7. *A grid graph is 12-representable if it contains no cycle of length $2n$ for $n \geq 4$ and no graph in Figure 6 as an induced subgraph.*

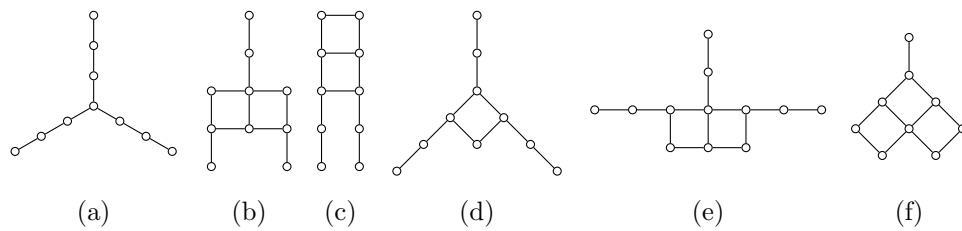


Figure 6. Forbidden induced subgraphs of 12-representable grid graphs.

Proof. It is easy to verify that the other graphs in the list of forbidden induced subgraphs for 12-representable bipartite graphs (see [17] for figures) are not induced subgraphs of a rectangular grid graph. ■

Remark 8. Chen and Kitaev [2] presented certain non-12-representable graphs and conjectured that these graphs would give us a forbidden induced subgraph characterization of 12-representable line grid graphs, see [2, Conjecture 3.6 and Figure 3.26]. Corollary 7 indicates that the graphs in [2, Conjecture 3.6] are not sufficient to characterize 12-representable line grid graphs. For example, the graph in Figure 6(b) is a proper induced subgraph of G_i , $i \in \{3, 4, 5\}$ in [2, Figure 3.26] and the graph in Figure 6(c) is a proper induced subgraph of G_6 in [2, Figure 3.26].

Interval containment bigraphs can be recognized in $O(n^2)$ time [17] because their complements (i.e., circular-arc graphs that can be partitioned into two cliques) can be recognized in $O(n^2)$ time [4], [18, Section 13.3]. Thus, Theorem 5 yields the following.

Corollary 9. *12-representable bipartite graphs can be recognized in $O(n^2)$ time.*

A graph G is a *circular-arc graph* if there is a circular arc A_v on a circle for each vertex $v \in V(G)$ such that for any $u, v \in V(G)$, $uv \in E(G)$ if and only if A_u intersects A_v . The set $\{A_v : v \in V(G)\}$ is called a *model* or *representation* of G . If the given bipartite graph G is the complement of a circular-arc graph, the recognition algorithm [4], [18, Section 13.3] provides a model of \overline{G} . The model can

be easily transformed into a model of interval containment bigraphs [9]. Thus, we have the following from Theorem 5.

Corollary 10. *A 12-representant of a bipartite graph can be obtained in $O(n^2)$ time if the graph is 12-representable.*

4. SIMPLE-TRIANGLE GRAPHS

This section shows the equivalence of 12-representable graphs and complements of simple-triangle graphs and its consequences.

Theorem 11. *Let G be a graph. The following statements are equivalent:*

- (i) G is 12-representable;
- (ii) there is an $\{I_3, J_4, Q_4\}$ -free labeling of G ;
- (iii) the complement \overline{G} of G is a simple-triangle graph.

Proof. The implications (i) \implies (ii) and (ii) \implies (iii) follow from Theorems 1 and 3, respectively. To prove (iii) \implies (i), we construct a labeling and a 12-representant of the complement of a simple-triangle graph. See Example 12 for an instance of construction.

Recall that L_1 and L_2 are two horizontal lines in the plane with L_1 above L_2 . Let G be a simple-triangle graph such that there is a triangle T_v for each $v \in V(G)$ and $uv \in E(G) \iff T_u \cap T_v \neq \emptyset$ for any $u, v \in V(G)$. Without loss of generality, we can assume that the endpoints of the triangles are distinct. Let p_v and I_v be the point on L_1 and the interval on L_2 of T_v , respectively. We assign a label i to a vertex $v \in V(G)$ if p_v is the i th point on L_1 from left to right.

We form a word w using the endpoints of the intervals on L_2 so that the i th letter of w is the label of a vertex v if, among all endpoints of the intervals (i.e., both left and right endpoints) *from right to left*, the i th endpoint is of I_v . We claim that w is a 12-representant of the complement \overline{G} of G . Let u and v be two vertices of G with labels i and j , respectively. Without loss of generality, we assume $i < j$, that is, $p_u < p_v$. It is easy to see that $w_{\{i,j\}} = jjii$ if and only if I_u lies entirely to the left of I_v . Thus, $uv \in E(\overline{G})$ if and only if $w_{\{i,j\}}$ has no 12-match. ■

Example 12. The graph G_2 in Figure 3(a) is a simple-triangle graph. The vertices are labeled based on the points on L_1 in Figure 3(b). By reading the labels of the endpoints on L_2 *from right to left*, we obtain the word $w = 464365235121$. It is straightforward to check that the word w is a 12-representant of the complement $\overline{G_2}$ of G_2 .

By Theorems 3 and 11, we have the following.

Corollary 13. *From an $\{I_3, J_4, Q_4\}$ -free labeling of a 12-representable graph G , a 12-representant of G can be obtained in $O(n^2)$ time without relabeling of G .*

Proof. An $\{I_3, J_4, Q_4\}$ -free labeling of a graph G can be viewed as a vertex ordering σ , which does not contain any pattern in Figure 4. Thus, by Theorem 3, we can obtain a model of the complement \overline{G} of G in $O(n^2)$ time such that σ coincides with the ordering of the points on L_1 . A 12-representant of G can be obtained from the model, as shown in the proof of Theorem 11. ■

Theorem 11 also yields the following, since simple-triangle graphs can be recognized in $O(nm)$ time [20] and the complement of a graph can be obtained in $O(n^2)$ time.

Corollary 14. *12-representable graphs can be recognized in $O(n(\bar{m} + n))$ time, where \bar{m} is the number of edges of the complement of the given graph.*

The recognition algorithm [20] provides a vertex ordering which does not contain any pattern in Figure 4, and we have the following from Corollary 13.

Corollary 15. *A 12-representant of a graph can be obtained in $O(n(\bar{m} + n))$ time if the graph is 12-representable.*

5. CONCLUDING REMARKS

The 12-representants constructed in the proof of Theorems 5 and 11 are of length $2n$, but they are not necessarily optimal (shortest possible). Indeed, for example, as shown in [2, Theorem 2.18], the graph G_1 in Figure 2(a) can be 12-represented by a word of length $n + 1$ (the labeling used in [2, Theorem 2.18] is different from that shown in Figure 2(a)). It is still an open question to improve the upper bound of the length of 12-representants of graphs.

Section 3 gives a forbidden induced subgraph characterization for 12-representable bipartite graphs and grid graphs from the equivalence between 12-representable bipartite graphs and interval containment bigraphs. Although the characterization has been known for interval containment bigraphs, no such characterization is known for simple-triangle graphs [20]. Thus, it is still an open question to characterize the class of 12-representable graphs in terms of forbidden induced subgraphs.

In this paper, we obtained some results on 12-representable graphs from the known facts on interval containment bigraphs and simple-triangle graphs. Studying these graphs via 12-representability is a possible direction for further research.

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