Discussiones Mathematicae
Graph Theory 44 (2024) 953–970
https://doi.org/10.7151/dmgt.2483

RAINBOW DISJOINT UNION OF CLIQUE AND MATCHING IN EDGE-COLORED COMPLETE GRAPH

Zemin Jin and Junqi Gu[1]

Department of Mathematics
Zhejiang Normal University
Jinhua 321004, China

e-mail: zeminjin@zjnu.cn
3250066378@qq.com

Abstract

Given an edge-coloring of a graph G, G is said to be *rainbow* if any two edges of G receive different colors. The anti-Ramsey number $AR(G, H)$ is defined to be the maximum integer k such that there exists a k-edge-coloring of G avoiding rainbow copies of H. The anti-Ramsey number for graphs, especially matchings, have been studied in several graph classes. Gilboa and Roditty focused on the anti-Ramsey number of graphs with small components, especially including a matching. In this paper, we continue the work in this direct and determine the exact value of the anti-Ramsey number of $K_4 \cup tP_2$ in complete graphs. Also, we improve the bound and obtain the exact value of $AR(K_n, C_3 \cup tP_2)$ for all $n \geq 2t + 3$.

Keywords: rainbow matching, anti-Ramsey number, clique.

2020 Mathematics Subject Classification: 05C55, 05C70, 05D10.

1. Introduction

An edge-colored graph is called *rainbow* if all the colors on its edges are distinct. Let H be a graph and G be a host graph. The anti-Ramsey number $AR(G, H)$ is the maximum number of colors in an edge-coloring of G which has no rainbow copy of H.

Anti-Ramsey number was introduced by Erdős, Simonovits and Sós in [5] and considered in the classical case when G is K_n. It has been shown that the anti-Ramsey number is closely related to the Turán number. Originally, the complete

[1]Corresponding author.

graph was the host graph of the anti-Ramsey numbers. Later, the host graphs are generalized to other graphs. The anti-Ramsey numbers of many graph classes have been studied extensively in the complete graph, including cycles [1, 5, 19], paths [24], cliques [23] etc.

Here we are mainly concerned with the anti-Ramsey problem for disconnected graphs, especially the graphs with small components. One of the most important such graph classes is matchings. Anti-Ramsey numbers for matchings in complete graph K_n have been determined independently by Schiermeyer [23] and Chen, Li and Tu [4]. Haas and Young [8] determined the anti-Ramsey number of perfect matchings in a complete graph. The complete split graphs contain complete graphs as a subclass and the anti-Ramsey problem in it was determined by Jin, Ye, Sun and Chen [16]. Jahanbekam and West [9] obtained the anti-Ramsey numbers for t edge-disjoint perfect matchings in K_n for $n \geq 4t+10$. Researchers have also obtained many results for matchings in bipartite graphs (see for example [11, 12, 17, 18] for some recent results). Besides of complete graphs and bipartite graphs as host graphs, the anti-Ramsey number of matchings has also been studied extensively in planar graphs [3, 10, 15, 20, 21] and hypergraphs [6, 13, 22, 25].

Gilboa and Roditty [7] and Bialostocki, Gilboa and Roditty [2] focused on the anti-Ramsey number of graphs with small components. Bialostocki, Gilboa and Roditty [2] determined the anti-Ramsey numbers for all graphs having at most four edges. Gilboa and Roditty [7] determined the anti-Ramsey numbers of several graphs, including $P_3 \cup tP_2$, $P_4 \cup tP_2$ and $C_3 \cup tP_2$, in K_n for large enough n. In this paper, we continue the study in this direction and determine the anti-Ramsey number of $K_4 \cup tP_2$. Note that Gilboa and Roditty [7] determined the value of $AR(K_n, C_3 \cup tP_2)$ for $n \geq \frac{5}{2}t+5$. Here we improve the bound and obtain the value of $AR(K_n, C_3 \cup tP_2)$ for all $n \geq 2t+3$.

Below we present some notions and definitions necessary in the paper. Given an edge-colored graph G, denote by $c(G)$ the set of colors of all edges of G and $c(e)$ the color of edge e. Also, given a subset $E^{'}$ of $E(G)$, we use $c(E^{'})$ to denote the set of colors of all edges in $E^{'}$. Let S and T be two disjoint subsets of $V(G)$; denote by $[S,T]$ the set of all edges between S and T in G. If $S=\{v\}$, we write $[v,T]$ for short.

In this paper, we consider the anti-Ramsey problem in complete graphs. Given an edge-colored graph K_n, denote by $l(D)$ the set of colors only appearing at the edges incident with vertices in $D \subseteq V(K_n)$, namely, $l(D)=c(K_n)\backslash c(K_n - D)$. When $D=\{v\}$, we write $l(v)$ for short and the number of colors in $l(v)$ is called the *saturated degree* of v. If $c(uv) \in l(v)$, then we say that u *saturates* v.

Given an edge-colored graph K_n without rainbow $H_2 \cup tP_2$. Suppose that K_n contains a rainbow subgraph H, a disjoint union of H_1 and H_2, where $H_1 = (t-1)P_2$. Let $D = V(K_n) \setminus V(H)$.

It is easy to check that $c(K_n[D]) \subseteq c(H)$. For each vertex $v \in D$, let

$l_1(H,v) = l(v) \setminus c([v, V(H_2)])$ and $l_2(H,v) = l(v) \setminus l_1(H,v)$. For each vertex $v \in D$, denote by E_v the set of edges between v and H_1 such that each color of $l_1(H,v)$ is assigned to exactly one edge of it.

For each vertex $v \in D$ and an edge set E_v, let $A_v = \{xy | xy \in E(H_1), \{vx, vy\} \subseteq E_v\}$ and $B_v = \{xy | xy \in E(H_1), |\{vx, vy\} \cap E_v| = 1\}$.

2. Result for $K_4 \cup tP_2$

Firstly, we present the result for K_4.

Theorem 1 [23]. *$AR(K_n, K_4) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ for any integer $n \geq 4$.*

Theorem 2. *$AR(K_6, K_4 \cup P_2) = 11$.*

Proof. In order to show the lower bound, first we find an edge-coloring of K_6 without rainbow $K_4 \cup P_2$. Let V be the vertex set of K_6. Choose a set $V_1 = \{v_1, v_2, v_3, v_4\} \subseteq V$ of cardinality 4 and let G be the complete subgraph of K_6 on the vertex set V_1. Let $V_2 = V \setminus V_1 = \{v, w\}$. Color all edges of G and $vv_1, vv_2, vv_3, wv_2, wv_3$ by distinct colors, color the edges vv_4, wv_4, wv_1 by $c(vv_1)$ and color the edge vw by $c(v_1v_4)$. Notice that there is no rainbow $K_4 \cup P_2$ in the coloring (see Figure 1). And we can see that the number of colors is 11. This implies that $AR(K_6, K_4 \cup P_2) \geq 11$.

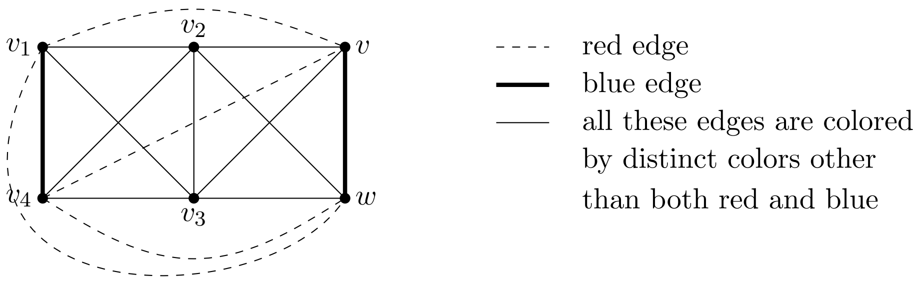

Figure 1. Edge-coloring of K_6.

Now we prove the inequality $AR(K_6, K_4 \cup P_2) \leq 11$. We only need to show that any 12-edge-coloring of K_6 contains a rainbow $K_4 \cup P_2$. On the contrary, we assume that there is a 12-edge-coloring of K_6 without any rainbow $K_4 \cup P_2$. By Theorem 1, $AR(K_6, K_4) = 10$. So it is easy to get the graph K_6 contains a rainbow subgraph H isomorphic to K_4. Let $V(H) = \{v_1, v_2, v_3, v_4\}$ and $D = V(K_6) \setminus V(H) = \{v, w\}$. Hence we have $c(vw) \subseteq c(H)$ and $|l(D)| = 6$. It is obvious that K_6 contains a rainbow $K_4 \cup P_2$ when there exists a vertex $x \in D$ such that $|l(x)| = 4$. Hence $|l(v)| = |l(w)| = 3$. If there is a vertex v_i for some $1 \leq i \leq 4$

which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup P_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Notice that K_6 does not contain any rainbow $K_4 \cup P_2$. Consider the subgraph $K_6[v_1, v_3, v_4, w] \cup K_6[v, v_2]$, a copy of $K_4 \cup P_2$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\})$. Similarly, consider the subgraph $K_6[v_2, v_3, v_4, w] \cup K_6[v, v_1]$, we have $c(wv_4) \in c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\})$. Consider the subgraph $K_6[v_1, v_2, v_4, w] \cup K_6[v, v_3]$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, vv_3\})$. This easily yields a contradiction, since $c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\}) \cap c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, vv_3\}) = \emptyset$. Hence any 12-edge-coloring of K_6 contains a rainbow $K_4 \cup P_2$, i.e., $AR(K_6, K_4 \cup P_2) \leq 11$. ■

Lemma 3. *For any integer $t \geq 1$ and $n \geq \max\{7, 2t+4\}$, $AR(K_n, K_4 \cup tP_2) \geq \max\left\{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, (t+1)(n-t-1) + \binom{t+1}{2} + 1\right\}$.*

Proof. In order to show the lower bound, we find an edge-coloring of K_n without rainbow $K_4 \cup tP_2$. Given a complete graph K_n, color the edges of a spanning subgraph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by distinct colors and then color all the other edges by a same new color. So it is obvious that the edge-coloring contains $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ colors and does not contain any rainbow K_4. Hence, $AR(K_n, K_4 \cup tP_2) \geq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$.

Let V be the vertex set of K_n. Choose a set $V_1 \subseteq V$ of cardinality $n-t-1$ and let G be the complete subgraph of K_n on the vertex set V_1. Color all the edges of G by the same color and then color all the other edges by distinct new colors. Then we get a $((t+1)(n-t-1) + \binom{t+1}{2} + 1)$-edge-coloring of K_n. Clearly, any rainbow K_4 contains at most two vertices of V_1. Take a rainbow K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$. Suppose that it contains two vertices of V_1, say $v_1, v_2 \in V \setminus V_1$ and $v_3, v_4 \in V_1$. Then any rainbow tP_2 in the graph $K_n - \{v_1, v_2, v_3, v_4\}$ contains an edge with the same color as v_3v_4. The other cases can be verified in the same way. In a word, there is no rainbow $K_4 \cup tP_2$ in the coloring. This implies that $AR(K_n, K_4 \cup tP_2) \geq (t+1)(n-t-1) + \binom{t+1}{2} + 1$. ■

Let t_n be the integer part of smaller root of $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1 = (t+1)(n-t-1) + \binom{(t+1)}{2} + 1$. By calculation, we can get

$$t_n = \begin{cases} \left\lfloor \frac{(2n-3)-\sqrt{2(n-1)^2-1}}{2} \right\rfloor = \frac{(2n-3)-\sqrt{2(n-1)^2-1}}{2} - \alpha_1, & \text{if } n = 2k; \\ \left\lfloor \frac{(2n-3)-\sqrt{2(n-1)^2+1}}{2} \right\rfloor = \frac{(2n-3)-\sqrt{2(n-1)^2+1}}{2} - \alpha_2, & \text{if } n = 2k+1, \end{cases}$$

where $0 \leq \alpha_1 < 1, 0 \leq \alpha_2 < 1$. We have the following result.

Theorem 4. *For any integer* $n \geq \max\{7, 2t+4\}$, $AR(K_n, K_4 \cup tP_2) = f(n,t)$, *where*

$$f(n,t) = \begin{cases} \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, & \text{if } t \leq t_n; \\ (t+1)(n-t-1) + \binom{t+1}{2} + 1, & \text{if } t > t_n. \end{cases}$$

Next we prove this theorem by induction on t and n. From Lemma 3, we only need to show that each $(f(n,t)+1)$-edge-coloring of K_n contains a rainbow $K_4 \cup tP_2$. We divide the proof into the following subsections.

2.1. Proof for the case $t = 1$

By Lemma 3, we have $AR(K_n, K_4 \cup P_2) \geq \max\left\{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, 2n-2\right\} = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. Now we prove the inequality $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. When $n = 7$, we assume that there is a 14-edge-coloring of K_7 which doesn't contain any rainbow $K_4 \cup P_2$. By Theorem 1, it is easy to get that the graph K_7 contains a rainbow subgraph H isomorphic to K_4. Let $V(H) = \{v_1, v_2, v_3, v_4\}$ and $D = V(K_7) \setminus V(H) = \{u, v, w\}$. Hence we have $c(K_7[D]) \subseteq c(H)$ and $|l(D)| = 8$. It is obvious that K_7 contains a rainbow $K_4 \cup P_2$ when there exists a vertex $x \in D$ such that $|l(x)| = 4$. Hence, without loss of generality, we can assume that $|l(v)| = |l(w)| = 3$ and $|l(u)| = 2$. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup P_2$. Hence we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Notice that K_7 does not contain any rainbow $K_4 \cup P_2$. Consider the subgraph $K_7[v_1, v_3, v_4, w] \cup K_7[v, v_2]$, a copy of $K_4 \cup P_2$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\})$. Similarly, consider the subgraph $K_7[v_2, v_3, v_4, w] \cup K_7[v, v_1]$, we have $c(wv_4) \in c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\})$. Consider the subgraph $K_7[v_1, v_2, v_4, w] \cup K_7[v, v_3]$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, vv_3\})$. This easily yields a contradiction, since $c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\}) \cap c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, vv_3\}) = \emptyset$. Hence any 14-edge-coloring of K_7 contains a rainbow $K_4 \cup P_2$, i.e., $AR(K_7, K_4 \cup P_2) \leq 14$. This completes the proof of theorem when $n = 7$ and $t = 1$.

Let $n \geq 8$ and c be a $\left(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2\right)$-edge-coloring of K_n. We assume that c does not contain any rainbow $K_4 \cup P_2$. Notice that the graph K_n contains a rainbow subgraph H isomorphic to K_4. Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that $|c(K_{n-1})| \leq \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1$. Hence the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \geq \left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2\right) - \left(\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 1\right).$$

Notice that if $n = 2k$, the saturated degree of each vertex of K_n is at least $k+1 = \frac{n}{2}+1$. If $n = 2k+1$, the saturated degree of each vertex of K_n is at least

$k+1 = \frac{n+1}{2}$. Hence we only need to check the cases $n = 2k$ and $n = 2k+1$ and below we distinguish these two cases.

Case 1. $n = 2k$. It is obvious that K_n contains a rainbow $K_4 \cup P_2$ when there exists a vertex $v \in D$ such that $|l(v)| = 4$. Hence for any $v \in D$, we have $k+1 \leq |l(v)| \leq 3$. Hence $n \leq 4$, a contradiction. So when $n = 2k$, $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$.

Case 2. $n = 2k+1$. It is obvious that K_n contains a rainbow $K_4 \cup P_2$ when there exists a vertex $v \in D$ such that $|l(v)| = 4$. Hence for any $v \in D$, we have $k+1 \leq |l(v)| \leq 3$. Hence $n \leq 5$, a contradiction. So when $n = 2k+1$, $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. This completes the proof of the theorem.

2.2. Proof for the case $2 \leq t \leq t_n$

Now we prove the inequality $AR(K_n, K_4 \cup tP_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. Let c be a $\left(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2\right)$-edge-coloring of K_n. Then we assume that c does not contain any rainbow $K_4 \cup tP_2$. Since $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2 > AR(K_n, K_4 \cup (t-1)P_2)$, we have that c contains a rainbow subgraph H, a disjoint union of H_1 and H_2, where $H_1 = (t-1)P_2$ and $H_2 = K_4$. Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that

$$AR(K_{n-1}, K_4 \cup tP_2) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1, & \text{if } t \leq t_{n-1}; \\ (t+1)(n-t-2) + \binom{t+1}{2} + 1, & \text{if } t > t_{n-1}. \end{cases}$$

Notice that when $n = 2k$,

$$\frac{(2n-3) - \sqrt{2(n-1)^2}}{2} \leq \frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} \leq \frac{(2n-3) - (\sqrt{2n^2} - \frac{3}{2})}{2}.$$

When $n = 2k+1$,

$$\frac{(2n-3) - (\sqrt{2n^2} - 1)}{2} \leq \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} \leq \frac{(2n-3) - \sqrt{2(n-1)^2}}{2}.$$

Hence we can get when $n = 2k$,

$$\frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2 + 1}}{2}$$

$$\geq \frac{(2n-3) - \sqrt{2(n-1)^2}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2}}{2} = \frac{2 - \sqrt{2}}{2} > 0.$$

When $n = 2k+1$, we have

$$\frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2 - 1}}{2}$$

$$\geq \frac{(2n-3)-(\sqrt{2n^2}-1)}{2}-\frac{(2(n-1)-3)-(\sqrt{2(n-1)^2}-\frac{3}{2})}{2}=\frac{\frac{3}{2}-\sqrt{2}}{2}>0.$$

Hence when $n=2k$ or $n=2k+1$, we have $t_n-t_{n-1}\geq 0$.

Suppose

$$t_{n-1}=\begin{cases}\left\lfloor\frac{(2n-5)-\sqrt{2(n-2)^2-1}}{2}\right\rfloor=\frac{(2n-5)-\sqrt{2(n-2)^2-1}}{2}-\beta_1, & \text{if } n-1=2k;\\ \left\lfloor\frac{(2n-5)-\sqrt{2(n-2)^2+1}}{2}\right\rfloor=\frac{(2n-5)-\sqrt{2(n-2)^2+1}}{2}-\beta_2, & \text{if } n-1=2k-1,\end{cases}$$

where $0\leq\beta_1<1$ and $0\leq\beta_2<1$. Therefore when $n=2k$, we can get

$$\begin{aligned}t_n-t_{n-1}&\leq\left(\frac{(2n-3)-\left(\sqrt{2n^2}-\frac{3}{2}\right)}{2}-\alpha_1\right)\\&\quad-\left(\frac{(2(n-1)-3)-\left(\sqrt{2(n-1)^2}-1\right)}{2}-\beta_2\right)\\&\leq\frac{1+\frac{3}{2}-\sqrt{2}}{2}+\beta_2-\alpha_1<2.\end{aligned}$$

When $n=2k+1$, we can get

$$\begin{aligned}t_n-t_{n-1}&\leq\frac{(2n-3)-\sqrt{2(n-1)^2}}{2}-\alpha_2)-\frac{(2(n-1)-3)-\sqrt{2(n-2)^2}}{2}-\beta_1)\\&\leq\frac{2-\sqrt{2}}{2}+\beta_1-\alpha_2<2.\end{aligned}$$

Hence we get $0\leq t_n-t_{n-1}\leq 1$.

Notice that if $t\leq t_{n-1}$, then $AR(K_{n-1},K_4\cup tP_2)=\lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil+1$. If $t>t_{n-1}$, then $AR(K_{n-1},K_4\cup tP_2)=(t+1)(n-t-1)+\binom{t+1}{2}+1$. Hence we only need to check the cases $t\leq t_{n-1}$ and $t>t_{n-1}$ and below we distinguish these two cases.

Case 1. $t\leq t_{n-1}$. Notice that $AR(K_{n-1},K_4\cup tP_2)=\lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil+1$ for any $t\leq t_{n-1}$. Since c does not contain any rainbow $K_4\cup tP_2$, we have $|c(K_{n-1})|\leq\lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil+1$. Hence the saturated degree of each vertex of K_n is at least $\left(\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil+2\right)-\left(\lfloor\frac{n-1}{2}\rfloor\lceil\frac{n-1}{2}\rceil+1\right)$. Notice that the lower bound above is different when n is even or odd. Hence we need to check the cases $n=2k$ and $n=2k+1$ and below we distinguish these two cases.

Subcase 1.1. $n=2k$. Since $n\geq 4+2t$, we can get $|D|\geq 2$. In this case, the saturated degree of each vertex of K_n is at least

$$\left(\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil+2\right)-\left(\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil+1\right)=k+1=\frac{n}{2}+1.$$

Also it follows from $|l(v)| \geq \frac{n}{2}+1 > 4$ that $|l_1(H,v)| > 0$ for each $v \in D$. For each $v \in D$ and E_v, by $|l(v)| \geq k+1$, we have $2|A_v|+|B_v| \geq (k+1)-|V(H_2)| = k-3$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $l(v) \geq k+1$, we have $|B_v| \geq k-3$. So $|V(H)| \geq 2|B_v|+|V(H_2)| \geq 2k-6+4 = n-2$. This implies that $|D| = 2$ and hence $n = 2t+4$. Let $D = \{v, w\}$. Hence the equalities above must hold. This implies that $|B_v| = k-3$ and $|V(H)| = 2t+2 = n-2 = 2k-2$. Then $|B_v| = k-3 = t-1$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$ and $c(vv_i) \neq c(vv_j)$ for $i \neq j$. If there is a set E_w such that $A_w \neq \emptyset$, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_w = \emptyset$ for any set E_w. Since $l(w) \geq k+1$ and $|B_w| \leq t-1 = k-3$, we have that $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$. Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x.

Suppose that $B_x = \emptyset$ for each vertex $x \in D$ and any set E_x. Hence for each vertex $x \in D$ and any set E_x, by $|l(x)| \geq k+1$, we have $2|A_x| \geq k-3$. Take two vertices $v, w \in D$. When $A_v \cap A_w \neq \emptyset$, it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_v \cap A_w = \emptyset$, and then

$$|V(H)| \geq 2|A_v| + 2|A_w| + |V(H_2)| \geq (k-3) + (k-3) + 4 = n-2.$$

This implies that $|D| = 2$, i.e., $D = \{v, w\}$. Hence the equalities above must hold. This implies that $2|A_v| = 2|A_w| = k-3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Hence there is a vertex $v \in D$ and a set E_v such that $B_v \neq \emptyset$. Take any vertex $w \in D$, $w \neq v$, and any set E_w. When $(A_v \cup B_v) \cap (A_w \cup B_w) = \emptyset$, we have

$$\begin{aligned}
|V(H)| &\geq 2|A_v| + 2|B_v| + 2|A_w| + 2|B_w| + |V(H_2)| \\
&\geq (k-3+|B_v|) + (k-3+|B_w|) + 4 \\
&= n-2+|B_v|+|B_w| > n-2,
\end{aligned}$$

a contradiction. This implies that $(A_v \cup B_v) \cap (A_w \cup B_w) \neq \emptyset$. If $A_v \cap (A_w \cup B_w) \neq \emptyset$ or $A_w \cap (A_v \cup B_v) \neq \emptyset$, we can easily find a rainbow $K_4 \cup tP_2$. Hence $A_v \cap (A_w \cup B_w) = \emptyset$ and $A_w \cap (A_v \cup B_v) = \emptyset$. This implies that $A_v \cap A_w = \emptyset$, $A_v \cap B_w = \emptyset$, $B_v \cap A_w = \emptyset$ and $B_v \cap B_w \neq \emptyset$. When there exists an edge in $B_v \cap B_w$ such that E_v and E_w cover distinct vertices of it, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that E_v and E_w cover one identical vertex of each edge in $B_v \cap B_w$, then we have

$$\begin{aligned}
|V(H)| &\geq 2|A_v| + 2|B_v| + 2|A_w| + 2(|B_w| - |B_v \cap B_w|) + |V(H_2)| \\
&\geq (k-3+|B_v|) + (k-3+|B_w| - 2|B_v \cap B_w|) + 4 \\
&= n-2+|B_v|+|B_w| - 2|B_v \cap B_w| \geq n-2.
\end{aligned}$$

Since $|D| \geq 2$, it follows that $|D| = 2$. Hence $D = \{v, w\}$ and $|B_v| = |B_w| = |B_v \cap B_w|$. Hence the equalities above must hold. This implies that $2|A_v|+|B_v| = k-3$ and $2|A_w|+|B_w| = k-3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Subcase 1.2. $n = 2k+1$. Together with $n \geq 4+2t$, we can get $|D| \geq 3$. In this case, the saturated degree of each vertex of K_n is at least

$$\left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2\right) - \left(\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 1\right) = k+1 = \frac{n+1}{2}.$$

Also it follows from $|l(v)| \geq \frac{n+1}{2} > 4$ that $|l_1(H, v)| > 0$ for any $v \in D$. For each $v \in D$ and any set E_v, by $|l(v)| \geq k+1$, we have $2|A_v|+|B_v| \geq (k+1)-|V(H_2)| = k-3$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $l(v) \geq k+1$, we have $|B_v| \geq k-3$. So $|V(H)| \geq 2|B_v|+|V(H_2)| \geq 2k-6+4 = n-3$. This implies that $|D| = 3$. Hence the equalities above must hold. This implies that $|B_v| = k-3$ and $|V(H)| = 2t+2 = n-3 = 2k-2$. Then $|B_v| = k-3 = t-1$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$ and $c(vv_i) \neq c(vv_j)$ for $i \neq j$. Take any vertex $w \in D$ and $w \neq v$. If there is a set E_w such that $A_w \neq \emptyset$, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_w = \emptyset$ for any set E_w. Since $l(w) \geq k+1$ and $|B_w| \leq t-1 = k-3$, we have that $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$. Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x.

Suppose that $B_x = \emptyset$ for each vertex $x \in D$ and any set E_x. Hence for any vertex $x \in D$ and any set E_x, by $|l(x)| \geq k+1$, we have $2|A_x| \geq k-3$. Take two vertices $v, w \in D$. When $A_v \cap A_w \neq \emptyset$, it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_v \cap A_w = \emptyset$, and then

$$|V(H)| \geq 2|A_v| + 2|A_w| + |V(H_2)| \geq (k-3) + (k-3) + 4 = n-3.$$

This implies that $|D| = 3$. Hence the equalities above must hold. This implies that $2|A_v| = 2|A_w| = k-3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v), c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Hence there is a vertex $v \in D$ and a set E_v such that $B_v \neq \emptyset$. Take any vertex $w \in D$, $w \neq v$, and any set E_w. When $(A_v \cup B_v) \cap (A_w \cup B_w) = \emptyset$, we have

$$\begin{aligned} |V(H)| &\geq 2|A_v| + 2|B_v| + 2|A_w| + 2|B_w| + |V(H_2)| \\ &\geq (k-3+|B_v|) + (k-3+|B_w|) + 4 \\ &= n-3+|B_v|+|B_w| > n-3, \end{aligned}$$

a contradiction. This implies that $(A_v \cup B_v) \cap (A_w \cup B_w) \neq \emptyset$. If $A_v \cap (A_w \cup B_w) \neq \emptyset$ or $A_w \cap (A_v \cup B_v) \neq \emptyset$, we can easily find a rainbow $K_4 \cup tP_2$. Hence $A_v \cap (A_w \cup B_w) = \emptyset$ and $A_w \cap (A_v \cup B_v) = \emptyset$. This implies that $A_v \cap A_w = \emptyset$, $A_v \cap B_w = \emptyset$, $B_v \cap A_w = \emptyset$ and $B_v \cap B_w \neq \emptyset$. When there exists an edge in $B_v \cap B_w$ such that E_v and E_w cover distinct vertices of it, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that E_v and E_w cover one identical vertex of each edge in $B_v \cap B_w$, then we have

$$\begin{aligned}|V(H)| &\geq 2|A_v| + 2|B_v| + 2|A_w| + 2(|B_w| - |B_v \cap B_w|) + |V(H_2)| \\ &\geq (k - 3 + |B_v|) + (k - 3 + |B_w| - 2|B_v \cap B_w|) + 4 \\ &= n - 3 + |B_v| + |B_w| - 2|B_v \cap B_w| \geq n - 3.\end{aligned}$$

Since $|D| \geq 3$, it follows that $|D| = 3$. So $|B_v| = |B_w| = |B_v \cap B_w|$. Hence the equalities above must hold. This implies that $2|A_v| + |B_v| = k - 3$ and $2|A_w| + |B_w| = k - 3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Case 2. $t > t_{n-1}$. From $0 \leq t_n - t_{n-1} \leq 1$ and $t \leq t_n$, we have $t = t_n = t_{n-1} + 1$. Then $AR(K_{n-1}, K_4 \cup tP_2) = (t+1)(n-2-t) + \binom{t+1}{2} + 1$. So $|c(K_{n-1})| \leq (t+1)(n-2-t) + \binom{t+1}{2} + 1$. Hence the saturated degree of each vertex of K_n is at least $\left(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2\right) - \left((t+1)(n-2-t) + \binom{t+1}{2} + 1\right)$.

When $n = 2k$,

$$t = t_n = \frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} - \alpha_1.$$

We can get

$$\frac{n^2}{4} - n - nt - n\alpha_1 + \frac{3t}{2} + \frac{3\alpha_1}{2} + \frac{t^2}{2} + t\alpha_1 + \frac{\alpha_1^2}{2} + 1 = 0.$$

Hence we have that for each vertex $v \in V(K_n)$,

$$\begin{aligned}|l(v)| &\geq \left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2\right) - \left((t+1)(n-2-t) + \binom{t+1}{2} + 1\right) \\ &= \frac{n^2}{4} - n - nt + \frac{5t}{2} + \frac{t^2}{2} + 3 = t + n\alpha_1 - \frac{3\alpha_1}{2} - t\alpha_1 - \frac{\alpha_1^2}{2} + 2 \geq t + 2.\end{aligned}$$

When $n = 2k + 1$,

$$t = t_n = \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} - \alpha_2.$$

We can get

$$\frac{n^2}{4} - n - nt - n\alpha_2 + \frac{3t}{2} + \frac{3\alpha_2}{2} + \frac{t^2}{2} + t\alpha_2 + \frac{\alpha_2^2}{2} + \frac{3}{4} = 0.$$

Hence we have that for each vertex $v \in V(K_n)$,

$$\begin{aligned}|l(v)| &\geq \left(\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil + 2\right) - \left((t+1)(n-2-t) + \binom{t+1}{2} + 1\right)\\ &= \frac{n^2}{4} - n - nt + \frac{5t}{2} + \frac{t^2}{2} + 3 - \frac{1}{4}\\ &= t + n\alpha_2 - \frac{3\alpha_2}{2} - t\alpha_2 - \frac{\alpha_2^2}{2} + 2 \geq t+2.\end{aligned}$$

Hence we have the saturated degree of each vertex v of K_n is at least $t+2$. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have that for $v, w \in D$ and sets E_v, E_w, $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Subcase 2.1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| \leq 1$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| \leq 1$, we have $|l_1(H, v)| \geq (t+2) - 1 = t+1$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-3$. Hence $|l_2(H, w)| \geq 5$, a contradiction.

Subcase 2.2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 2$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 2$, we have $|l_1(H, v)| \geq (t+2) - 2 = t$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-2$. Hence $|l_2(H, w)| = 4$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Subcase 2.3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 4$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 4$, we have $|l_1(H, v)| \geq (t+2) - 4 = t-2$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t$. Hence $|l_2(H, w)| \geq 2$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Subcase 2.4. For any $x \in D$, $|l_2(H, x)| = 3$. Then $|l_1(H, x)| \geq t-1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H, v)| \geq t-1$, we have $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-1$. Then $|l_1(H, w)| = t-1$ and hence $|l_1(H, x)| = t-1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H, v)| = t-1$, we have $|B_v| = t-1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w. Since $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$, by $|B_v| = t-1$, we have $|A_w| = \emptyset$ and $|B_w| = t-1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that $c(vw) = c(xy)$, $xy \in H_1$ with $x \in V(H_1)\backslash(V(B_v)\cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t+1$, a contradiction.

Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $K_4 \cup tP_2$. Consider the subgraph $K_n[v_2, v_3, v, w] \cup K_n[v_1, v_4] \cup H_1$, a copy of $K_4 \cup tP_2$, we have $c(vw) \in \{c(v_2v_3), c(v_1v_4)\}$. Similarly, consider the subgraph $K_n[v_1, v_2, v, w] \cup K_n[v_3, v_4] \cup H_1$, we have $c(vw) \in \{c(v_1v_2), c(v_3v_4)\}$. This easily yields a contradiction, since $\{c(v_1v_4), c(v_2v_3)\} \cap \{c(v_1v_2), c(v_3v_4)\} = \emptyset$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x. Take two vertices $v, w \in D$. Notice that for any sets E_v, E_w, $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. By $|l_1(H, v)| = |l_1(H, w)| = t-1$, we can deduce that $|A_v| = |A_w|$ and $|B_v| = |B_w|$. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Then for any edge $xy \in A_v$ and any edge xz with $c(xz) \in l(x)$, we have $z \in (V(A_v)\setminus\{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(A_v)\setminus\{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t+2$. Hence the equality holds, which implies that $l(x) = c(\{xz | z \in (V(A_v)\setminus\{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Similarly, $l(y) = c(\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Hence we can get that $K_n[v_1, x, y, v] \cup K_n[v_2, v_4] \cup K_n[v_3, w] \cup (H_1 - xy)$ is a rainbow $K_4 \cup tP_2$.

2.3. Proof for the case $t > t_n$

Now we prove the inequality $AR(K_n, K_4 \cup tP_2) \leq (t+1)(n-1-t) + \binom{t+1}{2} + 1$. By the induction hypothesis we have

$$AR(K_n, K_4 \cup (t-1)P_2) = \begin{cases} \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, & \text{if } t-1 \leq t_n; \\ t(n-t) + \binom{t}{2} + 1, & \text{if } t-1 > t_n. \end{cases}$$

Let c be a $((t+1)(n-1-t) + \binom{t+1}{2} + 2)$-edge-coloring of K_n. We assume that c does not contain any rainbow $K_4 \cup tP_2$. Since $(t+1)(n-1-t) + \binom{t+1}{2} + 2 > AR(K_n, K_4 \cup (t-1)P_2)$ when $t-1 = t_n$ or $t-1 > t_n$, so it is obvious c contains a rainbow subgraph H, a disjoint union of H_1 and H_2, where $H_1 = (t-1)P_2$ and

$H_2 = K_4$. Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that $|c(K_{n-1})| \leq (t+1)(n-2-t) + \binom{t+1}{2} + 1$. Hence the saturated degree of each vertex v of K_n satisfies

$$l(v) \geq ((t+1)(n-1-t) + \binom{t+1}{2} + 2) - ((t+1)(n-2-t) + \binom{t+1}{2} + 1) = t+2.$$

Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have that for $v, w \in D$ and sets E_v, E_w, $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Case 1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| \leq 1$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| \leq 1$, we have $|l_1(H, v)| \geq (t+2) - 1 = t+1$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-3$. Hence $|l_2(H, w)| \geq 5$, a contradiction.

Case 2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 2$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 2$, we have $|l_1(H, v)| \geq (t+2) - 2 = t$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(v)| \leq t-2$. Hence $|l_2(H, w)| = 4$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Case 3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 4$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 4$, we have $|l_1(H, v)| \geq (t+2) - 4 = t-2$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t$. Hence $|l_2(H, w)| \geq 2$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Case 4. For any $x \in D$, $|l_2(H, x)| = 3$. Then $|l_1(H, x)| \geq t-1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H, v)| \geq t-1$, we have $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-1$. Then $|l_1(H, w)| = t-1$ and hence $|l_1(H, x)| = t-1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H, v)| = t-1$, we have $|B_v| = t-1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w. Since $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$, by $|B_v| = t-1$, we have $|A_w| = \emptyset$ and $|B_w| = t-1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that $c(vw) = c(xy)$, $xy \in H_1$ with $x \in V(H_1) \setminus (V(B_v) \cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t+1$, a contradiction.

Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $K_4 \cup tP_2$. Consider the subgraph $K_n[v_2, v_3, v, w] \cup K_n[v_1, v_4] \cup H_1$, a copy of $K_4 \cup tP_2$, we have $c(vw) \in \{c(v_2v_3), c(v_1v_4)\}$. Similarly, consider the subgraph $K_n[v_1, v_2, v, w] \cup K_n[v_3, v_4] \cup H_1$, we have $c(vw) \in \{c(v_1v_2), c(v_3v_4)\}$. This easily yields a contradiction, since $\{c(v_1v_4), c(v_2v_3)\} \cap \{c(v_1v_2), c(v_3v_4)\} = \emptyset$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x. Take two vertices $v, w \in D$. Notice that for any sets E_v, E_w, $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. By $|l_1(H, v)| = |l_1(H, w)| = t - 1$, we can deduce that $|A_v| = |A_w|$ and $|B_v| = |B_w|$. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Then for any edge $xy \in A_v$ and any edge xz with $c(xz) \in l(x)$, we have $z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t + 2$. Hence the equality holds, which implies that $l(x) = c(\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Similarly, $l(y) = c(\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Hence we can get that $K_n[v_1, x, y, v] \cup K_n[v_2, v_4] \cup K_n[v_3, w] \cup (H_1 - xy)$ is a rainbow $K_4 \cup tP_2$, a contradiction. This completes the proof of the result.

3. Result for $C_3 \cup tP_2$

Bialostocki, Gilboa and Roditty [2] proved the following theorem for $C_3 \cup P_2$.

Theorem 5 [2]. *$AR(K_n, C_3 \cup P_2) = \max\{n, 6\}$ for any $n \geq 5$.*

Moreover, Gilboa and Roditty [7] obtained the result for $C_3 \cup tP_2$, $t \geq 1$, in K_n for large enough n.

Theorem 6 [7]. *For any integers $t \geq 1$ and $n \geq \frac{5}{2}t + 5$, $AR(K_n, C_3 \cup tP_2) = t(n-t) + \binom{t}{2} + 1$.*

Here, we show that the formula in the theorem above holds for all $n \geq \max\{6, 2t + 3\}$.

Theorem 7. *For any integers $t \geq 2$ and $n \geq 2t + 3$, $AR(K_n, C_3 \cup tP_2) = t(n-t) + \binom{t}{2} + 1$.*

Proof. In order to show the lower bound, we find an edge-coloring of K_n without rainbow $C_3 \cup tP_2$. Let V be the vertex set of K_n. Choose a set $V_1 \subseteq V$ of cardinality $n - t$ and let G be the complete subgraph of K_n on the vertex set

V_1. Color all the edges of G by the same color and then color all the other edges by distinct new colors. Then we get a $(t(n-t)+\binom{t}{2}+1)$-edge-coloring of K_n. Clearly, any rainbow C_3 contains at most two vertices of V_1. Take a rainbow C_3 with vertex set $\{v_1, v_2, v_3\}$. Suppose that it contains two vertices of V_1, say $v_1 \in V \setminus V_1$ and $v_2, v_3 \in V_1$. Then any rainbow tP_2 in the graph $K_n - \{v_1, v_2, v_3\}$ contains an edge with the same color as v_2v_3. The other cases can be verified in the same way. In a word, there is no rainbow $K_4 \cup tP_2$ in the coloring. This implies that $AR(K_n, C_3 \cup tP_2) \geq t(n-t)+\binom{t}{2}+1$.

We prove this by induction on t and n. When $t = 1$, by Theorem 5, $AR(K_n, C_3 \cup P_2) = n$ for any $n \geq 6$. Now let $t \geq 2$ and c be a $(t(n-t)+\binom{t}{2}+2)$-edge-coloring of K_n. We need to show that c contains a rainbow $C_3 \cup tP_2$. On the contrary, we assume that c does not contain any rainbow $C_3 \cup tP_2$. By the induction hypothesis on t, it is obvious that c contains a rainbow subgraph H, a disjoint union of H_1 and H_2, where $H_1 = (t-1)P_2$ and $H_2 = C_3$. Let $D = V(K_n) \setminus V(H)$. Clearly, any subgraph K_{n-1} has no rainbow $C_3 \cup tP_2$. Then by the induction hypothesis on n, we have $|c(K_{n-1})| \leq t(n-1-t)+\binom{t}{2}+1$. Hence the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \geq (t(n-t)+\binom{t}{2}+2)-(t(n-1-t)+\binom{t}{2}+1) = t+1.$$

Since K_n does not contain any rainbow $C_3 \cup tP_2$, we have that for $v, w \in D$ and sets E_v, E_w, $c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Case 1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 0$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 0$, we have $|l_1(H, v)| \geq t+1$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-3$. Hence $|l_2(H, w)| \geq 4$, a contradiction.

Case 2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 1$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 1$, we have $|l_1(H, v)| \geq (t+1)-1 = t$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-2$. Hence $|l_2(H, w)| = 3$. It is obvious that we can find a rainbow $C_3 \cup tP_2$, a contradiction.

Case 3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 3$. Let $w \in D$ and $w \neq v$ and take a set E_w. Since $|l_2(H, v)| = 3$, we have $|l_1(H, v)| \geq (t+1)-3 = t-2$. Then $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t$. Hence $|l_2(H, w)| \geq 1$. It is obvious that we can find a rainbow $C_3 \cup tP_2$, a contradiction.

Case 4. For any $x \in D$, $|l_2(H, x)| = 2$. Then $|l_1(H, x)| \geq t-1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H, v)| \geq t-1$, we have $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)| \leq t-1$. Then $|l_1(H, w)| = t-1$ and hence $|l_1(H, x)| = t-1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H,v)| = t-1$, we have $|B_v| = t-1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w. Since $c(yw) \notin l_1(H,w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$, by $|B_v| = t-1$, we have $|A_w| = \emptyset$ and $|B_w| = t-1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1. If there is a vertex v_i for some $1 \leq i \leq 3$ which saturates exactly one of v and w, then it is easy to find a rainbow $C_3 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2\}$ saturates both v and w. Since K_n does not contain any rainbow $C_3 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that $c(vw) = c(xy)$, $xy \in H_1$ with $x \in V(H_1) \setminus (V(B_v) \cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2\}$, otherwise we can get a rainbow $C_3 \cup tP_2$. So $t+1 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2\}| = t$, a contradiction.

Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $C_3 \cup tP_2$. Consider the subgraph $K_n[v_2, v, w] \cup K_n[v_1, v_3] \cup H_1$, a copy of $C_3 \cup tP_2$, we have $c(vw) = c(v_1v_3)$. Similarly, consider the subgraph $K_n[v_1, v, w] \cup K_n[v_2, v_3] \cup H_1$, we have $c(vw) = c(v_2v_3)$. This easily yields a contradiction, since $c(v_1v_3) \neq c(v_2v_3)$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x. Take two vertices $v, w \in D$ and let $xy \in A_v$. If there is a vertex v_i for some $1 \leq i \leq 3$ which saturates exactly one of v and w, then it is easy to find a rainbow $C_3 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2\}$ saturates both v and w. Then we can get that $K_n[x, y, v] \cup K_n[w, v_1] \cup K_n[v_2, v_3] \cup (H_1 - xy)$ is a rainbow $C_3 \cup tP_2$, a contradiction. This completes the proof of the result. ∎

Acknowledgement

This research was supported by Natural Science Foundation of China under Grants No.11571320 and No.12071440 and Zhejiang Provincial Natural Science Foundation of China under Grant No.LY19A010018.

References

[1] N. Alon, *On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey theorems*, J. Graph Theory **7** (1983) 91–94. https://doi.org/10.1002/jgt.3190070112

[2] A. Bialostocki, S. Gilboa and Y. Roditty, *Anti-Ramsey numbers of small graphs*, Ars Combin. **123** (2015) 41–53.

[3] G. Chen, Y.X. Lan and Z.X. Song, *Planar anti-Ramsey numbers of matchings*, Discrete Math. **342** (2019) 2106–2111. https://doi.org/10.1016/j.disc.2019.04.005

[4] H. Chen, X.L. Li and J.H. Tu, *Complete solution for the rainbow numbers of matchings*, Discrete Math. **309** (2009) 3370–3380.
https://doi.org/10.1016/j.disc.2008.10.002

[5] P. Erdős, M. Simonovits and V.T. Sós, *Anti-Ramsey theorems*, in: Infinite and Finite Sets, Vol. II, A. Hajnal, R. Rado, and V.T. Sós (Ed(s)) (North-Holland, Amsterdam, 1975) 633–643.

[6] P. Frankl and A. Kupavskii, *Two problems on matchings in set families—In the footsteps of Erdős and Kleitman*, J. Combin. Theory, Ser. B **138** (2019) 286–313.
https://doi.org/10.1016/j.jctb.2019.02.004

[7] S. Gilboa and Y. Roditty, *Anti-Ramsey numbers of graphs with small connected components*, Graphs Combin. **32** (2016) 649–662.
https://doi.org/10.1007/s00373-015-1581-y

[8] R. Haas and M. Young, *The anti-Ramsey number of perfect matching*, Discrete Math. **312** (2012) 933–937.
https://doi.org/10.1016/j.disc.2011.10.017

[9] S. Jahanbekam and D.B. West, *Anti-Ramsey problems for t edge-disjoint rainbow spanning subgraphs: cycles, matchings, or trees*, J. Graph Theory **82** (2016) 75–89.
https://doi.org/10.1002/jgt.21888

[10] S. Jendrol', I. Schiermeyer and J.H. Tu, *Rainbow numbers for matchings in plane triangulations*, Discrete Math. **331** (2014) 158–164.
https://doi.org/10.1016/j.disc.2014.05.012

[11] Y.X. Jia, M. Lu and Y. Zhang, *Anti-Ramsey problems in complete bipartite graphs for t edge-disjoint rainbow spanning subgraphs: cycles and matchings*, Graphs Combin. **35** (2019) 1011–1021.
https://doi.org/10.1007/s00373-019-02053-y

[12] Z.M. Jin, *Anti-Ramsey numbers for matchings in 3-regular bipartite graphs*, Appl. Math. Comput. **292** (2017) 114–119.
https://doi.org/10.1016/j.amc.2016.07.037

[13] Z.M. Jin, *Anti-Ramsey number of matchings in a hypergraph*, Discrete Math. **344(12)** (2021) 112594.
https://doi.org/10.1016/j.disc.2021.112594

[14] Z.M. Jin, H.W. Ma and R. Yu, *Rainbow matchings in an edge-colored planar bipartite graph*, Appl. Math. Comput. **432** (2022) 127356.
https://doi.org/10.1016/j.amc.2022.127356

[15] Z.M. Jin and K. Ye, *Rainbow number of matchings in planar graphs*, Discrete Math. **341** (2018) 2846–2858.
https://doi.org/10.1016/j.disc.2018.06.044

[16] Z.M. Jin, K.C. Ye, Y.F. Sun and H. Chen, *Rainbow matchings in edge-colored complete split graphs*, European J. Combin. **70** (2018) 297–316.
https://doi.org/10.1016/j.ejc.2018.01.010

[17] X.L. Li, J.H. Tu and Z.M. Jin, *Bipartite rainbow numbers of matchings*, Discrete Math. **309** (2009) 2575–2578.
https://doi.org/10.1016/j.disc.2008.05.011

[18] X.L. Li and Z.X. Xu, *The rainbow number of matchings in regular bipartite graphs*, Appl. Math. Lett. **22** (2009) 1525–1528.
https://doi.org/10.1016/j.aml.2009.03.019

[19] J.J. Montellano-Ballesteros and V. Neumann-Lara, *An anti-Ramsey theorem on cycles*, Graphs Combin. **21** (2005) 343–354.
https://doi.org/10.1007/s00373-005-0619-y

[20] Z.M. Qin, Y.X. Lan and Y.T. Shi, *Improved bounds for rainbow numbers of matchings in plane triangulations*, Discrete Math. **342** (2019) 221–225.
https://doi.org/10.1016/j.disc.2018.09.031

[21] Z.M. Qin, Y.X. Lan, Y.T. Shi and J. Yue, *Exact rainbow numbers for matchings in plane triangulations*, Discrete Math. **344(4)** (2021) 112301.
https://doi.org/10.1016/j.disc.2021.112301

[22] L. Özkahya and M. Young, *Anti-Ramsey number of matchings in hypergraphs*, Discrete Math. **313** (2013) 2359–2364.
https://doi.org/10.1016/j.disc.2013.06.015

[23] I. Schiermeyer, *Rainbow numbers for matchings and complete graphs*, Discrete Math. **286** (2004) 157–162.
https://doi.org/10.1016/j.disc.2003.11.057

[24] M. Simonovits and V.T. Sós, *On restricting colorings of K_n*, Combinatorica **4** (1984) 101–110.
https://doi.org/10.1007/BF02579162

[25] Y.S. Xue, E.F. Shan and L.Y. Kang, *Anti-Ramsey number of matchings in r-partite r-uniform hypergraphs*, Discrete Math. **345(4)** (2022) 112782.
https://doi.org/10.1016/j.disc.2021.112782

Received 28 May 2022
Revised 9 November 2022
Accepted 12 November 2022
Available online 29 January 2023

This article is distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License https://creativecommons.org/licenses/by-nc-nd/4.0/