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SPECTRAL BOUNDS FOR THE ZERO FORCING NUMBER OF A GRAPH

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Abstract

Let $Z(G)$ be the zero forcing number of a simple connected graph G. In this paper, we study the relationship between the zero forcing number of a graph and its (normalized) Laplacian eigenvalues. We provide the upper and lower bounds on $Z(G)$ in terms of its (normalized) Laplacian eigenvalues, respectively. Our bounds extend the existing bounds for regular graphs.

Keywords: zero forcing number, eigenvalue, bound.

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1. Introduction

All graphs considered in this paper are undirected and simple (i.e., without loops or multiple edges). Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Its *order* is $|V(G)|$, denoted by n, and its *size* is $|E(G)|$, denoted by m. For $v_i \in V(G)$, let $d_G(v_i)$ and $N_G(v_i)$ (or $d(v_i)$ and $N(v_i)$ for short) be the degree and the set of neighbors of v_i, respectively. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$ (or δ and Δ for short), respectively. We say G is *regular* if $\delta = \Delta = d$ and also call G a d-regular graph. For $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. Other undefined notations can be found in [3].

For a graph G, its vertices are colored with two different colors (white and black). Let $S \subseteq V(G)$ be the set of black vertices in G. If $u \in S$ and v is the only white neighbor of u, then u forces v to turn into black (color change rule). The set S is said to be a *zero forcing set* of G if by iteratively applying the color change rule, all vertices of G becomes black. The *zero forcing number* of G is the minimum cardinality of zero forcing sets of G, denoted by $Z(G)$.

The zero forcing number of G was introduced in [2] as the bounding of the minimum rank $mr(G)$ (and so the maximum nullity $M(G)$) of G. It was shown in [2] that $Z(G) \geq n - mr(G)$ (or $Z(G) \geq M(G)$) for any graph G, where $mr(G)$ is the smallest possible rank over all symmetric real matrices whose ij-th entry (for $i \neq j$) is nonzero whenever $ij \in E(G)$ in G and is zero otherwise and $M(G) = n - mr(G)$. This parameter has been extensively studied in over half a century, largely due to its connection to inverse eigenvalue problems for graphs and its applications to other problems. Up to now, there have been lots of research work on bounding the zero forcing number of a graph in terms of its various parameters, such as connected domination number [1], perfect domination number [9], degree sequence [7, 10], girth [11] and chromatic number [12], etc. We will not list them all here, but we will focus primarily on those related to the spectral bounds for the zero forcing number.

Very recently, Kalinowski, Kamčev and Sudakov [11] studied the relationship between the zero forcing number of a graph and its occurrence of a witness, where the witness in a graph is defined as follows.

Definition. For a graph G of order n, a k-*witness* (or a *witness of order* k) in G is a pair of ordered vertex k-tuples $\left((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\right)$ such that $s_i, t_i \in V(G)$, $s_i \sim t_i$ for each i, and $s_i \nsim t_j$ for $i < j$.

The above definition requires $s_i \neq s_j$ and $t_i \neq t_j$ for $i \neq j$, but it might happen that $s_i = t_j$ for some $i > j$. We will use the shortened notation $\mathbf{s} = (s_i)_{i\in[k]}$ and denote the image of this k-tuple by $\mathbf{s}[k] = \{s_i : i \in [k]\}$. We say a k-witness in G is *maximal* if G contains a k-witness but does not contain any $(k+1)$-witness.

Let $\mathcal{G}_n$ be the set of connected graphs of order n. We now divide $\mathcal{G}_n$ into the following two classes of graphs according to the parity of their maximal witness. Let $\mathcal{G}_n^e$ (or $\mathcal{G}_n^o$) be the set of connected graphs of order n containing a maximal k-witness with even k (or odd k), where $1 \leq k \leq n-1$. Clearly, $\mathcal{G}_n = \mathcal{G}_n^e \cup \mathcal{G}_n^o$.

For $G \in \mathcal{G}_n$, let $A(G)$ be the adjacency matrix of G. The eigenvalues of G are the eigenvalues of $A(G)$, and are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ (or $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ for short). An (n, d, λ)-*graph* is a d-regular graph of order n for which $|\lambda_i| \leq \lambda$ for $i = 2, 3, \ldots, n$.

By estimating the edge distribution of an (n, d, λ)-graph in terms of λ, Kalinowski, Kamčev and Sudakov [11] established the upper and lower bounds on $Z(G)$ in terms of λ when G is an (n, d, λ)-graph, respectively. Indeed, in their proof of the lower bound of $Z(G)$ [11, Theorem 1.4(i)], they would like to find the maximum possible value of k for a maximal k-witness in any $G \in \mathcal{G}_n$. From their proof, the k should be even. That is among all graphs $G \in \mathcal{G}_n^e$, they found the maximum possible value of k for a maximal k-witness in G. So we restate their result as follows.

Theorem 1 [11]**.** *For an (n, d, λ)-graph G, we have*

(i) *if $G \in \mathcal{G}_n^e$, then*

$$Z(G) \geq n\left(1 - \frac{2\lambda}{d+\lambda}\right); \tag{1}$$

(ii)

$$Z(G) \leq n\left(1 - \frac{1}{2(d-\lambda)}\log\left(\frac{d-\lambda}{2\lambda+1}\right)\right). \tag{2}$$

Moreover, the bound (1) *is tight, and the bound* (2) *is tight up to a constant factor.*

Moreover, Zhang, Wang, Wang and Ji [13] determined the graphs (respectively, trees) with maximum spectral radius among all graphs (respectively, trees) with zero forcing number at most k. By their results they also provided a sharp lower bound for the zero forcing number of graphs involving its spectral radius.

Inspired by the above mentioned works, in this paper, we further study the relationship between the zero forcing number of G and its another kind of eigenvalues. Before stating our results, we need some necessary notations and terminology. Recall that the *normalized Laplacian matrix* of G is defined as $\mathcal{L}(G) = D(G)^{-\frac{1}{2}}(D(G) - A(G))D(G)^{-\frac{1}{2}}$, where $D(G)$ is the diagonal degree matrix of G. The *normalized Laplacian eigenvalues* of G are the eigenvalues of $\mathcal{L}(G)$, denoted by $\xi_1(G) \geq \xi_2(G) \geq \cdots \geq \xi_{n-1}(G) \geq \xi_n(G) = 0$ (or $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \xi_n = 0$ for short). Let $\vartheta = \frac{\xi_1 - \xi_{n-1}}{\xi_1 + \xi_{n-1}}$ and $\xi = \max\{|1-\xi_1|, |1-\xi_{n-1}|\}$. We extend Theorem 1 to the setting of general graphs as follows.

Theorem 2. *Let G be a graph of order n with m edges and minimum degree δ.*

(i) *If $G \in \mathcal{G}_n^e$, then*

$$Z(G) \geq n - 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor; \tag{3}$$

(ii) *if $G \in \mathcal{G}_n^o$, then*

$$Z(G) \geq n - 2\left\lfloor \frac{\delta(\vartheta^2 - 1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor - 1. \tag{4}$$

Theorem 3. *Let G be a graph of order n with m edges, minimum degree δ and maximum degree Δ. Then*

$$Z(G) \leq n - \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right) \right\rceil. \tag{5}$$

Remark 4. When G is an (n,d,λ)-graph, note that $\Delta=\delta=d$, $nd=2m$, $\xi_i=1-\frac{\lambda_{n-i+1}}{d}$ and $\xi=\frac{\lambda}{d}$ [6]. Then we have

$$\begin{aligned} Z(G) &\geq n-2\left\lfloor\frac{2m\vartheta}{\delta(1+\vartheta)}\right\rfloor \geq n-\frac{4m}{\delta}\cdot\frac{\vartheta}{1+\vartheta}=n-\frac{4m}{\delta}\cdot\frac{\frac{\lambda_2-\lambda_n}{2d-\lambda_2-\lambda_n}}{\frac{2d-2\lambda_n}{2d-\lambda_2-\lambda_n}} \\ &= n-n\cdot\frac{\lambda_2-\lambda_n}{d-\lambda_n}\geq n-n\cdot\frac{2\lambda}{d-\lambda_n}\geq n-n\cdot\frac{2\lambda}{d+\lambda}. \end{aligned}$$

Similarly, it is easy to check that $\frac{m}{(1-\xi)\Delta^2}=\frac{n}{2(d-\lambda)}$ and $\frac{\delta n(1-\xi)}{2m\xi+\frac{2m}{\Delta^2}(\xi\Delta+1)\delta}=\frac{d-\lambda}{2\lambda+1}$. Then we have

$$\begin{aligned} Z(G) &\leq n-\left\lceil\frac{m}{\Delta^2(1-\xi)}\log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi+\frac{\delta}{\Delta^2}\right]}\right)\right\rceil \\ &\leq n-\frac{m}{\Delta^2(1-\xi)}\log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi+\frac{\delta}{\Delta^2}\right]}\right) \\ &= n-\frac{n}{2(d-\lambda)}\log\left(\frac{d-\lambda}{2\lambda+1}\right)=n\left(1-\frac{1}{2(d-\lambda)}\log\left(\frac{d-\lambda}{2\lambda+1}\right)\right). \end{aligned}$$

Those suggest that Theorems 2 and 3 can be viewed as a slight generalization of Theorem 1, respectively.

Remark 5. As mentioned above, for $G\in\mathcal{G}_n^o$, the lower bound of (1) is not tight, but our bound (4) is tight. This is shown by the following example. If $G\cong K_n$, one can check that $Z(K_n)=n-1$ and K_n contains a maximal 1-witness. Then $K_n\in\mathcal{G}_n^o$. Moreover, it is known that the eigenvalues and normalized Laplacian eigenvalues of K_n are $n-1,\underbrace{-1,\ldots,-1}_{n-1}$ and $\underbrace{\frac{n}{n-1},\ldots,\frac{n}{n-1}}_{n-1},0$, respectively. Thus $\lambda=1$ and $\vartheta=\frac{\xi_1-\xi_{n-1}}{\xi_1+\xi_{n-1}}=0$. By (1) and (4), we then have

$$n-1=Z(K_n)\geq n\left(1-\frac{2\lambda}{d+\lambda}\right)=n\left(1-\frac{2}{n}\right)=n-2,$$

and

$$\begin{aligned} n-1=Z(G) &\geq n-2\left\lfloor\frac{\delta(\vartheta^2-1)-4m\vartheta^2+\sqrt{\delta^2(1-\vartheta^2)^2+16m^2\vartheta^2}}{2\delta(1-\vartheta^2)}\right\rfloor-1 \\ &= n-1-2\left\lfloor\frac{-(n-1)+(n-1)}{2(n-1)}\right\rfloor=n-1. \end{aligned}$$

Remark 6. Moreover, the lower and upper bounds in Theorems 2 and 3 are also tight for some non-regular graphs. For example, if $G = K_{2,3}$, one can check that $Z(K_{2,3}) = 3$ and $K_{2,3}$ contains a maximal 2-witness. Then $K_{2,3} \in \mathcal{G}_5^e$. By a little calculation, we have that the normalized Laplacian eigenvalues of $K_{2,3}$ are $2, 1, 1, 1, 0$. Thus $\vartheta = \frac{\xi_1 - \xi_{n-1}}{\xi_1 + \xi_{n-1}} = \frac{1}{3}$ and $\frac{2m\vartheta}{\delta(1+\vartheta)} = \frac{3}{2}$. By (3), we have

$$3 = Z(K_{2,3}) \geq n - 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor = 5 - 2\left\lfloor \frac{3}{2} \right\rfloor = 3.$$

This shows that the bound of (3) is tight for $K_{2,3}$.

If $G = K_{3,4}^+$, where $K_{3,4}^+$ is the graph obtained from $K_{3,4}$ by adding a new edge e to one part of $K_{3,4}$ with 3 vertices, one can check that $Z(K_{3,4}^+) = 4$ and $K_{3,4}^+$ contains a maximal 3-witness. Then $K_{3,4}^+ \in \mathcal{G}_7^o$. By a little calculation, we have that the normalized Laplacian eigenvalues of $K_{3,4}^+$ are $\frac{21+\sqrt{51}}{15}, \frac{6}{5}, 1, 1, 1, \frac{21-\sqrt{51}}{15}, 0$. Thus $\vartheta = \frac{\xi_1 - \xi_{n-1}}{\xi_1 + \xi_{n-1}} = \sqrt{\frac{17}{147}}$ and $\frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} = \frac{49}{260}\left(-\frac{26}{3} + 26\sqrt{\frac{3407}{7203}}\right)$. By (4), we have

$$\begin{aligned} 4 = Z(K_{3,4}^+) &\geq n - 1 - 2\left\lfloor \frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor \\ &= 7 - 1 - 2\left\lfloor \frac{49}{260}\left(-\frac{26}{3} + 26\sqrt{\frac{3407}{7203}}\right) \right\rfloor = 4. \end{aligned}$$

This shows that the bound of (4) is tight for $K_{3,4}^+$.

If $G = K_{35}^-$, where K_{35}^- is the graph obtained from K_{35} by deleting any $e \in E(K_{35})$ from K_{35}, one can check that $Z(K_{35}^-) = 33$. By a little calculation, we have that the normalized Laplacian eigenvalues of K_{35}^- are $\frac{18}{17}, \underbrace{\frac{35}{34}, \ldots, \frac{35}{34}}_{32}, 1, 0$.

Thus $\xi = \max\{|1 - \xi_1|, |1 - \xi_{34}|\} = \frac{1}{17}$ and $\frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right) = \frac{297}{544}\log\left(\frac{9520}{1503}\right)$. By (5), we have

$$\begin{aligned} 33 = Z(K_{35}^-) &\leq n - \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right) \right\rceil \\ &= 35 - \left\lceil \frac{297}{544}\log\left(\frac{9520}{1503}\right) \right\rceil = 33. \end{aligned}$$

This shows that the bound of (5) is tight for K_{35}^-. In fact, it could be check that the bound of (5) is also tight for K_n^- for $n = 36, \ldots, 264$ by a little calculation, respectively.

2. Preliminaries

In [11], the relationship between the zero forcing number of a graph and the occurrence of a witness was explored as follows.

Lemma 7. *Let G be a graph of order n and $k \in \mathbb{N}$. If $Z(G) \leq n-k$, then G contains a k-witness. Moreover, if G contains a k-witness $(\mathbf{s}, \mathbf{t})$ with $\mathbf{s}[k] \cap \mathbf{t}[k] = \emptyset$, then $Z(G) \leq n-k$.*

Remark 8. Lemma 7 means that if G does not contain any k-witness, then $Z(G) > n-k$. Otherwise, if $Z(G) \leq n-k$, then G contains a k-witness, a contradiction.

For $X \subseteq V(G)$, let $\mathrm{Vol}(X) = \sum_{v \in X} d(v)$. Clearly, $\mathrm{Vol}(G) = \mathrm{Vol}(V(G)) = \sum_{v \in V(G)} d(v) = 2m$. For any $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, $E(X,Y)$ denotes the set of edges between X and Y and $e(X,Y) = |E(X,Y)|$. The following theorem is an irregular expander mixing lemma, which can be found in [8]. The expander mixing lemma is a truly remarkable result, connecting edge distribution and graph eigenvalues, and providing a very good quantitative handle for the uniformity of edge distribution based on graph eigenvalues.

Lemma 9. *Let G be a graph of order n. For any two subsets $X, Y \subseteq V(G)$, we have*

$$\left| e(X,Y) - \frac{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}{\mathrm{Vol}(G)} \right| \leq \xi \sqrt{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)\left(1 - \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(G)}\right)\left(1 - \frac{\mathrm{Vol}(Y)}{\mathrm{Vol}(G)}\right)}. \tag{6}$$

In particular,

$$\left| 2e(X) - \frac{\mathrm{Vol}^2(X)}{\mathrm{Vol}(G)} \right| \leq \xi \cdot \mathrm{Vol}(X)\left(1 - \frac{\mathrm{Vol}(X)}{\mathrm{Vol}(G)}\right).$$

The following separation inequality from [6] provides a bridge between graph parameters and its normalized Laplacian eigenvalues.

Lemma 10. *Let G be a graph of order n. For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$ and $e(X,Y) = 0$, we have*

$$\frac{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}{\mathrm{Vol}(\overline{X})\,\mathrm{Vol}(\overline{Y})} \leq \vartheta^2,$$

where $\overline{X} = V \backslash X$ and $\overline{Y} = V \backslash Y$.

3. Proofs of Theorems 2 and 3

We now give the proofs of Theorems 2 and 3, respectively.

***Proof of Theorem* 2.** Firstly, it is known that for any graph $G \in \mathcal{G}_n$, $Z(G) \leq n-1$ with equality holds if and only if $G \cong K_n$. That is $Z(K_n) = n-1$. Moreover, note that the normalized Laplacian eigenvalues of K_n are $\underbrace{\frac{n}{n-1}, \dots, \frac{n}{n-1}}_{n-1}, 0$, and K_n contains a maximal 1-witness. Then $\vartheta = \frac{\xi_1 - \xi_{n-1}}{\xi_1 + \xi_{n-1}} = 0$, which implies that the inequality (4) holds for $G = K_n$. In what follows, we assume that $G \neq K_n$. That is $Z(G) \leq n-2$. Thus Lemma 7 implies that G contains a maximal k-witness with $k \geq 2$. Without loss of generality, we assume that $\left((s_i)_{i\in[k]}, (t_i)_{i\in[k]}\right)$ is a maximal k-witness in G. In what follows, we may estimate the maximum possible value of k among all graphs $G \in \mathcal{G}_n$. Let $X = \left\{s_1, s_2, \dots, s_{\lfloor \frac{k}{2} \rfloor}\right\}$ and $Y = \left\{t_{\lfloor \frac{k}{2} \rfloor+1}, t_{\lfloor \frac{k}{2} \rfloor+2}, \dots, t_k\right\}$. According to the definition of k-witness, we have $X \cap Y = \emptyset$ and $e(X,Y) = 0$. Let $\vartheta = \frac{\xi_1 - \xi_{n-1}}{\xi_1 + \xi_{n-1}}$. Note that $\mathrm{Vol}(X) \geq \delta|X| = \delta \left\lfloor \frac{k}{2} \right\rfloor$ and $\mathrm{Vol}(Y) \geq \delta|Y| = \delta \left\lceil \frac{k}{2} \right\rceil$. Then Lemma 10 implies that

$$\frac{\delta^2 \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{k}{2} \right\rceil}{\left(2m - \delta \left\lfloor \frac{k}{2} \right\rfloor\right)\left(2m - \delta \left\lceil \frac{k}{2} \right\rceil\right)} \leq \frac{\mathrm{Vol}(X)\,\mathrm{Vol}(Y)}{(\mathrm{Vol}(G) - \mathrm{Vol}(X))(\mathrm{Vol}(G) - \mathrm{Vol}(Y))} \leq \vartheta^2. \tag{7}$$

We now consider the following two cases according to the parity of k.

Case 1. k is even. Let $\left\lfloor \frac{k}{2} \right\rfloor = \left\lceil \frac{k}{2} \right\rceil = t$. Then from (7), we have $\frac{\delta^2 \cdot t^2}{(2m-\delta t)^2} \leq \vartheta^2$. It follows that $t \leq \left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor$. Hence $k = 2t \leq 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor$.

Case 2. k is odd. Let $\left\lfloor \frac{k}{2} \right\rfloor = t$. Then $\left\lceil \frac{k}{2} \right\rceil = t+1$. From (7), we have $\frac{\delta^2 \cdot t(t+1)}{(2m-\delta t)(2m-\delta(t+1))} \leq \vartheta^2$. It follows that $t \leq \left\lfloor \frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor$. Hence

$$k = 2t+1 \leq 2\left\lfloor \frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor + 1.$$

From the above arguments, we know that the maximum possible value of k is $2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor$ when $G \in \mathcal{G}_n^e$ and $2\left\lfloor \frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor + 1$ when $G \in \mathcal{G}_n^o$. This means that G does not contain any k-witness with $k > 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor$ when $G \in \mathcal{G}_n^e$ and $k > 2\left\lfloor \frac{\delta(\vartheta^2-1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor + 1$ when $G \in \mathcal{G}_n^o$. Then Lemma 7 and Remark 8 imply that $Z(G) \geq n - 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor$ for $G \in \mathcal{G}_n^e$, and

$Z(G) \geq n - 2\left\lfloor \frac{\delta(\vartheta^2-1)-4m\vartheta^2+\sqrt{\delta^2(1-\vartheta^2)^2+16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor - 1$ for $G \in \mathcal{G}_n^o$. This completes the proof of Theorem 2. ■

Now we are in a position to give the proof of Theorem 3. Our strategy for proving Theorem 3 is employed the similar argument as that was used in [11].

***Proof of Theorem* 3.** Firstly, we greedily construct a witness. Start with $U_0 = V$, the vertex set of G. In each step i, we will select vertices s_i, $t_i \in U_{i-1}$ and a set $U_i \subseteq U_{i-1}$. Assuming that the steps $1, \ldots, i-1$ were executed. Let s_i be any vertex in U_{i-1} satisfying $1 \leq d_{G[U_i]}(s_i) \leq (1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta$. We fix any $t_i \in N_G(s_i) \cap U_{i-1}$, and set $U_i = U_{i-1} \setminus (N_G(s_i) \cup \{s_i\})$. The algorithm continues as long as $|U_i| > \frac{2m\xi}{\delta(1+\xi)}$. Denote the total number of steps by k.

By the above construction, the pair $(\mathbf{s}, \mathbf{t})$ is a k-witness in G. In what follows, we will show that there is a choice for s_i throughout the algorithm, and that

$$k \geq \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]} \right) \right\rceil.$$

Firstly, we have the following claim.

Claim 11. *If $|U_i| > \frac{2m\xi}{\delta(1+\xi)}$, then the induced subgraph $G[U_i]$ contains at lest one vertex u a vertex u satisfying $1 \leq d_{G[U_i]}(u) \leq (1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta$.*

Proof. Suppose to the contrary that there is some set U_i with $|U_i| > \frac{2m\xi}{\delta(1+\xi)}$, but for each $u \in U_i$, $d_{G[U_i]}(u) = 0$ or $d_{G[U_i]}(u) > (1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta$. Note that Lemma 9 implies that U_i is not an independent set as $|U_i| > \frac{2m\xi}{\delta(1+\xi)}$. It follows that there are some vertices $u \in U_i$ with $d_{G[U_i]}(u) > (1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta$. Moreover, by removing the isolated vertices in $G[U_i]$, we get a nonempty set $W \subseteq U_i$ in which every vertex $u \in W$ satisfies $d_{G[U_i]}(u) = d_{G[W]}(u) > (1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta$. In particular,

$$e(W, W) = \sum_{u \in W} d_{G[W]}(u) > |W| \left((1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta \right),$$

recalling that each edge in $E(W, W)$ is counted twice in $e(W, W)$. On the other hand, Lemma 9 implies that

$$\begin{aligned} e(W, W) &\leq \mathrm{Vol}(W) \left[\frac{\mathrm{Vol}(W)}{\mathrm{Vol}(G)} + \xi \left(1 - \frac{\mathrm{Vol}(W)}{\mathrm{Vol}(G)} \right) \right] \\ &\leq \mathrm{Vol}(W) \left((1-\xi)\frac{\mathrm{Vol}(U_i)}{\mathrm{Vol}(G)} + \xi \right) \leq |W| \left((1-\xi)\frac{|U_i|\Delta^2}{2m} + \xi\Delta \right), \end{aligned}$$

a contradiction. This completes the proof of Claim 11. □

Let $a_i = \frac{|U_i|}{n}$. By the construction of U_i, $a_0 = 1$, and for $i \geqslant 1$,

$$\begin{aligned} a_i &= \frac{|U_{i-1}| - d_{G[U_{i-1}]}(s_i) - 1}{n} \geq \frac{1}{n}\left[|U_{i-1}| - \left((1-\xi)\frac{|U_{i-1}|\Delta^2}{2m} + \xi\Delta\right) - 1\right] \\ &= \left(1 - \frac{(1-\xi)\Delta^2}{2m}\right) a_{i-1} - \frac{\xi\Delta + 1}{n}. \end{aligned} \tag{8}$$

By induction on i, it follows that for all i,

$$a_i \geq \left(1 + \frac{2m}{n\Delta^2} \cdot \frac{\xi\Delta + 1}{1-\xi}\right)\left(1 - \frac{(1-\xi)\Delta^2}{2m}\right)^i - \frac{2m}{n\Delta^2} \cdot \frac{\xi\Delta + 1}{1-\xi}.$$

We further have the following claim.

Claim 12. $a_i > \frac{2m\xi}{\delta n(1+\xi)}$ *for* $i \leq \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right\rceil$.

Proof. We estimate a_i for $i \leq \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right\rceil$, ignoring the constant $\left(1 + \frac{2m}{n\Delta^2} \cdot \frac{\xi\Delta+1}{1-\xi}\right)$ and using the inequality

$$1 - \frac{(1-\xi)\Delta^2}{2m} \geq e^{(-2)\cdot\frac{(1-\xi)\Delta^2}{2m}} = e^{-\frac{(1-\xi)\Delta^2}{m}}$$

for $\frac{(1-\xi)\Delta^2}{2m} < 1/2$. We have

$$\begin{aligned} a_i &\geq \exp\left(-\frac{(1-\xi)\Delta^2}{m} \cdot \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right\rceil\right) - \frac{2m}{n\Delta^2} \cdot \frac{\xi\Delta+1}{1-\xi} \\ &\geq \exp\left(-\frac{(1-\xi)\Delta^2}{m} \cdot \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right) - \frac{2m}{n\Delta^2} \cdot \frac{\xi\Delta+1}{1-\xi} \\ &= \frac{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}{\delta n(1-\xi)} - \frac{2m\delta(\frac{\xi}{\Delta} + \frac{1}{\Delta^2})}{\delta n(1-\xi)} = \frac{2m\xi}{\delta n(1-\xi)} > \frac{2m\xi}{\delta n(1+\xi)}, \end{aligned}$$

as required. □

We conclude that the algorithm continues for at least $\left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right\rceil$ steps, at the same time, G contains a k-witness $(\mathbf{s}, \mathbf{t})$ with $\mathbf{s}[k] \cap \mathbf{t}[k] = \emptyset$ by the above construction. Then by Lemma 7, we have

$$n - Z(G) \geq \left\lceil \frac{m}{\Delta^2(1-\xi)} \log\left(\frac{\delta n(1-\xi)}{2m\left[(1+\frac{\delta}{\Delta})\xi + \frac{\delta}{\Delta^2}\right]}\right)\right\rceil.$$

This completes the proof of Theorem 3. ■

4. Related to Laplacian Eigenvalues

In this section, using the separation inequality concerning Laplacian eigenvalues (Lemma 13) and a similar argument as that in the proof of Theorem 2, we also derive the lower bound for $Z(G)$ in terms of its Laplacian eigenvalues. Recall that the Laplacian eigenvalues of a graph G are the eigenvalues of $L(G) = D(G) - A(G)$, denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ (or $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ for short). Let $\zeta = \frac{\mu_1 - \mu_{n-1}}{\mu_1 + \mu_{n-1}}$. One of the main tools for connecting the zero forcing number of a graph to its Laplacian eigenvalues is the following separation inequality.

Lemma 13 [4]**.** *Let G be a graph of order n. For $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$ and $e(X,Y) = 0$, we have*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \zeta^2.$$

This together with a similar argument as that in the proof of Theorem 2 gives the following lower bound for $Z(G)$ in terms of its Laplacian eigenvalues.

Theorem 14. *For any $G \in \mathcal{G}_n$,*
(i) *if $G \in \mathcal{G}_n^e$, then*

$$Z(G) \geq n - 2\left\lfloor \frac{\zeta n}{1+\zeta} \right\rfloor; \tag{9}$$

(ii) *if $G \in \mathcal{G}_n^o$, then*

$$Z(G) \geq n - 2\left\lfloor \frac{(1-2n)\zeta^2 - 1 + \sqrt{(1-\zeta^2)^2 + 4n^2\zeta^2}}{2(1-\zeta^2)} \right\rfloor - 1. \tag{10}$$

Moreover, the above bounds are sharp.

Remark 15. When G is a d-regular graph, we have $\Delta = \delta = d$, $nd = 2m$, $\xi_i = \frac{\mu_i}{d}$ and $\zeta = \vartheta$. Then we have

$$n - 2\left\lfloor \frac{\zeta n}{1+\zeta} \right\rfloor = n - 2\left\lfloor \frac{2m\vartheta}{\delta(1+\vartheta)} \right\rfloor,$$

and

$$\begin{aligned}
& n - 2\left\lfloor \frac{(1-2n)\zeta^2 - 1 + \sqrt{(1-\zeta^2)^2 + 4n^2\zeta^2}}{2(1-\zeta^2)} \right\rfloor - 1 \\
&= n - 2\left\lfloor \frac{(1-2n)\delta\zeta^2 - \delta + \sqrt{\delta^2(1-\zeta^2)^2 + 4n^2\delta^2\zeta^2}}{2\delta(1-\zeta^2)} \right\rfloor - 1 \\
&= n - 2\left\lfloor \frac{\delta(\vartheta^2 - 1) - 4m\vartheta^2 + \sqrt{\delta^2(1-\vartheta^2)^2 + 16m^2\vartheta^2}}{2\delta(1-\vartheta^2)} \right\rfloor - 1.
\end{aligned}$$

Those show that the bounds of (3) and (9), (4) and (10) are consistent when G is a regular graph, respectively.

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