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A STUDY OF A COMBINATION OF DISTANCE DOMINATION AND RESOLVABILITY IN GRAPHS

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Abstract

For $k \geq 1$, in a graph $G = (V, E)$, a set of vertices D is a distance k-dominating set of G, if any vertex in $V \setminus D$ is at distance at most k from some vertex in D. The minimum cardinality of a distance k-dominating set of G is the distance k-domination number, denoted by $\gamma_k(G)$. An ordered set of vertices $W = \{w_1, w_2, \ldots, w_r\}$ is a resolving set of G, if for any two distinct vertices x and y in $V \setminus W$, there exists $1 \leq i \leq r$ such that

$d_G(x, w_i) \neq d_G(y, w_i)$. The minimum cardinality of a resolving set of G is the metric dimension of the graph G, denoted by $\dim(G)$. In this paper, we introduce the distance k-resolving dominating set which is a subset of V that is both a distance k-dominating set and a resolving set of G. The minimum cardinality of a distance k-resolving dominating set of G is called the distance k-resolving domination number and is denoted by $\gamma_k^r(G)$. We give several bounds for $\gamma_k^r(G)$, some in terms of the metric dimension $\dim(G)$ and the distance k-domination number $\gamma_k(G)$. We determine $\gamma_k^r(G)$ when G is a path or a cycle. Afterwards, we characterize the connected graphs of order n having $\gamma_k^r(G)$ equal to 1, $n-2$, and $n-1$, for $k \geq 2$. Then, we construct graphs realizing all the possible triples $(\dim(G), \gamma_k(G), \gamma_k^r(G))$, for all $k \geq 2$. Later, we determine the maximum order of a graph G having distance k-resolving domination number $\gamma_k^r(G) = \gamma_k^r \geq 1$, we provide graphs achieving this maximum order for any positive integers k and γ_k^r. Then, we establish Nordhaus-Gaddum bounds for $\gamma_k^r(G)$, for $k \geq 2$. Finally, we give relations between $\gamma_k^r(G)$ and the k-truncated metric dimension of graphs and give some directions for future work.

Keywords: resolving set, metric dimension, distance k-domination, distance k-resolving domination.

2020 Mathematics Subject Classification: 05C12, 05C69.

1. Introduction

In this paper, we study finite, simple, and undirected graphs. For graph terminology, we refer to [9].

In 1975, Meir and Moon [28] studied a combination of two concepts distance and domination in graphs. For $k \geq 1$, we call a *distance k-dominating set* in a graph $G = (V, E)$, a subset D of the vertex set V such that for any vertex $v \in V \setminus D$, we have $d_G(v, D) = \min\{d_G(v, x) : x \in D\} \leq k$, where $d_G(v, x)$ is the distance in G between the vertex v and x. The minimum cardinality overall distance k-dominating sets of G, is the *distance k-domination number* and is denoted by $\gamma_k(G)$. When $k = 1$, the distance 1-domination number is the well-known domination number of the graph denoted by $\gamma(G)$. Distance k-dominating sets find multiple applications in problems involving graphs like communication networks [31], geometric problems [26], facility location problems [19]. Results about this well-studied concept can be found surveyed in a recent book chapter [18].

Another concept associated with distance in graphs is resolvability and the metric dimension of graphs, introduced by Harary and Melter [17] and Slater [30]. Let $W = \{w_1, w_2, \ldots, w_r\}$ be an ordered set of vertices in a graph G, the *metric representation* of v with respect to W is the r-vector $c(v|W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_r))$. The set W is a *resolving set* of G, if for every two distinct vertices $v, u \in V \setminus W$, $c(v|W) \neq c(u|W)$. The minimum cardinality of a resolving set of G is the *metric dimension* of G, and is denoted $\dim(G)$. Due to their important role from both a theoretical and a practical point of view, resolving sets and the metric dimension of graphs attracted attention these past years (see surveys [2, 32]). Resolving sets find many applications in several areas like network verification [3], robot navigation [25], pharmaceutical chemistry [8], coin weighing problems, Mastermind game (see references in [6, 23]) and more.

The problems of finding $\gamma_k(G)$ and $\dim(G)$ are both NP-Hard problems in general, see respectively [7] and [25].

To join the utility of resolving sets and distance k-dominating sets, we study a set satisfying the two properties.

Definition 1.1. A distance k-resolving dominating set is a set $S \subseteq V$, where S is both a resolving set and a distance k-dominating set of G. The distance k-resolving domination number, denoted by $\gamma_k^r(G)$, is the minimum cardinality of a distance k-resolving dominating set of G, i.e., $\gamma_k^r(G) = \min\{|S| : S$ is a distance k-resolving dominating set of $G\}$.

A situation where the uses of resolving sets and distance k-dominating sets are both needed could represent a possible application of distance k-resolving dominating sets. For example, consider a network of vehicles. We want to identify the position of each vehicle, where the detection range within the network is not limited, and every position must be within a specific distance of a station that provides a service, such as energy supply or maintenance. The distance k-resolving dominating sets are required.

Resolving sets that satisfy additional properties are known and studied. For example, independent resolving set [11], is a resolving set that is also an independent set. Connected resolving set [29], is a resolving set that is also a connected set. For $k = 1$, the distance 1-resolving dominating set is a resolving set that is also a dominating set, the minimum cardinality of such set was first studied under the name of resolving domination number in [4], while it appeared as metric-location-domination number in [20]. More studies were done about that case relating it with other graph parameters, see for example [5, 16, 22]. Here we use the name resolving domination number and denote by $\gamma^r(G)$.

For $k \geq 1$ and $v, u \in V$, let $d_k(v, u) = \min\{d_G(v, u), k + 1\}$. A variation of the metric dimension that could be related to $\gamma_k^r(G)$ is the *k-truncated metric dimension*, $\dim_k(G)$, defined as the minimum cardinality of a *k-truncated resolving set* of G, which is a set $W \subseteq V$ verifying for any two distinct vertices $v, u \in V$, there exists a vertex x in W such that $d_k(v, x) \neq d_k(u, x)$. The k-truncated metric dimension was first studied when $k = 1$ in [24], also called adjacency dimension, where it was used to investigate the metric dimension of lexicographic product of graphs. For $k \geq 1$, the k-truncated metric dimension coincides with the $(1, k+1)$-metric dimension of graphs in [13]. Results on $\dim_k(G)$ can be found in [14, 15, 33].

In Section 2, we give sharp bounds for $\gamma_k^r(G)$ in terms of the metric dimension, the distance k-domination number, the order, the diameter, the radius, and the girth of the graph. Also, we give the distance k-resolving domination number of the families of paths and cycles. In Section 3, for all $k \geq 1$, we show that $\gamma_k^r(G)$ is equal to 1 if and only if G is a path of order at most $k+1$. For $k \geq 2$, we show an equivalence between $\gamma_k^r(G)$ and $\dim(G)$, which we use to characterize all graphs of order n having $\gamma_k^r(G)$ equal to $n-1$ and $n-2$. In Section 4, we determine all the realizable triples of positive integers (β, γ, α) by a graph G having $\dim(G) = \beta$, $\gamma_k(G) = \gamma$, and $\gamma_k^r(G) = \alpha$ when $k \geq 2$, in particular the graphs we construct realizing these values are all trees. In Section 5, for all $k \geq 1$, we show that a graph G having distance k-resolving domination number $\gamma_k^r(G) = \gamma_k^r \geq 1$, has a maximum order of $\gamma_k^r + \gamma_k^r \sum_{p=1}^{k} (2p+1)^{\gamma_k^r - 1}$. Also, we construct graphs attaining this maximum order for any arbitrary positive integers k and γ_k^r. Section 6 is devoted to Nordhaus-Gaddum bounds for the distance k-resolving domination number of graphs for $k \geq 2$. Finally, in Section 7, we discuss relations between $\gamma_k^r(G)$ and $\dim_k(G)$, we then conclude with some open questions.

2. Preliminary Results and Bounds for $\gamma_k^r(G)$

Every superset of a distance k-dominating set is a distance k-dominating set. It is true also for resolving sets. This means that every superset of a distance k-resolving dominating set is also a distance k-resolving dominating set. We give the following bounds that extend bounds given for k equal to 1 and 2, in [5] and [34] respectively to all $k \geq 1$.

Proposition 2.1. *Let G be a connected graph of order $n \geq 2$. For $k \geq 1$, we have*

$$\max\{\gamma_k(G), \dim(G)\} \leq \gamma_k^r(G) \leq \min\{\gamma_k(G) + \dim(G), n-1\}.$$

Proof. Let S be a minimum distance k-resolving dominating set of G. Since S is both a resolving set and a distance k-dominating set, then $\dim(G) \leq |S|$, and $\gamma_k(G) \leq |S|$. Thus $\max\{\gamma_k(G), \dim(G)\} \leq \gamma_k^r(G)$.

Let D and W be respectively a minimum distance k-dominating set and a minimum resolving set of G. The set $S = D \cup W$ is a distance k-resolving dominating set of cardinality $|S| = \gamma_k(G) + \dim(G)$. Also, any subset of V of cardinality $n-1$ is both a resolving set and a distance k-dominating set. Then we have $\gamma_k^r(G) \leq \min\{\gamma_k(G) + \dim(G), n-1\}$. ■

For any two positive integer k and k' such that $k \geq k' \geq 1$, every distance k'-dominating set is a distance k-dominating set. Therefore any distance k'-resolving dominating set is a distance k-resolving dominating set.

Observation 2.2. *For $k \geq k' \geq 1$, if G is a connected graph, then we have* $\dim(G) \leq \gamma_k^r(G) \leq \gamma_{k'}^r(G) \leq \gamma^r(G)$.

The *eccentricity* of a vertex v in G is the maximum distance between v and any other vertex in G. The maximum and minimum eccentricity in G are respectively the *diameter* and the *radius* of G denoted respectively $diam(G)$ and $rad(G)$.

Lemma 2.3. *Let G be a connected graph. For $k \geq diam(G)$, $\gamma_k^r(G) = \dim(G)$.*

Proof. If $k \geq diam(G)$, then any non-empty set of vertices in V is a distance k-dominating set. Hence any resolving set is also a distance k-dominating set of G. Therefore, $\gamma_k^r(G) \leq \dim(G)$. From Proposition 2.1 it follows that $\gamma_k^r(G) = \dim(G)$. ∎

Let P_n denote the path graph with $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{i(i+1) : 1 \leq i \leq n-1\}$. It is proved that $\dim(P_n) = 1$ [8], and for $k \geq 1$, $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$ [12]. The values of the distance k-resolving domination number of P_n for k equal to 1 and 2 are given respectively in [4] and [34]. In the following we give $\gamma_k^r(P_n)$ for all $k \geq 1$.

Proposition 2.4. *For $k \geq 1$ and $n \geq 2$,*

$$\gamma_k^r(P_n) = \begin{cases} 1, & \text{if } k \geq n-1, \\ 2, & \text{if } \lfloor \frac{n}{2} \rfloor \leq k \leq n-2, \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1. \end{cases}$$

Proof. In [8], we have $\dim(P_n) = 1$. So by Proposition 2.1, $\gamma_k(P_n) \leq \gamma_k^r(P_n) \leq \gamma_k(P_n) + 1$. Also, for $1 \leq i, j \leq n$, with $i \neq j$, we have $d_{P_n}(i, j) = |i - j|$. Then any resolving set of cardinality 1 must be $\{1\}$ or $\{n\}$.

- For $k \geq n-1$, since $diam(P_n) = n-1$, it follows from Lemma 2.3 that $\gamma_k^r(P_n) = \dim(P_n) = 1$.
- For $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$, based on [12] $\gamma_k(P_n) = 1$, then $\gamma_k^r(P_n)$ is equal to 1 or 2. It is clear that an end-vertex is not distance k-dominating. Thus, $\gamma_k^r(P_n) = \gamma_k(P_n) + 1 = 2$.
- For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, in [12] we have $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil \geq 2$. Also, any set S consisting of two or more distinct vertices in $V(P_n)$ is a resolving set of P_n. Thus, $\gamma_k^r(P_n) = \gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$. ∎

The path P_n is a graph achieving the bounds in Proposition 2.1. For $k \geq n-1$, we have $\gamma_k^r(P_n) = \dim(P_n)$. For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, $\gamma_k^r(P_n) = \gamma_k(P_n)$, and for $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$, $\gamma_k^r(P_n) = \gamma_k(P_n) + \dim(P_n)$.

Let C_n denote the cycle graph with $n \geq 3$, where $V(C_n) = \{0, 1, \ldots, n-1\}$ and $E(C_n) = \{i(i+1) \pmod{n} : 0 \leq i \leq n-1\}$. We have $\dim(C_n) = 2$ [10], and for $k \geq 1$, $\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$ [12].

Proposition 2.5. *For $k \geq 1$ and $n \geq 3$,*

$$\gamma_k^r(C_n) = \begin{cases} 2, & \text{if } 4k+1 \geq n, \\ 3, & \text{if } 4k+2 = n, \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } 4k+3 \leq n. \end{cases}$$

Proof. We have $d_{C_n}(i,j) = \min\{|i-j|, n-|i-j|\}$.

Claim 2.6. *For $n \geq 2k+2$ and $n \neq 4k+2$, the set of vertices $W = \{0, 2k+1\}$ is a resolving set of C_n.*

Proof. Let $i, j \in V(C_n) \setminus W$, with $i \neq j$. If $d_{C_n}(i,0) \neq d_{C_n}(j,0)$, then S is a resolving set. We suppose that $d_{C_n}(i,0) = d_{C_n}(j,0)$, then either $d_{C_n}(i,0) = i$ and $d_{C_n}(j,0) = n-j$ or $d_{C_n}(i,0) = n-i$ and $d_{C_n}(j,0) = j$. Without loss of generality we suppose that $d_{C_n}(i,0) = i$ and $d_{C_n}(j,0) = n-j$, which means that $i+j = n$. If $d_{C_n}(i, 2k+1) = d_{C_n}(j, 2k+1)$, then $\min\{|2k+1-i|, n-|2k+1-i|\} = \min\{|2k+1-j|, n-|2k+1-j|\}$. Since $\min\{x,y\} = \frac{x+y-|x-y|}{2}$, it follows that $|n-2(|2k+1-i|)| = |n-2(|2k+1-j|)|$.

We suppose that $n - 2|2k+1-i| = n - 2|2k+1-j|$, which means that $|2k+1-i| = |2k+1-j|$. Since $i \neq j$, necessarly $2k+1-i = j-2k-1$. It follows that $i+j = 4k+2 = n$, a contradiction since $n \neq 4k+2$.

Otherwise if $n - 2|2k+1-i| = 2|2k+1-j| - n$, then $n = |2k+1-i| + |2k+1-j|$. If $|2k+1-i| = 2k+1-i$ and $|2k+1-j| = 2k+1-j$, then $n = 2k+1-i+2k+1-j$. Assuming that $i+j = n$, it means that $n = 2k+1$, a contradiction.

Now if $|2k+1-i| = i-(2k+1)$ and $|2k+1-j| = j-(2k+1)$, then $n = i+j-2(2k+1)$. Since $i+j = n$, it means that $k = 0$, a contradiction.

Finally if $|2k+1-i| = i-(2k+1)$ or $|2k+1-j| = j-(2k+1)$, we suppose that $|2k+1-i| = i-(2k+1)$ and $|2k+1-j| = 2k+1-j$. Then we get that $n = i-j$, again a contradiction.

It follows that $d_{C_n}(i, 2k+1) \neq d_{C_n}(j, 2k+1)$. So for $i, j \in V(C_n) \setminus W$, if $i \neq j$, then $c(i|W) \neq c(j|W)$. $\square$

• If $2k+1 \geq n$, then $k \geq diam(C_n)$. By Lemma 2.3, $\gamma_k^r(C_n) = \dim(C_n)$. Since $\dim(C_n) = 2$, we have $\gamma_k^r(C_n) = 2$.

If $4k+1 \geq n \geq 2k+2$, we have $\gamma_k^r(C_n) \geq \dim(C_n) = 2$. From Claim 2.6, the set $\{0, 2k+1\}$ is a resolving set of C_n, it is also a distance k-dominating set of C_n for $4k+1 \geq n \geq 2k+2$. Therefore $\gamma_k^r(C_n) = 2$.

• If $4k+2 = n$, based on [12] we have $\gamma_k(C_{4k+2}) = 2$, then by Proposition 2.1, $\gamma_k^r(C_{4k+2}) \geq 2$. By using contradiction we suppose that $\gamma_k^r(C_{4k+2}) = 2$, and let

S be a distance k-resolving dominating set of cardinality 2. Since all the vertices have degree 2, if a vertex i is in a distance k-dominating set of cardinality 2, then the set contains necessarily $i + 2k + 1 \pmod n$. Since the cycle C_n is vertex-transitive, we suppose without loss of generality that $S = \{0, 2k + 1\}$. If we take the vertices 1 and $4k + 1$, then clearly $c(1|S) = c(4k + 1|S)$. It follows that S is not a resolving set of C_{4k+2}. Hence $\gamma_k^r(C_{4k+2}) > 2$.

Now, let us consider the set $S = \{0, 1, 2k+1\}$, we will show first that $\{0, 1\} \subset S$ is a resolving set of C_{4k+2}. For $i \in V(C_n) \setminus S$, we have $c(i|\{0, 1\}) = (\min\{i, n-i\}, \min\{i-1, n-i+1\})$. For $i, j \in V(C_n) \setminus S$, if $c(i|\{0, 1\}) = c(j|\{0, 1\})$, it means that $\min\{i, n-i\} = \min\{j, n-j\}$ and $\min\{i-1, n-i+1\} = \min\{j-1, n-j+1\}$. Since $\min\{x, y\} = \frac{x+y-|x-y|}{2}$, it follows that $|n - 2i| = |n - 2j|$ and $|n - 2(i - 1)| = |n - 2(j - 1)|$. Assuming that $i \neq j$, then necessarily $n - 2i = 2j - n$ and $n - 2(i - 1) = 2(j - 1) - n$, which is impossible. Then if $i \neq j$, we have $c(i|\{0, 1\}) \neq c(j|\{0, 1\})$. Therefore $\{0, 1\}$ is a resolving set of C_{4k+2}.

Since $\{0, 2k + 1\}$ is a distance k-dominating set of C_{4k+2}, it follows that $S = \{0, 1, 2k + 1\}$ is a distance k-resolving dominating set of C_{4k+2}. Therefore $\gamma_k^r(C_{4k+2}) = 3$.

- If $4k + 3 \leq n$, in [12] we have $\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$. Let us consider the set $S = \{i(2k+1) : 0 \leq i \leq \lceil \frac{n}{2k+1} \rceil - 1\}$, we have $|S| = \lceil \frac{n}{2k+1} \rceil$. Claim 2.6 shows that the set $\{0, 2k + 1\} \subset S$ is a resolving set of C_n. Also, it is easy to see that the set S is a distance k-dominating set of C_n. It follows that $\gamma_k^r(C_n) = \lceil \frac{n}{2k+1} \rceil$. ■

Proposition 2.7. *For $k \geq 1$, let G be a connected graph such that $rad(G) \leq k$ or $diam(G) = k + 1$. Then we have* $\dim(G) \leq \gamma_k^r(G) \leq \dim(G) + 1$.

Proof. Let G be a connected graph with $rad(G) \leq k$. This means that $\gamma_k(G) = 1$. Then by Proposition 2.1, we have $\dim(G) \leq \gamma_k^r(G) \leq \dim(G) + 1$.

If $diam(G) = k + 1$, let $W \subset V$ be a minimum resolving set of G. Let $v \in V \setminus dom_k(W)$, where $dom_k(W) = \{v \in V : d_G(v, W) \leq k\}$. Then v must be at distance greater or equal to $k+1$ from all the vertices of W. Since $diam(G) = k+1$, the only possible metric representation with respect to W of a vertex v such that $d_G(v, W) \geq k + 1$, is a vector having $k + 1$ as a value in all its coordinates. Since W is a resolving set, then there is at most one such vertex in G. Hence, $\dim(G) \leq \gamma_k^r(G) \leq \dim(G) + 1$. ■

For all $k \geq 1$ both bounds in Proposition 2.7 can be achieved. For $\lfloor \frac{n}{2} \rfloor \leq k \leq n - 2$, the graph P_n has $rad(P_n) \leq k$, from Proposition 2.4, $\gamma_k^r(P_n) = \dim(P_n)+1$. From Lemma 2.3, if $rad(G) \leq diam(G) \leq k$, then for any G we have $\gamma_k^r(G) = \dim(G)$. The cycle graphs C_{2k+2} or C_{2k+3} according to Proposition 2.5 are examples of graphs with $diam(G) = k + 1$ having $\gamma_k^r(G) = \dim(G)$. Also from Proposition 2.4, the path P_{k+2} is a graph of $diam(G) = k + 1$ having $\gamma_k^r(G) = \dim(G) + 1$.

Lemma 2.8 [21]**.** *For $k \geq 1$, let G be a connected graph of order $n \geq k+1$ and diameter $diam(G) \geq k$. Then there exists a minimum distance k-dominating set D of G satisfying for every vertex $v \in D$ there is a vertex $x \in V \setminus D$ such that $d_G(v,x) = k$ and $N_k(x) \cap D = \{v\}$.*

The following upper bound proved for $\dim(G)$ in [4] is true also for $\gamma_k^r(G)$, the proofs are similar.

Proposition 2.9. *For $k \geq 1$, let G be a connected graph of order $n \geq k+1$ with $diam(G) \geq k$. Then $\gamma_k^r(G) \leq n - k\gamma_k(G)$, and this upper bound is achieved for any positive integers k and $\gamma_k(G)$.*

Proof. Suppose that $\gamma_k(G) = \gamma$. Based on Lemma 2.8, let us consider $D = \{1, 2, \ldots, \gamma\}$ a minimum distance k-dominating set such that for all $1 \leq i \leq \gamma$, there exists a vertex $w_{i,k}$ verifying that $d_G(i, w_{i,k}) = k$, and for $j \neq i$, $d_G(j, w_{i,k}) > k$. Now let $P_i = iw_{i,1}w_{i,2} \cdots w_{i,k}$ be a shortest $(i, w_{i,k})$-path. We can see that for $1 \leq p \leq k$, we have $d_G(i, w_{i,p}) = p$ and $d_G(j, w_{i,p}) > p$. For any two different vertices $w_{i,p}$, $w_{j,q}$, with $1 \leq i, j \leq \gamma$ and $1 \leq p, q \leq k$, we will check the vector of distances with respect to the set D, we discuss the following two cases.

(i) If $i \neq j$, we suppose without loss of generality that $q \geq p$. We have $d_G(i, w_{i,p}) = p$ and $d_G(i, w_{j,q}) \geq q + 1 > p$.

(ii) If $i = j$ and $p \neq q$, we have $d_G(i, w_{i,p}) = p$ and $d_G(i, w_{i,q}) = q \neq p$.

It follows that the set D resolves all the vertices $w_{i,p}$, where $1 \leq i \leq \gamma$, and $1 \leq p \leq k$. Then the set $S = V \setminus \bigcup_{i=1}^{\gamma} \{w_{i,j}\}_{j=1}^{k}$ is both a distance k-dominating set and a resolving set. Hence $\gamma_k^r(G) \leq |S| = |V \setminus \bigcup_{i=1}^{\gamma} \{w_{i,j}\}_{j=1}^{k}| = n - k\gamma = n - k\gamma_k(G)$.

The family of trees $\{T_\gamma : \gamma \geq 1\}$ illustrated as an example in Figure 1 has $\gamma_k^r(T_\gamma) = n - k\gamma$, for $k, \gamma \geq 1$, where $\gamma_k(T_\gamma) = \gamma$. We have any distance k-dominating set in T_γ must contain at least one vertex in each branch $iw_{i,1} \cdots w_{i,k}$, with $1 \leq i \leq \gamma$. Also, the set of vertices $\{1, 2, \ldots, \gamma\}$ is a distance k-dominating set of T_γ. Then clearly $\gamma_k(T_\gamma) = \gamma$. We can check as above that the set of vertices $\{1, 2, \ldots, \gamma\}$ is a resolving set of T_γ. It follows from Proposition 2.1 that it is a minimum distance k-resolving dominating set of T_γ of cardinality $n - k\gamma = n - k\gamma_k(T_\gamma)$. ∎

For a connected graph G of order n and diameter d, we have $\dim(G) \leq n - d$ [8]. The graphs achieving equality are characterized in [23]. This type of bound involving the order and the diameter of the graph was provided for the resolving domination number in [5]. We give a general upper bound for all $k \geq 1$.

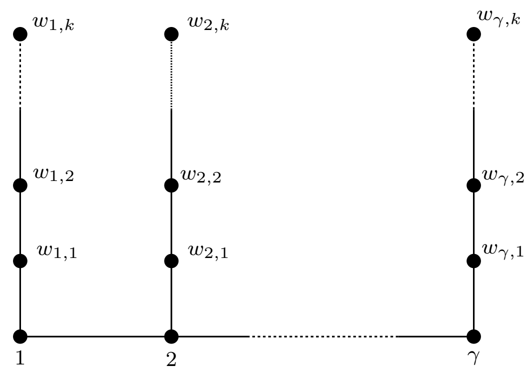

Figure 1. Tree graph T_γ having $\gamma_k^r(T_\gamma) = n - k\gamma_k(T_\gamma)$.

Proposition 2.10. *For $k \geq 1$, let G be a connected graph of order n and diameter d. Then*

$$\gamma_k^r(G) \leq \begin{cases} n-d, & \text{if } d \leq k,\\ n-d+1, & \text{if } k+1 \leq d \leq 2k,\\ n-d+\lfloor \frac{d}{2k+1} \rfloor, & \text{if } d \geq 2k+1. \end{cases}$$

These bounds are sharp.

Proof. Let $P = (0, 1, \ldots, d)$ be a diametral path in G, i.e., P is a shortest path of length d. For any two vertices i and j in P, we have $d_G(i, j) = |i - j|$.

If $d \leq k$, then by Lemma 2.3, $\gamma_k^r(G) = \dim(G)$. Based on [8], we have $\gamma_k^r(G) \leq n - d$.

If $k+1 \leq d \leq 2k$, we consider the set of vertices $\{k, d\}$. For $0 \leq l, m \leq d-1$, with $l \neq m$, we have $d_G(l, d) = |l - d| \neq |m - d| = d_G(m, d)$. Also, for any $0 \leq l \leq d$, we have $d_G(l, k) = |l - k| \leq k$. This means that the set $\{k, d\}$ is resolving and distance k-dominating of the vertices $i \notin \{k, d\}$. Now, let $S' = V \setminus \{i : i \notin \{k, d\}\}$. Then S' is a distance k-resolving dominating set of G. Hence, $\gamma_k^r(G) \leq |S'| = n - d + 1$.

If $d \geq 2k+1$, let us consider the set of vertices $S = \{k, k+(2k+1), \ldots, k+j(2k+1), \ldots, \min\{k + \lfloor \frac{d}{2k+1} \rfloor (2k+1), d\}\}$. Let l be a vertex in $P \setminus S$. If $\min\{k + \lfloor \frac{d}{2k+1} \rfloor (2k+1), d\} = k + \lfloor \frac{d}{2k+1} \rfloor (2k+1)$, then either $k + \lfloor \frac{d}{2k+1} \rfloor (2k+1) < l \leq d$ or there exists $1 \leq i \leq \lfloor \frac{d}{2k+1} \rfloor$ such that $k + (i-1)(2k+1) < l < k + i(2k+1)$, or $0 \leq l < k$. In all those cases there exists a vertex in S at distance less or equal to k from l. The same can be observed when $\min\{k + \lfloor \frac{d}{2k+1} \rfloor (2k+1), d\} = d$. Furthermore, since $|S| \geq 2$ and for $0 \leq i, j \leq d$, $d_G(i, j) = |i - j|$, it is straightforward that S resolves the vertices in $P \setminus S$.

If we consider the set $S' = V \setminus \{P \setminus S\}$, then S' is a distance k-resolving dominating set of the graph G. Hence, $\gamma_k^r(G) \leq |S'| = n - d + \lfloor \frac{d}{2k+1} \rfloor$.

The graph path P_n has diameter $n-1$. From Proposition 2.4 it is a graph achieving the upper bound $n-d$ for $n \leq k+1$. It achieves the upper bound $n-d+1$ when $k+2 \leq n \leq 2k+1$. The path graph P_n also achieves the upper bound $n-d+\lfloor \frac{d}{2k+1} \rfloor$ when $n \geq 2k+2$. ■

If $k=1$, for a connected graph of diameter $d \geq 3$, the upper bound in Proposition 2.10 is precisely the bound given in terms of the order and the diameter in [5].

The girth of the graph is the length of a shortest cycle in the graph. The following lower bounds proved in [12] for $\gamma_k(G)$ holds also for $\gamma_k^r(G)$ and they are achieved.

Proposition 2.11. *For $k \geq 1$, let G be a connected graph having diameter d, radius r, and girth g. Then we have*

(1) $\gamma_k^r(G) \geq \frac{d+1}{2k+1}$;

(2) $\gamma_k^r(G) \geq \frac{2r}{2k+1}$;

(3) $\gamma_k^r(G) \geq \frac{g}{2k+1}$, *if* $g < \infty$.

These bounds are sharp.

Proof. In [12], it is shown that if G is a connected graph of diameter d, then $\gamma_k(G) \geq \frac{d+1}{2k+1}$. In the same paper we have if G has radius r, then $\gamma_k(G) \geq \frac{2r}{2k+1}$. Also in [12], for a connected graph of girth $g < \infty$, we have $\gamma_k(G) \geq \frac{g}{2k+1}$. Since $\gamma_k^r(G) \geq \gamma_k(G)$, the above lower bounds for $\gamma_k(G)$ are true also for $\gamma_k^r(G)$.

Some graphs in Proposition 2.4 and 2.5 are examples of graphs attaining these bounds. In (1) consider the path graph of order $n = p(2k+1)$ for $p \geq 2$, since $d = n-1$, we get that $\gamma_k^r(G) = \frac{d+1}{2k+1}$. In (2) consider the path graph of order $n = 2p(2k+1)$. We have $r = p(2k+1)$, then from proposition 2.4, $\gamma_k^r(G) = \frac{2r}{2k+1}$. In (3) take a cycle graph of order $n = p(2k+1)$ for $p \geq 3$, since $g = n$, then this is a graph having $\gamma_k^r(G) = \frac{g}{2k+1}$. ■

3. GRAPHS WITH $\gamma_k^r(G)$ EQUAL TO $1, n-2$, AND $n-1$

Further, let K_n denote *the complete graph* on n vertices, and let $K_{s,t}$ with $s, t \geq 1$ denote *the complete bipartite graph.* For two graphs G_1 and G_2 the *disjoint union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The *join graph* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained from $G_1 \cup G_2$ by joining each vertex from $V(G_1)$ to each vertex in $V(G_2)$. We denote by $\overline{G}$ the *complement graph* of G.

Theorem 3.1 [8]**.** *For a connected graph G of order $n \geq 2$, we have the following.*

- $\dim(G) = 1$ *if and only if* $G \cong P_n$.
- *If* $n \geq 4$, *then* $\dim(G) = n-2$ *if and only if* $G \in \{K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$.
- $\dim(G) = n-1$ *if and only if* $G \cong K_n$.

In a connected graph G of order $n \geq k+1$, any subset of V of order greater or equal to $n-k$ is a distance k-dominating set.

Lemma 3.2. *Let* $k \geq 2$. *For* $1 \leq i \leq k$, *if* G *is a connected graph of order* $n \geq i+2$ *that is not a path graph, then* $\gamma_k^r(G) = n-i$ *if and only if* $\dim(G) = n-i$.

Proof. For all $1 \leq i \leq k$, if $\dim(G) = n-i$, any subset of V of cardinality $n-i \geq n-k$ is a distance k-dominating set. Then a resolving set of cardinality $\dim(G) = n-i$ is also a distance k-dominating set. Therefore $\gamma_k^r(G) = \dim(G) = n-i$.

Conversely, if $\gamma_k^r(G) = n-i$, by Proposition 2.1, we have $\dim(G) \leq n-i$. If $n = i+2$, then $\gamma_k^r(G) = n-i = 2$. It follows that $\dim(G)$ is equal to 1 or 2. Based on Theorem 3.1, the only graphs with $\dim(G) = 1$ are path graphs, it follows that $\dim(G) = 2$.

If $n \geq i+3$, we suppose that $\dim(G) < n-i$. If $i \leq k-1$, then a resolving set of cardinality $n-(i+1) \geq n-k$ is also a distance k-dominating set. Thus $\gamma_k^r(G) \leq n-(i+1)$, which is impossible. Now if $i = k$, let $W \subseteq V$ be a resolving set of cardinality $n-(k+1)$, and let us denote $1, 2, \ldots, k+1$ the vertices in $V \setminus W$. Assuming that $\gamma_k^r(G) = n-k$, then there is at least one vertex v in $V \setminus W$ such that $d_G(v, W) = k+1$. Let $w \in W$ be such that $d_G(v, w) = d_G(v, W) = k+1$, and let Q be a shortest (v, w)-path. Since $d_G(v, w) = d_G(v, W)$, and G is a connected graph, the only vertex in $W \cap Q$ is w. We have $|Q| = k+2$ and $|W| = n-(k+1)$, which means that the subgraph induced by the vertices $1, 2, \ldots, k+1$ and w is the path Q. Without loss of generality, we suppose that the path Q is $(k+1)k \cdots 1w$. Now, let $S = (W \setminus \{w\}) \cup \{k\}$. We have $d_G(k, k+1) = d_G(k, k-1) = 1$, $d_G(k, w) = k \geq 2$, and if $k \geq 3$, for $1 \leq j \leq k-2$, we have $d_G(k, j) = k-j \geq 2$. Also $d_G(k+1, S \setminus \{k\}) \geq k+1$, since G is a connected graph and $n \geq k+3$, then there exists a vertex $u \in S \setminus \{k\}$ such that either or both 1 and w are adjacent to u. This means that $d_G(k-1, u) \leq k$. It follows that S is a resolving set of G. Since $d_G(k, i) \leq k$, for $1 \leq i \leq k+1$, $i \neq k$, and $d_G(k, w) = k$, it means that the set S is also a distance k-dominating set of G. Hence $\gamma_k^r(G) \leq |S| = n-(k+1)$, a contradiction. Therefore $\dim(G) = n-k$. ∎

By combining Theorem 3.1 and Lemma 3.2 with Proposition 2.4, we give the following characterizations.

Theorem 3.3. *For any graph* G *of order* $n \geq 2$, *the following statements hold.*

(a) *For all* $k \geq 1$, $\gamma_k^r(G) = 1$ *if and only if* $G \in \{P_i\}_{i=2}^{k+1}$.

(b) *If G is a connected graph of order $n \geq 4$, $\gamma_2^r(G) = n-2$ if and only if $G \in \{P_4, K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$. For all $k \geq 3$, $\gamma_k^r(G) = n-2$ if and only if $G \in \{K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$.*

(c) *If G is a connected graph, for any $k \geq 2$, $\gamma_k^r(G) = n-1$ if and only if $G \cong K_n$.*

Proof. (a) For $k \geq 1$, if $\gamma_k^r(G) = 1$, then G is a connected graph and from Proposition 2.1, $\dim(G) = 1$. The equivalence is completed by Theorem 3.1 and Proposition 2.4.

(b) If G is a connected graph of order $n \geq 4$ different from a path graph, then by Lemma 3.2 we have $\gamma_k^r(G) = n-2$ if and only if $\dim(G) = n-2$. Which means by Theorem 3.1 that it is equivalent to $G \in \{K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$. From Proposition 2.4, we have $\gamma_k^r(P_n) = n-2$, it occurs only when $k = 2$ and $n = 4$. Then $\gamma_2^r(G) = n-2$ if and only if $G \in \{P_4, K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$. Also, for $k \geq 3$, $\gamma_k^r(G) = n-2$ if and only if $G \in \{K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$.

(c) The only connected graphs of order 2 and 3 are respectively K_2 and P_3 or K_3. For $k \geq 2$, from Proposition 2.4 and Theorem 3.1, we have $\gamma_k^r(K_2) = 1$, $\gamma_k^r(P_3) = 1$, and $\gamma_k^r(K_3) = 2$. If G has order $n \geq 4$, then by Lemma 3.2 and Theorem 3.1, we have $\gamma_k^r(G) = n-1$ if and only if $G \cong K_n$. ∎

For $k = 1$, we have $\gamma^r(G) = n-1$ if and only if $G \in \{K_{1,n-1}, K_n\}$ [4, 20]. The graphs having $\gamma^r(G)$ equal to 2 and $n-2$ are fully determined in [5] and [20], respectively.

4. Realizable Values for $\dim(G)$, $\gamma_k(G)$, and $\gamma_k^r(G)$.

In Proposition 2.1, we have $\max\{\gamma_k(G), \dim(G)\} \leq \gamma_k^r(G) \leq \gamma_k(G) + \dim(G)$. For $k = 1$, in [5] it is shown that for any three positive integers β, γ, and α, verifiying that $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$, and $(\beta, \gamma, \alpha) \notin \{(1, \gamma, \gamma+1) : \gamma \geq 2\}$, there is always a graph G having $\dim(G) = \beta$, $\gamma(G) = \gamma$, and $\gamma^r(G) = \alpha$. We give a similar result for $\dim(G)$, $\gamma_k(G)$, and $\gamma_k^r(G)$, for all $k \geq 2$.

The graph families we provide in Theorem 4.2 are all trees. To determine $\gamma_k^r(G)$ of some of these graphs, we will need the next formula for the metric dimension of trees that appeared in [8, 17, 30]. We will recall some terminology given in [8]. In a tree T for $v \in V$, if the degree $deg(v) \geq 3$, then v is called a *major vertex.* A leaf l, i.e., a vertex of degree one, in T is a *terminal vertex* of a major vertex v, if v is the closest major vertex in terms of distance to l, i.e., for u a major vertex in T different from v, we have $d_T(v,l) < d_T(u,l)$. If v is a major vertex having at least one terminal vertex, then v is called an *exterior* *major vertex.* Let $L(T)$ and $EX(T)$ denote respectively the number of leaves and the number of exterior major vertices in a tree T.

Theorem 4.1 [8, 17, 30]**.** *If T is a tree that is not a path graph, then $\dim(T) = L(T) - EX(T)$. Also, any resolving set of T must contain at least one vertex from each branch at an exterior major vertex containing its terminal vertices with at most one exception.*

Theorem 4.2. *For any three positive integers β, γ, and α such that $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$ and $(\beta, \gamma, \alpha) \notin \{(1, \gamma, \gamma + 1) : \gamma \geq 2\}$, and for all $k \geq 2$, there always exists a tree graph T having $\dim(T) = \beta$, $\gamma_k(T) = \gamma$, and $\gamma_k^r(T) = \alpha$. There is no graph realizing the triples $\{(1, \gamma, \gamma + 1) : \gamma \geq 2\}$.*

Proof. Let $\beta, \gamma, \alpha \geq 1$ be such that $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$. We discuss the possible values for the triple $(\dim(G), \gamma_k(G), \gamma_k^r(G)) = (\beta, \gamma, \alpha)$, according to the following cases.

• If $\beta = 1$, then $\gamma \leq \alpha \leq \gamma + 1$. Also by Theorem 3.1 we have the path graphs are the only graphs having the metric dimension equal to 1. For $k \geq 2$, in a path graph any subset of vertices of order greater or equal to 2 is a resolving set. Then if $\gamma \geq 2$, we have $\alpha = \gamma$. This means that the triple $(1, \gamma, \gamma + 1)$ is not realizable by any graph for $\gamma \geq 2$. Also, according to Proposition 2.4 the path graphs realizes the following cases. (i) If $k + 1 \geq n$, then we have $\gamma = \beta = \alpha = 1$. (ii) If $\lfloor \frac{n}{2} \rfloor \leq k \leq n - 2$, then $\gamma = \beta = 1$ and $\alpha = 2 = \gamma + \beta$. (iii) If $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$, then $\beta = 1 < \gamma = \alpha = \lceil \frac{n}{2k+1} \rceil \geq 2$.

• If $\gamma = 1$, for any $\beta \geq 2$, then we have $\beta \leq \alpha \leq \beta + 1$. The star graph $K_{1,\beta+1}$ has $\gamma_k(K_{1,\beta+1}) = 1$, and from Theorem 3.1 and Theorem 3.3 we have $\gamma_k^r(K_{1,\beta+1}) = \dim(K_{1,\beta+1}) = \beta$, for any $k \geq 2$. This means that for $k \geq 2$, the triple $(\beta, 1, \beta)$ is realized for all $\beta \geq 2$. For the case of the triple $(\beta, 1, \beta + 1)$, we consider the spider tree graph, denoted by $S_{\beta+1,k}$, having one vertex v_0 of degree $\beta + 1$ with $\beta + 1$ leaves l_i, $1 \leq i \leq \beta + 1$, at distance k from v_0. Note that all the vertices of $S_{\beta+1,k}$ are of degree less or equal to 2 except v_0. Clearly $\gamma_k(S_{\beta+1,k}) = 1$, and based on Theorem 4.1, we have $\dim(S_{\beta+1,k}) = \beta$. Also any resolving set must contain at least one vertex in all but one of the (v_0, l_i)-paths, where $1 \leq i \leq \beta + 1$. By using contradiction, we suppose that $\gamma_k^r(S_{\beta+1,k}) = \beta$. From Theorem 4.1, we consider that a minimum distance k-resolving dominating set W of $S_{\beta+1,k}$ having cardinality β contains one vertex in any of the (v_0, l_i)-paths, with $1 \leq i \leq \beta$. We have the vertex $l_{\beta+1}$ is at distance greater than k from the vertices in W. This means that W is not a distance k-dominating set, a contradiction. Hence, $\gamma_k^r(S_{\beta+1,k}) = \beta + 1$.

• If $\beta \geq 2$ and $\gamma \geq 2$, with $\max(\gamma, \beta) \leq \alpha \leq \gamma + \beta$, then the realizable values for the triple (β, γ, α) are considered depending on the following five subcases.

(i) If $2 \leq \beta = \gamma < \alpha$, then the trees $T^1 = \{T^1_{k,m,l} : m \geq 0,\ l \geq 1,\ k \geq 2\}$ in Figure 2 illustrate graphs realizing this case.

Claim 4.3. *We have* $\gamma_k(T^1_{k,m,l}) = \dim(T^1_{k,m,l}) = m + l$, *and* $\gamma^r_k(T^1_{k,m,l}) = m + 2l$.

Proof. Suppose that $\gamma_k(T^1_{k,m,l}) = \gamma$, $\dim(T^1_{k,m,l}) = \beta$, and $\gamma^r_k(T^1_{k,m,l}) = \alpha$. It is clear that $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l$ is a minimum distance k-dominating set. Then $\gamma = m + l$. Based on Theorem 4.1, we have $\beta = m + l$, and for each $1 \leq i \leq l$, a resolving set must contain one vertex from the set of vertices $\{v_{i,j}\}_{j=0}^k$. Also, for each $1 \leq i \leq m$, a resolving set must contain one vertex from the set of vertices $\{w_{i,j}, w'_{i,j}\}_{j=0}^k$. Now, let S be a minimum distance k-resolving dominating set of cardinality α. We suppose without loss of generality, that S contain a vertex from each $\{v_{i,j}\}_{j=0}^k$ with $1 \leq i \leq m$, and one vertex from each $\{w_{i,j}\}_{j=0}^k$ with $1 \leq i \leq l$. Since $d_G(w_i, w'_{i,k}) = k$, and for $x \notin \{w_i, w'_{i,j}\}$ we have $d_G(x, w'_{i,k}) > k$. Then to be a distance k-dominating set, S must contain for each $1 \leq i \leq l$, at least w_i or a vertex in $\{w'_{i,j}\}_{j=0}^k$. Hence $\alpha \geq m + 2l$. It is easy to check that the set of vertices $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{w_i\}_{i=1}^l$ is a distance k-resolving dominating set. Thus $\alpha \leq m + 2l$. It follows that $\alpha = m + 2l$. $\square$

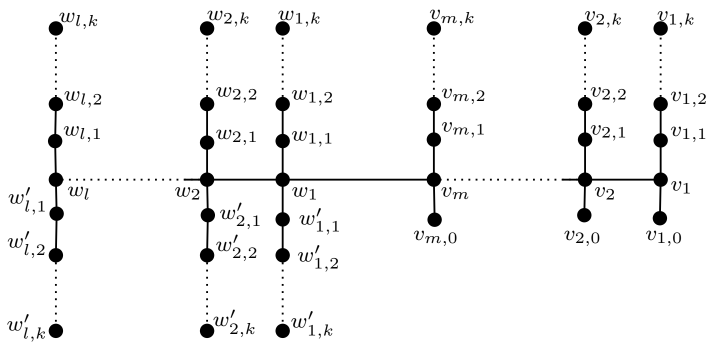

Figure 2. Tree $T^1_{k,m,l}$.

The proofs for the remaining cases use similar arguments as in the proof of Claim 4.3. In the following, we only provide examples of minimum distance k-dominating sets, minimum resolving sets, and minimum distance k-resolving dominating sets for each family of trees.

(ii) If $2 \leq \gamma \leq \beta = \alpha$, then the family of trees $T^2 = \{T^2_{k,m,l} : m \geq 1, l \geq 1, k \geq 2\}$ represented in Figure 3 realizes this case. The set of vertices $\{v_i\}_{i=1}^m \cup \{w\}$ is a minimum distance k-dominating set of cardinality $m+1$. Also, the set of vertices $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,1}\}_{i=1}^l$ is both a minimum resolving set and a minimum distance k-resolving dominating set of cardinality $m + l$.

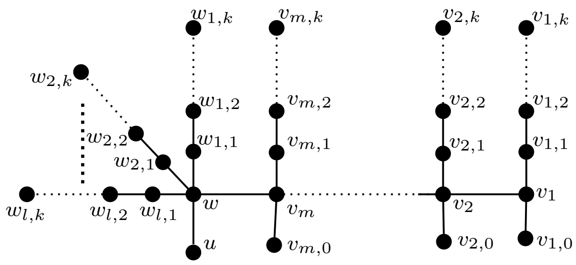

Figure 3. Tree $T^2_{k,m,l}$.

(iii) If $2 \leq \beta < \gamma = \alpha$, then the family of trees $T^3 = \{T^3_{k,m,l} : m \geq 1,\ l \geq 1,\ k \geq 2\}$ represented in Figure 4 realizes this case. From Theorem 4.1, we have $\dim(T^3_{k,m,l}) = m + 1$. Also, the set $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u\}$ is a minimum distance k-dominating set of $T^3_{k,m,l}$ of cardinality $m + l + 1$. Finally, the set $\{v_{i,1}\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u_1\}$ is a distance k-resolving dominating set of cardinality $m + l + 1$. It follows that $\gamma_k^r(T^3_{k,m,l}) = \gamma_k(T^3_{k,m,l}) = m + l + 1$.

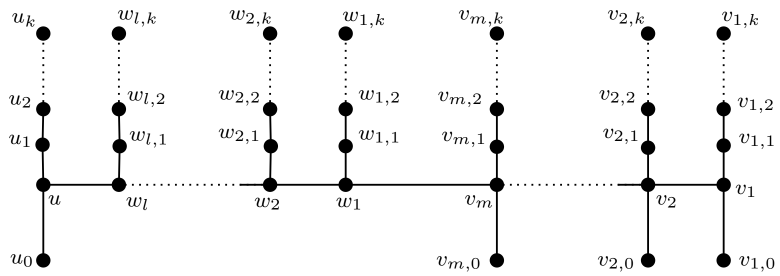

Figure 4. Trees $T^3_{k,m,l}$.

(iv) If $2 \leq \gamma < \beta < \alpha$, then the family of trees $T^4 = \{T^4_{k,m,l,r} : m \geq 0,\ l \geq 0,\ r \geq 3,\ k \geq 2\}$ represented in Figure 5 illustrates graphs realizing this case, where $(m, l) \neq (0, 0)$. The set of vertices $\{v_{i,k}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{u_{i,k}\}_{i=1}^{r-1}$ is a minimum resolving set of cardinality $m+l+r-1$. The set of vertices $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u\}$ is a minimum distance k-dominating set of cardinality $m + l + 1$. The set of vertices $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,k}, w_i\}_{i=1}^l \cup \{u_{i,k}\}_{i=1}^{r-1} \cup \{u\}$ is a minimum distance k-resolving dominating set of cardinality $m + 2l + r$.

(v) If $2 \leq \beta < \gamma < \alpha$, then Figure 6 illustrates a family of trees $T^5 = \{T^5_{k,m,l,r} : m \geq 0,\ l \geq 0,\ r \geq 2,\ k \geq 2\}$ realizing this case, where $(m, l) \neq (0, 0)$. The set of vertices $\{v_{i,k}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{u_{r,k}\}$ is a minimum resolving set of cardinality

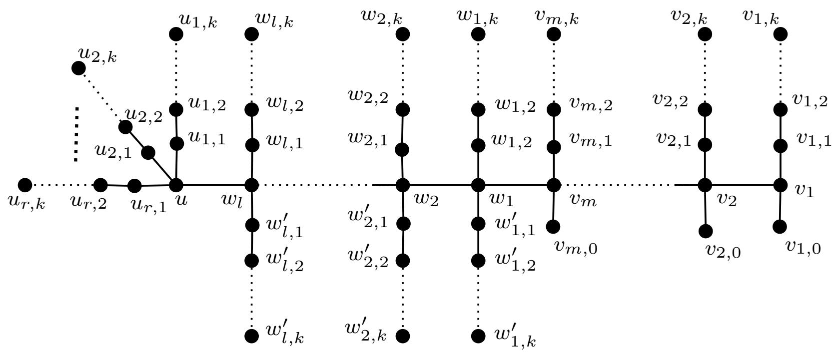

Figure 5. Tree $T^4_{k,m,l,r}$.

$m + l + 1$. The set of vertices $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u_i\}_{i=1}^r$ is a minimum distance k-dominating set of cardinality $m+l+r$. The set of vertices $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{w_i\}_{i=1}^l \cup \{u_i\}_{i=1}^r \cup \{u_{r,k}\}$ is a minimum distance k-resolving dominating set of cardinality $m + 2l + r + 1$.

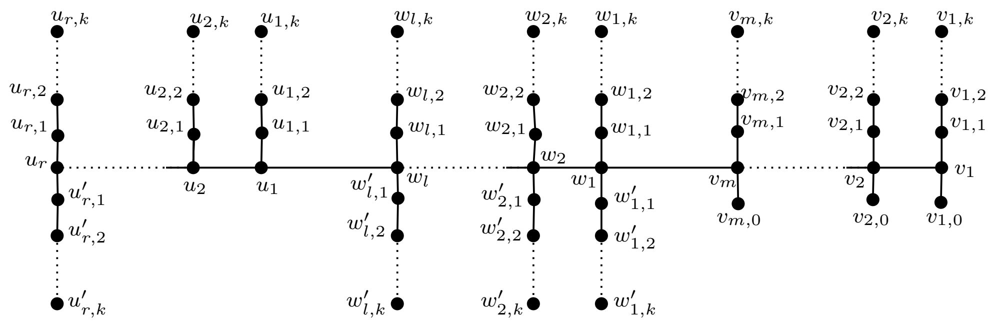

Figure 6. Tree $T^5_{k,m,l,r}$. ∎

5. Maximum Order Graphs

The maximum order n of a graph G having diameter d and metric dimension $\dim(G) = \beta$, was shown to be $\beta + d^\beta$ [8, 25]. This was proved by considering the maximum possible number of distinct metric representations with respect to a minimum resolving set. But this maximum order is only achieved when $d \leq 3$ or $\beta = 1$. Later, Hernando *et al.* [23] proved a stronger result by showing that $n \leq \left(\lfloor \frac{2d}{3} \rfloor + 1\right)^\beta + \beta \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i-1)^{\beta-1}$, where the maximum order is achieved for any arbitrary positive integers d and β.

Cáceres *et al.* [5] showed that for a graph G of order n having $\gamma^r(G) = \gamma^r$, then $n \leq \gamma^r + \gamma^r \cdot 3^{\gamma^r - 1}$. They also provided graphs achieving this maximum order. Next, we generalize this result for $\gamma_k^r(G)$ for all $k \geq 1$.

Theorem 5.1. *For $k \geq 1$, the maximum order of a connected graph G having distance k-resolving domination number γ_k^r is $\gamma_k^r + \gamma_k^r \sum_{p=1}^{k}(2p+1)^{\gamma_k^r - 1}$. This maximum order is achieved for any $k, \gamma_k^r \geq 1$.*

Proof. Let G be a graph of order n and let S be a minimum distance k-resolving dominating set of G. For any vertex $x \in V \setminus S$, let us consider v_i a vertex in S such that $d_G(x, v_i) = p \leq k$. If $\gamma_k^r(G) = \gamma_k^r \geq 2$, for any vertex v_j from S different from v_i, the triangle inequality gives $|d_G(x, v_j) - d_G(v_i, v_j)| \leq d_G(x, v_i) = p$. It follows that the metric representation of x with respect to S has the coordinate corresponding to v_i equal to p and for the other coordinates there are at most $2p+1$ possible values in each of the other $\gamma_k^r - 1$ coordinates. Therefore, there are at most $(2p+1)^{\gamma_k^r - 1}$ possible metric representations of x with respect to the set S. Since $1 \leq p \leq k$, there are at most $\sum_{p=1}^{k}(2p+1)^{\gamma_k^r - 1}$ distinct metric representations for the vertices at distance less or equal to k from v_i. Since $|S| = \gamma_k^r$, we have $n \leq \gamma_k^r + \gamma_k^r \sum_{p=1}^{k}(2p+1)^{\gamma_k^r - 1}$.

Let k and γ_k^r be two arbitrary positive integers, we will prove that there exists a graph having distance k-resolving domination number γ_k^r and order $\gamma_k^r + \gamma_k^r \sum_{p=1}^{k}(2p+1)^{\gamma_k^r - 1}$.

If $\gamma_k^r = 1$, then from Theorem 3.3 the graph G is a path graph of maximum order $k+1$, which coincides with the maximum order bound. If $\gamma_k^r = r \geq 2$, we consider the following subsets of $\mathbb{Z}^r$,

$$\begin{aligned} Q_0 = \ &\{(0, 2k+1, 2k+1, \ldots, 2k+1), (2k+1, 0, 2k+1, \ldots, 2k+1), \ldots, \\ &(2k+1, 2k+1, \ldots, 2k+1, 0)\}. \end{aligned}$$

For all $1 \leq i \leq r$,

$$Q_i = \{(q_1, q_2, \ldots, q_r) : 1 \leq q_i \leq k, \text{ and for } j \neq i,\ 2k - q_i + 1 \leq q_j \leq 2k + q_i + 1\}.$$

Let G_r be the graph whose vertex set is $V(G_r) = \bigcup_{i=0}^{r} Q_i$. For which two vertices $q = (q_1, q_2, \ldots, q_r)$ and $q' = (q'_1, q'_2, \ldots, q'_r)$ are adjacent if and only if $|q_j - q'_j| \leq 1$ for each $1 \leq j \leq r$.

Claim 5.2. *The graph G_r is a connected graph.*

Proof. If $q_{i,0}, q_{j,0} \in Q_0$, where $q_{i,0}$ has the i-th element equal to 0 and $q_{j,0}$ has the j-th element equal to 0, we construct a $(q_{i,0}, q_{j,0})$-path as following,

$$(2k+1,\dots,\underset{i}{0},2k+1,\dots,2k\underset{j}{+}1,\dots,2k+1)(2k+1,\dots,\underset{i}{1},2k+1,\dots,\underset{j}{2k},2k+1,\dots,2k+1)$$
$$(2k+1,\dots,\underset{i}{2},2k+1,\dots,2k\underset{j}{-}1,2k+1,\dots,2k+1)\dots\dots$$
$$(2k+1,\dots,\underset{i}{k},2k+1,\dots,k\underset{j}{+}1,2k+1,\dots,2k+1)(2k+1,\dots,k\underset{i}{+}1,2k+1,\dots,\underset{j}{k},2k+1,\dots,2k+1)$$
$$(2k+1,\dots,k\underset{i}{+}2,2k+1,\dots,k\underset{j}{-}1,2k+1,\dots,2k+1)\dots\dots$$
$$(2k+1,\dots,\underset{i}{2k},2k+1\dots,\underset{j}{1},2k+1\dots,2k+1)(2k+1,\dots,2k\underset{i}{+}1,2k+1,\dots,\underset{j}{0},2k+1,\dots,2k+1).$$

Also, for each $1 \le i \le r$, if $q = (q_1, q_2, \dots, q_r) \in Q_i$, it is easy to see from the definition of the adjacency in G_r, that there is a $(q, q_{i,0})$-path. Hence, the graph G_r is a connected graph. $\square$

For $1 \le i \le r$ and $q \in V(G_r) \setminus Q_0$, we denote $L_i(q) = (f_i(q_1), f_i(q_2), \dots, f_i(q_r))$, where f_i is an integer-valued function defined as following.

If $q = (q_1, q_2, \dots, q_r) \in Q_s$, with $s \neq i$.

- For $j \notin \{s, i\}$, $f_i(q_j) = \begin{cases} q_j, & if\ q_j = 2k+1, \\ q_j - 1, & if\ q_j > 2k+1, \\ q_j + 1, & if\ q_j < 2k+1. \end{cases}$
- $f_i(q_s) = \begin{cases} q_s, & if\ q_s = k, \\ q_s + 1, & if\ q_s < k \text{ or } q_i = k+1. \end{cases}$
- $f_i(q_i) = q_i - 1$.

If $q = (q_1, q_2, \dots, q_r) \in Q_i$.

- For $j \neq i$, $f_i(q_j) = \begin{cases} q_j, & if\ q_j = 2k+1, \\ q_j - 1, & if\ q_j > 2k+1, \\ q_j + 1, & if\ q_j < 2k+1. \end{cases}$
- $f_i(q_i) = q_i - 1$.

For $t \ge 1$, we define $L_i^t(q)$ with $L_i^1(q) = L_i(q)$. For $t \ge 2$, $L_i^t(q) = L_i(L_i^{t-1}(q)) = (f_i^t(q_1), f_i^t(q_2), \dots, f_i^t(q_r))$, where f_i^t is the t-th iterated function of f_i, i.e., $f_i^t = \underbrace{f_i \circ f_i \circ \dots \circ f_i}_{\text{t times}}$.

Claim 5.3. *For all $1 \le i \le r$, for any vertex $q = (q_1, q_2, \dots, q_r) \in V(G_r) \setminus Q_0$, we have $L_i(q) \in V(G_r)$. Also, $L_i(q)$ is adjacent in G_r to q, and $L_i^{q_i}(q) = q_{0,i}$.*

Proof. Let $q = (q_1, q_2, \dots, q_r) \in V(G_r) \setminus Q_0$. For $1 \le i \le r$, we have $L_i(q) = (f_i(q_1), f_i(q_2), \dots, f_i(q_r))$. If $q \in Q_s$, where $s \neq i$, for $j \neq s$, we have $2k - q_s + 1 \le q_j \le 2k + q_s + 1$ and $1 \le q_s \le k$. We discuss the membership of $L_i(q)$ according to the following cases.

(i) If $q_s < k$, then we have $f_i(q_s) = q_s + 1 \le k$, $f_i(q_i) = q_i - 1 \ge 2k - q_s$, and for $j \notin \{i, s\}$, $2k - q_s + 2 \le f_i(q_j) \le 2k + q_s$. So $L_i(q) = (f_i(q_1), f_i(q_2), \dots, f_i(q_r)) \in Q_s$.

(ii) If $q_s = k$ and $q_i > k+1$, then $f_i(q_i) = q_i - 1 \geq k+1$, $f_i(q_s) = k$, and for $j \notin \{i, s\}$, $k+1 \leq f_i(q_j) \leq 3k+1$. So $L_i(q) = (f_i(q_1), f_i(q_2), \ldots, f_i(q_r)) \in Q_s$.

(iii) If $q_i = k+1$, then $q_s = k$. It follows that $f_i(q_i) = k$, $f_i(q_s) = k+1$, and for $j \notin \{i, s\}$, $k+1 \leq f_i(q_j) \leq 3k+1$. Therefore, $L_i(q) \in Q_i$.

Now, if $q \in Q_i$, from the definition of f it is easy to see that $L_i(q) \in Q_i$. Hence, for any vertex $q \in V(G_r) \setminus Q_0$, we have $L_i(q) \in V(G_r)$. Moreover, for $q \in V(G_r) \setminus Q_0$, and all $1 \leq i, j \leq r$, we have $|f_i(q_j) - q_j| \leq 1$, $f_i^{q_i}(q_i) = 0$, and for $j \neq i$, $f_i^{q_i}(q_j) = 2k+1$. Thus, $L_i(q)q \in E(G_r)$, and $L_i^{q_i}(q) = q_{0,i}$. □

Claim 5.4. *For all $1 \leq i \leq r$, for any vertex $q = (q_1, q_2, \ldots, q_r) \in V(G_r) \setminus Q_0$, $d_{G^r}(q, q_{0,i}) = q_i$.*

Proof. Based on Claim 5.3 for $1 \leq i \leq r$, we have $qL_i(q)L_i^2(q) \cdots L_i^{q_i}(q) = q_{0,i}$ is a $(q, q_{0,i})$-path in G_r of length q_i. Hence $d_{G^r}(q, q_{0,i}) \leq q_i$. Since $q_{0,i}$ and q are vertices having respectively 0 and q_i at the i-th coordinate and any two vertices in G_r can be adjacent only if the difference between the respective coordinates is at most 1, it follows that $d_{G^r}(q, q_{0,i}) \geq q_i$. Therefore, $d_{G^r}(q, q_{0,i}) = q_i$. □

From above we can conclude that for any two different vertices q and q' in $V(G_r) \setminus Q_0$, there exists $1 \leq i \leq r$ such that $d_{G^r}(q, q_{0,i}) = q_i \neq d_{G^r}(q', q_{0,i}) = q'_i$. It follows that the set of vertices Q_0 is a resolving set of G_r. Also, for all $1 \leq i \leq r$, and any vertex $q \in Q_i$, $d_{G^r}(q, q_{0,i}) = q_i \leq k$. Hence, the set Q_0 is as well a distance k-dominating set of G_r. Hence, $\gamma_k^r(G_r) \leq |Q_0| = r$.

Suppose that $\gamma_k^r(G_r) \leq r-1$. We have the order of the graph G_r is $|G_r| = r + r\sum_{p=1}^{k}(2p+1)^{r-1}$. Also the maximum order of a graph having $\gamma_k^r(G_r) \leq r-1$ was previously proved to be less or equal to $\gamma_k^r(G_r) + \gamma_k^r(G_r)\sum_{p=1}^{k}(2p+1)^{\gamma_k^r(G_r)-1} \leq (r-1) + (r-1)\sum_{p=1}^{k}(2p+1)^{r-2}$, it is a contradiction. Therefore, $\gamma_k^r(G_r) = r$. ■

For $k = 1$, the maximum order in Theorem 5.1 is precisely the maximum order given in [5].

6. NORDHAUS-GADDUM TYPE BOUNDS

Nordhaus-Gaddum bounds are sharp bounds on the sum or the product of a parameter of a graph G and its complement $\overline{G}$. The survey [1] contains a bibliography of these types of bounds for some graph parameters. Hernando *et al.* [22] found Nordhaus-Gaddum type of bounds for the metric dimension and the resolving domination number. We provide those bounds for the distance k-resolving domination number for $k \geq 2$.

Theorem 6.1. *For any graph G of order $n \geq 2$, we have the following.*

- *If $k = 2$, then*

$$3 \leq \gamma_2^r(G) + \gamma_2^r(\overline{G}) \leq 2n - 1 \text{ and } 2 \leq \gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \leq n(n-1).$$

The lower bounds are attained if and only if $G \in \{K_2, \overline{K}_2, P_3, \overline{P}_3\}$.
The upper bounds are attained if and only if $G \in \{K_n, \overline{K}_n\}$.

- *If $k \geq 3$, then*

$$2 \leq \gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n - 1 \text{ and } 1 \leq \gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq n(n-1).$$

The lower bounds are attained if and only if $G \cong P_4$.
The upper bounds are attained if and only if $G \in \{K_n, \overline{K}_n\}$.

Proof. If $k = 2$, then we have from Theorem 3.3 (a), $\gamma_2^r(G) = 1$ if and only if G is K_2 or P_3. Also, for any other graph G, we have $\gamma_2^r(G) \geq 2$. This means that $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \geq 3$ and $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \geq 2$. Since $\gamma_2^r(\overline{K}_2) = 2$ and $\gamma_2^r(\overline{P}_3) = 2$, we can conclude that these lower bounds are attained if and only if $G \in \{K_2, \overline{K}_2, P_3, \overline{P}_3\}$.

If $k \geq 3$, then based on Theorem 3.3 (a), we have $\gamma_k^r(G) = 1$ if and only if $G \in \{P_2, P_3, \ldots, P_{k+1}\}$. The graph P_4 is a self-complementary graph, i.e., $\overline{P}_4 \cong P_4$, we have $\gamma_k^r(\overline{P}_4) = \gamma_k^r(P_4) = 1$. Also, P_4 is the only graph whose complement is also a path and has a distance k-resolving domination number equal to 1. Therefore $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \geq 2$ and $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \geq 1$, also these lower bounds are achieved if and only if G is P_4.

Otherwise, for $k \geq 2$, we have $\gamma_k^r(G) = n$ if and only if G is the empty graph on n vertices $\overline{K}_n$, whose complement graph is the complete graph K_n. According to Theorem 3.3 (c), we have $\gamma_k^r(K_n) = n - 1$. Therefore, for any graph G of order $n \geq 2$, for $k \geq 2$, we have $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n - 1$ and $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq n(n-1)$. Moreover, these upper bounds are achieved if and only if $G \in \{K_n, \overline{K}_n\}$. ∎

Let G be a connected graph with $V(G) = \{1, 2, \ldots, n\}$. The graph $G[H^i]$ is the graph obtained from G by replacing the vertex i with a graph H and joining each vertex of H to every vertex adjacent to i in G. Let H_1 and H_2 be two graphs, the graph $G[H_1^i, H_2^j]$ is the graph obtained from G by replacing the vertex i (respectively, j) with the graph H_1 (respectively, H_2) and joining each vertex of H_1 (respectively, H_2) to every vertex adjacent to i (respectively, j) in G. If i and j are adjacent in G, join every vertex of H_1 to every vertex of H_2. The Bull graph B is the graph with vertex set $V(B) = \{1, 2, 3, 4, 5\}$ and edge set $V(B) = \{12, 13, 23, 14, 25\}$. The graph B is a self-complementary graph, i.e., $\overline{B} \cong B$.

Theorem 6.2. *If G and $\overline{G}$ are both connected graphs of order $n \geq 4$, we have the following.*

- *If $k = 2$, then*

$$4 \leq \gamma_2^r(G) + \gamma_2^r(\overline{G}) \leq 2n - 4 \text{ and } 4 \leq \gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \leq (n-2)^2.$$

The upper bounds are attained if and only if $G \cong P_4$.

- *If $k \geq 3$, then*

$$2 \leq \gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n - 6 \text{ and } 1 \leq \gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq (n-3)^2.$$

The lower bounds are attained if and only if $G \cong P_4$.
The upper bounds are attained if and only if $G \in \{P_4, C_5, B\} \cup \{P_4[K^1_{n-3}], P_4[\overline{K}^1_{n-3}], P_4[K^2_{n-3}], P_4[\overline{K}^2_{n-3}]\} \cup \{P_4[K^1_r, K^2_{n-r-2}] : 1 \leq r \leq n-3\} \cup \{P_4[\overline{K}^1_r, \overline{K}^3_{n-r-2}] : 1 \leq r \leq n-3\}$.

Proof. For $k = 2$, let G be a graph such that G and $\overline{G}$ are connected graphs. From Theorem 3.3 (a), $\gamma_2^r(G) = 1$ if and only if G is either K_2 or P_3. Then both G and $\overline{G}$ have distance 2-resolving domination number greater or equal to 2. Hence, $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \geq 4$ and $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \geq 4$. Also based on Proposition 2.9 we have $\gamma_2^r(P_4) = \gamma_2^r(\overline{P}_4) = 2$, then the lower bounds are sharp.

Otherwise, we have from Theorem 3.3 (c), K_n is the only connected graph with distance 2-resolving domination number equal to $n-1$. Since the complement of the complete graph is disconnected, it follows that $\gamma_2^r(G) \leq n-2$. Moreover, from Theorem 3.3 (b), for $n \geq 4$, $\gamma_2^r(G) = n-2$ if and only if G is either P_4, $K_{s,t}(s, t \geq 1)$, $K_s + \overline{K}_t(s \geq 1, t \geq 2)$, or $K_s + (K_1 \cup K_t)(s, t \geq 1)$. The only graph from these graphs whose complement graph is also connected is the path P_4. Since P_4 is self-complementary, we can conclude that $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \leq 2n-4$ and $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \leq (n-2)^2$, where the equality holds if and only if $G \cong P_4$.

For $k \geq 3$, we have $\gamma_k^r(P_4) = 1$. The graph P_4 is self-complementary and is the only graph in Theorem 3.3 (a) whose complement is a path graph having $\gamma_k^r(\overline{G}) = 1$. Then $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \geq 2$ and $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \geq 1$, and these lower bounds are achieved if and only if G is P_4.

Otherwise, we have from Theorem 3.3 (c), $\gamma_k^r(G) = n-1$ if and only if G is a complete graph. It follows that $\gamma_k^r(G) \leq n-2$. Furthermore, in Theorem 3.3 (b), $\gamma_k^r(G) = n-2$ if and only if G is either $K_{s,t}(s, t \geq 1)$, $K_s + \overline{K}_t(s \geq 1, t \geq 2)$, or $K_s + (K_1 \cup K_t)(s, t \geq 1)$. Since the complements of these graphs are all disconnected, it follows that $\gamma_k^r(G) \leq n-3$ and $\gamma_k^r(\overline{G}) \leq n-3$. Therefore, for $k \geq 3$, $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n-6$ and $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq (n-3)^2$. The only connected graph of order 4 whose complement graph is also a connected graph is P_4, we have $\gamma_k^r(P_4) = \gamma_k^r(\overline{P}_4) = 1$. Also for $n \geq 5$, based on Lemma 3.2, we have $\gamma_k^r(G) = \gamma_k^r(\overline{G}) = n-3$ if and only if $\dim(G) = \dim(\overline{G}) = n-3$. It follows that $\gamma_k^r(G) + \gamma_k^r(\overline{G}) = 2n-6$ if and only if $\dim(G) + \dim(\overline{G}) = 2n-6$. In [22], if G and $\overline{G}$ are both conncted graphs, we have $\dim(G) + \dim(\overline{G}) = 2n-6$ if and only if

$G \in \{P_4, C_5, B\} \cup \{P_4[K^1_{n-3}], P_4[\overline{K}^1_{n-3}], P_4[K^2_{n-3}], P_4[\overline{K}^2_{n-3}]\} \cup \{P_4[K^1_r, K^2_{n-r-2}] : 1 \leq r \leq n-3\} \cup \{P_4[\overline{K}^1_r, \overline{K}^3_{n-r-2}] : 1 \leq r \leq n-3\}$. ■

7. Some Relations Between $\gamma^r_k(G)$ and $\dim_k(G)$

For $k \geq 1$, let W be a k-truncated resolving set of a graph G. For any two distinct vertices $v, u \in V$, there exists a vertex x in W such that $d_k(v,x) = \min\{d_G(v,x), k+1\} \neq d_k(u,x) = \min\{d_G(u,x), k+1\}$. We have W is a resolving set of G. Also, at least one of u and v is at distance at most k from x. Based on this observation we get the following upper bound for $\gamma^r_k(G)$ in terms of $\dim_k(G)$.

Proposition 7.1. *For $k \geq 1$, let G be a connected graph. Then we have $\gamma^r_k(G) \leq \dim_k(G) + 1$.*

Proof. Let W be a minimum k-truncated resolving set of G. Then there is at most one vertex v in V such that $d_G(v, W) > k$. Otherwise, if v and u are two distinct vertices at distance greater than k from W, then $d_k(v,x) = d_k(u,x) = k+1$, for every $x \in W$. Now, suppose that there exists a vertex v such that $d_G(v,W) > k$, then the set $W \cup \{v\}$ is a distance k-dominating set of G. Since W is a resolving set of G, we have $W \cup \{v\}$ is a distance k-resolving dominating set of G. Thus, $\gamma^r_k(G) \leq |W| + 1 = \dim_k(G) + 1$. ■

If there exists a minimum k-truncated resolving set W of a connected graph G such that $d_G(v,W) \leq k$ for any $v \in V$, then necessarily $\gamma^r_k(G) \leq \dim_k(G)$.

In the following, we show that every k-truncated resolving set is a distance $(k+1)$-resolving dominating set.

Proposition 7.2. *For $k \geq 1$, let G be a connected graph. Then we have $\gamma^r_{k+1}(G) \leq \dim_k(G)$.*

Proof. Let W be a minimum k-truncated resolving set of G. Suppose that there is a vertex v in V such that $d_G(v,W) \geq k+2$. Let u be a vertex adjacent to v. Then necessarily $d_G(u,W) \geq k+1$, otherwise $d_G(v,W) \leq k+1$. This means that $d_k(v,x) = d_k(u,x) = k+1$, for all $x \in W$, a contradiction. Therefore $d_G(v,W) \leq k+1$, for any vertex v in V. Thus W is a distance $(k+1)$-resolving dominating set. Hence $\gamma^r_{k+1}(G) \leq |W| = \dim_k(G)$. ■

For $k \geq 1$, for a connected graph G of order n, we have $1 \leq \dim_k(G) \leq n-1$. A characterization of connected graphs of order n having $\dim_k(G) \in \{1, n-2, n-1\}$ is given in the following.

Theorem 7.3. *For a connected graph G of order $n \geq 2$, the following statements hold.*

(a) [13] *For* $k \geq 1$, $\dim_k(G) = 1$ *if and only if* $G \in \{P_i\}_{i=2}^{k+2}$.

(b) [14] *For* $n \geq 4$, $\dim_1(G) = n-2$ *if and only if* $G \in \{P_4, K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$. *For* $k \geq 2$ *and* $n \geq 4$, $\dim_k(G) = n-2$ *if and only if* $G \in \{K_{s,t}(s,t \geq 1), K_s + \overline{K}_t(s \geq 1, t \geq 2), K_s + (K_1 \cup K_t)(s,t \geq 1)\}$.

(c) [14] *For* $k \geq 1$, $\dim_k(G) = n-1$ *if and only if* $G \cong K_n$.

In Theorems 3.3 and 7.3, if G is a connected graph of order $n \geq 2$, we can see that for $k \geq 1$ and $r \in \{1, n-2, n-1\}$, we have $\dim_k(G) = r$ if and only if $\gamma_{k+1}^r(G) = r$.

Proposition 7.4. *For* $k \geq 1$, *and any positive integers* $\beta \geq 1$ *and* $\beta \leq \gamma \leq \beta+1$, *there exists a connected graph* G *having* $\dim_k(G) = \beta$ *and* $\gamma_k^r(G) = \gamma$. *For* $\beta \geq 2$, *the pair* $(\beta, 1)$ *is not realizable.*

Proof. Let $\beta \geq 1$ and $\beta \leq \gamma \leq \beta+1$. If $\gamma = \beta \geq 1$, we have $\gamma^r(K_{\beta+1}) = \beta$ [4]. By Theorem 3.3 and Theorem 7.3, for $k \geq 1$, $\dim_k(K_{\beta+1}) = \gamma_k^r(K_{\beta+1}) = \beta$.

If $\gamma = \beta + 1$, then for $k \geq 1$, if $\beta = 1$, according to Theorem 7.3 and Proposition 2.4, the path graph P_{k+2} has $\dim_k(P_{k+2}) = 1$ and $\gamma_k^r(P_{k+2}) = 2$. Now let $k \geq 1$ and $\beta \geq 2$. Let $S_{\beta+1,k}$ be the spider tree graph considered in the proof of Theorem 4.2 having one vertex v_0 of degree $\beta+1$ and $\beta+1$ leaves at distance k from v_0. As shown previously in Theorem 4.2, we have $\dim(S_{\beta+1,k}) = \beta$ and $\gamma_k^r(S_{\beta+1,k}) = \beta+1$. Let $v_1, v_2, \ldots, v_{\beta+1}$ be the neighbors of v_0 in $S_{\beta+1,k}$ and let $W = \{v_1, v_2, \ldots, v_\beta\}$. It is easy to check that W is a k-truncated resolving set of $S_{\beta+1,k}$. Therefore, $\dim_k(S_{\beta+1,k}) \leq |W| = \beta$. Since $\dim_k(S_{\beta+1,k}) \geq \dim(S_{\beta+1,k}) = \beta$, it follows that $\dim_k(S_{\beta+1,k}) = \beta$.

From Theorem 3.3, we have $\gamma_k^r(G) = 1$ if and only if G is a path graph of order at most $k+1$. If $n \leq k+1$, in Theorem 7.3, we have $\dim_k(P_n) = 1$. Therefore, there is no connected graph G having $\gamma_k^r(G) = 1$ and $\dim_k(G) \geq 2$. ∎

The case $\gamma = \beta + 1$, in Proposition 7.4, proves the sharpness of the upper bound in Proposition 7.1.

To provide examples of connected graphs having $\dim_k(G) > \gamma_k^r(G)$, we give the k-truncated metric dimension of path graphs which appeared in [14].

Theorem 7.5 [14]**.** *For* $k \geq 1$, *we have*

- $\dim_k(P_n) = 1$ *for* $2 \leq n \leq k+2$;
- $\dim_k(P_n) = 2$ *for* $k+3 \leq n \leq 3k+3$;
- *for* $n \geq 3k+4$, *we have*

$$\dim_k(P_n) = \begin{cases} \left\lfloor \frac{2n+3k-1}{3k+2} \right\rfloor, & \text{if } n \equiv 0, 1, \ldots, k+2 \pmod{(3k+2)}, \\ \left\lfloor \frac{2n+4k-1}{3k+2} \right\rfloor, & \text{if } n \equiv k+3, \ldots, \left\lceil \frac{3k+5}{2} \right\rceil - 1 \pmod{(3k+2)}, \\ \left\lfloor \frac{2n+3k-1}{3k+2} \right\rfloor, & \text{if } n \equiv \left\lceil \frac{3k+5}{2} \right\rceil, \ldots, 3k+1 \pmod{(3k+2)}. \end{cases}$$

From Theorem 7.5 and Proposition 2.4, we can see, for example, that if G is a path graph of order $6k+3$, then $\dim_k(G) = 4 > \gamma_k^r(G) = 3$. Moreover, we remark that the difference $\dim_k(G) - \gamma_k^r(G)$ can be arbitrarily large.

Proposition 7.6. *Let $k \geq 1$. For any positive integer N there exists a connected graph G with $\dim_k(G) - \gamma_k^r(G) > N$.*

Proof. For $k \geq 1$, let G be a path graph of order $n = i(3k+2)$ where $i \geq 1$. Based on Theorem 7.5, we have $\dim_k(G) = 2i$. From Proposition 2.4, $\gamma_k^r(G) = \left\lceil \frac{i(3k+2)}{2k+1} \right\rceil < \frac{i(3k+2)}{2k+1} + 1 \leq \frac{5}{3}i + 1$. It follows that $\dim_k(G) - \gamma_k^r(G) > 2i - \frac{5}{3}i - 1 = \frac{1}{3}i - 1 \to \infty$ as $i \to \infty$. ■

The upper bound in Proposition 2.9 holds for $\dim_k(G)$ the proofs are similar.

Proposition 7.7. *For $k \geq 1$, let G be a connected graph of order $n \geq k+1$, with diam$(G) \geq k$. Then $\dim_k(G) \leq n - k\gamma_k(G)$.*

8. Concluding Remarks

The study of the distance k-resolving domination number could be extended to other graph families and operations on graphs not discussed here. For example for trees, a formula in [20] is provided to compute efficiently $\gamma^r(T)$ for any tree T. We ask if it would be possible also for $\gamma_k^r(T)$ when $k \geq 2$. Also, it would be interesting to investigate the following questions.

- Is there a characterization of graphs achieving the bounds in Proposition 2.1?
- For $k \geq 1$ and $2 \leq \gamma \leq n-3$, can we characterize the connected graphs G of order n having $\gamma_k^r(G) = \gamma$?

A characterization of connected graphs G with $\gamma_2^r(G) = 2$ will provide all the graphs having $\gamma_2^r(G) + \gamma_2^r(\overline{G}) = 4$ and $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) = 4$ in Theorem 6.2, where G and $\overline{G}$ are both connected graphs.

In view of the discussion in Section 7 the following questions naturally arise.

- What is a sharp upper bound for $\dim_k(G)$ in terms of $\gamma_k^r(G)$ and what can be said about the ratio $\frac{\dim_k(G)}{\gamma_k^r(G)}$ for a connected graph G?

- Is there a characterization of graphs G having $\gamma_k^r(G) = \dim_k(G) + 1$ or $\gamma_{k+1}^r(G) = \dim_k(G)$?
- For which pair β, γ of positive integers with $\gamma < \beta$ does there exist a connected graph G such that $\dim_k(G) = \beta$ and $\gamma_k^r(G) = \gamma$?

For $k \geq 1$, we denote $N_k(v) = \{x \in V : 0 < d_G(v,x) \leq k\}$, the *open k-neighborhood* of a vertex v in V. The *k-locating-dominating set* defined as a set $X \subseteq V$, verifying for every $v, u \in V \setminus X$, we have $\emptyset \neq N_k(v) \cap X \neq N_k(u) \cap X \neq \emptyset$. The minimum cardinality of such set is called the *k-locating-domination number* denoted by $LD_k(G)$. Results about the k-locating-domination number can be found surveyed in [27]. Necessarily every k-locating-dominating set is a distance k-resolving dominating set, the opposite is not true. Therefore for all $k \geq 1$, we have $\gamma_k^r(G) \leq LD_k(G)$. For $k = 1$, in [5] it is shown that $LD_1(T) \leq 2\gamma^r(T) - 2$ for any tree T different from P_6. In [16], it is proved that $LD_1(G) \leq (\gamma^r(G))^2$ for any graph G not containing C_4 or C_6 as a subgraph. Finding an upper bound for $LD_1(G)$ in terms of $\gamma^r(G)$ for graphs in general is still open, it is shown [16] that such an upper bound is at least exponential in terms of $\gamma^r(G)$. Is it possible to find upper bounds for $LD_k(G)$ in terms of $\gamma_k^r(G)$ when $k \geq 2$ for graphs?

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