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# A STUDY OF A COMBINATION OF DISTANCE DOMINATION AND RESOLVABILITY IN GRAPHS

## DWI AGUSTIN RETNOWARDANI

### Mohammad Imam Utoyo

Mathematics Department, University of Airlangga Surabaya 60115, Indonesia

> e-mail: 2i.agustin@ikipjember.ac.id m.i.utoyo@fst.unair.ac.id

## Dafik

Mathematics Education Department, University of Jember Jember 68121, Indonesia

e-mail: d.dafik@unej.ac.id

LILIEK SUSILOWATI

Mathematics Department, University of Airlangga Surabaya 60115, Indonesia

e-mail: liliek-s@fst.unair.ac.id

AND

# KAMAL DLIOU

National School of Applied Sciences (ENSA) Ibn Zohr University B.P. 1136, Agadir, Morocco e-mail: dlioukamal@gmail.com

### Abstract

For  $k \geq 1$ , in a graph G = (V, E), a set of vertices D is a distance kdominating set of G, if any vertex in  $V \setminus D$  is at distance at most k from some vertex in D. The minimum cardinality of a distance k-dominating set of G is the distance k-domination number, denoted by  $\gamma_k(G)$ . An ordered set of vertices  $W = \{w_1, w_2, \ldots, w_r\}$  is a resolving set of G, if for any two distinct vertices x and y in  $V \setminus W$ , there exists  $1 \leq i \leq r$  such that

 $d_G(x, w_i) \neq d_G(y, w_i)$ . The minimum cardinality of a resolving set of G is the metric dimension of the graph G, denoted by  $\dim(G)$ . In this paper, we introduce the distance k-resolving dominating set which is a subset of V that is both a distance k-dominating set and a resolving set of G. The minimum cardinality of a distance k-resolving dominating set of G is called the distance k-resolving domination number and is denoted by  $\gamma_k^r(G)$ . We give several bounds for  $\gamma_k^r(G)$ , some in terms of the metric dimension dim(G)and the distance k-domination number  $\gamma_k(G)$ . We determine  $\gamma_k^r(G)$  when G is a path or a cycle. Afterwards, we characterize the connected graphs of order n having  $\gamma_k^r(G)$  equal to 1, n-2, and n-1, for  $k \geq 2$ . Then, we construct graphs realizing all the possible triples  $(\dim(G), \gamma_k(G), \gamma_k^r(G)),$ for all  $k \geq 2$ . Later, we determine the maximum order of a graph G having distance k-resolving domination number  $\gamma_k^r(G) = \gamma_k^r \ge 1$ , we provide graphs achieving this maximum order for any positive integers k and  $\gamma_k^r$ . Then, we establish Nordhaus-Gaddum bounds for  $\gamma_k^r(G)$ , for  $k \ge 2$ . Finally, we give relations between  $\gamma_k^r(G)$  and the k-truncated metric dimension of graphs and give some directions for future work.

**Keywords:** resolving set, metric dimension, distance k-domination, distance k-resolving domination.

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### 1. INTRODUCTION

In this paper, we study finite, simple, and undirected graphs. For graph terminology, we refer to [9].

In 1975, Meir and Moon [28] studied a combination of two concepts distance and domination in graphs. For  $k \geq 1$ , we call a *distance* k-dominating set in a graph G = (V, E), a subset D of the vertex set V such that for any vertex  $v \in V \setminus D$ , we have  $d_G(v, D) = \min\{d_G(v, x) : x \in D\} \leq k$ , where  $d_G(v, x)$ is the distance in G between the vertex v and x. The minimum cardinality overall distance k-dominating sets of G, is the distance k-domination number and is denoted by  $\gamma_k(G)$ . When k = 1, the distance 1-domination number is the well-known domination number of the graph denoted by  $\gamma(G)$ . Distance k-dominating sets find multiple applications in problems involving graphs like communication networks [31], geometric problems [26], facility location problems [19]. Results about this well-studied concept can be found surveyed in a recent book chapter [18].

Another concept associated with distance in graphs is resolvability and the metric dimension of graphs, introduced by Harary and Melter [17] and Slater [30]. Let  $W = \{w_1, w_2, \ldots, w_r\}$  be an ordered set of vertices in a graph G, the metric representation of v with respect to W is the r-vector  $c(v|W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_r))$ . The set W is a resolving set of G, if for

every two distinct vertices  $v, u \in V \setminus W$ ,  $c(v|W) \neq c(u|W)$ . The minimum cardinality of a resolving set of G is the *metric dimension* of G, and is denoted dim(G). Due to their important role from both a theoretical and a practical point of view, resolving sets and the metric dimension of graphs attracted attention these past years (see surveys [2, 32]). Resolving sets find many applications in several areas like network verification [3], robot navigation [25], pharmaceutical chemistry [8], coin weighing problems, Mastermind game (see references in [6, 23]) and more.

The problems of finding  $\gamma_k(G)$  and  $\dim(G)$  are both NP-Hard problems in general, see respectively [7] and [25].

To join the utility of resolving sets and distance k-dominating sets, we study a set satisfying the two properties.

**Definition 1.1.** A distance k-resolving dominating set is a set  $S \subseteq V$ , where S is both a resolving set and a distance k-dominating set of G. The distance k-resolving domination number, denoted by  $\gamma_k^r(G)$ , is the minimum cardinality of a distance k-resolving dominating set of G, i.e.,  $\gamma_k^r(G) = \min\{|S| : S \text{ is a distance } k\text{-resolving dominating set of } G\}$ .

A situation where the uses of resolving sets and distance k-dominating sets are both needed could represent a possible application of distance k-resolving dominating sets. For example, consider a network of vehicles. We want to identify the position of each vehicle, where the detection range within the network is not limited, and every position must be within a specific distance of a station that provides a service, such as energy supply or maintenance. The distance k-resolving dominating sets are required.

Resolving sets that satisfy additional properties are known and studied. For example, independent resolving set [11], is a resolving set that is also an independent set. Connected resolving set [29], is a resolving set that is also a connected set. For k = 1, the distance 1-resolving dominating set is a resolving set that is also a dominating set, the minimum cardinality of such set was first studied under the name of resolving domination number in [4], while it appeared as metric-location-domination number in [20]. More studies were done about that case relating it with other graph parameters, see for example [5, 16, 22]. Here we use the name resolving domination number and denote by  $\gamma^r(G)$ .

For  $k \geq 1$  and  $v, u \in V$ , let  $d_k(v, u) = \min\{d_G(v, u), k+1\}$ . A variation of the metric dimension that could be related to  $\gamma_k^r(G)$  is the k-truncated metric dimension,  $\dim_k(G)$ , defined as the minimum cardinality of a k-truncated resolving set of G, which is a set  $W \subseteq V$  verifying for any two distinct vertices  $v, u \in V$ , there exists a vertex x in W such that  $d_k(v, x) \neq d_k(u, x)$ . The k-truncated metric dimension was first studied when k = 1 in [24], also called adjacency dimension, where it was used to investigate the metric dimension of lexicographic product of graphs. For  $k \geq 1$ , the k-truncated metric dimension coincides with the (1, k + 1)-metric dimension of graphs in [13]. Results on  $\dim_k(G)$  can be found in [14, 15, 33].

In Section 2, we give sharp bounds for  $\gamma_k^r(G)$  in terms of the metric dimension, the distance k-domination number, the order, the diameter, the radius, and the girth of the graph. Also, we give the distance k-resolving domination number of the families of paths and cycles. In Section 3, for all  $k \ge 1$ , we show that  $\gamma_k^r(G)$  is equal to 1 if and only if G is a path of order at most k+1. For  $k \ge 2$ , we show an equivalence between  $\gamma_k^r(G)$  and dim(G), which we use to characterize all graphs of order n having  $\gamma_k^r(G)$  equal to n-1 and n-2. In Section 4, we determine all the realizable triples of positive integers  $(\beta, \gamma, \alpha)$  by a graph G having dim $(G) = \beta$ ,  $\gamma_k(G) = \gamma$ , and  $\gamma_k^r(G) = \alpha$  when  $k \ge 2$ , in particular the graphs we construct realizing these values are all trees. In Section 5, for all  $k \ge 1$ , we show that a graph G having distance k-resolving domination number  $\gamma_k^r(G) = \gamma_k^r \ge 1$ , has a maximum order of  $\gamma_k^r + \gamma_k^r \sum_{p=1}^k (2p+1)^{\gamma_k^r-1}$ . Also, we construct graphs attaining this maximum order for any arbitrary positive integers k and  $\gamma_k^r$ . Section 6 is devoted to Nordhaus-Gaddum bounds for the distance k-resolving domination number of graphs for  $k \ge 2$ . Finally, in Section 7, we discuss relations between  $\gamma_k^r(G)$  and dim<sub>k</sub>(G), we then conclude with some open questions.

# 2. Preliminary Results and Bounds for $\gamma_k^r(G)$

Every superset of a distance k-dominating set is a distance k-dominating set. It is true also for resolving sets. This means that every superset of a distance k-resolving dominating set is also a distance k-resolving dominating set. We give the following bounds that extend bounds given for k equal to 1 and 2, in [5] and [34] respectively to all  $k \ge 1$ .

**Proposition 2.1.** Let G be a connected graph of order  $n \ge 2$ . For  $k \ge 1$ , we have

$$\max\{\gamma_k(G), \dim(G)\} \le \gamma_k^r(G) \le \min\{\gamma_k(G) + \dim(G), n-1\}.$$

**Proof.** Let S be a minimum distance k-resolving dominating set of G. Since S is both a resolving set and a distance k-dominating set, then  $\dim(G) \leq |S|$ , and  $\gamma_k(G) \leq |S|$ . Thus  $\max\{\gamma_k(G), \dim(G)\} \leq \gamma_k^r(G)$ .

Let D and W be respectively a minimum distance k-dominating set and a minimum resolving set of G. The set  $S = D \cup W$  is a distance k-resolving dominating set of cardinality  $|S| = \gamma_k(G) + \dim(G)$ . Also, any subset of V of cardinality n - 1 is both a resolving set and a distance k-dominating set. Then we have  $\gamma_k^r(G) \leq \min\{\gamma_k(G) + \dim(G), n - 1\}$ .

For any two positive integer k and k' such that  $k \ge k' \ge 1$ , every distance k'dominating set is a distance k-dominating set. Therefore any distance k'-resolving dominating set is a distance k-resolving dominating set.

**Observation 2.2.** For  $k \ge k' \ge 1$ , if G is a connected graph, then we have  $\dim(G) \le \gamma_k^r(G) \le \gamma_{k'}^r(G) \le \gamma^r(G)$ .

The eccentricity of a vertex v in G is the maximum distance between v and any other vertex in G. The maximum and minimum eccentricity in G are respectively the *diameter* and the *radius* of G denoted respectively diam(G) and rad(G).

**Lemma 2.3.** Let G be a connected graph. For  $k \ge diam(G)$ ,  $\gamma_k^r(G) = \dim(G)$ .

**Proof.** If  $k \ge diam(G)$ , then any non-empty set of vertices in V is a distance k-dominating set. Hence any resolving set is also a distance k-dominating set of G. Therefore,  $\gamma_k^r(G) \le \dim(G)$ . From Proposition 2.1 it follows that  $\gamma_k^r(G) = \dim(G)$ .

Let  $P_n$  denote the path graph with  $V(P_n) = \{1, 2, ..., n\}$  and  $E(P_n) = \{i(i+1) : 1 \le i \le n-1\}$ . It is proved that  $\dim(P_n) = 1$  [8], and for  $k \ge 1$ ,  $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$  [12]. The values of the distance k-resolving domination number of  $P_n$  for k equal to 1 and 2 are given respectively in [4] and [34]. In the following we give  $\gamma_k^r(P_n)$  for all  $k \ge 1$ .

**Proposition 2.4.** For  $k \ge 1$  and  $n \ge 2$ ,

$$\gamma_k^r(P_n) = \begin{cases} 1, & \text{if } k \ge n-1, \\ 2, & \text{if } \lfloor \frac{n}{2} \rfloor \le k \le n-2, \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } 1 \le k \le \lfloor \frac{n}{2} \rfloor - 1. \end{cases}$$

**Proof.** In [8], we have dim $(P_n) = 1$ . So by Proposition 2.1,  $\gamma_k(P_n) \leq \gamma_k^r(P_n) \leq \gamma_k(P_n) + 1$ . Also, for  $1 \leq i, j \leq n$ , with  $i \neq j$ , we have  $d_{P_n}(i, j) = |i - j|$ . Then any resolving set of cardinality 1 must be  $\{1\}$  or  $\{n\}$ .

- For  $k \ge n-1$ , since  $diam(P_n) = n-1$ , it follows from Lemma 2.3 that  $\gamma_k^r(P_n) = \dim(P_n) = 1$ .
- For  $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$ , based on [12]  $\gamma_k(P_n) = 1$ , then  $\gamma_k^r(P_n)$  is equal to 1 or 2. It is clear that an end-vertex is not distance k-dominating. Thus,  $\gamma_k^r(P_n) = \gamma_k(P_n) + 1 = 2$ .
- For  $1 \le k \le \lfloor \frac{n}{2} \rfloor 1$ , in [12] we have  $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil \ge 2$ . Also, any set S consisting of two or more distinct vertices in  $V(P_n)$  is a resolving set of  $P_n$ . Thus,  $\gamma_k^r(P_n) = \gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$ .

The path  $P_n$  is a graph achieving the bounds in Proposition 2.1. For  $k \ge n-1$ , we have  $\gamma_k^r(P_n) = \dim(P_n)$ . For  $1 \le k \le \lfloor \frac{n}{2} \rfloor - 1$ ,  $\gamma_k^r(P_n) = \gamma_k(P_n)$ , and for  $\lfloor \frac{n}{2} \rfloor \le k \le n-2$ ,  $\gamma_k^r(P_n) = \gamma_k(P_n) + \dim(P_n)$ .

Let  $C_n$  denote the cycle graph with  $n \ge 3$ , where  $V(C_n) = \{0, 1, \ldots, n-1\}$ and  $E(C_n) = \{i(i+1) \pmod{n} : 0 \le i \le n-1\}$ . We have  $\dim(C_n) = 2$  [10], and for  $k \ge 1$ ,  $\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$  [12].

**Proposition 2.5.** For  $k \ge 1$  and  $n \ge 3$ ,

$$\gamma_k^r(C_n) = \begin{cases} 2, & \text{if } 4k+1 \ge n, \\ 3, & \text{if } 4k+2 = n, \\ \left\lceil \frac{n}{2k+1} \right\rceil, & \text{if } 4k+3 \le n. \end{cases}$$

**Proof.** We have  $d_{C_n}(i, j) = \min\{|i - j|, n - |i - j|\}$ .

**Claim 2.6.** For  $n \ge 2k + 2$  and  $n \ne 4k + 2$ , the set of vertices  $W = \{0, 2k + 1\}$  is a resolving set of  $C_n$ .

**Proof.** Let  $i, j \in V(C_n) \setminus W$ , with  $i \neq j$ . If  $d_{C_n}(i, 0) \neq d_{C_n}(j, 0)$ , then S is a resolving set. We suppose that  $d_{C_n}(i, 0) = d_{C_n}(j, 0)$ , then either  $d_{C_n}(i, 0) = i$  and  $d_{C_n}(j, 0) = n - j$  or  $d_{C_n}(i, 0) = n - i$  and  $d_{C_n}(j, 0) = j$ . Without loss of generality we suppose that  $d_{C_n}(i, 0) = i$  and  $d_{C_n}(j, 0) = n - j$ , which means that i+j = n. If  $d_{C_n}(i, 2k+1) = d_{C_n}(j, 2k+1)$ , then  $\min\{|2k+1-i|, n-|2k+1-i|\} = \min\{|2k+1-j|, n-|2k+1-j|\}$ . Since  $\min\{x, y\} = \frac{x+y-|x-y|}{2}$ , it follows that |n-2(|2k+1-i|)| = |n-2(|2k+1-j|)|.

We suppose that n-2|2k+1-i| = n-2|2k+1-j|, which means that |2k+1-i| = |2k+1-j|. Since  $i \neq j$ , necessarily 2k+1-i = j-2k-1. It follows that i+j = 4k+2 = n, a contradiction since  $n \neq 4k+2$ .

Otherwise if n-2|2k+1-i| = 2|2k+1-j|-n, then n = |2k+1-i|+|2k+1-j|. If |2k+1-i| = 2k+1-i and |2k+1-j| = 2k+1-j, then n = 2k+1-i+2k+1-j. Assuming that i+j=n, it means that n = 2k+1, a contradiction.

Now if |2k + 1 - i| = i - (2k + 1) and |2k + 1 - j| = j - (2k + 1), then n = i + j - 2(2k + 1). Since i + j = n, it means that k = 0, a contradiction.

Finally if |2k+1-i| = i - (2k+1) or |2k+1-j| = j - (2k+1), we suppose that |2k+1-i| = i - (2k+1) and |2k+1-j| = 2k+1-j. Then we get that n = i - j, again a contradiction.

It follows that  $d_{C_n}(i, 2k+1) \neq d_{C_n}(j, 2k+1)$ . So for  $i, j \in V(C_n) \setminus W$ , if  $i \neq j$ , then  $c(i|W) \neq c(j|W)$ .

• If  $2k + 1 \ge n$ , then  $k \ge diam(C_n)$ . By Lemma 2.3,  $\gamma_k^r(C_n) = \dim(C_n)$ . Since  $\dim(C_n) = 2$ , we have  $\gamma_k^r(C_n) = 2$ .

If  $4k + 1 \ge n \ge 2k + 2$ , we have  $\gamma_k^r(C_n) \ge \dim(C_n) = 2$ . From Claim 2.6, the set  $\{0, 2k + 1\}$  is a resolving set of  $C_n$ , it is also a distance k-dominating set of  $C_n$  for  $4k + 1 \ge n \ge 2k + 2$ . Therefore  $\gamma_k^r(C_n) = 2$ .

• If 4k + 2 = n, based on [12] we have  $\gamma_k(C_{4k+2}) = 2$ , then by Proposition 2.1,  $\gamma_k^r(C_{4k+2}) \ge 2$ . By using contradiction we suppose that  $\gamma_k^r(C_{4k+2}) = 2$ , and let

S be a distance k-resolving dominating set of cardinality 2. Since all the vertices have degree 2, if a vertex i is in a distance k-dominating set of cardinality 2, then the set contains necessarily  $i + 2k + 1 \pmod{n}$ . Since the cycle  $C_n$  is vertextransitive, we suppose without loss of generality that  $S = \{0, 2k + 1\}$ . If we take the vertices 1 and 4k + 1, then clearly c(1|S) = c(4k + 1|S). It follows that S is not a resolving set of  $C_{4k+2}$ . Hence  $\gamma_k^r(C_{4k+2}) > 2$ .

Now, let us consider the set  $S = \{0, 1, 2k+1\}$ , we will show first that  $\{0, 1\} \subset S$  is a resolving set of  $C_{4k+2}$ . For  $i \in V(C_n) \setminus S$ , we have  $c(i|\{0,1\}) = (\min\{i, n-i\}, \min\{i-1, n-i+1\})$ . For  $i, j \in V(C_n) \setminus S$ , if  $c(i|\{0,1\}) = c(j|\{0,1\})$ , it means that  $\min\{i, n-i\} = \min\{j, n-j\}$  and  $\min\{i-1, n-i+1\} = \min\{j-1, n-j+1\}$ . Since  $\min\{x, y\} = \frac{x+y-|x-y|}{2}$ , it follows that |n-2i| = |n-2j| and |n-2(i-1)| = |n-2(j-1)|. Assuming that  $i \neq j$ , then necessarily n-2i = 2j - n and n-2(i-1) = 2(j-1) - n, which is impossible. Then if  $i \neq j$ , we have  $c(i|\{0,1\}) \neq c(j|\{0,1\})$ . Therefore  $\{0,1\}$  is a resolving set of  $C_{4k+2}$ .

Since  $\{0, 2k + 1\}$  is a distance k-dominating set of  $C_{4k+2}$ , it follows that  $S = \{0, 1, 2k + 1\}$  is a distance k-resolving dominating set of  $C_{4k+2}$ . Therefore  $\gamma_k^r(C_{4k+2}) = 3$ .

• If  $4k + 3 \leq n$ , in [12] we have  $\gamma_k(C_n) = \lceil \frac{n}{2k+1} \rceil$ . Let us consider the set  $S = \{i(2k+1): 0 \leq i \leq \lceil \frac{n}{2k+1} \rceil - 1\}$ , we have  $|S| = \lceil \frac{n}{2k+1} \rceil$ . Claim 2.6 shows that the set  $\{0, 2k + 1\} \subset S$  is a resolving set of  $C_n$ . Also, it is easy to see that the set S is a distance k-dominating set of  $C_n$ . It follows that  $\gamma_k^r(C_n) = \lceil \frac{n}{2k+1} \rceil$ .

**Proposition 2.7.** For  $k \ge 1$ , let G be a connected graph such that  $rad(G) \le k$ or diam(G) = k + 1. Then we have  $\dim(G) \le \gamma_k^r(G) \le \dim(G) + 1$ .

**Proof.** Let G be a connected graph with  $rad(G) \leq k$ . This means that  $\gamma_k(G) = 1$ . Then by Proposition 2.1, we have  $\dim(G) \leq \gamma_k^r(G) \leq \dim(G) + 1$ .

If diam(G) = k + 1, let  $W \subset V$  be a minimum resolving set of G. Let  $v \in V \setminus dom_k(W)$ , where  $dom_k(W) = \{v \in V : d_G(v, W) \leq k\}$ . Then v must be at distance greater or equal to k + 1 from all the vertices of W. Since diam(G) = k+1, the only possible metric representation with respect to W of a vertex v such that  $d_G(v, W) \geq k + 1$ , is a vector having k + 1 as a value in all its coordinates. Since W is a resolving set, then there is at most one such vertex in G. Hence,  $\dim(G) \leq \gamma_k^r(G) \leq \dim(G) + 1$ .

For all  $k \geq 1$  both bounds in Proposition 2.7 can be achieved. For  $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$ , the graph  $P_n$  has  $rad(P_n) \leq k$ , from Proposition 2.4,  $\gamma_k^r(P_n) = \dim(P_n)+1$ . From Lemma 2.3, if  $rad(G) \leq diam(G) \leq k$ , then for any G we have  $\gamma_k^r(G) = \dim(G)$ . The cycle graphs  $C_{2k+2}$  or  $C_{2k+3}$  according to Proposition 2.5 are examples of graphs with diam(G) = k + 1 having  $\gamma_k^r(G) = \dim(G)$ . Also from Proposition 2.4, the path  $P_{k+2}$  is a graph of diam(G) = k + 1 having  $\gamma_k^r(G) = \dim(G) + 1$ .

**Lemma 2.8** [21]. For  $k \ge 1$ , let G be a connected graph of order  $n \ge k + 1$  and diameter diam $(G) \ge k$ . Then there exists a minimum distance k-dominating set D of G satisfying for every vertex  $v \in D$  there is a vertex  $x \in V \setminus D$  such that  $d_G(v, x) = k$  and  $N_k(x) \cap D = \{v\}$ .

The following upper bound proved for dim(G) in [4] is true also for  $\gamma_k^r(G)$ , the proofs are similar.

**Proposition 2.9.** For  $k \ge 1$ , let G be a connected graph of order  $n \ge k+1$  with  $diam(G) \ge k$ . Then  $\gamma_k^r(G) \le n - k\gamma_k(G)$ , and this upper bound is achieved for any positive integers k and  $\gamma_k(G)$ .

**Proof.** Suppose that  $\gamma_k(G) = \gamma$ . Based on Lemma 2.8, let us consider  $D = \{1, 2, \ldots, \gamma\}$  a minimum distance k-dominating set such that for all  $1 \leq i \leq \gamma$ , there exists a vertex  $w_{i,k}$  verifying that  $d_G(i, w_{i,k}) = k$ , and for  $j \neq i$ ,  $d_G(j, w_{i,k}) > k$ . Now let  $P_i = iw_{i,1}w_{i,2}\cdots w_{i,k}$  be a shortest  $(i, w_{i,k})$ -path. We can see that for  $1 \leq p \leq k$ , we have  $d_G(i, w_{i,p}) = p$  and  $d_G(j, w_{i,p}) > p$ . For any two different vertices  $w_{i,p}$ ,  $w_{j,q}$ , with  $1 \leq i, j \leq \gamma$  and  $1 \leq p, q \leq k$ , we will check the vector of distances with respect to the set D, we discuss the following two cases.

- (i) If  $i \neq j$ , we suppose without loss of generality that  $q \geq p$ . We have  $d_G(i, w_{i,p}) = p$  and  $d_G(i, w_{j,q}) \geq q + 1 > p$ .
- (ii) If i = j and  $p \neq q$ , we have  $d_G(i, w_{i,p}) = p$  and  $d_G(i, w_{i,q}) = q \neq p$ .

It follows that the set D resolves all the vertices  $w_{i,p}$ , where  $1 \leq i \leq \gamma$ , and  $1 \leq p \leq k$ . Then the set  $S = V \setminus \bigcup_{i=1}^{\gamma} \{w_{i,j}\}_{j=1}^{k}$  is both a distance k-dominating set and a resolving set. Hence  $\gamma_k^r(G) \leq |S| = |V \setminus \bigcup_{i=1}^{\gamma} \{w_{i,j}\}_{j=1}^{k}| = n - k\gamma = n - k\gamma_k(G)$ .

The family of trees  $\{T_{\gamma} : \gamma \geq 1\}$  illustrated as an example in Figure 1 has  $\gamma_k^r(T_{\gamma}) = n - k\gamma$ , for  $k, \gamma \geq 1$ , where  $\gamma_k(T_{\gamma}) = \gamma$ . We have any distance kdominating set in  $T_{\gamma}$  must contain at least one vertex in each branch  $iw_{i,1} \cdots w_{i,k}$ , with  $1 \leq i \leq \gamma$ . Also, the set of vertices  $\{1, 2, \ldots, \gamma\}$  is a distance k-dominating set of  $T_{\gamma}$ . Then clearly  $\gamma_k(T_{\gamma}) = \gamma$ . We can check as above that the set of vertices  $\{1, 2, \ldots, \gamma\}$  is a resolving set of  $T_{\gamma}$ . It follows from Proposition 2.1 that it is a minimum distance k-resolving dominating set of  $T_{\gamma}$  of cardinality  $n - k\gamma = n - k\gamma_k(T_{\gamma})$ .

For a connected graph G of order n and diameter d, we have  $\dim(G) \leq n-d$ [8]. The graphs achieving equality are characterized in [23]. This type of bound involving the order and the diameter of the graph was provided for the resolving domination number in [5]. We give a general upper bound for all  $k \geq 1$ .



Figure 1. Tree graph  $T_{\gamma}$  having  $\gamma_k^r(T_{\gamma}) = n - k \gamma_k(T_{\gamma})$ .

**Proposition 2.10.** For  $k \ge 1$ , let G be a connected graph of order n and diameter d. Then

$$\gamma_k^r(G) \le \begin{cases} n - d, & \text{if } d \le k, \\ n - d + 1, & \text{if } k + 1 \le d \le 2k, \\ n - d + \lfloor \frac{d}{2k + 1} \rfloor, & \text{if } d \ge 2k + 1. \end{cases}$$

These bounds are sharp.

**Proof.** Let P = (0, 1, ..., d) be a diametral path in G, i.e., P is a shortest path of length d. For any two vertices i and j in P, we have  $d_G(i, j) = |i - j|$ .

If  $d \leq k$ , then by Lemma 2.3,  $\gamma_k^r(G) = \dim(G)$ . Based on [8], we have  $\gamma_k^r(G) \leq n - d$ .

If  $k+1 \leq d \leq 2k$ , we consider the set of vertices  $\{k, d\}$ . For  $0 \leq l, m \leq d-1$ , with  $l \neq m$ , we have  $d_G(l, d) = |l - d| \neq |m - d| = d_G(m, d)$ . Also, for any  $0 \leq l \leq d$ , we have  $d_G(l, k) = |l - k| \leq k$ . This means that the set  $\{k, d\}$  is resolving and distance k-dominating of the vertices  $i \notin \{k, d\}$ . Now, let  $S' = V \setminus \{i : i \notin \{k, d\}\}$ . Then S' is a distance k-resolving dominating set of G. Hence,  $\gamma_k^r(G) \leq |S'| = n - d + 1$ .

If  $d \geq 2k+1$ , let us consider the set of vertices  $S = \{k, k+(2k+1), \ldots, k+j(2k+1), \ldots, \min\{k+\lfloor \frac{d}{2k+1} \rfloor (2k+1), d\}\}$ . Let l be a vertex in  $P \setminus S$ . If  $\min\{k+\lfloor \frac{d}{2k+1} \rfloor (2k+1), d\} = k+\lfloor \frac{d}{2k+1} \rfloor (2k+1)$ , then either  $k+\lfloor \frac{d}{2k+1} \rfloor (2k+1) < l \leq d$  or there exists  $1 \leq i \leq \lfloor \frac{d}{2k+1} \rfloor$  such that k+(i-1)(2k+1) < l < k+i(2k+1), or  $0 \leq l < k$ . In all those cases there exists a vertex in S at distance less or equal to k from l. The same can be observed when  $\min\{k+\lfloor \frac{d}{2k+1} \rfloor (2k+1), d\} = d$ . Furthermore, since  $|S| \geq 2$  and for  $0 \leq i, j \leq d$ ,  $d_G(i, j) = |i-j|$ , it is straightforward that S resolves the vertices in  $P \setminus S$ .

If we consider the set  $S' = V \setminus \{P \setminus S\}$ , then S' is a distance k-resolving dominating set of the graph G. Hence,  $\gamma_k^r(G) \leq |S'| = n - d + \lfloor \frac{d}{2k+1} \rfloor$ .

The graph path  $P_n$  has diameter n-1. From Proposition 2.4 it is a graph achieving the upper bound n-d for  $n \leq k+1$ . It achieves the upper bound n-d+1 when  $k+2 \leq n \leq 2k+1$ . The path graph  $P_n$  also achieves the upper bound  $n-d+\lfloor \frac{d}{2k+1} \rfloor$  when  $n \geq 2k+2$ .

If k = 1, for a connected graph of diameter  $d \ge 3$ , the upper bound in Proposition 2.10 is precisely the bound given in terms of the order and the diameter in [5].

The girth of the graph is the length of a shortest cycle in the graph. The following lower bounds proved in [12] for  $\gamma_k(G)$  holds also for  $\gamma_k^r(G)$  and they are achieved.

**Proposition 2.11.** For  $k \ge 1$ , let G be a connected graph having diameter d, radius r, and girth g. Then we have

- (1)  $\gamma_k^r(G) \ge \frac{d+1}{2k+1};$
- (2)  $\gamma_k^r(G) \ge \frac{2r}{2k+1};$
- (3)  $\gamma_k^r(G) \ge \frac{g}{2k+1}$ , if  $g < \infty$ .

These bounds are sharp.

**Proof.** In [12], it is shown that if G is a connected graph of diameter d, then  $\gamma_k(G) \geq \frac{d+1}{2k+1}$ . In the same paper we have if G has radius r, then  $\gamma_k(G) \geq \frac{2r}{2k+1}$ . Also in [12], for a connected graph of girth  $g < \infty$ , we have  $\gamma_k(G) \geq \frac{g}{2k+1}$ . Since  $\gamma_k^r(G) \geq \gamma_k(G)$ , the above lower bounds for  $\gamma_k(G)$  are true also for  $\gamma_k^r(G)$ .

Some graphs in Proposition 2.4 and 2.5 are examples of graphs attaining these bounds. In (1) consider the path graph of order n = p(2k + 1) for  $p \ge 2$ , since d = n - 1, we get that  $\gamma_k^r(G) = \frac{d+1}{2k+1}$ . In (2) consider the path graph of order n = 2p(2k+1). We have r = p(2k+1), then from proposition 2.4,  $\gamma_k^r(G) = \frac{2r}{2k+1}$ . In (3) take a cycle graph of order n = p(2k+1) for  $p \ge 3$ , since g = n, then this is a graph having  $\gamma_k^r(G) = \frac{g}{2k+1}$ .

# 3. Graphs with $\gamma_k^r(G)$ Equal to 1, n-2, and n-1

Further, let  $K_n$  denote the complete graph on n vertices, and let  $K_{s,t}$  with  $s, t \ge 1$ denote the complete bipartite graph. For two graphs  $G_1$  and  $G_2$  the disjoint union of  $G_1$  and  $G_2$ , denoted by  $G_1 \cup G_2$ , is the graph with vertex set  $V(G_1 \cup G_2) =$  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ . The join graph of  $G_1$ and  $G_2$ , denoted by  $G_1 + G_2$ , is the graph obtained from  $G_1 \cup G_2$  by joining each vertex from  $V(G_1)$  to each vertex in  $V(G_2)$ . We denote by  $\overline{G}$  the complement graph of G.

**Theorem 3.1** [8]. For a connected graph G of order  $n \ge 2$ , we have the following.

- $\dim(G) = 1$  if and only if  $G \cong P_n$ .
- If  $n \ge 4$ , then  $\dim(G) = n 2$  if and only if  $G \in \{K_{s,t}(s, t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s, t \ge 1)\}.$
- $\dim(G) = n 1$  if and only if  $G \cong K_n$ .

In a connected graph G of order  $n \ge k+1$ , any subset of V of order greater or equal to n-k is a distance k-dominating set.

**Lemma 3.2.** Let  $k \ge 2$ . For  $1 \le i \le k$ , if G is a connected graph of order  $n \ge i+2$  that is not a path graph, then  $\gamma_k^r(G) = n-i$  if and only if  $\dim(G) = n-i$ .

**Proof.** For all  $1 \leq i \leq k$ , if  $\dim(G) = n - i$ , any subset of V of cardinality  $n - i \geq n - k$  is a distance k-dominating set. Then a resolving set of cardinality  $\dim(G) = n - i$  is also a distance k-dominating set. Therefore  $\gamma_k^r(G) = \dim(G) = n - i$ .

Conversely, if  $\gamma_k^r(G) = n - i$ , by Proposition 2.1, we have  $\dim(G) \leq n - i$ . If n = i + 2, then  $\gamma_k^r(G) = n - i = 2$ . It follows that  $\dim(G)$  is equal to 1 or 2. Based on Theorem 3.1, the only graphs with  $\dim(G) = 1$  are path graphs, it follows that  $\dim(G) = 2$ .

If  $n \ge i+3$ , we suppose that  $\dim(G) < n-i$ . If  $i \le k-1$ , then a resolving set of cardinality  $n - (i + 1) \ge n - k$  is also a distance k-dominating set. Thus  $\gamma_{k}^{r}(G) \leq n - (i+1)$ , which is impossible. Now if i = k, let  $W \subseteq V$  be a resolving set of cardinality n - (k+1), and let us denote  $1, 2, \ldots, k+1$  the vertices in  $V \setminus W$ . Assuming that  $\gamma_k^r(G) = n - k$ , then there is at least one vertex v in  $V \setminus W$  such that  $d_G(v, W) = k+1$ . Let  $w \in W$  be such that  $d_G(v, w) = d_G(v, W) = k+1$ , and let Q be a shortest (v, w)-path. Since  $d_G(v, w) = d_G(v, W)$ , and G is a connected graph, the only vertex in  $W \cap Q$  is w. We have |Q| = k+2 and |W| = n - (k+1), which means that the subgraph induced by the vertices  $1, 2, \ldots, k+1$  and w is the path Q. Without loss of generality, we suppose that the path Q is  $(k+1)k\cdots 1w$ . Now, let  $S = (W \setminus \{w\}) \cup \{k\}$ . We have  $d_G(k, k+1) = d_G(k, k-1) = 1$ ,  $d_G(k,w) = k \ge 2$ , and if  $k \ge 3$ , for  $1 \le j \le k-2$ , we have  $d_G(k,j) = k-j \ge 2$ . Also  $d_G(k+1, S \setminus \{k\}) \ge k+1$ , since G is a connected graph and  $n \ge k+3$ , then there exists a vertex  $u \in S \setminus \{k\}$  such that either or both 1 and w are adjacent to u. This means that  $d_G(k-1, u) \leq k$ . It follows that S is a resolving set of G. Since  $d_G(k,i) \leq k$ , for  $1 \leq i \leq k+1$ ,  $i \neq k$ , and  $d_G(k,w) = k$ , it means that the set S is also a distance k-dominating set of G. Hence  $\gamma_k^r(G) \leq |S| = n - (k+1)$ , a contradiction. Therefore  $\dim(G) = n - k$ .

By combining Theorem 3.1 and Lemma 3.2 with Proposition 2.4, we give the following characterizations.

**Theorem 3.3.** For any graph G of order  $n \ge 2$ , the following statements hold. (a) For all  $k \ge 1$ ,  $\gamma_k^r(G) = 1$  if and only if  $G \in \{P_i\}_{i=2}^{k+1}$ .

- (b) If G is a connected graph of order  $n \ge 4$ ,  $\gamma_2^r(G) = n 2$  if and only if  $G \in \{P_4, K_{s,t}(s, t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s, t \ge 1)\}$ . For all  $k \ge 3$ ,  $\gamma_k^r(G) = n 2$  if and only if  $G \in \{K_{s,t}(s, t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s, t \ge 1)\}$ .
- (c) If G is a connected graph, for any  $k \ge 2$ ,  $\gamma_k^r(G) = n-1$  if and only if  $G \cong K_n$ .

**Proof.** (a) For  $k \ge 1$ , if  $\gamma_k^r(G) = 1$ , then G is a connected graph and from Proposition 2.1, dim(G) = 1. The equivalence is completed by Theorem 3.1 and Proposition 2.4.

(b) If G is a connected graph of order  $n \ge 4$  different from a path graph, then by Lemma 3.2 we have  $\gamma_k^r(G) = n-2$  if and only if dim(G) = n-2. Which means by Theorem 3.1 that it is equivalent to  $G \in \{K_{s,t}(s,t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s,t \ge 1)\}$ . From Proposition 2.4, we have  $\gamma_k^r(P_n) = n-2$ , it occurs only when k = 2 and n = 4. Then  $\gamma_2^r(G) = n-2$  if and only if  $G \in \{P_4, K_{s,t}(s,t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s,t \ge 1)\}$ . Also, for  $k \ge 3$ ,  $\gamma_k^r(G) = n-2$  if and only if  $G \in \{K_{s,t}(s,t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s,t \ge 1)\}$ .

(c) The only connected graphs of order 2 and 3 are respectively  $K_2$  and  $P_3$  or  $K_3$ . For  $k \ge 2$ , from Proposition 2.4 and Theorem 3.1, we have  $\gamma_k^r(K_2) = 1$ ,  $\gamma_k^r(P_3) = 1$ , and  $\gamma_k^r(K_3) = 2$ . If G has order  $n \ge 4$ , then by Lemma 3.2 and Theorem 3.1, we have  $\gamma_k^r(G) = n - 1$  if and only if  $G \cong K_n$ .

For k = 1, we have  $\gamma^r(G) = n - 1$  if and only if  $G \in \{K_{1,n-1}, K_n\}$  [4, 20]. The graphs having  $\gamma^r(G)$  equal to 2 and n - 2 are fully determined in [5] and [20], respectively.

## 4. Realizable Values for dim(G), $\gamma_k(G)$ , and $\gamma_k^r(G)$ .

In Proposition 2.1, we have  $\max\{\gamma_k(G), \dim(G)\} \leq \gamma_k^r(G) \leq \gamma_k(G) + \dim(G)$ . For k = 1, in [5] it is shown that for any three positive integers  $\beta$ ,  $\gamma$ , and  $\alpha$ , verifying that  $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$ , and  $(\beta, \gamma, \alpha) \notin \{(1, \gamma, \gamma + 1) : \gamma \geq 2\}$ , there is always a graph G having  $\dim(G) = \beta$ ,  $\gamma(G) = \gamma$ , and  $\gamma^r(G) = \alpha$ . We give a similar result for  $\dim(G)$ ,  $\gamma_k(G)$ , and  $\gamma_k^r(G)$ , for all  $k \geq 2$ .

The graph families we provide in Theorem 4.2 are all trees. To determine  $\gamma_k^r(G)$  of some of these graphs, we will need the next formula for the metric dimension of trees that appeared in [8, 17, 30]. We will recall some terminology given in [8]. In a tree T for  $v \in V$ , if the degree  $deg(v) \geq 3$ , then v is called a *major vertex*. A leaf l, i.e., a vertex of degree one, in T is a *terminal vertex* of a major vertex v, if v is the closest major vertex in terms of distance to l, i.e., for u a major vertex in T different from v, we have  $d_T(v, l) < d_T(u, l)$ . If v is a major vertex having at least one terminal vertex, then v is called an *exterior* 

*major vertex.* Let L(T) and EX(T) denote respectively the number of leaves and the number of exterior major vertices in a tree T.

**Theorem 4.1** [8, 17, 30]. If T is a tree that is not a path graph, then  $\dim(T) = L(T) - EX(T)$ . Also, any resolving set of T must contain at least one vertex from each branch at an exterior major vertex containing its terminal vertices with at most one exception.

**Theorem 4.2.** For any three positive integers  $\beta$ ,  $\gamma$ , and  $\alpha$  such that  $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$  and  $(\beta, \gamma, \alpha) \notin \{(1, \gamma, \gamma + 1) : \gamma \geq 2\}$ , and for all  $k \geq 2$ , there always exists a tree graph T having dim $(T) = \beta$ ,  $\gamma_k(T) = \gamma$ , and  $\gamma_k^r(T) = \alpha$ . There is no graph realizing the triples  $\{(1, \gamma, \gamma + 1) : \gamma \geq 2\}$ .

**Proof.** Let  $\beta, \gamma, \alpha \geq 1$  be such that  $\max\{\gamma, \beta\} \leq \alpha \leq \gamma + \beta$ . We discuss the possible values for the triple  $(\dim(G), \gamma_k(G), \gamma_k^r(G)) = (\beta, \gamma, \alpha)$ , according to the following cases.

• If  $\beta = 1$ , then  $\gamma \leq \alpha \leq \gamma + 1$ . Also by Theorem 3.1 we have the path graphs are the only graphs having the metric dimension equal to 1. For  $k \geq 2$ , in a path graph any subset of vertices of order greater or equal to 2 is a resolving set. Then if  $\gamma \geq 2$ , we have  $\alpha = \gamma$ . This means that the triple  $(1, \gamma, \gamma + 1)$  is not realizable by any graph for  $\gamma \geq 2$ . Also, according to Proposition 2.4 the path graphs realizes the following cases. (i) If  $k + 1 \geq n$ , then we have  $\gamma = \beta = \alpha = 1$ . (ii) If  $\lfloor \frac{n}{2} \rfloor \leq k \leq n-2$ , then  $\gamma = \beta = 1$  and  $\alpha = 2 = \gamma + \beta$ . (iii) If  $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ , then  $\beta = 1 < \gamma = \alpha = \lfloor \frac{n}{2k+1} \rfloor \geq 2$ .

• If  $\gamma = 1$ , for any  $\beta \geq 2$ , then we have  $\beta \leq \alpha \leq \beta + 1$ . The star graph  $K_{1,\beta+1}$  has  $\gamma_k(K_{1,\beta+1}) = 1$ , and from Theorem 3.1 and Theorem 3.3 we have  $\gamma_k^r(K_{1,\beta+1}) = \dim(K_{1,\beta+1}) = \beta$ , for any  $k \geq 2$ . This means that for  $k \geq 2$ , the triple  $(\beta, 1, \beta)$  is realized for all  $\beta \geq 2$ . For the case of the triple  $(\beta, 1, \beta + 1)$ , we consider the spider tree graph, denoted by  $S_{\beta+1,k}$ , having one vertex  $v_0$  of degree  $\beta + 1$  with  $\beta + 1$  leaves  $l_i$ ,  $1 \leq i \leq \beta + 1$ , at distance k from  $v_0$ . Note that all the vertices of  $S_{\beta+1,k}$  are of degree less or equal to 2 except  $v_0$ . Clearly  $\gamma_k(S_{\beta+1,k}) = 1$ , and based on Theorem 4.1, we have  $\dim(S_{\beta+1,k}) = \beta$ . Also any resolving set must contain at least one vertex in all but one of the  $(v_0, l_i)$ -paths, where  $1 \leq i \leq \beta + 1$ . By using contradiction, we suppose that  $\gamma_k^r(S_{\beta+1,k}) = \beta$ . From Theorem 4.1, we consider that a minimum distance k-resolving dominating set W of  $S_{\beta+1,k}$  having cardinality  $\beta$  contains one vertex in any of the  $(v_0, l_i)$ -paths, with  $1 \leq i \leq \beta$ . We have the vertex  $l_{\beta+1}$  is at distance greater than k from the vertices in W. This means that W is not a distance k-dominating set, a contradiction. Hence,  $\gamma_k^r(S_{\beta+1,k}) = \beta + 1$ .

• If  $\beta \geq 2$  and  $\gamma \geq 2$ , with  $\max(\gamma, \beta) \leq \alpha \leq \gamma + \beta$ , then the realizable values for the triple  $(\beta, \gamma, \alpha)$  are considered depending on the following five subcases.

(i) If  $2 \leq \beta = \gamma < \alpha$ , then the trees  $T^1 = \{T^1_{k,m,l} : m \geq 0, l \geq 1, k \geq 2\}$  in Figure 2 illustrate graphs realizing this case.

**Claim 4.3.** We have  $\gamma_k(T_{k,m,l}^1) = \dim(T_{k,m,l}^1) = m + l$ , and  $\gamma_k^r(T_{k,m,l}^1) = m + 2l$ .

**Proof.** Suppose that  $\gamma_k(T_{k,m,l}^1) = \gamma$ ,  $\dim(T_{k,m,l}^1) = \beta$ , and  $\gamma_k^r(T_{k,m,l}^1) = \alpha$ . It is clear that  $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l$  is a minimum distance k-dominating set. Then  $\gamma = m + l$ . Based on Theorem 4.1, we have  $\beta = m + l$ , and for each  $1 \leq i \leq l$ , a resolving set must contain one vertex from the set of vertices  $\{v_{i,j}\}_{j=0}^k$ . Also, for each  $1 \leq i \leq m$ , a resolving set must contain one vertex from the set of vertices  $\{v_{i,j}\}_{j=0}^k$ . Also, for each  $1 \leq i \leq m$ , a resolving set must contain one vertex from the set of vertices  $\{w_{i,j}, w_{i,j}'\}_{j=0}^k$ . Now, let S be a minimum distance k-resolving dominating set of cardinality  $\alpha$ . We suppose without loss of generality, that S contain a vertex from each  $\{v_{i,j}\}_{j=0}^k$  with  $1 \leq i \leq m$ , and one vertex from each  $\{w_{i,j}\}_{j=0}^k$  with  $1 \leq i \leq m$ , and for  $x \notin \{w_i, w_{i,j}'\}$  we have  $d_G(x, w_{i,k}) > k$ . Then to be a distance k-dominating set, S must contain for each  $1 \leq i \leq l$ , at least  $w_i$  or a vertex in  $\{w_{i,j}\}_{j=0}^k$ . Hence  $\alpha \geq m + 2l$ . It is easy to check that the set of vertices  $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{w_i\}_{i=1}^l$  is a distance k-resolving dominating set. Thus  $\alpha \leq m + 2l$ . It follows that  $\alpha = m + 2l$ .



Figure 2. Tree  $T_{k\,m,l}^1$ .

The proofs for the remaining cases use similar arguments as in the proof of Claim 4.3. In the following, we only provide examples of minimum distance k-dominating sets, minimum resolving sets, and minimum distance k-resolving dominating sets for each family of trees.

(ii) If  $2 \leq \gamma \leq \beta = \alpha$ , then the family of trees  $T^2 = \{T_{k,m,l}^2 : m \geq 1, l \geq 1, k \geq 2\}$  represented in Figure 3 realizes this case. The set of vertices  $\{v_i\}_{i=1}^m \cup \{w\}$  is a minimum distance k-dominating set of cardinality m+1. Also, the set of vertices  $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,1}\}_{i=1}^l$  is both a minimum resolving set and a minimum distance k-resolving dominating set of cardinality m+l.



Figure 3. Tree  $T_{k,m,l}^2$ .

(iii) If  $2 \leq \beta < \gamma = \alpha$ , then the family of trees  $T^3 = \{T^3_{k,m,l} : m \geq 1, l \geq 1, k \geq 2\}$  represented in Figure 4 realizes this case. From Theorem 4.1, we have  $\dim(T^3_{k,m,l}) = m + 1$ . Also, the set  $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u\}$  is a minimum distance k-dominating set of  $T^3_{k,m,l}$  of cardinality m + l + 1. Finally, the set  $\{v_{i,1}\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u_1\}$  is a distance k-resolving dominating set of cardinality m + l + 1. It follows that  $\gamma^r_k(T^3_{k,m,l}) = \gamma_k(T^3_{k,m,l}) = m + l + 1$ .



Figure 4. Trees  $T_{k,m,l}^3$ .

(iv) If  $2 \leq \gamma < \beta < \alpha$ , then the family of trees  $T^4 = \{T_{k,m,l,r}^4 : m \geq 0, l \geq 0, r \geq 3, k \geq 2\}$  represented in Figure 5 illustrates graphs realizing this case, where  $(m,l) \neq (0,0)$ . The set of vertices  $\{v_{i,k}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{u_{i,k}\}_{i=1}^{r-1}$  is a minimum resolving set of cardinality m+l+r-1. The set of vertices  $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u_i\}_{i=1}^l \cup \{u_i\}_{i=1}^r \cup \{$ 

(v) If  $2 \leq \beta < \gamma < \alpha$ , then Figure 6 illustrates a family of trees  $T^5 = \{T_{k,m,l,r}^5 : m \geq 0, l \geq 0, r \geq 2, k \geq 2\}$  realizing this case, where  $(m, l) \neq (0, 0)$ . The set of vertices  $\{v_{i,k}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{u_{r,k}\}$  is a minimum resolving set of cardinality



Figure 5. Tree  $T_{k,m,l,r}^4$ .

m + l + 1. The set of vertices  $\{v_i\}_{i=1}^m \cup \{w_i\}_{i=1}^l \cup \{u_i\}_{i=1}^r$  is a minimum distance k-dominating set of cardinality m + l + r. The set of vertices  $\{v_{i,1}\}_{i=1}^m \cup \{w_{i,k}\}_{i=1}^l \cup \{w_i\}_{i=1}^l \cup \{u_i\}_{i=1}^r \cup \{u_{r,k}\}$  is a minimum distance k-resolving dominating set of cardinality m + 2l + r + 1.



## 5. MAXIMUM ORDER GRAPHS

The maximum order n of a graph G having diameter d and metric dimension  $\dim(G) = \beta$ , was shown to be  $\beta + d^{\beta}$  [8, 25]. This was proved by considering the maximum possible number of distinct metric representations with respect to a minimum resolving set. But this maximum order is only achieved when  $d \leq 3$  or  $\beta = 1$ . Later, Hernando *et al.* [23] proved a stronger result by showing that  $n \leq (\lfloor \frac{2d}{3} \rfloor + 1)^{\beta} + \beta \sum_{i=1}^{\lfloor \frac{d}{3} \rfloor} (2i-1)^{\beta-1}$ , where the maximum order is achieved for any

arbitrary positive integers d and  $\beta$ .

Cáceres *et al.* [5] showed that for a graph G of order n having  $\gamma^r(G) = \gamma^r$ , then  $n \leq \gamma^r + \gamma^r \cdot 3^{\gamma^r-1}$ . They also provided graphs achieving this maximum order. Next, we generalize this result for  $\gamma_k^r(G)$  for all  $k \geq 1$ .

**Theorem 5.1.** For  $k \ge 1$ , the maximum order of a connected graph G having distance k-resolving domination number  $\gamma_k^r$  is  $\gamma_k^r + \gamma_k^r \sum_{p=1}^k (2p+1)^{\gamma_k^r-1}$ . This maximum order is achieved for any  $k, \gamma_k^r \ge 1$ .

**Proof.** Let G be a graph of order n and let S be a minimum distance k-resolving dominating set of G. For any vertex  $x \in V \setminus S$ , let us consider  $v_i$  a vertex in S such that  $d_G(x, v_i) = p \leq k$ . If  $\gamma_k^r(G) = \gamma_k^r \geq 2$ , for any vertex  $v_j$  from S different from  $v_i$ , the triangle inequality gives  $|d_G(x, v_j) - d_G(v_i, v_j)| \leq d_G(x, v_i) = p$ . It follows that the metric representation of x with respect to S has the coordinate corresponding to  $v_i$  equal to p and for the other coordinates there are at most 2p + 1 possible values in each of the other  $\gamma_k^r - 1$  coordinates. Therefore, there are at most  $(2p+1)^{\gamma_k^r-1}$  possible metric representations of x with respect to the set S. Since  $1 \leq p \leq k$ , there are at most  $\sum_{p=1}^k (2p+1)^{\gamma_k^r-1}$  distinct metric representations for the vertices at distance less or equal to k from  $v_i$ . Since  $|S| = \gamma_k^r$ , we have  $n \leq \gamma_k^r + \gamma_k^r \sum_{p=1}^k (2p+1)^{\gamma_k^r-1}$ .

Let k and  $\gamma_k^r$  be two arbitrary positive integers, we will prove that there exists a graph having distance k-resolving domination number  $\gamma_k^r$  and order  $\gamma_k^r + \gamma_k^r \sum_{p=1}^k (2p+1)^{\gamma_k^r-1}$ .

If  $\gamma_k^r = 1$ , then from Theorem 3.3 the graph G is a path graph of maximum order k + 1, which coincides with the maximum order bound. If  $\gamma_k^r = r \ge 2$ , we consider the following subsets of  $\mathbb{Z}^r$ ,

$$Q_0 = \{(0, 2k+1, 2k+1, \dots, 2k+1), (2k+1, 0, 2k+1, \dots, 2k+1), \dots, (2k+1, 2k+1, \dots, 2k+1, 0)\}.$$

For all  $1 \leq i \leq r$ ,

$$Q_i = \{(q_1, q_2, \dots, q_r) : 1 \le q_i \le k, \text{ and for } j \ne i, 2k - q_i + 1 \le q_j \le 2k + q_i + 1\}.$$

Let  $G_r$  be the graph whose vertex set is  $V(G_r) = \bigcup_{i=0}^r Q_i$ . For which two vertices  $q = (q_1, q_2, \ldots, q_r)$  and  $q' = (q'_1, q'_2, \ldots, q'_r)$  are adjacent if and only if  $|q_j - q'_j| \le 1$  for each  $1 \le j \le r$ .

Claim 5.2. The graph  $G_r$  is a connected graph.

**Proof.** If  $q_{i,0}, q_{j,0} \in Q_0$ , where  $q_{i,0}$  has the *i*-th element equal to 0 and  $q_{j,0}$  has the *j*-th element equal to 0, we construct a  $(q_{i,0}, q_{j,0})$ -path as following,

$$(2k+1,\ldots,\underset{i}{0},2k+1,\ldots,2k+1,\ldots,2k+1)(2k+1,\ldots,\underset{i}{1},2k+1,\ldots,2k,2k+1,\ldots,2k+1) \\ (2k+1,\ldots,\underset{i}{2},2k+1,\ldots,2k_{j}-1,2k+1,\ldots,2k+1) \\ (2k+1,\ldots,\underset{i}{k},2k+1,\ldots,k_{j}+1,2k+1,\ldots,2k+1)(2k+1,\ldots,k_{i}+1,2k+1,\ldots,\underset{j}{k},2k+1,\ldots,2k+1) \\ (2k+1,\ldots,k_{i}+2,2k+1,\ldots,k_{j}-1,2k+1,\ldots,2k+1) \\ (2k+1,\ldots,2\underset{i}{k},2k+1,\ldots,\underset{j}{1},2k+1,\ldots,2k+1)(2k+1,\ldots,2k+1,2k+1,\ldots,\underset{j}{0},2k+1,\ldots,2k+1).$$

Also, for each  $1 \leq i \leq r$ , if  $q = (q_1, q_2, \ldots, q_r) \in Q_i$ , it is easy to see from the definition of the adjacency in  $G_r$ , that there is a  $(q, q_{i,0})$ -path. Hence, the graph  $G_r$  is a connected graph.

For  $1 \leq i \leq r$  and  $q \in V(G_r) \setminus Q_0$ , we denote  $L_i(q) = (f_i(q_1), f_i(q_2), \ldots, f_i(q_r))$ , where  $f_i$  is an integer-valued function defined as following. If  $q = (q_1, q_2, \ldots, q_r) \in Q_s$ , with  $s \neq i$ .

• For 
$$j \notin \{s, i\}$$
,  $f_i(q_j) = \begin{cases} q_j, & \text{if } q_j = 2k+1, \\ q_j - 1, & \text{if } q_j > 2k+1, \\ q_j + 1, & \text{if } q_j < 2k+1. \end{cases}$ 

• 
$$f_i(q_s) = \begin{cases} q_s, & \text{if } q_s = k, \\ q_s + 1, & \text{if } q_s < k \text{ or } q_i = k + 1 \end{cases}$$

• 
$$f_i(q_i) = q_i - 1.$$
  
If  $q = (q_1, q_2, \dots, q_r) \in Q_i.$   
• For  $j \neq i$ ,  $f_i(q_j) = \begin{cases} q_j, & \text{if } q_j = 2k + 1, \\ q_j - 1, & \text{if } q_j > 2k + 1, \\ q_j + 1, & \text{if } q_j < 2k + 1. \end{cases}$ 

•  $f_i(q_i) = q_i - 1.$ 

For  $t \ge 1$ , we define  $L_i^t(q)$  with  $L_i^1(q) = L_i(q)$ . For  $t \ge 2$ ,  $L_i^t(q) = L_i(L_i^{t-1}(q)) = (f_i^t(q_1), f_i^t(q_2), \dots, f_i^t(q_r))$ , where  $f_i^t$  is the t-th iterated function of  $f_i$ , i.e.,  $f_i^t = \underbrace{f_i \circ f_i \circ \cdots \circ f_i}_{\text{t times}}$ .

**Claim 5.3.** For all  $1 \leq i \leq r$ , for any vertex  $q = (q_1, q_2, \ldots, q_r) \in V(G_r) \setminus Q_0$ , we have  $L_i(q) \in V(G_r)$ . Also,  $L_i(q)$  is adjacent in  $G_r$  to q, and  $L_i^{q_i}(q) = q_{0,i}$ .

**Proof.** Let  $q = (q_1, q_2, \ldots, q_r) \in V(G_r) \setminus Q_0$ . For  $1 \le i \le r$ , we have  $L_i(q) = (f_i(q_1), f_i(q_2), \ldots, f_i(q_r))$ . If  $q \in Q_s$ , where  $s \ne i$ , for  $j \ne s$ , we have  $2k - q_s + 1 \le q_j \le 2k + q_s + 1$  and  $1 \le q_s \le k$ . We discuss the membership of  $L_i(q)$  according to the following cases.

(i) If  $q_s < k$ , then we have  $f_i(q_s) = q_s + 1 \le k$ ,  $f_i(q_i) = q_i - 1 \ge 2k - q_s$ , and for  $j \notin \{i, s\}, 2k - q_s + 2 \le f_i(q_j) \le 2k + q_s$ . So  $L_i(q) = (f_i(q_1), f_i(q_2), \dots, f_i(q_r)) \in Q_s$ .

- (ii) If  $q_s = k$  and  $q_i > k+1$ , then  $f_i(q_i) = q_i 1 \ge k+1$ ,  $f_i(q_s) = k$ , and for  $j \notin \{i, s\}, k+1 \le f_i(q_j) \le 3k+1$ . So  $L_i(q) = (f_i(q_1), f_i(q_2), \dots, f_i(q_r)) \in Q_s$ .
- (iii) If  $q_i = k + 1$ , then  $q_s = k$ . It follows that  $f_i(q_i) = k$ ,  $f_i(q_s) = k + 1$ , and for  $j \notin \{i, s\}, k + 1 \leq f_i(q_j) \leq 3k + 1$ . Therefore,  $L_i(q) \in Q_i$ .

Now, if  $q \in Q_i$ , from the definition of f it is easy to see that  $L_i(q) \in Q_i$ . Hence, for any vertex  $q \in V(G_r) \setminus Q_0$ , we have  $L_i(q) \in V(G_r)$ . Moreover, for  $q \in V(G_r) \setminus Q_0$ , and all  $1 \leq i, j \leq r$ , we have  $|f_i(q_j) - q_j| \leq 1$ ,  $f_i^{q_i}(q_i) = 0$ , and for  $j \neq i$ ,  $f_i^{q_i}(q_j) = 2k + 1$ . Thus,  $L_i(q)q \in E(G_r)$ , and  $L_i^{q_i}(q) = q_{0,i}$ .

**Claim 5.4.** For all  $1 \le i \le r$ , for any vertex  $q = (q_1, q_2, ..., q_r) \in V(G_r) \setminus Q_0$ ,  $d_{G^r}(q, q_{0,i}) = q_i$ .

**Proof.** Based on Claim 5.3 for  $1 \leq i \leq r$ , we have  $qL_i(q)L_i^2(q)\cdots L_i^{q_i}(q) = q_{0,i}$  is a  $(q, q_{0,i})$ -path in  $G_r$  of length  $q_i$ . Hence  $d_{G^r}(q, q_{0,i}) \leq q_i$ . Since  $q_{0,i}$  and q are vertices having respectively 0 and  $q_i$  at the *i*-th coordinate and any two vertices in  $G_r$  can be adjacent only if the difference between the respective coordinates is at most 1, it follows that  $d_{G^r}(q, q_{0,i}) \geq q_i$ . Therefore,  $d_{G^r}(q, q_{0,i}) = q_i$ .

From above we can conclude that for any two different vertices q and q' in  $V(G_r) \setminus Q_0$ , there exists  $1 \leq i \leq r$  such that  $d_{G^r}(q, q_{0,i}) = q_i \neq d_{G^r}(q', q_{0,i}) = q'_i$ . It follows that the set of vertices  $Q_0$  is a resolving set of  $G_r$ . Also, for all  $1 \leq i \leq r$ , and any vertex  $q \in Q_i$ ,  $d_{G^r}(q, q_{0,i}) = q_i \leq k$ . Hence, the set  $Q_0$  is as well a distance k-dominating set of  $G_r$ . Hence,  $\gamma_k^r(G_r) \leq |Q_0| = r$ .

Suppose that  $\gamma_k^r(G_r) \leq r-1$ . We have the order of the graph  $G_r$  is  $|G_r| = r + r \sum_{p=1}^k (2p+1)^{r-1}$ . Also the maximum order of a graph having  $\gamma_k^r(G_r) \leq r-1$  was previously proved to be less or equal to  $\gamma_k^r(G_r) + \gamma_k^r(G_r) \sum_{p=1}^k (2p+1)^{\gamma_k^r(G_r)-1} \leq (r-1) + (r-1) \sum_{p=1}^k (2p+1)^{r-2}$ , it is a contradiction. Therefore,  $\gamma_k^r(G_r) = r$ .

For k = 1, the maximum order in Theorem 5.1 is precisely the maximum order given in [5].

## 6. Nordhaus-Gaddum Type Bounds

Nordhaus-Gaddum bounds are sharp bounds on the sum or the product of a parameter of a graph G and its complement  $\overline{G}$ . The survey [1] contains a bibliography of these types of bounds for some graph parameters. Hernando *et al.* [22] found Nordhaus-Gaddum type of bounds for the metric dimension and the resolving domination number. We provide those bounds for the distance k-resolving domination number for  $k \geq 2$ .

**Theorem 6.1.** For any graph G of order  $n \ge 2$ , we have the following.

• If k = 2, then

$$3 \leq \gamma_2^r(G) + \gamma_2^r(\overline{G}) \leq 2n-1 \text{ and } 2 \leq \gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \leq n(n-1).$$

The lower bounds are attained if and only if  $G \in \{K_2, \overline{K}_2, P_3, \overline{P}_3\}$ . The upper bounds are attained if and only if  $G \in \{K_n, \overline{K}_n\}$ .

• If  $k \geq 3$ , then

$$2 \leq \gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n-1 \text{ and } 1 \leq \gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq n(n-1).$$

The lower bounds are attained if and only if  $G \cong P_4$ . The upper bounds are attained if and only if  $G \in \{K_n, \overline{K}_n\}$ .

**Proof.** If k = 2, then we have from Theorem 3.3 (a),  $\gamma_2^r(G) = 1$  if and only if G is  $K_2$  or  $P_3$ . Also, for any other graph G, we have  $\gamma_2^r(G) \ge 2$ . This means that  $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \ge 3$  and  $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \ge 2$ . Since  $\gamma_2^r(\overline{K}_2) = 2$  and  $\gamma_2^r(\overline{P}_3) = 2$ , we can conclude that these lower bounds are attained if and only if  $G \in \{K_2, \overline{K}_2, P_3, \overline{P}_3\}$ .

If  $k \geq 3$ , then based on Theorem 3.3 (a), we have  $\gamma_k^r(G) = 1$  if and only if  $G \in \{P_2, P_3, \ldots, P_{k+1}\}$ . The graph  $P_4$  is a self-complementary graph, i.e.,  $\overline{P}_4 \cong P_4$ , we have  $\gamma_k^r(\overline{P}_4) = \gamma_k^r(P_4) = 1$ . Also,  $P_4$  is the only graph whose complement is also a path and has a distance k-resolving domination number equal to 1. Therefore  $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \geq 2$  and  $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \geq 1$ , also these lower bounds are achieved if and only if G is  $P_4$ .

Otherwise, for  $k \geq 2$ , we have  $\gamma_k^r(G) = n$  if and only if G is the empty graph on n vertices  $\overline{K}_n$ , whose complement graph is the complete graph  $K_n$ . According to Theorem 3.3 (c), we have  $\gamma_k^r(K_n) = n - 1$ . Therefore, for any graph G of order  $n \geq 2$ , for  $k \geq 2$ , we have  $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n - 1$  and  $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq n(n-1)$ . Moreover, these upper bounds are achieved if and only if  $G \in \{K_n, \overline{K}_n\}$ .

Let G be a connected graph with  $V(G) = \{1, 2, ..., n\}$ . The graph  $G[H^i]$ is the graph obtained from G by replacing the vertex *i* with a graph H and joining each vertex of H to every vertex adjacent to *i* in G. Let  $H_1$  and  $H_2$  be two graphs, the graph  $G[H_1^i, H_2^j]$  is the graph obtained from G by replacing the vertex *i* (respectively, *j*) with the graph  $H_1$  (respectively,  $H_2$ ) and joining each vertex of  $H_1$  (respectively,  $H_2$ ) to every vertex adjacent to *i* (respectively, *j*) in G. If *i* and *j* are adjacent in G, join every vertex of  $H_1$  to every vertex of  $H_2$ . The Bull graph B is the graph with vertex set  $V(B) = \{1, 2, 3, 4, 5\}$  and edge set  $V(B) = \{12, 13, 23, 14, 25\}$ . The graph B is a self-complementary graph, i.e.,  $\overline{B} \cong B$ .

**Theorem 6.2.** If G and  $\overline{G}$  are both connected graphs of order  $n \ge 4$ , we have the following.

• If k = 2, then

$$4 \le \gamma_2^r(G) + \gamma_2^r(\overline{G}) \le 2n - 4 \text{ and } 4 \le \gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \le (n - 2)^2.$$

The upper bounds are attained if and only if  $G \cong P_4$ .

• If  $k \geq 3$ , then

$$2 \le \gamma_k^r(G) + \gamma_k^r(\overline{G}) \le 2n - 6 \text{ and } 1 \le \gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \le (n - 3)^2.$$

The lower bounds are attained if and only if  $G \cong P_4$ .

The upper bounds are attained if and only if  $G \in \{P_4, C_5, B\} \cup \{P_4[K_{n-3}^1], P_4[\overline{K}_{n-3}^1], P_4[\overline{K}_{n-3}^2], P_4[\overline{K}_{n-3}^2]\} \cup \{P_4[K_r^1, K_{n-r-2}^2] : 1 \le r \le n-3\} \cup \{P_4[\overline{K}_r^1, \overline{K}_{n-r-2}^3] : 1 \le r \le n-3\}.$ 

**Proof.** For k = 2, let G be a graph such that G and  $\overline{G}$  are connected graphs. From Theorem 3.3 (a),  $\gamma_2^r(G) = 1$  if and only if G is either  $K_2$  or  $P_3$ . Then both G and  $\overline{G}$  have distance 2-resolving domination number greater or equal to 2. Hence,  $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \ge 4$  and  $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \ge 4$ . Also based on Proposition 2.9 we have  $\gamma_2^r(P_4) = \gamma_2^r(\overline{P_4}) = 2$ , then the lower bounds are sharp.

Otherwise, we have from Theorem 3.3 (c),  $K_n$  is the only connected graph with distance 2-resolving domination number equal to n-1. Since the complement of the complete graph is disconnected, it follows that  $\gamma_2^r(G) \leq n-2$ . Moreover, from Theorem 3.3 (b), for  $n \geq 4$ ,  $\gamma_2^r(G) = n-2$  if and only if G is either  $P_4$ ,  $K_{s,t}(s,t \geq 1)$ ,  $K_s + \overline{K}_t(s \geq 1, t \geq 2)$ , or  $K_s + (K_1 \cup K_t)(s,t \geq 1)$ . The only graph from these graphs whose complement graph is also connected is the path  $P_4$ . Since  $P_4$  is self-complementary, we can conclude that  $\gamma_2^r(G) + \gamma_2^r(\overline{G}) \leq 2n-4$ and  $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) \leq (n-2)^2$ , where the equality holds if and only if  $G \cong P_4$ .

For  $k \geq 3$ , we have  $\gamma_k^r(P_4) = 1$ . The graph  $P_4$  is self-complementary and is the only graph in Theorem 3.3 (a) whose complement is a path graph having  $\gamma_k^r(\overline{G}) = 1$ . Then  $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \geq 2$  and  $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \geq 1$ , and these lower bounds are achieved if and only if G is  $P_4$ .

Otherwise, we have from Theorem 3.3 (c),  $\gamma_k^r(G) = n - 1$  if and only if G is a complete graph. It follows that  $\gamma_k^r(G) \leq n - 2$ . Furthermore, in Theorem 3.3 (b),  $\gamma_k^r(G) = n - 2$  if and only if G is either  $K_{s,t}(s, t \geq 1)$ ,  $K_s + \overline{K}_t(s \geq 1, t \geq 2)$ , or  $K_s + (K_1 \cup K_t)(s, t \geq 1)$ . Since the complements of these graphs are all disconnected, it follows that  $\gamma_k^r(G) \leq n - 3$  and  $\gamma_k^r(\overline{G}) \leq n - 3$ . Therefore, for  $k \geq 3$ ,  $\gamma_k^r(G) + \gamma_k^r(\overline{G}) \leq 2n - 6$  and  $\gamma_k^r(G) \cdot \gamma_k^r(\overline{G}) \leq (n - 3)^2$ . The only connected graph of order 4 whose complement graph is also a connected graph is  $P_4$ , we have  $\gamma_k^r(P_4) = \gamma_k^r(\overline{P}_4) = 1$ . Also for  $n \geq 5$ , based on Lemma 3.2, we have  $\gamma_k^r(G) = \gamma_k^r(\overline{G}) = n - 3$  if and only if  $\dim(G) = \dim(\overline{G}) = n - 3$ . It follows that  $\gamma_k^r(G) + \gamma_k^r(\overline{G}) = 2n - 6$  if and only if  $\dim(G) + \dim(\overline{G}) = 2n - 6$ . In [22], if G and  $\overline{G}$  are both connected graphs, we have  $\dim(G) + \dim(\overline{G}) = 2n - 6$  if and only if

$$G \in \{P_4, C_5, B\} \cup \{P_4[K_{n-3}^1], P_4[\overline{K}_{n-3}^1], P_4[K_{n-3}^2], P_4[\overline{K}_{n-3}^2]\} \cup \{P_4[K_r^1, K_{n-r-2}^2]: 1 \le r \le n-3\} \cup \{P_4[\overline{K}_r^1, \overline{K}_{n-r-2}^3]: 1 \le r \le n-3\}.$$

# 7. Some Relations Between $\gamma_k^r(G)$ and $\dim_k(G)$

For  $k \geq 1$ , let W be a k-truncated resolving set of a graph G. For any two distinct vertices  $v, u \in V$ , there exists a vertex x in W such that  $d_k(v, x) = \min\{d_G(v, x), k+1\} \neq d_k(u, x) = \min\{d_G(u, x), k+1\}$ . We have W is a resolving set of G. Also, at least one of u and v is at distance at most k from x. Based on this observation we get the following upper bound for  $\gamma_k^r(G)$  in terms of  $\dim_k(G)$ .

**Proposition 7.1.** For  $k \ge 1$ , let G be a connected graph. Then we have  $\gamma_k^r(G) \le \dim_k(G) + 1$ .

**Proof.** Let W be a minimum k-truncated resolving set of G. Then there is at most one vertex v in V such that  $d_G(v, W) > k$ . Otherwise, if v and u are two distinct vertices at distance greater than k from W, then  $d_k(v, x) = d_k(u, x) = k + 1$ , for every  $x \in W$ . Now, suppose that there exists a vertex v such that  $d_G(v, W) > k$ , then the set  $W \cup \{v\}$  is a distance k-dominating set of G. Since W is a resolving set of G, we have  $W \cup \{v\}$  is a distance k-resolving dominating set of G. Thus,  $\gamma_k^r(G) \leq |W| + 1 = \dim_k(G) + 1$ .

If there exists a minimum k-truncated resolving set W of a connected graph G such that  $d_G(v, W) \leq k$  for any  $v \in V$ , then necessarily  $\gamma_k^r(G) \leq \dim_k(G)$ .

In the following, we show that every k-truncated resolving set is a distance (k + 1)-resolving dominating set.

**Proposition 7.2.** For  $k \geq 1$ , let G be a connected graph. Then we have  $\gamma_{k+1}^r(G) \leq \dim_k(G)$ .

**Proof.** Let W be a minimum k-truncated resolving set of G. Suppose that there is a vertex v in V such that  $d_G(v, W) \ge k + 2$ . Let u be a vertex adjacent to v. Then necessarily  $d_G(u, W) \ge k + 1$ , otherwise  $d_G(v, W) \le k + 1$ . This means that  $d_k(v, x) = d_k(u, x) = k + 1$ , for all  $x \in W$ , a contradiction. Therefore  $d_G(v, W) \le k + 1$ , for any vertex v in V. Thus W is a distance (k + 1)-resolving dominating set. Hence  $\gamma_{k+1}^r(G) \le |W| = \dim_k(G)$ .

For  $k \ge 1$ , for a connected graph G of order n, we have  $1 \le \dim_k(G) \le n-1$ . A characterization of connected graphs of order n having  $\dim_k(G) \in \{1, n-2, n-1\}$  is given in the following.

**Theorem 7.3.** For a connected graph G of order  $n \ge 2$ , the following statements hold.

- (a) [13] For  $k \ge 1$ ,  $\dim_k(G) = 1$  if and only if  $G \in \{P_i\}_{i=2}^{k+2}$ .
- (b) [14] For  $n \ge 4$ , dim<sub>1</sub>(G) = n 2 if and only if  $G \in \{P_4, K_{s,t}(s, t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s, t \ge 1)\}$ . For  $k \ge 2$  and  $n \ge 4$ , dim<sub>k</sub>(G) = n 2 if and only if  $G \in \{K_{s,t}(s, t \ge 1), K_s + \overline{K}_t(s \ge 1, t \ge 2), K_s + (K_1 \cup K_t)(s, t \ge 1)\}$ .
- (c) [14] For  $k \ge 1$ ,  $\dim_k(G) = n 1$  if and only if  $G \cong K_n$ .

In Theorems 3.3 and 7.3, if G is a connected graph of order  $n \ge 2$ , we can see that for  $k \ge 1$  and  $r \in \{1, n-2, n-1\}$ , we have  $\dim_k(G) = r$  if and only if  $\gamma_{k+1}^r(G) = r$ .

**Proposition 7.4.** For  $k \ge 1$ , and any positive integers  $\beta \ge 1$  and  $\beta \le \gamma \le \beta + 1$ , there exists a connected graph G having  $\dim_k(G) = \beta$  and  $\gamma_k^r(G) = \gamma$ . For  $\beta \ge 2$ , the pair  $(\beta, 1)$  is not realizable.

**Proof.** Let  $\beta \geq 1$  and  $\beta \leq \gamma \leq \beta + 1$ . If  $\gamma = \beta \geq 1$ , we have  $\gamma^r(K_{\beta+1}) = \beta$  [4]. By Theorem 3.3 and Theorem 7.3, for  $k \geq 1$ ,  $\dim_k(K_{\beta+1}) = \gamma_k^r(K_{\beta+1}) = \beta$ .

If  $\gamma = \beta + 1$ , then for  $k \geq 1$ , if  $\beta = 1$ , according to Theorem 7.3 and Proposition 2.4, the path graph  $P_{k+2}$  has  $\dim_k(P_{k+2}) = 1$  and  $\gamma_k^r(P_{k+2}) = 2$ . Now let  $k \geq 1$  and  $\beta \geq 2$ . Let  $S_{\beta+1,k}$  be the spider tree graph considered in the proof of Theorem 4.2 having one vertex  $v_0$  of degree  $\beta + 1$  and  $\beta + 1$ leaves at distance k from  $v_0$ . As shown previously in Theorem 4.2, we have  $\dim(S_{\beta+1,k}) = \beta$  and  $\gamma_k^r(S_{\beta+1,k}) = \beta + 1$ . Let  $v_1, v_2, \ldots, v_{\beta+1}$  be the neighbors of  $v_0$  in  $S_{\beta+1,k}$  and let  $W = \{v_1, v_2, \ldots, v_\beta\}$ . It is easy to check that W is a k-truncated resolving set of  $S_{\beta+1,k}$ . Therefore,  $\dim_k(S_{\beta+1,k}) \leq |W| = \beta$ . Since  $\dim_k(S_{\beta+1,k}) \geq \dim(S_{\beta+1,k}) = \beta$ , it follows that  $\dim_k(S_{\beta+1,k}) = \beta$ .

From Theorem 3.3, we have  $\gamma_k^r(G) = 1$  if and only if G is a path graph of order at most k + 1. If  $n \leq k + 1$ , in Theorem 7.3, we have  $\dim_k(P_n) = 1$ . Therefore, there is no connected graph G having  $\gamma_k^r(G) = 1$  and  $\dim_k(G) \geq 2$ .

The case  $\gamma = \beta + 1$ , in Proposition 7.4, proves the sharpness of the upper bound in Proposition 7.1.

To provide examples of connected graphs having  $\dim_k(G) > \gamma_k^r(G)$ , we give the k-truncated metric dimension of path graphs which appeared in [14].

**Theorem 7.5** [14]. For  $k \ge 1$ , we have

- $\dim_k(P_n) = 1$  for  $2 \le n \le k+2$ ;
- $\dim_k(P_n) = 2 \text{ for } k + 3 \le n \le 3k + 3;$
- for  $n \ge 3k + 4$ , we have

$$\dim_k(P_n) = \begin{cases} \left\lfloor \frac{2n+3k-1}{3k+2} \right\rfloor, & \text{if } n \equiv 0, 1, \dots, k+2 \pmod{(3k+2)}, \\ \left\lfloor \frac{2n+4k-1}{3k+2} \right\rfloor, & \text{if } n \equiv k+3, \dots, \left\lceil \frac{3k+5}{2} \right\rceil - 1 \pmod{(3k+2)}, \\ \left\lfloor \frac{2n+3k-1}{3k+2} \right\rfloor, & \text{if } n \equiv \left\lceil \frac{3k+5}{2} \right\rceil, \dots, 3k+1 \pmod{(3k+2)}. \end{cases}$$

From Theorem 7.5 and Proposition 2.4, we can see, for example, that if G is a path graph of order 6k + 3, then  $\dim_k(G) = 4 > \gamma_k^r(G) = 3$ . Moreover, we remark that the difference  $\dim_k(G) - \gamma_k^r(G)$  can be arbitrarily large.

**Proposition 7.6.** Let  $k \ge 1$ . For any positive integer N there exists a connected graph G with  $\dim_k(G) - \gamma_k^r(G) > N$ .

**Proof.** For  $k \ge 1$ , let G be a path graph of order n = i(3k+2) where  $i \ge 1$ . Based on Theorem 7.5, we have  $\dim_k(G) = 2i$ . From Proposition 2.4,  $\gamma_k^r(G) = \left\lceil \frac{i(3k+2)}{2k+1} \right\rceil < \frac{i(3k+2)}{2k+1} + 1 \le \frac{5}{3}i + 1$ . It follows that  $\dim_k(G) - \gamma_k^r(G) > 2i - \frac{5}{3}i - 1 = \frac{1}{3}i - 1 \to \infty$  as  $i \to \infty$ .

The upper bound in Proposition 2.9 holds for  $\dim_k(G)$  the proofs are similar.

**Proposition 7.7.** For  $k \ge 1$ , let G be a connected graph of order  $n \ge k+1$ , with  $diam(G) \ge k$ . Then  $\dim_k(G) \le n - k\gamma_k(G)$ .

### 8. Concluding Remarks

The study of the distance k-resolving domination number could be extended to other graph families and operations on graphs not discussed here. For example for trees, a formula in [20] is provided to compute efficiently  $\gamma^r(T)$  for any tree T. We ask if it would be possible also for  $\gamma_k^r(T)$  when  $k \ge 2$ . Also, it would be interesting to investigate the following questions.

- Is there a characterization of graphs achieving the bounds in Proposition 2.1?
- For k ≥ 1 and 2 ≤ γ ≤ n − 3, can we characterize the connected graphs G of order n having γ<sup>r</sup><sub>k</sub>(G) = γ?

A characterization of connected graphs G with  $\gamma_2^r(G) = 2$  will provide all the graphs having  $\gamma_2^r(G) + \gamma_2^r(\overline{G}) = 4$  and  $\gamma_2^r(G) \cdot \gamma_2^r(\overline{G}) = 4$  in Theorem 6.2, where G and  $\overline{G}$  are both connected graphs.

In view of the discussion in Section 7 the following questions naturally arise.

• What is a sharp upper bound for  $\dim_k(G)$  in terms of  $\gamma_k^r(G)$  and what can be said about the ratio  $\frac{\dim_k(G)}{\gamma_k^r(G)}$  for a connected graph G?

- Is there a characterization of graphs G having  $\gamma_k^r(G) = \dim_k(G) + 1$  or  $\gamma_{k+1}^r(G) = \dim_k(G)$ ?
- For which pair  $\beta$ ,  $\gamma$  of positive integers with  $\gamma < \beta$  does there exist a connected graph G such that  $\dim_k(G) = \beta$  and  $\gamma_k^r(G) = \gamma$ ?

For  $k \geq 1$ , we denote  $N_k(v) = \{x \in V : 0 < d_G(v, x) \leq k\}$ , the open kneighborhood of a vertex v in V. The k-locating-dominating set defined as a set  $X \subseteq V$ , verifying for every  $v, u \in V \setminus X$ , we have  $\emptyset \neq N_k(v) \cap X \neq N_k(u) \cap X \neq \emptyset$ . The minimum cardinality of such set is called the k-locating-domination number denoted by  $LD_k(G)$ . Results about the k-locating-domination number can be found surveyed in [27]. Necessarily every k-locating-dominating set is a distance k-resolving dominating set, the opposite is not true. Therefore for all  $k \geq 1$ , we have  $\gamma_k^r(G) \leq LD_k(G)$ . For k = 1, in [5] it is shown that  $LD_1(T) \leq 2\gamma^r(T) - 2$ for any tree T different from  $P_6$ . In [16], it is proved that  $LD_1(G) \leq (\gamma^r(G))^2$  for any graph G not containing  $C_4$  or  $C_6$  as a subgraph. Finding an upper bound for  $LD_1(G)$  in terms of  $\gamma^r(G)$  for graphs in general is still open, it is shown [16] that such an upper bound is at least exponential in terms of  $\gamma^r(G)$ . Is it possible to find upper bounds for  $LD_k(G)$  in terms of  $\gamma_k^r(G)$  when  $k \geq 2$  for graphs?

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