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RAINBOW DISJOINT UNION OF CLIQUE AND MATCHING IN EDGE-COLORED COMPLETE GRAPH

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Abstract

Given an edge-coloring of a graph G, G is said to be rainbow if any two edges of G receive different colors. The anti-Ramsey number AR(G, H)is defined to be the maximum integer k such that there exists a k-edgecoloring of G avoiding rainbow copies of H. The anti-Ramsey number for graphs, especially matchings, have been studied in several graph classes. Gilboa and Roditty focused on the anti-Ramsey number of graphs with small components, especially including a matching. In this paper, we continue the work in this direct and determine the exact value of the anti-Ramsey number of $K_4 \cup tP_2$ in complete graphs. Also, we improve the bound and obtain the exact value of $AR(K_n, C_3 \cup tP_2)$ for all $n \ge 2t + 3$.

Keywords: rainbow matching, anti-Ramsey number, clique.

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1. INTRODUCTION

An edge-colored graph is called *rainbow* if all the colors on its edges are distinct. Let H be a graph and G be a host graph. The anti-Ramsey number AR(G, H) is the maximum number of colors in an edge-coloring of G which has no rainbow copy of H.

Anti-Ramsey number was introduced by Erdős, Simonovits and Sós in [5] and considered in the classical case when G is K_n . It has been shown that the anti-Ramsey number is closely related to the Turán number. Originally, the complete

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graph was the host graph of the anti-Ramsey numbers. Later, the host graphs are generalized to other graphs. The anti-Ramsey numbers of many graph classes have been studied extensively in the complete graph, including cycles [1, 5, 19], paths [24], cliques [23] etc.

Here we are mainly concerned with the anti-Ramsey problem for disconnected graphs, especially the graphs with small components. One of the most important such graph classes is matchings. Anti-Ramsey numbers for matchings in complete graph K_n have been determined independently by Schiermeyer [23] and Chen, Li and Tu [4]. Haas and Young [8] determined the anti-Ramsey number of perfect matchings in a complete graph. The complete split graphs contain complete graphs as a subclass and the anti-Ramsey problem in it was determined by Jin, Ye, Sun and Chen [16]. Jahanbekam and West [9] obtained the anti-Ramsey numbers for t edge-disjoint perfect matchings in K_n for $n \ge 4t + 10$. Researchers have also obtained many results for matchings in bipartite graphs (see for example [11, 12, 17, 18] for some recent results). Besides of complete graphs and bipartite graphs as host graphs, the anti-Ramsey number of matchings has also been studied extensively in planar graphs [3, 10, 15, 20, 21] and hypergraphs [6, 13, 22, 25].

Gilboa and Roditty [7] and Bialostocki, Gilboa and Roditty [2] focused on the anti-Ramsey number of graphs with small components. Bialostocki, Gilboa and Roditty [2] determined the anti-Ramsey numbers for all graphs having at most four edges. Gilboa and Roditty [7] determined the anti-Ramsey numbers of several graphs, including $P_3 \cup tP_2$, $P_4 \cup tP_2$ and $C_3 \cup tP_2$, in K_n for large enough n. In this paper, we continue the study in this direction and determine the anti-Ramsey number of $K_4 \cup tP_2$. Note that Gilboa and Roditty [7] determined the value of $AR(K_n, C_3 \cup tP_2)$ for $n \geq \frac{5}{2}t + 5$. Here we improve the bound and obtain the value of $AR(K_n, C_3 \cup tP_2)$ for all $n \geq 2t + 3$.

Below we present some notions and definitions necessary in the paper. Given an edge-colored graph G, denote by c(G) the set of colors of all edges of G and c(e) the color of edge e. Also, given a subset E' of E(G), we use c(E') to denote the set of colors of all edges in E'. Let S and T be two disjoint subsets of V(G); denote by [S, T] the set of all edges between S and T in G. If $S = \{v\}$, we write [v, T] for short.

In this paper, we consider the anti-Ramsey problem in complete graphs. Given an edge-colored graph K_n , denote by l(D) the set of colors only appearing at the edges incident with vertices in $D \subseteq V(K_n)$, namely, $l(D) = c(K_n) \setminus c(K_n - D)$. When $D = \{v\}$, we write l(v) for short and the number of colors in l(v) is called the *saturated degree* of v. If $c(uv) \in l(v)$, then we say that u saturates v.

Given an edge-colored graph K_n without rainbow $H_2 \cup tP_2$. Suppose that K_n contains a rainbow subgraph H, a disjoint union of H_1 and H_2 , where $H_1 = (t-1)P_2$. Let $D = V(K_n) \setminus V(H)$.

It is easy to check that $c(K_n[D]) \subseteq c(H)$. For each vertex $v \in D$, let

 $l_1(H,v) = l(v) \setminus c([v, V(H_2)])$ and $l_2(H,v) = l(v) \setminus l_1(H,v)$. For each vertex $v \in D$, denote by E_v the set of edges between v and H_1 such that each color of $l_1(H,v)$ is assigned to exactly one edge of it.

For each vertex $v \in D$ and an edge set E_v , let $A_v = \{xy | xy \in E(H_1), \{vx, vy\} \subseteq E_v\}$ and $B_v = \{xy | xy \in E(H_1), |\{vx, vy\} \cap E_v| = 1\}.$

2. Result for $K_4 \cup tP_2$

Firstly, we present the result for K_4 .

Theorem 1 [23]. $AR(K_n, K_4) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + 1$ for any integer $n \ge 4$.

Theorem 2. $AR(K_6, K_4 \cup P_2) = 11.$

Proof. In order to show the lower bound, first we find an edge-coloring of K_6 without rainbow $K_4 \cup P_2$. Let V be the vertex set of K_6 . Choose a set $V_1 = \{v_1, v_2, v_3, v_4\} \subseteq V$ of cardinality 4 and let G be the complete subgraph of K_6 on the vertex set V_1 . Let $V_2 = V \setminus V_1 = \{v, w\}$. Color all edges of G and $vv_1, vv_2, vv_3, wv_2, wv_3$ by distinct colors, color the edges vv_4, wv_4, wv_1 by $c(vv_1)$ and color the edge vw by $c(v_1v_4)$. Notice that there is no rainbow $K_4 \cup P_2$ in the coloring (see Figure 1). And we can see that the number of colors is 11. This implies that $AR(K_6, K_4 \cup P_2) \ge 11$.



Figure 1. Edge-coloring of K_6 .

Now we prove the inequality $AR(K_6, K_4 \cup P_2) \leq 11$. We only need to show that any 12-edge-coloring of K_6 contains a rainbow $K_4 \cup P_2$. On the contrary, we assume that there is a 12-edge-coloring of K_6 without any rainbow $K_4 \cup P_2$. By Theorem 1, $AR(K_6, K_4) = 10$. So it is easy to get the graph K_6 contains a rainbow subgraph H isomorphic to K_4 . Let $V(H) = \{v_1, v_2, v_3, v_4\}$ and D = $V(K_6) \setminus V(H) = \{v, w\}$. Hence we have $c(vw) \subseteq c(H)$ and |l(D)| = 6. It is obvious that K_6 contains a rainbow $K_4 \cup P_2$ when there exists a vertex $x \in D$ such that |l(x)| = 4. Hence |l(v)| = |l(w)| = 3. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup P_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Notice that K_6 does not contain any rainbow $K_4 \cup P_2$. Consider the subgraph $K_6[v_1, v_3, v_4, w] \cup K_6[v, v_2]$, a copy of $K_4 \cup P_2$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\})$. Similarly, consider the subgraph $K_6[v_2, v_3, v_4, w] \cup K_6[v, v_1]$, we have $c(wv_4) \in c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\})$. Consider the subgraph $K_6[v_1, v_2, v_4, w] \cup K_6[v, v_3]$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_4, v_1v_4, v_1v_3, vv_2, vv_3\})$. This easily yields a contradiction, since $c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\}) \cap c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, vv_3\}) = \emptyset$. Hence any 12-edge-coloring of K_6 contains a rainbow $K_4 \cup P_2$, i.e., $AR(K_6, K_4 \cup P_2) \leq 11$.

Lemma 3. For any integer $t \ge 1$ and $n \ge \max\{7, 2t+4\}$, $AR(K_n, K_4 \cup tP_2) \ge \max\{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, (t+1)(n-t-1) + \binom{t+1}{2} + 1\}.$

Proof. In order to show the lower bound, we find an edge-coloring of K_n without rainbow $K_4 \cup tP_2$. Given a complete graph K_n , color the edges of a spanning subgraph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ by distinct colors and then color all the other edges by a same new color. So it is obvious that the edge-coloring contains $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ colors and does not contain any rainbow K_4 . Hence, $AR(K_n, K_4 \cup tP_2) \ge \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$.

Let V be the vertex set of K_n . Choose a set $V_1 \subseteq V$ of cardinality n-t-1 and let G be the complete subgraph of K_n on the vertex set V_1 . Color all the edges of G by the same color and then color all the other edges by distinct new colors. Then we get a $((t+1)(n-t-1) + {t+1 \choose 2} + 1)$ -edge-coloring of K_n . Clearly, any rainbow K_4 contains at most two vertices of V_1 . Take a rainbow K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$. Suppose that it contains two vertices of V_1 , say $v_1, v_2 \in V \setminus V_1$ and $v_3, v_4 \in V_1$. Then any rainbow tP_2 in the graph $K_n - \{v_1, v_2, v_3, v_4\}$ contains an edge with the same color as v_3v_4 . The other cases can be verified in the same way. In a word, there is no rainbow $K_4 \cup tP_2$ in the coloring. This implies that $AR(K_n, K_4 \cup tP_2) \ge (t+1)(n-t-1) + {t+1 \choose 2} + 1$.

Let t_n be the integer part of smaller root of $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1 = (t+1)(n-t-1) + \binom{(t+1)}{2} + 1$. By calculation, we can get

$$t_n = \begin{cases} \left\lfloor \frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} \right\rfloor = \frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} - \alpha_1, & \text{if } n = 2k; \\ \left\lfloor \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} \right\rfloor = \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} - \alpha_2, & \text{if } n = 2k+1, \end{cases}$$

where $0 \le \alpha_1 < 1, 0 \le \alpha_2 < 1$. We have the following result.

Theorem 4. For any integer $n \ge \max\{7, 2t + 4\}$, $AR(K_n, K_4 \cup tP_2) = f(n, t)$, where

$$f(n,t) = \begin{cases} \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, & \text{if } t \le t_n; \\ (t+1)(n-t-1) + \binom{t+1}{2} + 1, & \text{if } t > t_n. \end{cases}$$

Next we prove this theorem by induction on t and n. From Lemma 3, we only need to show that each (f(n,t)+1)-edge-coloring of K_n contains a rainbow $K_4 \cup tP_2$. We divide the proof into the following subsections.

2.1. Proof for the case t = 1

By Lemma 3, we have $AR(K_n, K_4 \cup P_2) \ge \max\left\{ \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, 2n-2 \right\} = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ 1. Now we prove the inequality $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. When n = 7, we assume that there is a 14-edge-coloring of K_7 which doesn't contain any rainbow $K_4 \cup P_2$. By Theorem 1, it is easy to get that the graph K_7 contains a rainbow subgraph H isomorphic to K_4 . Let $V(H) = \{v_1, v_2, v_3, v_4\}$ and $D = V(K_7) \setminus V(H) = \{u, v, w\}$. Hence we have $c(K_7[D]) \subseteq c(H)$ and |l(D)| = 8. It is obvious that K_7 contains a rainbow $K_4 \cup P_2$ when there exists a vertex $x \in D$ such that |l(x)| = 4. Hence, without loss of generality, we can assume that |l(v)| = |l(w)| = 3 and |l(u)| = 2. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup P_2$. Hence we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Notice that K_7 does not contain any rainbow $K_4 \cup P_2$. Consider the subgraph $K_7[v_1, v_3, v_4, w] \cup K_7[v, v_2]$, a copy of $K_4 \cup P_2$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_3, v_3v_4, wv_1, wv_3, vv_2\})$. Similarly, consider the subgraph $K_7[v_2, v_3, v_4, w] \cup K_7[v, v_1]$, we have $c(wv_4) \in c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\})$. Consider the subgraph $K_7[v_1, v_2, v_4, w] \cup K_7[v, v_3]$, we have $c(wv_4) \in c(\{v_1v_4, v_1v_2, v_3\})$ v_2v_4, wv_1, wv_2, vv_3). This easily yields a contradiction, since $c(\{v_1v_4, v_1v_3, v_3v_4, v_1v_3, v_2v_4, v_2v_4, v_1v_3, v_2v_4, v_2v_4, v_1v_3, v_2v_4, v_$ $wv_1, wv_3, vv_2\}) \cap c(\{v_2v_4, v_2v_3, v_3v_4, wv_3, wv_2, vv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, wv_1, wv_2, wv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, wv_2, wv_1\}) \cap c(\{v_1v_4, v_1v_2, v_2v_4, wv_1, wv_2, wv_2, wv_1\})$ vv_3) = \emptyset . Hence any 14-edge-coloring of K_7 contains a rainbow $K_4 \cup P_2$, i.e., $AR(K_7, K_4 \cup P_2) \leq 14$. This completes the proof of theorem when n = 7 and t = 1.

Let $n \geq 8$ and c be a $\left(\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + 2\right)$ -edge-coloring of K_n . We assume that c does not contain any rainbow $K_4 \cup P_2$. Notice that the graph K_n contains a rainbow subgraph H isomorphic to K_4 . Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that $|c(K_{n-1})| \leq \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1$. Hence the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \ge \left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2 \right) - \left(\left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil + 1 \right).$$

Notice that if n = 2k, the saturated degree of each vertex of K_n is at least $k+1 = \frac{n}{2} + 1$. If n = 2k+1, the saturated degree of each vertex of K_n is at least

 $k+1 = \frac{n+1}{2}$. Hence we only need to check the cases n = 2k and n = 2k+1 and below we distinguish these two cases.

Case 1. n = 2k. It is obvious that K_n contains a rainbow $K_4 \cup P_2$ when there exists a vertex $v \in D$ such that |l(v)| = 4. Hence for any $v \in D$, we have $k + 1 \leq |l(v)| \leq 3$. Hence $n \leq 4$, a contradiction. So when n = 2k, $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1$.

Case 2. n = 2k + 1. It is obvious that K_n contains a rainbow $K_4 \cup P_2$ when there exists a vertex $v \in D$ such that |l(v)| = 4. Hence for any $v \in D$, we have $k + 1 \leq |l(v)| \leq 3$. Hence $n \leq 5$, a contradiction. So when n = 2k + 1, $AR(K_n, K_4 \cup P_2) \leq \lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1$. This completes the proof of the theorem.

2.2. Proof for the case $2 \le t \le t_n$

Now we prove the inequality $AR(K_n, K_4 \cup tP_2) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$. Let c be a $(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2)$ -edge-coloring of K_n . Then we assume that c does not contain any rainbow $K_4 \cup tP_2$. Since $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2 > AR(K_n, K_4 \cup (t-1)P_2)$, we have that c contains a rainbow subgraph H, a disjoint union of H_1 and H_2 , where $H_1 = (t-1)P_2$ and $H_2 = K_4$. Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that

$$AR(K_{n-1}, K_4 \cup tP_2) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1, & \text{if } t \le t_{n-1}; \\ (t+1)(n-t-2) + \binom{t+1}{2} + 1, & \text{if } t > t_{n-1}. \end{cases}$$

Notice that when n = 2k,

$$\frac{(2n-3)-\sqrt{2(n-1)^2}}{2} \le \frac{(2n-3)-\sqrt{2(n-1)^2-1}}{2} \le \frac{(2n-3)-(\sqrt{2n^2}-\frac{3}{2})}{2}.$$

When n = 2k + 1,

$$\frac{(2n-3) - (\sqrt{2n^2} - 1)}{2} \le \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} \le \frac{(2n-3) - \sqrt{2(n-1)^2}}{2}$$

Hence we can get when n = 2k,

$$\frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2 + 1}}{2}$$

$$\geq \frac{(2n-3) - \sqrt{2(n-1)^2}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2}}{2} = \frac{2 - \sqrt{2}}{2} > 0.$$

When n = 2k + 1, we have

$$\frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2 - 1}}{2}$$

$$\geq \frac{(2n-3) - (\sqrt{2n^2} - 1)}{2} - \frac{(2(n-1) - 3) - (\sqrt{2(n-1)^2} - \frac{3}{2})}{2} = \frac{\frac{3}{2} - \sqrt{2}}{2} > 0$$

Hence when n = 2k or n = 2k + 1, we have $t_n - t_{n-1} \ge 0$. Suppose

$$t_{n-1} = \begin{cases} \left\lfloor \frac{(2n-5)-\sqrt{2(n-2)^2-1}}{2} \right\rfloor = \frac{(2n-5)-\sqrt{2(n-2)^2-1}}{2} - \beta_1, & \text{if } n-1 = 2k; \\ \left\lfloor \frac{(2n-5)-\sqrt{2(n-2)^2+1}}{2} \right\rfloor = \frac{(2n-5)-\sqrt{2(n-2)^2+1}}{2} - \beta_2, & \text{if } n-1 = 2k-1, \end{cases}$$

where $0 \leq \beta_1 < 1$ and $0 \leq \beta_2 < 1$. Therefore when n = 2k, we can get

$$t_n - t_{n-1} \le \left(\frac{(2n-3) - \left(\sqrt{2n^2} - \frac{3}{2}\right)}{2} - \alpha_1\right) \\ - \left(\frac{(2(n-1)-3) - \left(\sqrt{2(n-1)^2} - 1\right)}{2} - \beta_2\right) \\ \le \frac{1 + \frac{3}{2} - \sqrt{2}}{2} + \beta_2 - \alpha_1 < 2.$$

When n = 2k + 1, we can get

$$t_n - t_{n-1} \le \frac{(2n-3) - \sqrt{2(n-1)^2}}{2} - \alpha_2) - \frac{(2(n-1)-3) - \sqrt{2(n-2)^2}}{2} - \beta_1)$$
$$\le \frac{2 - \sqrt{2}}{2} + \beta_1 - \alpha_2 < 2.$$

Hence we get $0 \le t_n - t_{n-1} \le 1$.

Notice that if $t \leq t_{n-1}$, then $AR(K_{n-1}, K_4 \cup tP_2) = \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1$. If $t > t_{n-1}$, then $AR(K_{n-1}, K_4 \cup tP_2) = (t+1)(n-t-1) + {t+1 \choose 2} + 1$. Hence we only need to check the cases $t \leq t_{n-1}$ and $t > t_{n-1}$ and below we distinguish these two cases.

Case 1. $t \leq t_{n-1}$. Notice that $AR(K_{n-1}, K_4 \cup tP_2) = \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1$ for any $t \leq t_{n-1}$. Since c does not contain any rainbow $K_4 \cup tP_2$, we have $|c(K_{n-1})| \leq \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1$. Hence the saturated degree of each vertex of K_n is at least $(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2) - (\lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil + 1)$. Notice that the lower bound above is different when n is even or odd. Hence we need to check the cases n = 2k and n = 2k + 1 and below we distinguish these two cases.

Subcase 1.1. n = 2k. Since $n \ge 4 + 2t$, we can get $|D| \ge 2$. In this case, the saturated degree of each vertex of K_n is at least

$$\left(\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil+2\right)-\left(\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil+1\right)=k+1=\frac{n}{2}+1.$$

Also it follows from $|l(v)| \ge \frac{n}{2} + 1 > 4$ that $|l_1(H, v)| > 0$ for each $v \in D$. For each $v \in D$ and E_v , by $|l(v)| \ge k+1$, we have $2|A_v|+|B_v| \ge (k+1)-|V(H_2)| = k-3$. Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $l(v) \ge k+1$, we have $|B_v| \ge k-3$. So $|V(H)| \ge 2|B_v|+|V(H_2)| \ge 2k-6+4 = n-2$. This implies that |D| = 2 and hence n = 2t + 4. Let $D = \{v, w\}$. Hence the equalities above must hold. This implies that $|B_v| = k - 3$ and |V(H)| = 2t + 2 = n - 2 = 2k - 2. Then $|B_v| = k - 3 = t - 1$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$ and $c(vv_i) \ne c(vv_j)$ for $i \ne j$. If there is a set E_w such that $A_w \ne \emptyset$, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_w = \emptyset$ for any set E_w . Since $l(w) \ge k + 1$ and $|B_w| \le t - 1 = k - 3$, we have that $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(wv_i) \ne c(wv_j)$ for $i \ne j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$. Hence we can assume that $A_x \ne \emptyset$ for each vertex $x \in D$ and any set E_x .

Suppose that $B_x = \emptyset$ for each vertex $x \in D$ and any set E_x . Hence for each vertex $x \in D$ and any set E_x , by $|l(x)| \ge k + 1$, we have $2|A_x| \ge k - 3$. Take two vertices $v, w \in D$. When $A_v \cap A_w \neq \emptyset$, it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_v \cap A_w = \emptyset$, and then

$$|V(H)| \ge 2|A_v| + 2|A_w| + |V(H_2)| \ge (k-3) + (k-3) + 4 = n-2.$$

This implies that |D| = 2, i.e., $D = \{v, w\}$. Hence the equalities above must hold. This implies that $2|A_v| = 2|A_w| = k-3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Hence there is a vertex $v \in D$ and a set E_v such that $B_v \neq \emptyset$. Take any vertex $w \in D$, $w \neq v$, and any set E_w . When $(A_v \cup B_v) \cap (A_w \cup B_w) = \emptyset$, we have

$$|V(H)| \ge 2|A_v| + 2|B_v| + 2|A_w| + 2|B_w| + |V(H_2)|$$

$$\ge (k - 3 + |B_v|) + (k - 3 + |B_w|) + 4$$

$$= n - 2 + |B_v| + |B_w| > n - 2,$$

a contradiction. This implies that $(A_v \cup B_v) \cap (A_w \cup B_w) \neq \emptyset$. If $A_v \cap (A_w \cup B_w) \neq \emptyset$ or $A_w \cap (A_v \cup B_v) \neq \emptyset$, we can easily find a rainbow $K_4 \cup tP_2$. Hence $A_v \cap (A_w \cup B_w) = \emptyset$ and $A_w \cap (A_v \cup B_v) = \emptyset$. This implies that $A_v \cap A_w = \emptyset$, $A_v \cap B_w = \emptyset$, $B_v \cap A_w = \emptyset$ and $B_v \cap B_w \neq \emptyset$. When there exists an edge in $B_v \cap B_w$ such that E_v and E_w cover distinct vertices of it, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that E_v and E_w cover one identical vertex of each edge in $B_v \cap B_w$, then we have

$$|V(H)| \ge 2|A_v| + 2|B_v| + 2|A_w| + 2(|B_w| - |B_v \cap B_w|) + |V(H_2)|$$

$$\ge (k - 3 + |B_v|) + (k - 3 + |B_w| - 2|B_v \cap B_w|) + 4$$

$$= n - 2 + |B_v| + |B_w| - 2|B_v \cap B_w| \ge n - 2.$$

Since $|D| \ge 2$, it follows that |D| = 2. Hence $D = \{v, w\}$ and $|B_v| = |B_w| = |B_v \cap B_w|$. Hence the equalities above must hold. This implies that $2|A_v| + |B_v| = k - 3$ and $2|A_w| + |B_w| = k - 3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \ne c(vv_j)$, $c(wv_i) \ne c(wv_j)$ for $i \ne j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Subcase 1.2. n = 2k + 1. Together with $n \ge 4 + 2t$, we can get $|D| \ge 3$. In this case, the saturated degree of each vertex of K_n is at least

$$\left(\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil+2\right)-\left(\left\lfloor\frac{n-1}{2}\right\rfloor\left\lceil\frac{n-1}{2}\right\rceil+1\right)=k+1=\frac{n+1}{2}$$

Also it follows from $|l(v)| \ge \frac{n+1}{2} > 4$ that $|l_1(H, v)| > 0$ for any $v \in D$. For each $v \in D$ and any set E_v , by $|l(v)| \ge k+1$, we have $2|A_v|+|B_v| \ge (k+1)-|V(H_2)| = k-3$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $l(v) \ge k+1$, we have $|B_v| \ge k-3$. So $|V(H)| \ge 2|B_v|+|V(H_2)| \ge 2k-6+4 = n-3$. This implies that |D| = 3. Hence the equalities above must hold. This implies that $|B_v| = k-3$ and |V(H)| = 2t+2 = n-3 = 2k-2. Then $|B_v| = k-3 = t-1$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$ and $c(vv_i) \ne c(vv_j)$ for $i \ne j$. Take any vertex $w \in D$ and $w \ne v$. If there is a set E_w such that $A_w \ne \emptyset$, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_w = \emptyset$ for any set E_w . Since $l(w) \ge k+1$ and $|B_w| \le t-1 = k-3$, we have that $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(wv_i) \ne c(wv_j)$ for $i \ne j$. Then we get that $K_n[v_1, v_2, v_3, w] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$. Hence we can assume that $A_x \ne \emptyset$ for each vertex $x \in D$ and any set E_x .

Suppose that $B_x = \emptyset$ for each vertex $x \in D$ and any set E_x . Hence for any vertex $x \in D$ and any set E_x , by $|l(x)| \ge k + 1$, we have $2|A_x| \ge k - 3$. Take two vertices $v, w \in D$. When $A_v \cap A_w \neq \emptyset$, it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that $A_v \cap A_w = \emptyset$, and then

$$|V(H)| \ge 2|A_v| + 2|A_w| + |V(H_2)| \ge (k-3) + (k-3) + 4 = n - 3.$$

This implies that |D| = 3. Hence the equalities above must hold. This implies that $2|A_v| = 2|A_w| = k - 3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v), c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j), c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Hence there is a vertex $v \in D$ and a set E_v such that $B_v \neq \emptyset$. Take any vertex $w \in D$, $w \neq v$, and any set E_w . When $(A_v \cup B_v) \cap (A_w \cup B_w) = \emptyset$, we have

$$\begin{aligned} |V(H)| &\geq 2|A_v| + 2|B_v| + 2|A_w| + 2|B_w| + |V(H_2)| \\ &\geq (k - 3 + |B_v|) + (k - 3 + |B_w|) + 4 \\ &= n - 3 + |B_v| + |B_w| > n - 3, \end{aligned}$$

a contradiction. This implies that $(A_v \cup B_v) \cap (A_w \cup B_w) \neq \emptyset$. If $A_v \cap (A_w \cup B_w) \neq \emptyset$ or $A_w \cap (A_v \cup B_v) \neq \emptyset$, we can easily find a rainbow $K_4 \cup tP_2$. Hence $A_v \cap (A_w \cup B_w) = \emptyset$ and $A_w \cap (A_v \cup B_v) = \emptyset$. This implies that $A_v \cap A_w = \emptyset$, $A_v \cap B_w = \emptyset$, $B_v \cap A_w = \emptyset$ and $B_v \cap B_w \neq \emptyset$. When there exists an edge in $B_v \cap B_w$ such that E_v and E_w cover distinct vertices of it, then it is easy to find a rainbow $K_4 \cup tP_2$. So we can assume that E_v and E_w cover one identical vertex of each edge in $B_v \cap B_w$, then we have

$$|V(H)| \ge 2|A_v| + 2|B_v| + 2|A_w| + 2(|B_w| - |B_v \cap B_w|) + |V(H_2)|$$

$$\ge (k - 3 + |B_v|) + (k - 3 + |B_w| - 2|B_v \cap B_w|) + 4$$

$$= n - 3 + |B_v| + |B_w| - 2|B_v \cap B_w| \ge n - 3.$$

Since $|D| \geq 3$, it follows that |D| = 3. So $|B_v| = |B_w| = |B_v \cap B_w|$. Hence the equalities above must hold. This implies that $2|A_v| + |B_v| = k - 3$ and $2|A_w| + |B_w| = k - 3$. Thus we have that $c(\{vv_1, vv_2, vv_3, vv_4\}) \subseteq l(v)$, $c(\{wv_1, wv_2, wv_3, wv_4\}) \subseteq l(w)$ and $c(vv_i) \neq c(vv_j)$, $c(wv_i) \neq c(wv_j)$ for $i \neq j$. Then we get that $K_n[v_1, v_2, v_3, v] \cup K_n[v_4, w] \cup H_1$ is a rainbow $K_4 \cup tP_2$.

Case 2. $t > t_{n-1}$. From $0 \le t_n - t_{n-1} \le 1$ and $t \le t_n$, we have $t = t_n = t_{n-1} + 1$. Then $AR(K_{n-1}, K_4 \cup tP_2) = (t+1)(n-2-t) + {t+1 \choose 2} + 1$. So $|c(K_{n-1})| \le (t+1)(n-2-t) + {t+1 \choose 2} + 1$. Hence the saturated degree of each vertex of K_n is at least $(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 2) - ((t+1)(n-2-t) + {t+1 \choose 2} + 1)$.

When n = 2k,

$$t = t_n = \frac{(2n-3) - \sqrt{2(n-1)^2 - 1}}{2} - \alpha_1.$$

We can get

$$\frac{n^2}{4} - n - nt - n\alpha_1 + \frac{3t}{2} + \frac{3\alpha_1}{2} + \frac{t^2}{2} + t\alpha_1 + \frac{\alpha_1^2}{2} + 1 = 0.$$

Hence we have that for each vertex $v \in V(K_n)$,

$$|l(v)| \ge \left(\left\lfloor\frac{n}{2}\right\rfloor \left\lceil\frac{n}{2}\right\rceil + 2\right) - \left((t+1)(n-2-t) + \binom{t+1}{2} + 1\right)$$
$$= \frac{n^2}{4} - n - nt + \frac{5t}{2} + \frac{t^2}{2} + 3 = t + n\alpha_1 - \frac{3\alpha_1}{2} - t\alpha_1 - \frac{\alpha_1^2}{2} + 2 \ge t + 2.$$

When n = 2k + 1,

$$t = t_n = \frac{(2n-3) - \sqrt{2(n-1)^2 + 1}}{2} - \alpha_2$$

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We can get

$$\frac{n^2}{4} - n - nt - n\alpha_2 + \frac{3t}{2} + \frac{3\alpha_2}{2} + \frac{t^2}{2} + t\alpha_2 + \frac{\alpha_2^2}{2} + \frac{3}{4} = 0.$$

Hence we have that for each vertex $v \in V(K_n)$,

$$|l(v)| \ge \left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2\right) - \left((t+1)(n-2-t) + \binom{t+1}{2} + 1\right)$$
$$= \frac{n^2}{4} - n - nt + \frac{5t}{2} + \frac{t^2}{2} + 3 - \frac{1}{4}$$
$$= t + n\alpha_2 - \frac{3\alpha_2}{2} - t\alpha_2 - \frac{\alpha_2^2}{2} + 2 \ge t + 2.$$

Hence we have the saturated degree of each vertex v of K_n is at least t + 2. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have that for $v, w \in D$ and sets $E_v, E_w, c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t - 1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Subcase 2.1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| \leq 1$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| \leq 1$, we have $|l_1(H,v)| \geq (t+2) - 1 = t+1$. Then $|l_1(H,w)| \leq 2(t-1) - |l_1(H,v)| \leq t-3$. Hence $|l_2(H,w)| \geq 5$, a contradiction.

Subcase 2.2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| = 2$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| = 2$, we have $|l_1(H,v)| \ge (t+2) - 2 = t$. Then $|l_1(H,w)| \le 2(t-1) - |l_1(H,v)| \le t-2$. Hence $|l_2(H,w)| = 4$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Subcase 2.3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| = 4$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| = 4$, we have $|l_1(H,v)| \geq (t+2) - 4 = t - 2$. Then $|l_1(H,w)| \leq 2(t-1) - |l_1(H,v)| \leq t$. Hence $|l_2(H,w)| \geq 2$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Subcase 2.4. For any $x \in D$, $|l_2(H,x)| = 3$. Then $|l_1(H,x)| \ge t-1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H,v)| \ge t-1$, we have $|l_1(H,w)| \le 2(t-1) - |l_1(H,v)| \le t-1$. Then $|l_1(H,w)| = t-1$ and hence $|l_1(H,x)| = t-1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H,v)| = t - 1$, we have $|B_v| = t - 1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w . Since $c(yw) \notin l_1(H,w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$, by $|B_v| = t - 1$, we have $|A_w| = \emptyset$ and $|B_w| = t - 1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1 . If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that c(vw) = c(xy), $xy \in H_1$ with $x \in V(H_1) \setminus (V(B_v) \cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t + 2 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t + 1$, a contradiction.

Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $K_4 \cup tP_2$. Consider the subgraph $K_n[v_2, v_3, v, w] \cup K_n[v_1, v_4] \cup H_1$, a copy of $K_4 \cup tP_2$, we have $c(vw) \in \{c(v_2v_3), c(v_1v_4)\}$. Similarly, consider the subgraph $K_n[v_1, v_2, v, w] \cup K_n[v_3, v_4] \cup H_1$, we have $c(vw) \in \{c(v_1v_2), c(v_3v_4)\}$. This easily yields a contradiction, since $\{c(v_1v_4), c(v_2v_3)\} \cap \{c(v_1v_2), c(v_3v_4)\} = \emptyset$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x . Take two vertices $v, w \in D$. Notice that for any sets $E_v, E_w, c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$. By $|l_1(H,v)| = |l_1(H,w)| = t - 1$, we can deduce that $|A_v| = |A_w|$ and $|B_v| = |B_w|$. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Then for any edge $xy \in A_v$ and any edge xzwith $c(xz) \in l(x)$, we have $z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(A_v) \setminus \{x\}) \cup \{v\} \cup \{v\}$ $(V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\} = t + 2$. Hence the equality holds, which implies that $l(x) = c(\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{xz|z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Similarly, $l(y) = c(\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Hence we can get that $K_n[v_1, x, y, v] \cup K_n[v_2, v_4] \cup K_n[v_3, w] \cup (H_1 - xy)$ is a rainbow $K_4 \cup tP_2$.

2.3. Proof for the case $t > t_n$

Now we prove the inequality $AR(K_n, K_4 \cup tP_2) \leq (t+1)(n-1-t) + {t+1 \choose 2} + 1$. By the induction hypothesis we have

$$AR(K_n, K_4 \cup (t-1)P_2) = \begin{cases} \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1, & \text{if } t-1 \le t_n; \\ t(n-t) + {t \choose 2} + 1, & \text{if } t-1 > t_n. \end{cases}$$

Let c be a $((t+1)(n-1-t) + {t+1 \choose 2} + 2)$ -edge-coloring of K_n . We assume that c does not contain any rainbow $K_4 \cup tP_2$. Since $(t+1)(n-1-t) + {t+1 \choose 2} + 2 > AR(K_n, K_4 \cup (t-1)P_2)$ when $t-1 = t_n$ or $t-1 > t_n$, so it is obvious c contains a rainbow subgraph H, a disjoint union of H_1 and H_2 , where $H_1 = (t-1)P_2$ and $H_2 = K_4$. Let $D = V(K_n) \setminus V(H)$ and it is easy to check that $c(K_n[D]) \subseteq c(H)$. Clearly, any subgraph K_{n-1} has no rainbow $K_4 \cup tP_2$. Then by the induction hypothesis on n, we have that $|c(K_{n-1})| \leq (t+1)(n-2-t) + {t+1 \choose 2} + 1$. Hence the saturated degree of each vertex v of K_n satisfies

$$l(v) \ge ((t+1)(n-1-t) + \binom{t+1}{2} + 2) - ((t+1)(n-2-t) + \binom{t+1}{2} + 1) = t+2.$$

Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have that for $v, w \in D$ and sets $E_v, E_w, c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Case 1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| \leq 1$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| \leq 1$, we have $|l_1(H,v)| \geq (t+2) - 1 = t+1$. Then $|l_1(H,w)| \leq 2(t-1) - |l_1(H,v)| \leq t-3$. Hence $|l_2(H,w)| \geq 5$, a contradiction.

Case 2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 2$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H, v)| = 2$, we have $|l_1(H, v)| \ge (t+2)-2=t$. Then $|l_1(H, w)| \le 2(t-1)-|l_1(v)| \le t-2$. Hence $|l_2(H, w)| = 4$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Case 3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 4$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H, v)| = 4$, we have $|l_1(H, v)| \ge (t+2)-4 = t-2$. Then $|l_1(H, w)| \le 2(t-1)-|l_1(H, v)| \le t$. Hence $|l_2(H, w)| \ge 2$. It is obvious that we can find a rainbow $K_4 \cup tP_2$, a contradiction.

Case 4. For any $x \in D$, $|l_2(H,x)| = 3$. Then $|l_1(H,x)| \ge t - 1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H,v)| \ge t - 1$, we have $|l_1(H,w)| \le 2(t-1) - |l_1(H,v)| \le t - 1$. Then $|l_1(H,w)| = t - 1$ and hence $|l_1(H,x)| = t - 1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H,v)| = t - 1$, we have $|B_v| = t - 1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w . Since $c(yw) \notin l_1(H,w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$, by $|B_v| = t - 1$, we have $|A_w| = \emptyset$ and $|B_w| = t - 1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1 . If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Since K_n does not contain any rainbow $K_4 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that c(vw) = c(xy), $xy \in H_1$ with $x \in V(H_1) \setminus (V(B_v) \cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t + 2 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2, v_3\}| = t + 1$, a contradiction. Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $K_4 \cup tP_2$. Consider the subgraph $K_n[v_2, v_3, v, w] \cup K_n[v_1, v_4] \cup H_1$, a copy of $K_4 \cup tP_2$, we have $c(vw) \in \{c(v_2v_3), c(v_1v_4)\}$. Similarly, consider the subgraph $K_n[v_1, v_2, v, w] \cup K_n[v_3, v_4] \cup H_1$, we have $c(vw) \in \{c(v_1v_2), c(v_3v_4)\}$. This easily yields a contradiction, since $\{c(v_1v_4), c(v_2v_3)\} \cap \{c(v_1v_2), c(v_3v_4)\} = \emptyset$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x . Take two vertices $v, w \in D$. Notice that for any sets $E_v, E_w, c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$. By $|l_1(H,v)| = |l_1(H,w)| = t-1$, we can deduce that $|A_v| = |A_w|$ and $|B_v| = |B_w|$. If there is a vertex v_i for some $1 \leq i \leq 4$ which saturates exactly one of v and w, then it is easy to find a rainbow $K_4 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2, v_3\}$ saturates both v and w. Then for any edge $xy \in A_v$ and any edge xzwith $c(xz) \in l(x)$, we have $z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}$, otherwise we can get a rainbow $K_4 \cup tP_2$. So $t+2 \leq |l(x)| \leq |(V(A_v) \setminus \{x\}) \cup \{v\} \cup \{v\}$ $(V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\} = t + 2$. Hence the equality holds, which implies that $l(x) = c(\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{xz | z \in (V(A_v) \setminus \{x\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Similarly, $l(y) = c(\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\})$ and the edge set $\{yz | z \in (V(A_v) \setminus \{y\}) \cup \{v\} \cup (V(B_v) \cap V(E_v)) \cup \{v_1, v_2, v_3\}\}$ is rainbow. Hence we can get that $K_n[v_1, x, y, v] \cup K_n[v_2, v_4] \cup K_n[v_3, w] \cup (H_1 - xy)$ is a rainbow $K_4 \cup tP_2$, a contradiction. This completes the proof of the result.

3. Result for $C_3 \cup tP_2$

Bialostocki, Gilboa and Roditty [2] proved the following theorem for $C_3 \cup P_2$.

Theorem 5 [2]. $AR(K_n, C_3 \cup P_2) = \max\{n, 6\}$ for any $n \ge 5$.

Moreover, Gilboa and Roditty [7] obtained the result for $C_3 \cup tP_2$, $t \ge 1$, in K_n for large enough n.

Theorem 6 [7]. For any integers $t \ge 1$ and $n \ge \frac{5}{2}t + 5$, $AR(K_n, C_3 \cup tP_2) = t(n-t) + {t \choose 2} + 1$.

Here, we show that the formula in the theorem above holds for all $n \ge \max\{6, 2t+3\}$.

Theorem 7. For any integers $t \ge 2$ and $n \ge 2t + 3$, $AR(K_n, C_3 \cup tP_2) = t(n-t) + {t \choose 2} + 1$.

Proof. In order to show the lower bound, we find an edge-coloring of K_n without rainbow $C_3 \cup tP_2$. Let V be the vertex set of K_n . Choose a set $V_1 \subseteq V$ of cardinality n - t and let G be the complete subgraph of K_n on the vertex set

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 V_1 . Color all the edges of G by the same color and then color all the other edges by distinct new colors. Then we get a $(t(n-t) + {t \choose 2} + 1)$ -edge-coloring of K_n . Clearly, any rainbow C_3 contains at most two vertices of V_1 . Take a rainbow C_3 with vertex set $\{v_1, v_2, v_3\}$. Suppose that it contains two vertices of V_1 , say $v_1 \in V \setminus V_1$ and $v_2, v_3 \in V_1$. Then any rainbow tP_2 in the graph $K_n - \{v_1, v_2, v_3\}$ contains an edge with the same color as v_2v_3 . The other cases can be verified in the same way. In a word, there is no rainbow $K_4 \cup tP_2$ in the coloring. This implies that $AR(K_n, C_3 \cup tP_2) \geq t(n-t) + {t \choose 2} + 1$.

We prove this by induction on t and n. When t = 1, by Theorem 5, $AR(K_n, C_3 \cup P_2) = n$ for any $n \ge 6$. Now let $t \ge 2$ and c be a $(t(n-t) + {t \choose 2} + 2)$ edge-coloring of K_n . We need to show that c contains a rainbow $C_3 \cup tP_2$. On the contrary, we assume that c does not contain any rainbow $C_3 \cup tP_2$. By the induction hypothesis on t, it is obvious that c contains a rainbow subgraph H, a disjoint union of H_1 and H_2 , where $H_1 = (t-1)P_2$ and $H_2 = C_3$. Let $D = V(K_n) \setminus V(H)$. Clearly, any subgraph K_{n-1} has no rainbow $C_3 \cup tP_2$. Then by the induction hypothesis on n, we have $|c(K_{n-1})| \le t(n-1-t) + {t \choose 2} + 1$. Hence the saturated degree of each vertex v of K_n satisfies

$$|l(v)| \ge (t(n-t) + \binom{t}{2} + 2) - (t(n-1-t) + \binom{t}{2} + 1) = t + 1.$$

Since K_n does not contain any rainbow $C_3 \cup tP_2$, we have that for $v, w \in D$ and sets $E_v, E_w, c(yw) \notin l_1(H, w)$ if $c(xv) \in l_1(H, v)$ for any $xy \in H_1$. This implies that $|l_1(H, w)| \leq 2(t-1) - |l_1(H, v)|$. Next we distinguish cases on $|l_2(H, v)|$ for $v \in D$.

Case 1. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| = 0$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| = 0$, we have $|l_1(H,v)| \geq t + 1$. Then $|l_1(H,w)| \leq 2(t-1) - |l_1(H,v)| \leq t - 3$. Hence $|l_2(H,w)| \geq 4$, a contradiction.

Case 2. There is a vertex $v \in D$ and a set E_v such that $|l_2(H, v)| = 1$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H, v)| = 1$, we have $|l_1(H, v)| \geq (t+1)-1 = t$. Then $|l_1(H, w)| \leq 2(t-1)-|l_1(H, v)| \leq t-2$. Hence $|l_2(H, w)| = 3$. It is obvious that we can find a rainbow $C_3 \cup tP_2$, a contradiction.

Case 3. There is a vertex $v \in D$ and a set E_v such that $|l_2(H,v)| = 3$. Let $w \in D$ and $w \neq v$ and take a set E_w . Since $|l_2(H,v)| = 3$, we have $|l_1(H,v)| \ge (t+1)-3 = t-2$. Then $|l_1(H,w)| \le 2(t-1)-|l_1(H,v)| \le t$. Hence $|l_2(H,w)| \ge 1$. It is obvious that we can find a rainbow $C_3 \cup tP_2$, a contradiction.

Case 4. For any $x \in D$, $|l_2(H,x)| = 2$. Then $|l_1(H,x)| \ge t - 1$ for each $x \in D$. Take any two vertices $v, w \in D$. Since $|l_1(H,v)| \ge t - 1$, we have $|l_1(H,w)| \le 2(t-1) - |l_1(H,v)| \le t - 1$. Then $|l_1(H,w)| = t - 1$ and hence $|l_1(H,x)| = t - 1$ for any $x \in D$.

Suppose that there exists a vertex $v \in D$ and a set E_v such that $A_v = \emptyset$. By $|l_1(H,v)| = t - 1$, we have $|B_v| = t - 1$. Take any vertex $w \in D$, $w \neq v$, and any set E_w . Since $c(yw) \notin l_1(H,w)$ if $c(xv) \in l_1(H,v)$ for any $xy \in H_1$, by $|B_v| = t - 1$, we have $|A_w| = \emptyset$ and $|B_w| = t - 1$. Moreover, both E_v and E_w cover exactly one identical vertex of each edge in H_1 . If there is a vertex v_i for some $1 \leq i \leq 3$ which saturates exactly one of v and w, then it is easy to find a rainbow $C_3 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2\}$ saturates both v and w. Since K_n does not contain any rainbow $C_3 \cup tP_2$, we have $c(vw) \in c(H)$.

Suppose that c(vw) = c(xy), $xy \in H_1$ with $x \in V(H_1) \setminus (V(B_v) \cap V(E_v))$. For any edge xz with $c(xz) \in l(x)$, we have $z \in (V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2\}$, otherwise we can get a rainbow $C_3 \cup tP_2$. So $t + 1 \leq |l(x)| \leq |(V(B_v \setminus \{xy\}) \cap V(E_v)) \cup \{v_1, v_2\}| = t$, a contradiction.

Hence $c(vw) \in c(H_2)$. Notice that K_n does not contain any rainbow $C_3 \cup tP_2$. Consider the subgraph $K_n[v_2, v, w] \cup K_n[v_1, v_3] \cup H_1$, a copy of $C_3 \cup tP_2$, we have $c(vw) = c(v_1v_3)$. Similarly, consider the subgraph $K_n[v_1, v, w] \cup K_n[v_2, v_3] \cup H_1$, we have $c(vw) = c(v_2v_3)$. This easily yields a contradiction, since $c(v_1v_3) \neq c(v_2v_3)$.

Hence we can assume that $A_x \neq \emptyset$ for each vertex $x \in D$ and any set E_x . Take two vertices $v, w \in D$ and let $xy \in A_v$. If there is a vertex v_i for some $1 \leq i \leq 3$ which saturates exactly one of v and w, then it is easy to find a rainbow $C_3 \cup tP_2$. Hence, without loss of generality, we can assume that each vertex of $\{v_1, v_2\}$ saturates both v and w. Then we can get that $K_n[x, y, v] \cup K_n[w, v_1] \cup K_n[v_2, v_3] \cup (H_1 - xy)$ is a rainbow $C_3 \cup tP_2$, a contradiction. This completes the proof of the result.

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