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AN EXTREMAL PROBLEM FOR THE NEIGHBORHOOD LIGHTS OUT GAME

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Abstract

Neighborhood Lights Out is a game played on graphs. Begin with a graph and a vertex labeling of the graph from the set $\{0, 1, 2, \ldots, \ell - 1\}$ for $\ell \in \mathbb{N}$. The game is played by toggling vertices: when a vertex is toggled, that vertex and each of its neighbors has its label increased by 1 (modulo ℓ). The game is won when every vertex has label 0. For any $n \geq 2$ it is clear that one cannot win the game on K_n unless the initial labeling assigns all vertices the same label. Given that K_n has the maximum number of edges of any simple graph on n vertices it is natural to ask how many edges can be in a graph so that the Neighborhood Lights Out game is winnable regardless of the initial labeling. We find the maximum number of edges a winnable n-vertex graph can have when at least one of n and ℓ is odd. When n and ℓ are both even we find the maximum size in two additional cases. The proofs of our results require us to introduce a new version of the Lights Out game that can be played given any square matrix.

Keywords: Lights Out, light-switching game, winnability, extremal graph theory, linear algebra.

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1. Introduction

The Lights Out game was originally created by Tiger Electronics. It has since been reimagined as a light-switching game on graphs. Several variations of the game have been developed (see, for example [8] and [13]), but all have some important elements in common. In each game, we begin with a graph G and a labeling of $V(G)$ with labels in $\mathbb{Z}_\ell$ for some $\ell \geq 2$. The vertices can be toggled so as to change the labels of some of the vertices, and there is some desired labeling (usually the labeling with all labels being 0, called the *zero labeling*) that marks the end of the game.

The most common variation of the Lights Out game is what we call the *Neighborhood Lights Out game*. This is a generalization of Sutner's σ^+-game (see [15]). Each time we toggle some $v \in V(G)$, the label of each vertex in the closed neighborhood of v, $N[v]$, is increased by 1 modulo ℓ. The game is won when the zero labeling is achieved. This game was developed independently in [10] and [3] and has been studied in [4, 5, 9, 14], and [6]. The Tiger Electronics Lights Out game is the Neighborhood Lights Out game on a grid graph with $\ell = 2$ and has been studied in [1, 11], and [15].

Much of the work on Lights Out games has centered on the conditions under which winning the game is possible. Winnability depends on the version of the game that is played, the graph on which the game is played, and on ℓ.

For each $n \geq 2$ the Neighborhood Lights Out game on K_n is impossible to win unless in the initial labeling every vertex has the same label. Since K_n has the most edges of any simple graph on n vertices it makes sense to ask, given $n, \ell \geq 2$, what is the maximum size of a simple graph on n vertices with labels in $\mathbb{Z}_\ell$ for which the Neighborhood Lights Out game can be won for every possible initial labeling? We call this maximum size $\max(n, \ell)$. In addition, we seek to classify the winnable graphs of maximum size among all graphs on n vertices with labels from $\mathbb{Z}_\ell$, which we call (n, ℓ)-*extremal graphs*.

Our main results are in Section 4, where we determine partial results on the classification of (n, ℓ)-extremal graphs. In the case of n odd, we show that all (n, ℓ)-extremal graphs are complements of near perfect matchings. We also classify all (n, ℓ)-extremal graphs when n is even and ℓ is odd. In the remaining case we have the following conjecture.

Conjecture 1. *If n, ℓ are even, then*

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + k\right),$$

where k is the smallest nonnegative integer such that $\gcd(n - 2k - 1, \ell) = 1$. In each case the (n, ℓ)-extremal graphs are precisely the graphs of order n that have size $\binom{n}{2} - \left(\frac{n}{2} + k\right)$ that are complements of pendant graphs (defined later).

By proving that the complements of pendant graphs can be won no matter the initial labeling, we conclude that $\max(n, \ell)$ is at least the quantity given in Conjecture 1. We also prove equality for $0 \leq k \leq 3$ and in the family of all graphs that have minimum degree at least $n - 3$.

To determine winnability, we depend heavily on linear algebra methods similar to those in [2, 5, 9], and [10]. We discuss these methods in Section 2. Our techniques differ in that we introduce how to play Lights Out given any square matrix. These tools allow us to determine winnability in some dense graphs by considering winnability in a modified Lights Out game in their sparse complements, which we discuss further in Section 3.

Throughout the paper, we assume the vertex labels of any labeling are from $\mathbb{Z}_\ell$ for some $\ell \in \mathbb{N}$.

2. Linear Algebra

While winnability can be determined by giving a strategy for toggling the vertices, it is often convenient to instead use linear algebra. We provide key ideas here. For more details, see [2] and [10].

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $N(G) = [N_{ij}]$ be the neighborhood matrix of G (where $N_{ij} = 1$ if and only if v_i is adjacent to v_j or $i = j$, and $N_{ij} = 0$ otherwise), and $\mathbf{b} \in \mathbb{Z}_\ell^n$ so that $\mathbf{b}[i]$ is the initial label of v_i. This Neighborhood Lights Out game is winnable if and only if there exists a vector $\mathbf{x} \in \mathbb{Z}_\ell^n$ such that $N(G)\mathbf{x} = -\mathbf{b}$ [10, Lemma 3.1].

We can generalize the game by using any square matrix in $M_n(\mathbb{Z}_\ell)$ (the set of $n \times n$ matrices with entries in $\mathbb{Z}_\ell$). For $M = [m_{ij}] \in M_n(\mathbb{Z}_\ell)$ define the *vertex set of* M as $V(M) = \{v_1, v_2, \ldots, v_n\}$. Label the elements of $V(M)$ with a vector $\mathbf{b} \in \mathbb{Z}_\ell^n$, so each v_i has label $\mathbf{b}[i]$. We play the game by toggling elements of $V(M)$. Each time v_j is toggled, we add m_{ij} to the label of v_i for all $1 \leq i \leq n$. As with the ordinary Lights Out game, we win the game when we achieve the labeling $\mathbf{0}$. We call this the M-Lights Out game. For $M = N(G)$ we get the Neighborhood Lights Out game. For $M = A(G)$ we get a generalization of the σ-game from Sutner (see [15]), where toggling a vertex v increases the label of each vertex in the open neighborhood of v by 1 modulo ℓ. We call these games the N-Lights Out game and the A-Lights Out game, respectively.

Definition. Let M and $V = V(M)$ be as above. We call a labeling π *M-winnable* if the M-Lights Out game can be won with initial labeling π. We say that V is *M-always winnable*, or *M-AW* for short, if all labelings of $V(M)$ are M-winnable.

The following lemma can be proved using basic linear algebra.

Lemma 2. *Let $M \in M_n(\mathbb{Z}_\ell)$ and $V(M) = \{v_1, v_2, \ldots, v_n\}$.*

(1) *Let π be a labeling of $V(M)$, and define $\boldsymbol{b}[i] = \pi(v_i)$. Then π is M-winnable with the toggles given by $\boldsymbol{x}$ if and only if $M\boldsymbol{x} = -\boldsymbol{b}$.*

(2) *The vertex set $V(M)$ is M-AW if and only if M is invertible over $\mathbb{Z}_\ell$.*

In this paper, we focus on whether or not a graph G is $N(G)$-AW, so we seek to determine whether or not a given neighborhood matrix is invertible. One straightforward way to apply linear algebra techniques is when two rows or columns of a matrix are identical. We call v and w *M-twins* if the rows or columns of M represented by v and w are identical.

Corollary 3. *Let $M \in M_n(\mathbb{Z}_\ell)$, and $v, w \in V(M)$. If $V(M)$ has M-twins, then $V(M)$ is not M-AW.*

The name twins comes from graph theory. Two vertices v and w are twins provided that have the same open neighborhood. Twin vertices that are adjacent in a graph result in identical rows in the neighborhood matrix and thus are N-twins. Twin vertices that are not adjacent result in identical rows in the adjacency matrix and thus are A-twins.

Recall a matrix is invertible if and only if its determinant is a unit [7, Corollary 2.21]. As in standard linear algebra, elementary row operations (multiplying a row by a unit in $\mathbb{Z}_\ell$, adding an integer multiple of one row to another, and switching two rows) leave the determinant unchanged or multiplied by a unit. In particular, the typical elementary row operations have no effect on whether or not the determinant is a unit.

We say that M is *row equivalent* to M', denoted $M \sim M'$, if M can be turned into M' by applying a sequence of elementary row operations. If $M \sim M'$, then a common vertex set V is M-AW if and only if V is M'-AW. Our general strategy will be to use elementary row operations to change $N(G)$ into a matrix whose Lights Out game is easy to play. Our first result using this technique will be for graphs that have a dominating vertex. Note $G \cup H$ is the disjoint union of the graphs and $\overline{G}$ is the complement graph.

Theorem 4. *Let G be a graph. Then $\overline{G \cup K_1}$ is N-AW if and only if G is A-AW.*

Proof. We have

$$N(\overline{G \cup K_1}) = \left[\begin{array}{c|c} N(\overline{G}) & 1 \\ \hline 1 & 1 \end{array}\right]$$

where the last row and column represent $V(K_1)$. We multiply each row except the last by the unit -1 and then add to each of those rows the last row. This turns every 1 of $N(\overline{G})$ into a 0 and vice versa, resulting in the adjacency matrix of G. So

$$N(\overline{G \cup K_1}) \sim M = \left[\begin{array}{c|c} A(G) & 0 \\ \hline 1 & 1 \end{array}\right].$$

Thus, it suffices to show that $\overline{G \cup K_1}$ is M-AW if and only if G is A-AW. Note that the M-Lights Out game is played as the A-Lights Out game on G, each vertex toggled in $V(G)$ adds 1 to the label of the vertex $v \in V(K_1)$, and toggling v increases its own label by 1 and has no other effect.

First suppose that G is A-AW, and let π be a labeling of $\overline{G \cup K_1}$. Since G is A-AW, we can toggle the vertices of G in a way that wins the $A(G)$-Lights Out game for the labeling $\pi\mid_{V(G)}$. At this point, every vertex has label 0 except v. We then toggle v until it has label 0. In the M-Lights Out game, toggling v has no effect on labels of other vertices, so this wins the M-Lights Out game. Thus $\overline{G \cup K_1}$ is M-AW.

Conversely, suppose that G is not A-AW. Let π be a labeling of $V(G)$ that is not A-winnable. In the M-Lights Out game the only vertices that affect the labels of $V(G)$ are the vertices in $V(G)$, so π is not M-winnable. Thus, $\overline{G \cup K_1}$ is not M-AW. ∎

3. Winnability in Dense Graphs

In proving Theorem 4, we use elementary row operations to convert the neighborhood Lights Out game on a dense graph into something resembling the adjacency Lights Out game on a sparse graph. Since the extremal problem we are working on seeks dense, winnable graphs and playing the game on sparse graphs is typically easier, this technique works to our advantage. The next result allows us to make a graph denser by removing an edge from the complement graph when the complement graph is combined with P_4.

Theorem 5. *Let G be a graph, $U \subseteq V(G)$ and v be an end vertex of P_4. Let H be the graph where $V(H) = V(G) \cup V(P_4)$ and $E(H) = E(G) \cup E(P_4) \cup \{uv : u \in U\}$. Then $\overline{H}$ is N-AW if and only if $\overline{G \cup P_4}$ is N-AW.*

Proof. Let $V = V(\overline{G \cup P_4}) = V(\overline{H})$, and let P_4 in both $G \cup P_4$ and H be given by $vv_2v_3v_4$. Note that $\overline{P_4}$ is the path given by $v_2v_4vv_3$. By [10, Thm. 4.3], P_4 is N-AW for all ℓ. It follows that in both $\overline{H}$ and $\overline{G \cup P_4}$, the subgraph induced by $\{v, v_2, v_3, v_4\}$ is N-AW. Thus, we can toggle the vertices of P_4 in such a way that each vertex in P_4 has label zero.

We first assume $\overline{H}$ is N-AW and show $\overline{G \cup P_4}$ is N-AW. To that end, we let $\pi : V \to \mathbb{Z}_\ell$ and show that π is winnable on $\overline{G \cup P_4}$. As discussed above, we can assume that $\pi\mid_{V(P_4)} = 0$. Since $\overline{H}$ is N-AW, π is winnable on $\overline{H}$. In this winning strategy, let each $w \in V(G)$ be toggled x_w times, and let v_2 be toggled x times. If we apply this strategy to $\overline{H}$ but refrain from toggling v, v_3, and v_4, this leaves v_2 and v_4 with label $x + \sum_{w \in V(G)} x_w$, v with label $\sum_{w \in V(G) \setminus U} x_w$, and v_3 with label $\sum_{w \in V(G)} x_w$. Since v_4 is the only remaining vertex adjacent to v_2, v_4 must be toggled $-x - \sum_{w \in V(G)} x_w$ times. This will leave both v_2 and v_4 with label zero. Since v is the only remaining vertex adjacent to v_4, this means we do not toggle v at all. Thus, v_3 (the only remaining untoggled vertex) must make its own label zero by being toggled $-\sum_{w \in V(G)} x_w$ times. This completes winning the game on $\overline{H}$. An important observation is that the vertices of P_4 are collectively toggled $-2\sum_{w\in V(G)} x_w$ times, and none of those toggles come from v. Since each of v_2, v_3, and v_4 is adjacent to every vertex in $V(\overline{G})$, this implies that toggling the vertices of P_4 adds $-2\sum_{w\in V(G)} x_w$ to the labeling of each vertex in $V(G)$. Looked at another way, if we only toggle the vertices in $V(G)$, this leaves each such vertex with label $2\sum_{w\in V(G)} x_w$.

With the initial labeling π, we now apply the above toggling strategy to $V(G)$ in $\overline{G \cup P_4}$. By the above, each vertex in $V(G)$ has label $2\sum_{w\in V(G)} x_w$. Since v and each of the v_i are adjacent to all vertices in $V(G)$, it follows that toggling the vertices in $V(G)$ leaves v and each v_i with label $\sum_{w\in V(G)} x_w$. Each of v_2 and v_3 is now toggled $-\sum_{w\in V(G)} x_w$ times. This makes the label of v and each v_i zero. In addition, it adds $-2\sum_{w\in V(G)} x_i$ to the labels of $V(G)$, which gives each of them label zero as well.

We proceed similarly for the converse. Assume $\overline{G \cup P_4}$ is N-AW, and let $\pi : V \to \mathbb{Z}_\ell$ be a labeling as above with $\pi\,|_{V(P_4)}= 0$. We need to prove that π is winnable on $\overline{H}$. As before, there is a winning toggling strategy for $\overline{G \cup P_4}$, where each $w \in V(G)$ is toggled x'_w times, and v_2 is toggled x' times. At this point, we determine the toggles for v and each remaining v_i as before, and it follows that the vertices are collectively toggled $-2\sum_{w\in V(G)} x'_w$ times. As before, this implies that toggling the vertices of $V(G)$ results in the label of each vertex in $V(G)$ being $2\sum_{w\in V(G)} x'_w$.

Again, we apply the above toggling strategy just to the vertices of $V(G)$ in $\overline{H}$. This leaves each of v_2, v_3, and v_4 with label $\sum_{w\in V(G)} x'_w$ and v with label $\sum_{w\in V(G)\setminus U} x'_w$. We then win the game as follows: v_2 is toggled $-2\sum_{w\in U} x'_w - \sum_{w\in V(G)\setminus U} x'_w$ times, v_3 is toggled $-\sum_{w\in V(G)} x'_w$ times, and v_4 is toggled $\sum_{w\in U} x'_w$ times. ■

We can apply this result to complements of graphs that include components that are path graphs. For $k \in \mathbb{N}$ and G a graph we use kG to denote k disjoint copies of G.

Corollary 6. *Let G be a graph of order n that is N-AW.*

(1) *No component of $\overline{G}$ can be P_k such that k is congruent to* 3 *mod* 4.

(2) *At most one component of $\overline{G}$ can be P_k such that k is congruent to* 1 *modulo* 4.

(3) *If $\overline{G}$ is an (n,ℓ)-extremal graph, then no component of G is a path of order more than* 4.

Proof. For (1), let P be a component of $\overline{G}$ that is a path of order $4k+3$ with $k \in \mathbb{N}\cup\{0\}$. By Lemma 5, if we replace P in $\overline{G}$ with $kP_4 \cup P_3$, the complement of the resulting graph is N-AW if and only if G is. Thus, we can assume $P = P_3$.

However, the end vertices of the P_3 component in $\overline{G}$ are $N(G)$-twins in G, so G is not N-AW by Corollary 3.

For (2), we apply Lemma 5 again. If we have more than one component of $\overline{G}$ is a path with order congruent to 1 modulo 4, we can assume that all such components are P_1. But the vertices of these components are all $N(G)$-twins, and so in order for G to be N-AW, $\overline{G}$ can have at most one component be a path of order congruent to 1 modulo 4.

Finally, (3) follows from the fact that if we replace the component of $\overline{G}$ that is P_k with $k > 4$ with $P_{k-4} \cup P_4$, the complement of the resulting graph will be N-AW with larger size, thus contradicting the assumption that $\overline{G}$ is (n, ℓ)-extremal. ∎

Given a matrix M, let π be a labeling of $V(M)$. For $U \subseteq V(M)$ and $r \in \mathbb{Z}_\ell$, we define the labeling $\pi_{U,r} : V(M) \to \mathbb{Z}_\ell$ as

$$\pi_{U,r}(v) = \begin{cases} \pi(v) + r & v \in U, \\ \pi(v) & v \notin U. \end{cases}$$

In the case $U = V(M)$, we write $\pi_{V(M),r} = \pi_r$.

When we encounter these labelings in the proof of Theorem 9, we are concerned not only if certain labelings are winnable, but also how many toggles can be used to win the game for these labelings. We use $\mathbf{0}$ to denote the zero labeling, which assigns to every vertex a label of 0.

Definition. Let $M \in M_n(\mathbb{Z}_\ell)$, $r \in \mathbb{Z}_\ell$ and $U \subseteq V(M)$. We define the set of *U-toggling numbers* $T_U^M(r) \subseteq \mathbb{Z}_\ell$ as follows. We say $t \in T_U^M(r)$ if the elements of $V(M)$ can be toggled to win the M-Lights Out game with initial labeling $\mathbf{0}_{U,r}$ in such a way that the vertices in U are collectively toggled t times.

Note that each number in $T_U^M(0)$ corresponds to a set of toggles that leaves the initial labeling unchanged. Such sets of toggles are called *null toggles*. Null toggles function very similarly to null spaces of a linear transformation. For instance, there exist two sets of toggles with t toggles and t' toggles of the vertices of U, respectively, to have the same effect on the labels of $V(M)$ if and only if $t' = t + q$ for some $q \in T_U^M(0)$.

In both the neighborhood and adjacency Lights Out games, winning a particular game is equivalent to winning the game on each individual connected component. This simplifies the computation of toggling numbers in these cases. Let G be a graph with $U \subseteq V(G)$ and M is $N(G)$ or $A(G)$. If $G_1, G_2, \ldots, G_c$ are the connected components of G, and $U_i = U \cap V(G_i)$, then $T_U^M(r) = \left\{ \sum_{i=1}^c t_i : t_i \in T_{U_i}^{M_i}(r) \right\}$, where M_i is $N(G_i)$ or $A(G_i)$, respectively.

Suppose we have two different sets of toggles and look at their effect individually on each vertex. For each $v \in V(M)$, suppose that the label of v is increased by r_v for the first set of toggles and is increased by s_v for the second set of toggles. Then combining the two sets of toggles increases each $v \in V(M)$ by $r_v + s_v$. We use this observation to prove the following.

Lemma 7. *Let $n \in \mathbb{N}$ and $M \in M_n(\mathbb{Z}_\ell)$, let $U \subseteq V(M)$, and let $r \in \mathbb{N}$ be minimal such that $T_U^M(r) \neq \emptyset$. Then $r \mid \ell$, and $T_U^M(s) \neq \emptyset$ if and only if $r \mid s$.*

Proof. It is easy to show that $\left\{s \in \mathbb{Z}_\ell : T_U^M(s) \neq \emptyset\right\}$ is an additive subgroup of $\mathbb{Z}_\ell$. The result follows easily. ∎

For graphs with a pendant vertex, it will be helpful to understand the relationship between winning the adjacency game on both the graph and a certain subgraph.

Lemma 8. *Let G be a graph with a pendant vertex p. Let v be the neighbor of p in G, let G' be the graph induced by $V(G) \setminus \{p, v\}$, let $U = N_G(v) \setminus \{p\}$. Finally, let π be a labeling of G, and define the labeling π' on G' by*

$$\pi'(w) = \begin{cases} \pi(w) - \pi(p) & w \in U, \\ \pi(w) & \textit{otherwise.} \end{cases}$$

Then

(1) *π' is $A(G')$-winnable with t toggles from $V(G') \setminus U$ (along with perhaps some toggles from U) if and only if π is $A(G)$-winnable with $t - \pi(v) - \pi(p)$ toggles from $V(G)$.*

(2) *If $s \in \mathbb{Z}_\ell$, then $T_{V(G)}^{A(G)}(s) = \left\{t - 2s : t \in T_{V(G') \setminus U}^{A(G')}(s)\right\}$.*

Proof. For (1), we first assume π' is $A(G')$-winnable with t toggles from $V(G') \setminus U$. If we begin with the labeling π on G, we begin by toggling the vertices as we would to win the adjacency game on G' with labeling π'. When we do this, we subtract $\pi(w)$ from the label of each $w \in V(G') \setminus U$ and subtract $\pi(w) - \pi(p)$ from each $w \in U$. This leaves each vertex in $V(G') \setminus U$ with label 0 and each vertex in U with label $\pi(p)$. If the vertices of U get toggled t_U times, it also leaves v with label $\pi(v) + t_U$. Then v is toggled $-\pi(p)$ times and p is toggled $-\pi(v) - t_U$ times to win the game. The total number of toggles is $t + t_U - \pi(p) - \pi(v) - t_U = t - \pi(p) - \pi(v)$.

If we assume π is $A(G)$-winnable with $t - \pi(p) - \pi(v)$ toggles, note that since v is the only neighbor to p in G, v must be toggled $-\pi(p)$ times to win the $A(G)$-Lights Out game with initial labeling π. This leaves each $w \in V(G')$ with label $\pi'(w)$. We then toggle the vertices of G' as we do for winning the adjacency game on G with initial labeling π. This will win the adjacency game on G' with initial labeling π'. Note that if t_U is the number of toggles among the vertices of U, then that leaves v with label $\pi(v) + t_U$. This requires p to be toggled $-\pi(v) - t_U$ times. If we let t' be the number of toggles among vertices in $V(G') \setminus U$, and if we total up the number of toggles altogether, we get $t - \pi(p) - \pi(v) = -\pi(p) + t' + t_U - \pi(v) - t_U = t' - \pi(p) - \pi(v)$. Thus, $t = t'$, which proves the result. Part (2) follows from letting $\pi(w) = s$ for all $w \in V(G)$. ■

Theorem 9. *Let G be a graph with a pendant vertex p. Let v be the neighbor of p in G, and let G' be the graph induced by $V(G) \setminus \{p, v\}$.*

(1) *G is $A(G)$-AW if and only if G' is $A(G')$-AW.*

(2) *Let $r \in \mathbb{N}$ be minimum such that $T^{A(G)}_{V(G)}(r) \neq \emptyset$, and let $t \in T^{A(G)}_{V(G)}(r)$. Then $\overline{G}$ is $N(\overline{G})$-AW if and only if*

   (a) *for each labeling π of $V(G)$, there is some $s \in \mathbb{Z}_\ell$ such that π_s is $A(G)$-winnable,*

   (b) *for each $z \in \mathbb{Z}_\ell$, there exists $q \in T^{A(G)}_{V(G)}(0)$ such that there is a solution to $(r + t)x \equiv z + q \pmod{\ell}$.*

Proof. For (1), we first assume G is $A(G)$-AW. Let G' have an arbitrary labeling. We extend this labeling to a labeling of G by giving each of p and v a label of 0. This labeling of G is winnable since G is $A(G)$-AW, so we toggle the vertices of G' as we would in a winning toggling of G. If not every label of G' becomes 0, then we have to toggle v to give G a zero labeling. However, this leaves p with a nonzero label. Since v is the only neighbor of p, this implies that toggling v makes the zero labeling on G impossible. Thus, the toggles we did for G' leaves all vertices in G' with label 0, and so G' is $A(G')$-AW.

If we assume G' is $A(G')$-AW and let G have an arbitrary labeling, we first toggle v so that p has label 0. The resulting labeling restricted to G' is winnable since G' is $A(G')$-AW. We can then toggle the vertices of G' so that all vertices of G' have label 0. This leaves all vertices with label 0, except perhaps v since v is the only vertex not in G' that is adjacent to a vertex in G'. We then toggle p until v has label 0, which wins that game. Thus, G is $A(G)$-AW.

For (2), let $U = N_G(v) \setminus \{p\}$. Then $N(\overline{G})$ looks like the following.

	$V(G') \setminus U$	U	v	p
$V(G') \setminus U$	$N(\overline{G' - U})$	$*$	1	1
U	$*$	$N(\overline{U})$	0	1
v	1	0	1	0
p	1	1	0	1

where $G' - U$ is the induced subgraph with vertex set $V(G') \setminus U$ and the $*$ blocks are the entries that make the four top-left blocks $N(\overline{G'})$. We multiply each row except the last by the unit -1 and then add to each of those rows the last row to get

$$M = \begin{array}{c|c|c|c|c|} & V(G')\setminus U & U & v & p \\ \hline V(G')\setminus U & A(G'-U) & \overline{*} & -1 & 0 \\ \hline U & \overline{*} & A(U) & 0 & 0 \\ \hline v & 0 & 1 & -1 & 1 \\ \hline p & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

where the $\overline{*}$ blocks are obtained from $*$ by changing the 1 entries to 0 and the 0 entries to 1. This makes the top-left four blocks $A(G')$. So the M-Lights Out game is played as the $A(G')$-Lights Out game on $V(G')$; toggling any vertex in $V(G')$ adds 1 to the label of p; toggling any vertex in U adds 1 to the label of v; toggling v adds -1 to every vertex in $(V(G)\setminus U)\cup\{v\}$; and toggling p adds 1 to v and p.

We first assume $\overline{G}$ is $N(\overline{G})$-AW. Since $M \sim N(\overline{G})$, $\overline{G}$ is M-AW. To prove (2a), consider the labeling giving v and p labels of 0, each $w \in U$ a label of $\pi(w) - \pi(p)$, and each $w \in V(G')\setminus U$ a label of $\pi(w)$, which is M-winnable by assumption. If we toggle v and p as part of a winning toggling, we get the following labeling of $V(G')$.

$$\lambda(w) = \begin{cases} \pi(w) - \pi(p) & w \in U, \\ \pi(w) + s & \text{otherwise}, \end{cases}$$

where v is toggled $-s$ times. We claim that π_s is $A(G)$-winnable. If we define π'_s similarly as π' in Lemma 8, then $\pi'_s = \lambda$, which we showed to be $A(G')$-winnable. By Lemma 8(1), π_s is winnable.

For (2b), let $z \in \mathbb{Z}_\ell$, and consider the labeling where p has label $-z$ and all other labels are 0. This labeling is M-winnable by assumption, so let y_1 be the number of times v is toggled and y_2 be the number of times p is toggled in order to win the M-Lights Out game with this labeling. This results in each vertex of $V(G')\setminus U$ having label $-y_1$, each vertex of U having label 0, v having label $y_2 - y_1$, and p having label $-z + y_2$.

At this point, we have only the vertices in $V(G')$ to toggle, which means the remaining toggles necessary to win the M-Lights Out game will also win the $A(G')$-Lights Out game with labeling $\mathbf{0}_{V(G')\setminus U,-y_1}$. Thus, $T^{A(G')}_{V(G')\setminus U}(-y_1) \neq \emptyset$. By Lemma 8(2), $T^{A(G)}_{V(G)}(-y_1) \neq \emptyset$, and so $-y_1 = rx$ for some $x \in \mathbb{Z}$ by Lemma 7. By assumption, $t \in T^{A(G)}_{V(G)}(r)$, and so $t = t' - 2r$ for some $t' \in T^{A(G')}_{V(G')\setminus U}(r)$ by Lemma 8(2). Thus, there exists $t_U \in \mathbb{Z}$ such that we can collectively toggle the vertices of U t_U times and the vertices of $V(G')\setminus U$ t' times to win the $A(G')$-Lights Out Game with labeling $\mathbf{0}_{V(G')\setminus U,r}$. By repeating x times the toggles we use for the labeling $\mathbf{0}_{V(G')\setminus U,r}$, we can toggle the vertices of U and $V(G')\setminus U$ xt_U and xt' times, respectively, to win the $A(G')$-Lights Out Game with labeling

$\mathbf{0}_{V(G')\setminus U, rx}$. Since toggling the vertices of G' to win the M-Lights Out game also must win the adjacency game on G' with labeling $\mathbf{0}_{V(G')\setminus U, rx}$, we must toggle the vertices of G' $xt_U + xt' + k$ for some $k \in T^{A(G')}_{V(G')}(0)$. If we let $k = q_1 + q_2$, where q_1 is the number of toggles from $V(G') \setminus U$ and q_2 is the number of toggles from U in the null toggle, we have $q_1 \in T^{A(G')}_{V(G')\setminus U}(0)$. Note that by negating all toggles in this null toggle, we still get a null toggle, and so $-q_1 \in T^{A(G')}_{V(G')\setminus U}(0)$. This leaves all vertices in $V(G')$ with label 0, v with label $y_2 + xr + xt_U + q_2$ (by setting $-y_1 = xr$), and p with label $-z + y_2 + xt_U + xt' + q_1 + q_2$.

All of the toggles have been accounted for, and so the labels of v and p must be 0. We then eliminate y_2 in the resulting system of equations to get $(r - t')x = -z + q_1$. Recall that $t = t' - 2r$, and so $t' = t + 2r$. This gives us $(-r - t)x = -z + q_1$, and so $(r + t)x = z - q_1$. Now let $q = -q_1$. As noted above, $q \in T^{A(G')}_{V(G')\setminus U}(0)$. By Lemma 8(2), $T^{A(G')}_{V(G')\setminus U}(0) = T^{A(G)}_{V(G)}(0)$, and so $q \in T^{A(G)}_{V(G)}(0)$. Since $(r + t)x = z + q$, this proves (2b).

Now we assume that (2a) and (2b) hold, and we prove that $\overline{G}$ is $N(\overline{G})$-AW. Since $M \sim N(\overline{G})$, we need only prove that $\overline{G}$ is M-AW. Let π be a labeling of $V(\overline{G})$. Consider the labeling λ of $V(G)$ that is 0 on p and v and π on $V(G')$. By (2a), λ_s is $A(G)$-winnable for some $s \in \mathbb{Z}_\ell$. If we define λ'_s as in Lemma 8(1), we get $\lambda'_s = \pi_{V(G')\setminus U, s}|_{V(G')}$. By Lemma 8(1), $\pi_{V(G')\setminus U, s}|_{V(G')}$ is $A(G')$-winnable. Then v can be toggled in the M-Lights Out game $-s$ times to obtain $\pi_{V(G')\setminus U, s}|_{V(G')}$ on $V(G')$, and we can then toggle the vertices of $V(G')$ to give every vertex in $V(G')$ a label of 0. This leaves v with label a and p with label b for some $a, b \in \mathbb{Z}_\ell$.

By Lemma 8(2), $t = t' - 2r$ for some $t' \in T^{A(G')}_{V(G')\setminus U}(r)$. Thus, there exists $t_U \in \mathbb{Z}_\ell$ such that we can toggle the vertices of $V(G') \setminus U$ t' times and the vertices of U t_U times to win the $A(G')$-Lights Out game with labeling $\mathbf{0}_{V(G')\setminus U, r}$. Lemma 8(2) implies that $T^{A(G)}_{V(G)}(0) = T^{A(G')}_{V(G')\setminus U}(0)$, and so $q \in T^{A(G')}_{V(G')\setminus U}(0)$. As we reasoned above, $-q \in T^{A(G')}_{V(G')\setminus U}(0)$, and so there exists $q_U \in T^{A}_{U}(0)$ such that $q_U - q \in T^{A(G')}_{V(G')}(0)$. Now let x be a solution to (2b), where $z = a - b$. This gives us $(r - t')x = b - a - q$. If v is toggled $-xr$ times and p is toggled $-b - x(t_U + t') + (q - q_U)$ times, this leaves each vertex of $V(G') - U$ with label xr, each vertex of U with label 0, v with label $a - b + xr - x(t_U + t') + (q - q_U)$, and p with label $-x(t_U + t') + (q - q_U)$. As we reasoned above, we can then win the $A(G')$-Lights Out game with labeling $\mathbf{0}_{V(G')\setminus U, xr}$ (and thus make the labels of $V(G')$ to be 0) by toggling the vertices of U xt_U times and the vertices of $V(G') \setminus U$ a total of xt' times. The vertices of $V(G')$ can be toggled in such a way that the vertices of U are toggled q_U times, the vertices of $V(G') \setminus U$ are toggled $-q$ times, and these toggles collectively have no effect on the labels of $V(G')$. So we combine these to toggle the vertices of U collectively $xt_U + q_U$ times and the vertices of $V(G') \setminus U$ collectively $xt' - q$ times. This leaves the vertices of $V(G')$ with label 0, p with label $(-x[t_U + t'] + [q - q_U]) + (xt_U + q_U + xt' - q) = 0$, and v with label

$$\begin{aligned} a - b + xr - x(t_U + t') + (q - q_U) + xt_U + q_U &= a - b + x(r - t') + q \\ &= a - b + (b - a - q) + q = 0. \end{aligned}$$

This wins the game and shows that $\overline{G}$ is $N(\overline{G})$-AW. ∎

In Theorem 9(2), if G is A-AW, then π_s is automatically $A(G)$-winnable for all $s \in \mathbb{Z}_\ell$. Thus, G satisfies Theorem 9(2a) and makes $r = 1$. Furthermore, $A(G)$ is invertible, so the only null toggle possible is where no buttons are pushed, making $T_{V(G)}^{A(G)}(0) = \{0\}$. This gives us the following.

Corollary 10. *Let G be an A-AW graph with a pendant vertex. Let $t \in T_{V(G)}^{A(G)}(1)$. Then $\overline{G}$ is N-AW if and only if* $\gcd(1 + t, \ell) = 1$.

Proof. Since G is A-AW, part (2a) of Theorem 9 is automatically satisfied. Moreover, since G is A-AW, A is invertible, which implies that $T_{V(G)}^{A(G)}(0) = \{0\}$. The result then follows directly from Theorem 9. ∎

Furthermore, for possible (n, ℓ)-extremal graphs with a dominating vertex, Theorem 9(1) gives us a way to eliminate most graphs with pendant vertices.

Corollary 11. *Let G be a graph with a dominating vertex. If $\overline{G}$ has a pendant vertex that is not part of a component isomorphic to P_2, then G is not (n, ℓ)-extremal for any n and ℓ.*

Proof. For contradiction, assume G is (n, ℓ)-extremal, that $\overline{G}$ has a pendant vertex p with neighbor v, and that p and v are not the only vertices in their connected component of $\overline{G}$. Thus, v has a neighbor other than p in $\overline{G}$. Let w be the dominating vertex in G, and let G' be the subgraph of $\overline{G}$ induced by $V(\overline{G}) \setminus \{p, v, w\}$. If we remove the edges in $\overline{G}$ incident to v but not p, we get the graph $H = G' \cup P_2 \cup P_1$. Note that H has size smaller than $\overline{G}$, and so $\overline{H}$ has size greater than G. To contradict the assumption that G is (n, ℓ)-extremal, it then suffices to prove that $\overline{H}$ is N-AW.

Since G is N-AW, Theorem 4 implies that $\overline{G} - \{w\}$ is A-AW. By Theorem 9(2), this implies that G' is A-AW. Since P_2 is A-AW for all ℓ, it follows that $G' \cup P_2 = H - \{w\}$ is A-AW. By Theorem 4, $\overline{H}$ is N-AW. This means G is not (n, ℓ)-extremal, a contradiction. ∎

A *pendant graph* is a graph where every non-pendant vertex is adjacent to a pendant vertex. As in [12] we write $H \odot K_1$ for the graph in which, for each vertex

v of H, we add a new vertex adjacent only to v. In the case that a pendant graph is a tree or a forest, we use the terms *pendant tree* or *pendant forest*, respectively. One nice property of pendant graphs is that it is really easy to play the A-Lights Out game on them. This is demonstrated in the following result.

Lemma 12. *Let H be a graph, and $G = H \odot K_1$. Then we have the following.*

(1) *G is A-AW for all $\ell \in \mathbb{N}$.*

(2) *If G has size m and order n, then $T_{V(G)}^{A(G)}(1) = \{2(m-n)\}$.*

(3) *If G is a pendant forest with c components, then $T_{V(G)}^{A(G)}(1) = \{-2c\}$.*

Proof. For (1), we have the following algorithm for winning any A-Lights Out game on G. Toggle each vertex in $V(H)$ until its pendant neighbor has label 0. Then toggle each vertex not in $V(H)$ until its neighbor has label 0. This results in the zero labeling, which makes G A-AW.

For (2), we begin with the labeling $\mathbf{0}_1$. Applying the above strategy, each vertex in $V(H)$ is toggled -1 times, giving a total of $-\frac{n}{2}$ toggles. Each vertex toggled also decreases by 1 the label of each adjacent vertex in H. Collectively, this decreases the labels of $V(H)$ by $2|E(H)| = 2\left(m - \frac{n}{2}\right)$. That means that when we toggle the pendant vertices, we must toggle -1 each for the initial label of 1 for each vertex in H plus $\left(m - \frac{n}{2}\right)$ for the decrease in labels from toggling $V(H)$. In total, we get $-\frac{n}{2} - \frac{n}{2} + 2\left(m - \frac{n}{2}\right) = 2(m-n)$, which completes the proof.

Finally, (3) follows from the fact that for a forest, we have $m = n - c$. ∎

We can now determine the N-winnability of the complements of pendant graphs. Interestingly, the issue of whether or not a pendant graph is N-AW depends entirely on the size and order of the pendant graph.

Lemma 13. *Let G be a pendant graph of size m and order n. Then $\overline{G}$ is N-AW if and only if $\gcd(2[n-m]-1, \ell) = 1$. Equivalently, if G is a graph of even order n and size $\binom{n}{2} - \left(\frac{n}{2} + k\right)$ such that $\overline{G}$ is a pendant graph, then G is N-AW if and only if $\gcd(n-2k-1, \ell) = 1$.*

Proof. By Lemma 12(1), G is A-AW, and so we can apply Corollary 10. By Lemma 12(2), $T_{V(G)}^{A(G)}(1) = \{2(m-n)\}$. By Corollary 10, $\overline{G}$ is N-AW if and only if $\gcd(2(m-n)+1, \ell) = 1$. The second part follows from substituting $m = \frac{n}{2} + k$ or $k = m - \frac{n}{2}$ to get $2(n-m) - 1 = n - 2k - 1$. ∎

If $\overline{G}$ is a forest, then $n - m$ is the number of components of $\overline{G}$. This along with Lemma 13 gives us the following.

Corollary 14. *Let G be a graph such that the components of $\overline{G}$ are all pendant trees. If c is the number of components of $\overline{G}$, then G is N-AW if and only if* $\gcd(2c-1,\ell)=1$.

When we are classifying (n,ℓ)-extremal graphs in Section 4, it will be helpful to replace connected components of the complement of one graph with another graph without affecting the N-winnability of the original graph. The following guarantees that the conditions of Theorem 9(2) are unaffected by the replacement.

Corollary 15. *Let G be a graph with a pendant vertex, and let C be a connected component of G that is A-AW. If there exists a graph C' such that*

(1) *C' is A-AW,*

(2) $T_{V(C')}^{A(C')}(1)=T_{V(C)}^{A(C)}(1)$,

(3) *C and C' have the same order,*

(4) *C' has smaller size than C,*

then $\overline{G}$ is not (n,ℓ)-extremal.

Proof. Let G' be the graph identical to G except that the component C is replaced with C'. The winnability of the adjacency game is determined by the winnability of the adjacency game on each connected component of a given graph. Since both C and C' are A-AW, then given a labeling π on G (respectively, a labeling π' on G'), we have that π_s is $A(G)$-winnable (respectively, π'_s is $A(G')$-winnable) if and only if π_s restricted to $G-C$ (respectively, π'_s restricted to $G'-C'$) is $A(G-C)$-winnable (respectively, $A(G'-C')$-winnable). Since $G-C=G'-C'$, this condition is identical for both G and G'. Thus, either both of G and G' satisfy Theorem 9(2a) or neither does. A similar argument gives us that either both of G and G' satisfy Theorem 9(2b) or neither does. Thus, $\overline{G}$ is N-AW if and only if $\overline{G'}$ is N-AW. Furthermore, since C and C' have the same order, so do $\overline{G}$ and $\overline{G'}$. Finally, since C' has smaller size than C, $\overline{G'}$ has larger size than $\overline{G}$. Since $\overline{G'}$ and $\overline{G}$ have the same order and same winnability but $\overline{G'}$ has larger size, $\overline{G}$ cannot be (n,ℓ)-extremal. ■

4. Extremal Graphs

Recall that $\max(n,\ell)$ is the maximum number of edges in an N-AW graph with n vertices and a graph is (n,ℓ)-extremal provided that it has order n, size $\max(n,\ell)$, and is N-AW. We begin with straightforward upper and lower bounds on $\max(n,\ell)$.

Proposition 16. *For any $n,\ell\in\mathbb{N}$ we have*

$$\binom{n}{2}-(n-1)\leq \max(n,\ell)\leq \binom{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor.$$

Proof. For the right inequality, if $|E(\overline{G})| < \left\lfloor \frac{n}{2} \right\rfloor$, at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ edges are removed from K_n to obtain G. Thus, at most $2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \leq n-2$ vertices of K_n can have their degrees reduced by one or more to obtain G. So, at least two vertices in G are dominating vertices. Two dominating vertices are N-twins, and such a G is not N-AW by Corollary 3. On the other hand we know $\max(n,\ell) \geq \binom{n}{2} - (n-1)$ since the complement of any pendant tree is N-AW for all ℓ by Corollary 14. ∎

To obtain the upper bound of Proposition 16, we need G to be a perfect or near-perfect matching. Let M_n be a perfect matching on n vertices when n is even and a near-perfect matching on n vertices when n is odd. The following two results show us when M_n is (n,ℓ)-extremal.

Proposition 17. *If n is odd, then*

$$\max(n,\ell) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$$

for all $\ell \in \mathbb{N}$. Moreover, $\overline{M_n}$ is the unique N-AW of maximum size on n vertices.

Proof. We have that M_{n-1} is a pendant graph, specifically $\left(\frac{n-1}{2}K_1\right) \odot K_1$, so by Lemma 12(1), M_{n-1} is A-AW for all ℓ. By Theorem 4, $\overline{M_n}$ is N-AW. So, $\max(n,\ell) \geq \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$ when n is odd. By Proposition 16, $\max(n,\ell) = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$. Any graph $G \neq \overline{M_n}$ with $\binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor$ edges must have at least two dominating vertices in G, which are N-twins. Thus, $\overline{M_n}$ is unique. ∎

However, when n is even, not all complements of perfect matchings give us (n,ℓ)-extremal graphs.

Proposition 18. *If n is even, then*

$$\max(n,\ell) = \binom{n}{2} - \frac{n}{2} \textit{ if and only if } \gcd(n-1,\ell) = 1.$$

If n is even and $\gcd(n-1,\ell) = 1$, then $\overline{M_n}$ is the unique N-AW graph of maximum size on n vertices.

Proof. Each component of M_n is a pendant tree. By Corollary 14, $\overline{M_n}$ is N-AW if and only if $\gcd\left(2\binom{\frac{n}{2}}{}- 1, \ell\right) = \gcd(n-1,\ell) = 1$. That $\overline{M_n}$ is the unique N-AW graph of maximum size on n vertices again follows from two dominating vertices being N-twins in any other case. ∎

It turns out that when n is even, finding $\max(n,\ell)$ is considerably more complicated, though we conjecture that almost all extremal graphs are complements of pendant graphs. In Proposition 19 we find the extremal graphs where n is even and ℓ is odd. This is the only situation we have found in which an (n,ℓ)-extremal graph is not the complement of a pendant graph.

Proposition 19. *Suppose that $n \geq 4$ is even. If ℓ is odd and* $\gcd(n-1,\ell) \neq 1$, *then* $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$. *In this case an (n,ℓ)-extremal graph is* $\overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$.

Proof. We first show that $H = \overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$, which has $\binom{n}{2} - \left(\frac{n}{2}+1\right)$ edges, is N-AW. By Theorem 4, we need only prove that $C_3 \cup \left(\frac{n-4}{2}\right) P_2$ is A-AW. Clearly, P_2 is A-AW for all $\ell \in \mathbb{N}$. We can see that C_3 is A-AW if and only if ℓ is odd by row reducing the adjacency matrix of C_3. By playing on each component, $C_3 \cup \left(\frac{n-4}{2}\right) P_2$ is A-AW, and so H is N-AW.

We now show that H is (n,ℓ)-extremal. Since $\gcd(n-1,\ell) \neq 1$, Proposition 18 implies that M_n is not (n,ℓ)-extremal. By the uniqueness of M_n, $\max(n,\ell) \leq \binom{n}{2} - \left(\frac{n}{2}+1\right)$. Since H is N-AW and $|E(\overline{H})| = \binom{n}{2} - \left(\frac{n}{2}+1\right)$ $\max(n,\ell) = \binom{n}{2} - \frac{n}{2} + 1$. ■

Since $\overline{M_n}$ is the (n,ℓ)-extremal graph in the case that n is odd and $\overline{C_3 \cup \left(\frac{n-4}{2}\right) P_2 \cup K_1}$ is an (n,ℓ)-extremal graph in the case that n is even and ℓ is odd, from here on we consider only cases where n and ℓ are both even. In this case we find the quantity given in Conjecture 1 is a lower bound.

Proposition 20. *If n and ℓ are both even, then*

$$\max(n,\ell) \geq \binom{n}{2} - \left(\frac{n}{2}+k\right),$$

where k is the smallest nonnegative integer such that $\gcd(n-2k-1,\ell) = 1$.

Proof. Suppose that k is the smallest nonnegative integer such that $\gcd(n-2k-1,\ell) = 1$. Let $G = kP_4 \cup \left(\frac{n}{2} - 2k\right) P_2$. Then G is a pendant graph and $\overline{G}$ has size $\binom{n}{2} - \left(\frac{n}{2}+k\right)$. So by Lemma 13 we have $\overline{G}$ is N-AW and the result follows. ■

Note when $k = \frac{n}{2}-1$ we have $n-2k-1 = 1$ and so $\gcd(n-2k-1,\ell) = 1$. This gives us the lower bound in Proposition 16. In the following two subsections we find $\max(n,\ell)$ in two cases: finding all graphs with minimum degree $n-2$ or $n-3$ that are (n,ℓ)-extremal for any ℓ in Section 4.1, and finding all combinations of n and ℓ such that the (n,ℓ)-extremal graph has $\binom{n}{2} - \left(\frac{n}{2}+k\right)$ edges for $0 \leq k \leq 3$ in Section 4.2. In both perspectives we are led to pendant graphs, which supports Conjecture 1.

4.1. Extremal graphs with a given minimum degree

The minimum degree of an (n,ℓ)-extremal graph cannot be $n-1$ because K_n is not N-AW. Moreover, if the minimum degree is $n-2$ then, to avoid twins, the complement graph must be M_n. So, Propositions 17 and 18 tell us that if G is an (n,ℓ)-extremal graph with minimum degree $n-2$, then $G = \overline{M_n}$, which is the complement of a pendant graph when n is even. Thus, in this section we find $\max(n,\ell)$ among all graphs with minimum degree $n-3$. Recall we can assume n and ℓ are even.

If G has minimum degree $n-3$, the complement has maximum degree 2. So the components of the complement graph are paths and cycles. We denote the cycle graph C_k by $V(C_k)=\{v_i : 1\le i\le k\}$, $E(C_k)=\{v_iv_{i+1}, v_kv_1 : 1\le i\le k-1\}$. Our approach to the A-Lights Out game on C_k is similar to our approach to the N-Lights Out game in [10]. We first reduce an arbitrary labeling to a canonical labeling, and then determine when these canonical labelings can be won.

To that end, for $a,b\in\mathbb{Z}_\ell$ we define $\lambda_{a,b}$ to be the labeling where v_1 has label a, v_2 has label b, and the other vertices have label 0. By a straightforward induction proof, given any initial labeling of C_k in the A-Lights Out game, the vertices can be toggled to achieve the $\lambda_{a,b}$ labeling for some $a,b\in\mathbb{Z}_\ell$. These are our canonical labelings. The following lemma shows how we deal with the labelings $\lambda_{a,b}$ and $(\lambda_{a,b})_s$.

Lemma 21. *Let π be a labeling of $V(C_k)$, and let ℓ be even.*

(1) *The labeling $\lambda_{a,b}$ is A-winnable precisely in the following circumstances.*
   - *When $n\equiv 0 \pmod 4$ and $a=b=0$.*
   - *When $n\equiv 1,3 \pmod 4$ and a and b have the same parity.*
   - *When $n\equiv 2 \pmod 4$ and a and b are both even.*

(2) *The labeling $(\lambda_{a,b})_s$ can be toggled in the A-Lights Out game to obtain the following labelings.*
   - *When $n\equiv 0 \pmod 4$, $\lambda_{a,b}$.*
   - *When $n\equiv 1 \pmod 4$, $\lambda_{a,b-s}$.*
   - *When $n\equiv 2 \pmod 4$, $\lambda_{a-s,b-s}$.*
   - *When $n\equiv 3 \pmod 4$, $\lambda_{a-s,b}$.*

Proof. For (1), let t_i be the number of times we toggle v_i. By a straightforward induction proof, it follows that $\lambda_{a,b}$ is A-winnable if and only if $t_{n-1}+t_1=0$, $t_n+a+t_2=0$, and for $2\le i\le n$ we have

$$t_i=\begin{cases}-t_2 & i\equiv 0 \pmod 4,\\ b+t_1 & i\equiv 1 \pmod 4,\\ t_2 & i\equiv 2 \pmod 4,\\ -b-t_1 & i\equiv 3 \pmod 4\,.\end{cases}$$

Then (1) follows from using the equations above with $i=n-1$ and $i=n$.

For (2), we begin with the labeling $(\lambda_{a,b})_s$, and then each v_i with $2 \leq i < 4\left\lfloor \frac{n}{4} \right\rfloor$ and $i \equiv 2, 3 \pmod 4$ is toggled $-s$ times. This results in the labeling where each v_i with $1 \leq i \leq 4\left\lfloor \frac{n}{4} \right\rfloor$ has label $\lambda_{a,b}(v_i)$ and each v_i with $i > 4\left\lfloor \frac{n}{4} \right\rfloor$ has label $(\lambda_{a,b})_s(v_i)$. This gives us the $n \equiv 0 \pmod 4$ case, and the other cases follow from appropriately toggling some combination of v_1, v_{n-1}, and v_n. ■

The next result helps us see how the presence of cycle components in a graph can affect how we apply Theorem 9(2).

Lemma 22. *Let G be a graph.*

(1) *If G has a connected component that is a cycle of even order, then G has a labeling π such that π_s is not A-winnable for all $s \in \mathbb{Z}_\ell$.*

(2) *If G has two connected components that are cycles, then G has a labeling π such that π_s is not A-winnable for all $s \in \mathbb{Z}_\ell$.*

Proof. For (1), let C be a cycle component of G with even order, and define a labeling that is $\lambda_{1,0}$ on C and arbitrary on the remaining vertices of G. Since a labeling is winnable on a graph if and only if it is winnable on each connected component, it suffices to prove that $(\lambda_{1,0})_s$ is not winnable on C for all $s \in \mathbb{Z}_\ell$. If C has order divisible by 4, then Lemma 21(2) implies that the vertices of C can be toggled to achieve $\lambda_{1,0}$, which is not winnable by Lemma 21(1). If C has order not divisible by 4, then by Lemma 21(2), the vertices can be toggled to achieve the labeling $\lambda_{1-s,-s}$. Since $1-s$ and s can never both be even, Lemma 21(1) implies that $\lambda_{1-s,-s}$ is not winnable for all $s \in \mathbb{Z}_\ell$. In either case, $(\lambda_{1,0})_s$ is not winnable on C for all $s \in \mathbb{Z}_\ell$, and so $(\lambda_{1,0})_s$ is not A-winnable for all $s \in \mathbb{Z}_\ell$.

For (2), let C and C' be two cycle components of G. By (1), we can assume each of C and C' has odd order. We claim that for any labeling π that restricts to $\lambda_{1,0}$ on C and $\lambda_{0,0}$ on C', π_s is not A-winnable for all $s \in \mathbb{Z}_\ell$. By Lemma 21(2), with initial labeling π_s, we can toggle the vertices of G to obtain a labeling that restricts either to $\lambda_{1-s,0}$ or $\lambda_{1,-s}$ on C and restricts either to $\lambda_{-s,0}$ or $\lambda_{0,-s}$ on C'. If s is even, then $1-s$ and 0 as well as 1 and $-s$ have opposite parity. If s is odd, then $-s$ and 0 have opposite parity. In any case, Lemma 21(1) implies that π_s is not winnable, and so π is not A-winnable for all $s \in \mathbb{Z}_\ell$. ■

Our next lemma helps us when we want to apply Theorem 4 to graphs with both a dominating vertex and a cycle component in its complement.

Lemma 23. *If ℓ is even, then every cycle graph is not A-AW.*

Proof. By Lemma 21(1), if $a, b \in \mathbb{Z}_\ell$ have opposite parity, then $\lambda_{a,b}$ is not A-winnable. The result follows. ■

The following theorem gives us a connection between (n, ℓ)-extremal graphs and pendant graphs, in support of Conjecture 1. We use $\Delta(G)$ to denote the maximum degree of a graph G.

Theorem 24. *Let ℓ be even, and let G be an (n, ℓ)-extremal graph of even order with $\Delta(\overline{G}) \leq 2$. Then each connected component of $\overline{G}$ is either P_2 or P_4.*

Proof. Suppose $\Delta(\overline{G}) = 1$. All dominating vertices in G are N-twins so by Corollary 3, G has at most 1 dominating vertex. However, G cannot have only one dominating vertex since G has even order. Thus, G has no dominating vertices, and so $\overline{G}$ has no isolated vertices. It follows that each connected component of G is P_2.

In the case $\Delta(\overline{G}) = 2$, we first prove that $\overline{G}$ has at least one path component. If not, all connected components are cycles, and so $|E(\overline{G})| = |V(\overline{G})|$. However, note that any pendant tree of order $|V(G)|$ is N-AW for all ℓ by Corollary 14. Since the pendant tree has size $|V(G)| - 1$, this implies that G is not (n, ℓ)-extremal. Thus, $\overline{G}$ has at least one path component (possibly P_1).

By Corollary 6, no component of $\overline{G}$ is P_k with $k \geq 5$ or $k = 3$. Furthermore two components of P_1 in $\overline{G}$ would be N-twins in G, which is prohibited by Corollary 3. If we have one component of P_1, Theorem 4 implies that all other connected components of $\overline{G}$ are A-AW. This excludes cycles by Lemma 23. Since the remaining paths have even order, this would force G to have odd order, which is a contradiction.

So G is N-AW and $\overline{G}$ has a pendant tree (P_2 or P_4) as a component. Thus, $\overline{G}$ has a pendant vertex, so we can use Theorem 9(2). This implies that $\overline{G}$ is (A, ℓ, s)-winnable for some $s \in \mathbb{Z}_\ell$. However, Lemma 22 implies that this cannot happen if either $\overline{G}$ has more than one cycle component or if $\overline{G}$ has a cycle component of even order. Moreover, if $\overline{G}$ has precisely one cycle component, and if that connected component has odd order, this implies that G has odd order, which is a contradiction. Thus, $\overline{G}$ has no cycle components, and so each connected component is either P_2 or P_4, which completes the proof. ∎

Note that the (n, ℓ)-extremal graphs given in Theorem 24 are pendant graphs. By Lemma 13, $kP_4 \cup \frac{n-4k}{2}P_2$ is N-AW if and only if $\gcd(n - 2k - 1, \ell) = 1$. This implies Conjecture 1 for the family of graphs that have minimum degree at least $n - 3$.

4.2. Extremal graphs with $\binom{n}{2} - \left(\frac{n}{2} + k\right)$ edges

In this section we prove Conjecture 1 for $0 \leq k \leq 3$. We state Theorem 25 in the language of that conjecture.

Theorem 25. *For n, ℓ even and $0 \le k \le 3$*

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + k\right),$$

where k is the smallest nonnegative integer such that $\gcd(n - 2k - 1, \ell) = 1$. In each case the (n, ℓ)-extremal graphs are precisely the complements of pendant graphs of order n that have size $\binom{n}{2} - \left(\frac{n}{2} + k\right)$.

We will prove this result using separate propositions for each k. When $k = 0$, Proposition 18 implies Theorem 25. The following lemma will help us for the cases $1 \le k \le 3$.

Lemma 26. *Let $n \in \mathbb{N}$ be even, let $\ell \in \mathbb{N}$, and let G be a N-AW graph with $|E(\overline{G})| = \frac{n}{2} + t$, where $t \ge 1$. Then $\Delta(\overline{G}) \le t + 1$ where $\Delta(\overline{G})$ is the maximum degree of $\overline{G}$.*

Proof. We let $v \in V(\overline{G})$ and show $\deg(v) \le t + 1$, where $\deg(v)$ is the degree of v in $\overline{G}$. Let $W = V(\overline{G}) \setminus N_{\overline{G}}[v]$, and note that $\deg(v) = |N_{\overline{G}}(v)|$. Then $|W| = n - \deg(v) - 1$. In the graph $\overline{G}$, let k be the number of edges incident only to vertices in $N_{\overline{G}}(v)$, let r be the number of edges incident only to vertices in W, and let s be the number of edges between a vertex in $N_{\overline{G}}(v)$ and a vertex in W. Since $|E(\overline{G})| = \frac{n}{2} + t$, we have $\frac{n}{2} + t = \deg(v) + k + r + s$, and so $k + r + s = \frac{n}{2} + t - \deg(v)$.

Since G is N-AW, it cannot have any N-twins. Thus, no vertices in $N_{\overline{G}}(v)$ can be N-twins, so we can have at most one vertex in $N_{\overline{G}}(v)$ that is adjacent in G to every vertex except v. In other words, there are at least $\deg(v) - 1$ vertices in W that are adjacent in $\overline{G}$ to vertices other than v. There can be at most two such vertices for each of the k edges in $\overline{G}$ incident with two vertices in $N_{\overline{G}}(v)$, and at most one such vertex for each of the s edges between vertices in $N_{\overline{G}}(v)$ and W. This means that there are at most $2k + s$ such vertices in $N_{\overline{G}}(v)$. It follows that $\deg(v) - 1 \le 2k + s$, and so $\deg(v) \le 2k + s + 1$.

In order to prevent any vertices in W from becoming N-twins, we can have at most one vertex in W that is adjacent to every vertex in G. In other words, there are at least $|W| - 1 = n - \deg(v) - 2$ vertices in W with nonzero degree in $\overline{G}$. Similar reasoning as in the previous paragraph implies that there are at most $2r + s$ such vertices in W, and so $n - \deg(v) - 2 \le 2r + s$. Thus, $\deg(v) \ge n - 2r - s - 2$.

Since we have $n - 2r - s - 2 \le \deg(v) \le 2k + s + 1$, it follows that $n - 2r - s - 2 \le 2k + s + 1$. This gives us $n - 2k - 2r - 2s \le 3$. Since the left side of the equation is even, this actually gives us $n - 2k - 2r - 2s \le 2$, and so $\frac{n}{2} - k - r - s \le 1$. Rearranging this a bit gives us $k + r + s \ge \frac{n}{2} - 1$.

Now we use the fact $k + r + s = \frac{n}{2} + t - \deg(v)$ to get $\frac{n}{2} + t - \deg(v) \ge \frac{n}{2} - 1$. Solving for $\deg(v)$ gives $\deg(v) \le t + 1$. ■

In the next proposition, we resolve the $k=1$ case of Theorem 25.

Proposition 27. *Suppose that n and ℓ are even and $n \geq 4$. Then*

$\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$ *if and only if* $\gcd(n-1,\ell) \neq 1$ *and* $\gcd(n-3,\ell) = 1$.

Moreover, the only (n,ℓ)-extremal graph in this case is the complement of the unique pendant graph of order n and size $\binom{n}{2} - \left(\frac{n}{2}+1\right)$, which is $\overline{P_4 \cup \left(\frac{n}{2}-2\right) P_2}$.

Proof. Suppose $\gcd(n-1,\ell) \neq 1$ and $\gcd(n-3,\ell) = 1$. Consider $H = P_4 \cup \left(\frac{n}{2}-2\right) P_2$. Note that H is a pendant graph with n vertices and $\frac{n}{2}+1$ edges. By Lemma 13, $\overline{H}$ is N-AW if and only if $\gcd(n-3,\ell)=1$. Since $\gcd(n-1,\ell) \neq 1$ it follows from Proposition 18 that $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$.

Suppose $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+1\right)$. Then $\gcd(n-1,\ell) \neq 1$, since otherwise $\max(n,\ell) = \binom{n}{2} - \frac{n}{2}$ by Proposition 18. By Lemma 26, if G is N-AW with $|E(\overline{G})| = \frac{n}{2}+1$, then $\Delta(\overline{G}) \leq 2$. So by Theorem 24 each connected component of $\overline{G}$ is either P_2 or P_4. The only such graph with $\frac{n}{2}+1$ edges is H. Thus $\gcd(n-3,\ell)=1$. It is clear that H is the only pendant graph of order $\frac{n}{2}+1$. ■

In the next proposition, we resolve the $k=2$ case of Theorem 25. The proof considers the possible degree sequences of the complements of (n,ℓ)-extremal graphs. To ease our explanation we introduce a notation. Let a *d-vertex* refer to a vertex of degree d. A *d^+-vertex* is a vertex of degree d or more.

Lemma 28. *Suppose G is an N-AW graph. Then any d-vertex in $\overline{G}$ with $d \geq 2$ must have at least $d-1$ neighbors that are 2^+-vertices.*

Proof. Suppose that v is a d-vertex in $\overline{G}$ with $d \geq 2$ and that v has fewer than $d-1$ neighbors that are 2^+ vertices. Then v has two neighbors of degree 1 in $\overline{G}$, which results in G having N-twins. By Corollary 3, $\overline{G}$ is not N-AW. ■

Proposition 29. *Let $n, \ell \in \mathbb{N}$ be even and $n \geq 6$. Then*

$$\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right) \text{ if and only if } \gcd(n-1,\ell) \neq 1,\ \gcd(n-3,\ell) \neq 1, \text{ and } \gcd(n-5,\ell) = 1.$$

Moreover, the (n,ℓ)-extremal graphs in this case are precisely the complements of pendant graphs of order n and size $\frac{n}{2}+2$: $\overline{(P_3 \odot K_1) \cup \frac{n-6}{2} P_2}$ and $\overline{2P_4 \cup \frac{n-8}{2} P_2}$ with the latter only possible when $n \geq 8$.

Proof. Suppose $\gcd(n-1,\ell) \neq 1$, $\gcd(n-3,\ell) \neq 1$ and $\gcd(n-5,\ell) = 1$. By Proposition 18 and Proposition 27, $\max(n,\ell) \leq \binom{n}{2} - \left(\frac{n}{2}+2\right)$. Consider $H = (P_3 \odot K_1) \cup \frac{n-6}{2} P_2$ which has size $\frac{n}{2}+2$. By Lemma 13, $\overline{H}$ is N-AW if and only if $\gcd(n-5,\ell)=1$. So, $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right)$.

Now suppose that $\max(n,\ell) = \binom{n}{2} - \left(\frac{n}{2}+2\right)$. Then $\gcd(n-1,\ell) \neq 1$ and $\gcd(n-3,\ell) \neq 1$ by Propositions 18 and 27. We will describe all G such that

G is N-AW and $E(\overline{G}) = \frac{n}{2} + 2$ and show either that these graphs are not (n, ℓ)-extremal or that they are N-AW if and only if $\gcd(n-5, \ell) = 1$.

Suppose that G is N-AW with $E(\overline{G}) = \frac{n}{2} + 2$. The degree sum of $\overline{G}$ is $n+4$. By Lemma 26, $\Delta(\overline{G}) \leq 3$. To avoid N-twins in G, $\overline{G}$ can have at most one 0-vertex. Thus the only possible degree sequences for $\overline{G}$ are $d_0 = (3, 3, 1, 1, \ldots, 1)$, $d_1 = (3, 2, 2, 1, 1, \ldots, 1)$, $d_2 = (2, 2, 2, 2, 1, 1, \ldots, 1)$, $d_3 = (3, 3, 2, 1, 1, \ldots, 1, 0)$, $d_4 = (3, 2, 2, 2, 1, 1, \ldots, 1, 0)$, and $d_5 = (2, 2, 2, 2, 2, 1, 1, \ldots, 1, 0)$. For a graph with degree sequence d_0 note that each of the 3-vertices must have at least two pendant neighbors. So by Lemma 28, no graph with degree sequence d_0 is N-AW.

If $\overline{G}$ has degree sequence d_1, Lemma 28 implies that all 2^+-vertices are in the same component. The other components of $\overline{G}$ must be a matching. So our options are $G' \cup \frac{n-4}{2} P_2$ where G' is shown in Figure 1 or $H = (P_3 \odot K_1) \cup \frac{n-6}{2} P_2$. Note $\overline{H}$ is N-AW if and only if $\gcd(n-5, \ell) = 1$ as shown above. The graph G' has order 4 and size 4. Moreover, given the initial labeling $\mathbf{0_1}$ we can achieve the 0 labeling by toggling the vertices b and c each -1 times. Thus, $T^A_{V(G_1)}(1) = \{-2\}$. By Lemma 2, G' is A-AW because the adjacency matrix is invertible. The graph $P_4 = P_2 \odot K_1$ has order 4, size 3, is A-AW by Lemma 12(1), and, by Lemma 12(2), $T^A_{V(P_4)}(1) = \{-2\}$. Thus, $\overline{G' \cup \frac{n-4}{2} P_2}$ is not (n, ℓ)-extremal by Corollary 15.

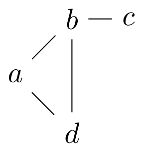

Figure 1. One option for the non-matching component of a graph with degree sequence d_1 in the proof of Proposition 29.

If $\overline{G}$ has degree sequence d_2, then $\Delta(\overline{G}) = 2$ and so, by Theorem 24, each component is P_2 or P_4. This leaves just $2P_4 \cup \frac{n-8}{2} P_2$ which is a pendant graph and thus, by Lemma 13, N-AW if and only if $\gcd(n-5, \ell) = 1$.

Suppose $\overline{G}$ has degree sequence d_3, d_4 or d_5. In these cases $\overline{G}$ has an isolated vertex so by Corollary 11, any component with non-pendant vertices has no pendant vertices. Degree sequence d_3 is impossible because there are not enough 2^+ vertices to be in a component with a 3-vertex. If $\overline{G}$ has degree sequence d_4, this implies one of the components must have odd degree sum which is impossible. If $\overline{G}$ has degree sequence d_5, then $\Delta(\overline{G}) = 2$. By Theorem 24, if G is (n, ℓ)-extremal, then each component of $\overline{G}$ is either P_2 or P_4. Since the number of 2-vertices is odd no such graph exists.

Thus if $\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 2\right)$, then $\gcd(n-5, \ell) = 1$. Moreover the unique (n, ℓ)-extremal graphs are $\overline{(P_3 \odot K_1) \cup \frac{n-6}{2} P_2}$ and $\overline{2P_4 \cup \frac{n-8}{2} P_2}$ which are the complements of the only pendant graphs of order n and size $\frac{n}{2} + 2$. ∎

In the next proposition, we resolve the case $k = 3$ of Theorem 25.

Proposition 30. *Let $n, \ell \in \mathbb{N}$ be even and $n \geq 8$. Then*

$$\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 3\right) \text{ if and only if } \gcd(n - 2k - 1, \ell) \neq 1 \text{ for } 0 \leq k \leq 2, \text{ and } \gcd(n - 7, \ell) = 1.$$

In this case the unique (n, ℓ)-extremal examples are exactly those graphs whose complements are pendant graphs with $\frac{n}{2} + 3$ edges : $(C_3 \odot K_1) \cup \frac{n-6}{2} P_2$, $(P_4 \odot K_1) \cup \frac{n-8}{2} P_2$, $(K_{1,3} \odot K_1) \cup \frac{n-8}{2} P_2$, $(P_3 \odot K_1) \cup P_4 \cup \frac{n-10}{2} P_2$, and $3P_4 \cup \frac{n-12}{2} P_2$.

Proof. Since the proof is a case analysis using the same techniques as in Proposition 29, we offer a summary of the proof here. The proof of the forward direction is essentially the same. Suppose that $\max(n, \ell) = \binom{n}{2} - \left(\frac{n}{2} + 3\right)$. We describe all G such that G is (N, ℓ)-AW with $E(\overline{G}) = \frac{n}{2} + 3$ and show that these graphs are either not (n, ℓ)-extremal or are (N, ℓ)-AW if and only if $\gcd(n - 7, \ell) = 1$. To avoid N-twins in G, $\overline{G}$ can have at most one 0-vertex. We consider two cases: $\overline{G}$ having a 0-vertex or having no 0-vertices.

If $\overline{G}$ has a 0-vertex, using Corollary 11 and Lemma 26, the possible degree sequences are $d_0 = (4, 4, 2, 1, 1, \ldots, 1, 0)$; $d_1 = (4, 3, 3, 1, 1, \ldots, 1, 0)$; $d_2 = (4, 2, 2, 2, 2, 1, \ldots, 1, 0)$; $d_3 = (3, 3, 2, 2, 2, 1, \ldots, 1, 0)$; and $d_4 = (2, 2, 2, 2, 2, 2, 2, 1, \ldots, 1, 0)$. Using a combination of Corollary 11 and Lemma 28 we find the possible graphs associated with these degree sequences. Then we use Theorem 4 and row reduction of the resulting adjacency matrix to show these graphs are never (N, ℓ)-AW or not extremal.

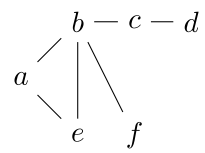

Figure 2. One option for the non-matching component for a graph with degree sequence d_7 in the proof of Proposition 30.

Suppose $\overline{G}$ has no 0-vertex. To get a degree sum of $n + 6$ we add all integer partitions of 6 that have parts of size at most 3 (because the max degree is 4 by Lemma 26) and add these partitions to $(1, 1, \ldots, 1)$. We get the following degree sequences: $d_5 = (4, 4, 1, 1, \ldots, 1)$; $d_6 = (4, 3, 2, 1, \ldots, 1)$; $d_7 = (4, 2, 2, 2, 1, \ldots, 1)$; $d_8 = (3, 3, 3, 1, \ldots, 1)$; $d_9 = (3, 3, 2, 2, 1, \ldots, 1)$; $d_{10} = (3, 2, 2, 2, 2, 1, \ldots, 1)$; $d_{11} = (2, 2, 2, 2, 2, 2, 1, \ldots, 1)$. We can eliminate d_5 and d_6 using Lemma 28. Considering the possible adjacencies among the 2-vertices in d_7 the possible graphs are $K_{1,3} \odot K_1 \cup \frac{n-4}{2} P_2$ and the graph G in Figure 2. This graph has order 6, size 6, and $T_G^A(1) = -2$ (by toggling b and c each -1 times). However, $P_3 \odot K_1$ is

(A,ℓ)-AW by Lemma 12(1), has order 6, size 5, and has $T^{A}_{V(P_3\odot K_1)} = -2$ by Lemma 12(3). Thus by Corollary 15, $\overline{G_2 \cup \frac{n-6}{2}P_2}$ is not (n,ℓ)-extremal.

For degree sequence d_8, by Lemma 28 the only possibility is the pendant graph $(C_3\odot K_1) \cup \frac{n-6}{2}P_2$ which is (N,ℓ)-AW if and only if $\gcd(n-7,\ell) = 1$ by Lemma 13. For degree sequences d_9 and d_{10} we generate all possible graphs using Lemma 28. Then we can eliminate all possibilities that are not pendant graphs using Corollary 3 and Corollary 15. Finally, for degree sequence d_{11} we apply Theorem 24 to get the only possibility is a pendant graph which is always winnable if and only if $\gcd(n-7,\ell) = 1$ by Lemma 13. ■

Note that we have now proved Theorem 25 by Propositions 18, 27, 29, 30.

5. Open Problems

We close with three open problems related to our results.

(1) *Does Theorem* 25 *hold for* $k \geq 4$? We made much progress on this result by considering the possible degree sequences. However, when $k = 4$, there are 37 partitions of 7 and 8. Even with the additional restriction of Lemma 26 there are 23 different degree sequences to consider. Thus, we need an alternative method to solve the general problem.

(2) *What are the graphs of maximum size that are* (N,ℓ)*-AW for all* ℓ? This would be the graphs with neighborhood adjacency matrices that have determinant 1 or -1. The best candidates we have found are complements of pendant trees, which have size $\binom{n}{2} - (n-1)$. They are all (N,ℓ)-AW for all ℓ, but it is not clear that they are (n,ℓ)-extremal.

(3) *What are the* (n,ℓ)*-extremal graphs for other Lights Out games, such as the adjacency game*?

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