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STRONG SUBGRAPH 2-ARC-CONNECTIVITY AND ARC-STRONG CONNECTIVITY OF CARTESIAN PRODUCT OF DIGRAPHS

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Abstract

Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. A strong subgraph H of D is called an S-strong subgraph if $S \subseteq V(H)$. A pair of S-strong subgraphs D_1 and D_2 are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Let $\lambda_S(D)$ be the maximum number of pairwise arc-disjoint S-strong subgraphs in D. The strong subgraph k-arc-connectivity is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

The parameter $\lambda_k(D)$ can be seen as a generalization of classical edge-connectivity of undirected graphs.

In this paper, we first obtain a formula for the arc-connectivity of Cartesian product $\lambda(G\Box H)$ of two digraphs G and H generalizing a formula for edge-connectivity of Cartesian product of two undirected graphs obtained by Xu and Yang (2006). Using this formula, we get a new formula for the arc-connectivity of Cartesion product of $k \geq 2$ copies of a strong digraph G: $\lambda(G^k) = k \cdot \min\{\delta^+(G), \delta^-(G)\}$. Then we study the strong

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subgraph 2-arc-connectivity of Cartesian product $\lambda_2(G\Box H)$ and prove that $\min\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G)+\delta^+(H),$
$\delta^-(G)+\delta^-(H)\} \geq \lambda_2(G\Box H) \geq \lambda_2(G)+\lambda_2(H)-1$. The upper bound for $\lambda_2(G\Box H)$ is sharp and is a simple corollary of the formula for $\lambda(G\Box H)$. The lower bound for $\lambda_2(G\Box H)$ is either sharp or almost sharp i.e., differs by 1 from the sharp bound. We improve the lower bound under an additional condition and prove its sharpness by showing that $\lambda_2(G\Box H) \geq \lambda_2(G)+\lambda_2(H)$, where G and H are two strong digraphs such that $\delta^+(H) > \lambda_2(H)$. We also obtain exact values for $\lambda_2(G\Box H)$, where G and H are digraphs from some digraph families.

Keywords: connectivity, strong subgraph arc-connectivity, Cartesian product, tree connectivity.

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1. Introduction

For a graph $G=(V,E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or, simply, an *S-tree* is a subgraph T of G which is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *edge-disjoint* if $E(T_1)\cap E(T_2)=\emptyset$. Two arc-disjoint S-trees T_1 and T_2 are said to be *internally disjoint* if $V(T_1)\cap V(T_2)=S$. The *generalized local connectivity* $\kappa_S(G)$ is the maximum number of pairwise internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* [3] is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Similarly, the *generalized local edge-connectivity* $\lambda_S(G)$ is the maximum number of pairwise edge-disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* [8] is defined as

$$\lambda_k(G) = \min\{\lambda_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Let $\kappa(G)$ and $\lambda(G)$ denote the classical vertex-connectivity and edge-connectivity of an undirected graph G. Observe that $\kappa_2(G)=\kappa(G)$ and $\lambda_2(G)=\lambda(G)$, hence, these two parameters are generalizations of classical connectivity of undirected graphs and are also called tree connectivity. Now the topic of tree connectivity has become an established area in graph theory, see a recent monograph [7] by Li and Mao on this topic.

To extend generalized k-connectivity to directed graphs, Sun, Gutin, Yeo and Zhang [12] observed that in the definition of $\kappa_S(G)$, one can replace "an S-tree" by "a connected subgraph of G containing S." Therefore, they defined *strong subgraph k-connectivity* by replacing "connected" with "strongly connected" (or, simply, "strong") as follows. Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. An *S-strong subgraph* is a strong subgraph H of D such that $S \subseteq V(H)$. S-strong subgraphs $D_1, \ldots, D_p$ are said to be *internally disjoint* if $V(D_i) \cap V(D_j) = S$ and $A(D_i) \cap A(D_j) = \emptyset$ for all $1 \leq i < j \leq p$. Let $\kappa_S(D)$ be the maximum number of pairwise internally disjoint S-strong digraphs in D. The *strong subgraph k-connectivity* [12] is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$

As a natural counterpart of the strong subgraph k-connectivity, Sun and Gutin [11] introduced the concept of strong subgraph k-arc-connectivity. Let $D = (V(D), A(D))$ be a digraph of order n, $S \subseteq V$ a k-subset of $V(D)$ and $2 \leq k \leq n$. Let $\lambda_S(D)$ be the maximum number of pairwise arc-disjoint S-strong digraphs in D. The *strong subgraph k-arc-connectivity* is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

Note that $\kappa_k(D)$ and $\lambda_k(D)$ are not only natural extensions of tree connectivity, but also could be seen as generalizations of connectivity and edge-connectivity of undirected graphs as $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$ [12] and $\lambda_2(\overleftrightarrow{G}) = \lambda(G)$ [11, 13]. For more information on the topic of strong subgraph connectivity of digraphs, the readers can see [10] for a recent survey.

In this paper, we continue to do research on strong subgraph arc-connectivity and focus on the strong subgraph 2-arc-connectivity of Cartesian products of digraphs. It is well known that Cartesian products of digraphs are of interest in graph theory and its applications; see a recent survey chapter by Hammack [4] considering many results on Cartesian products of digraphs.

In the next section we introduce terminology and notation on digraphs and give a simple yet useful upper bound on $\lambda_2(D)$, where D is Cartesian product of any digraphs G and H i.e., $D = G\Box H$.

For a strong digraph $D = (V, A)$, a set of arcs $W \subseteq A$ is a *cut* (or a *cutset*) if $D - A$ is not strong. A digraph D is *k-arc-strong* (or *k-arc-strongly connected*) if D has no cut with less than k arcs. The *arc-strong connectivity* of D, denoted by $\lambda(D)$, is the largest integer k such that D is k-arc-strongly connected. In Section 3, we prove that

$$\lambda(G\Box H) = \min\left\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G) + \delta^+(H), \delta^-(G) + \delta^-(H)\right\}$$

for every pair G and H of strong digraphs, each of order at least 2,[2]. Also, we get a formula for arc-connectivity of a strong digraph G on at least two vertices

[2]Note that the case of at least one of two digraphs having just one vertex in $\lambda(G\Box H)$ is trivial. Thus, we will henceforth assume that each of the two digraphs is of order at least 2. The same holds for $\lambda_2(G\Box H)$.

with itself by proving that for any $k \geq 2$, $\lambda(G^k) = k\delta^0(G)$ (Theorem 3.2), where $\delta^0 = \min\{\delta^+(G), \delta^-(G)\}$.

In Section 4 we prove that

$$\min\left\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G) + \delta^+(H), \delta^-(G) + \delta^-(H)\right\}$$

and $\lambda_2(G) + \lambda_2(H) - 1$ are an upper bound and a lower bound, respectively, for $\lambda_2(G\Box H)$ (Theorems 4.1 and 4.2). The upper bound follows from the formula for $\lambda(G\Box H)$ and thus it is tight. Unfortunately, we do not know whether this lower bound is tight or not, but by Theorem 5.5 (mentioned below), the gap with a tight bound is at most 1. Furthermore, we improve the bound under an additional condition and prove its sharpness by proving that $\lambda_2(G\Box H) \geq \lambda_2(G) + \lambda_2(H)$, where G and H are two strong digraphs such that $\delta^+(H) > \lambda_2(H)$ (Theorem 4.3).

In Section 5, we obtain exact values for the strong subgraph 2-arc-connectivity of Cartesian products of some digraph classes; our results are collated in Theorem 5.5. For the classes of strong digraphs considered in Theorem 5.5, we have $\lambda_2(G\Box H) = \lambda_2(G) + \lambda_2(H)$.

2. Additional Terminology and Notation

We refer the readers to [1,2] for graph theoretical notation and terminology not given here. Note that all digraphs considered in this paper have no parallel arcs or loops. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. A digraph D is *symmetric* if it can be obtained from its underlying undirected graph G by replacing each edge of G with the corresponding arcs of both directions, that is, $D = \overleftrightarrow{G}$. The order $|G|$ of a (di)graph G is the number of vertices in G. Let $\overleftrightarrow{T}_n$ be the symmetric digraph whose underlying undirected graph is a tree of order n. We use $\overrightarrow{C}_n$ and $\overleftrightarrow{K}_n$ to denote the cycle and complete digraph of order n, respectively.

Let $D = (V, A)$ be a digraph. For an arc (u, v), $u, v \in V$, we define that u *dominates* v and denote it by $u \rightarrow v$. For a pair $X, Y \subseteq V(D)$, we define $(X, Y)_D = \{xy \in A(D) \mid x \in X, y \in Y\}$. For $X, Y \subseteq V(D)$ and $X \cap Y = \emptyset$, we use $X \Rightarrow Y$ to denote that every vertex of X dominates every vertex of Y and $(Y, X)_D = \emptyset$. The *minimum out-degree* (*minimum in-degree*) of D is $\delta^+(D) = \min\{d^+(D) \mid x \in V(D)\}$ ($\delta^-(D) = \min\{d^-(D) \mid x \in V(D)\}$). The *minimum semi-degree* of D is $\delta^0(D) = \min\{d^+(D), d^-(D)\}$. We use $N_D^+(u)$ to denote that set of all out-neighbours of u, say *out-neighbourhood*, that is, $N_D^+(u) = \{v \in V - u | uv \in A\}$.

For a strongly connected digraph D, a *strong component* of D is a maximal induced subdigraph of D which is strong. If $D_1, \ldots, D_t$ are the strong components of D, then $V(D_1) \cup \cdots \cup V(D_t) = V(D)$ and we can get $V(D_i) \cap V(D_j) = \emptyset$

$(i \neq j)$. *The strong component digraph* $SC(D)$ of D is obtained by contracting each strong component of D to a vertex and deleting any parallel arcs obtained in this process. The strong components of D can be labelled $D_1, D_2, \ldots, D_t$ such that there is no arc from D_j to D_i unless $j < i$. We call such an ordering an *acyclic ordering* of the strong components of D. A set $B \subseteq A$ is *one-way* if there is a pair of sets $X, Y \subset V$ such that $B = (X, Y)_D$, that is, B is the set of arcs from X to Y.

Let G and H be two digraphs with $V(G) = \{u_i \mid 1 \leq i \leq n\}$ and $V(H) = \{v_j \mid 1 \leq j \leq m\}$. The *Cartesian product* $G \Box H$ of two digraphs G and H is a digraph with vertex set

$$V(G \Box H) = V(G) \times V(H) = \{(x, x') \mid x \in V(G), x' \in V(H)\}$$

and arc set

$$A(G \Box H) = \{(x, x')(y, y') \mid xy \in A(G),\ x' = y',\ or\ x = y,\ x'y' \in A(H)\}.$$

We will use $u_{i,j}$ to denote (u_i, v_j) in the rest of the paper. By definition, we know the Cartesian product is associative and commutative, and $G \Box H$ is strongly connected if and only if both G and H are strongly connected [4].

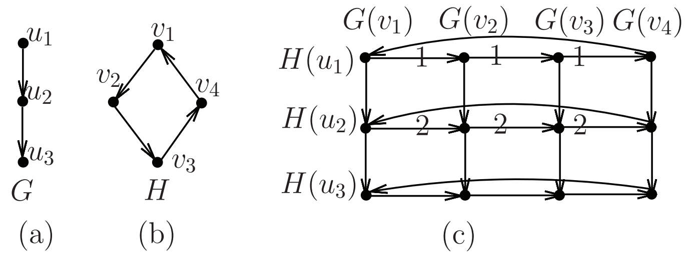

Figure 1. Two digraphs G, H and their Cartesian product.

We use $G(v_j)$ to denote the subgraph of $G \Box H$ induced by vertex set $\{u_{i,j} \mid 1 \leq i \leq n\}$ where $1 \leq j \leq m$, and use $H(u_i)$ to denote the subgraph of $G \Box H$ induced by vertex set $\{u_{i,j} \mid 1 \leq j \leq m\}$ where $1 \leq i \leq n$. Clearly, we have $G(v_j) \cong G$ and $H(u_i) \cong H$. (For example, as shown in Figure 1, $G(v_j) \cong G$ for $1 \leq j \leq 4$ and $H(u_i) \cong H$ for $1 \leq i \leq 3$.) For $1 \leq j_1 \neq j_2 \leq m$, the vertices u_{i,j_1} and u_{i,j_2} belong to the same digraph $H(u_i)$ where $u_i \in V(G)$; we call u_{i,j_2} the *vertex corresponding to* u_{i,j_1} in $G(v_{j_2})$; for $1 \leq i_1 \neq i_2 \leq n$, we call $u_{i_2,j}$ the vertex corresponding to $u_{i_1,j}$ in $H(u_{i_2})$. Similarly, we can define the subgraph *corresponding* to some subgraph. For example, in the digraph (c) of Figure 1, let P_1 (P_2) be the path labelled 1 (2) in $H(u_1)$ $(H(u_2))$, then P_2 is called the path *corresponding* to P_1 in $H(u_2)$.

It follows from the definition of strong subgraph 2-arc-connectivity that for any digraph D, $\lambda_2(D) \leq \min\{\delta^+(D), \delta^-(D)\}$ [11]. We will use this inequality in Section 5. Note that if $D = G\Box H$ then $\delta^+(D) = \delta^+(G) + \delta^+(H)$ and $\delta^-(D) = \delta^-(G) + \delta^-(H)$.

3. Formula for Arc-Connectivity of Cartesian Product of Two Digraphs and Its Corollaries

Xu and Yang [14] (see also [6, 9] and [5, Theorem 5.5]) proved that

$$\lambda(G\Box H) = \min\{\lambda(G)|V(H)|,\ \lambda(H)|V(G)|, \delta(G) + \delta(H)\} \tag{1}$$

for all connected undirected graphs G and H, each with at least two vertices. Since $\lambda(\overleftrightarrow{Q}) = \lambda(Q)$ for every undirected graph Q, (1) can be easily extended to symmetric digraphs. In this section, we generalise (1) to all strong digraphs.

Clearly, $\lambda(D) \leq \min\{\delta^+(D), \delta^-(D)\}$ for every digraph D. Hence, for any two strong digraphs G and H, we have

$$\begin{aligned}\lambda(G\Box H) &\leq \min\{\delta^+(G\Box H),\ \delta^-(G\Box H)\}\\ &= \min\{\delta^+(G) + \delta^+(H),\ \delta^-(G) + \delta^-(H)\}.\end{aligned} \tag{2}$$

Furthermore, by the definitions of arc-strong connectivity and Cartesian product of digraphs, we have

$$\lambda(G\Box H) \leq \lambda(G)|H| \tag{3}$$

and

$$\lambda(G\Box H) \leq \lambda(H)|G|. \tag{4}$$

The inequalities (2), (3) and (4) imply that

$$\lambda(G\Box H) \leq \min\left\{\lambda(G)|H|,\ \lambda(H)|G|,\ \delta^+(G) + \delta^+(H),\ \delta^-(G) + \delta^-(H)\right\}. \tag{5}$$

In fact, we will now prove that the equality holds and it could be seen as a digraph extension of (1). Note that the proof of Theorem 3.1 follows the lines of the proof of (1) in [6].

Theorem 3.1. *Let G and H be two strong digraphs, each of order at least 2. Then*

$$\lambda(G\Box H) = \min\left\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G) + \delta^+(H), \delta^-(G) + \delta^-(H)\right\}.$$

Proof. Let $S \subseteq A(G\Box H)$ be an arc-cut set of $G\Box H$ with $|S| = \lambda(G\Box H)$. By (5) it suffices to show that

$$|S| \geq \min\left\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G) + \delta^+(H), \delta^-(G) + \delta^-(H)\right\}.$$

If $|S| \geq \min\{\lambda(G)|H|, \lambda(H)|G|\}$, then the inequality clearly holds.

Therefore, we assume that $|S| < \min\{\lambda(G)|H|, \lambda(H)|G|\}$ in the following argument and in this case it suffices to show that $|S| \geq \delta^+(G\Box H)$ or $|S| \geq \delta^-(G\Box H)$.

Now there must exist a strong component B of $G\Box H - S$ which contains some $G(v_j)$, say $G(v_1)$, (as $|S| < \lambda(G)|H|$) and some $H(u_i)$, say $H(u_1)$, in $G\Box H - S$ (as $|S| < \lambda(H)|G|$). Let $(u,v) \in V(G\Box H) \setminus V(B)$.

We want to prove that $|S| \geq d^+((u,v))$ by the following operation that assigns each out-neighbor of (u,v) in $G\Box H$ a unique arc from S.

We first consider out-neighbors of (u,v) in $G(v)$. Let (u',v) be an out-neighbor of (u,v) in $G(v)$. If the arc $a = (u,v)(u',v) \in S$, we assign a to (u',v). Otherwise, we must have $(u',v) \notin B$.

We next consider out-neighbors of (u,v) in $H(u)$. Let (u,v') be an out-neighbor of (u,v) in $H(u)$. If $a' = (u,v)(u,v') \in S$, we assign a' to (u,v'). Otherwise, we must have $(u,v') \notin B$. Therefore, the subdigraph of $G\Box H - S$ induced by $V(G(v'))$ is not strong and so $G(v')$ contains at least one arc from S, and we assign this arc to (u,v').

The above operations mean that $|S| \geq d^+((u,v)) \geq \delta^+(G\Box H)$. With a similar argument, we can prove that $|S| \geq \delta^-(G\Box H)$. This completes the proof. ■

In many cases, we have $\lambda(G\Box H) = \min\{\delta^+(G) + \delta^+(H), \delta^-(G) + \delta^-(H)\}$ in Theorem 3.1. This holds for the Cartesian product of a strong digraph of at least two vertices and itself, as the following theorem asserts. We use G^k to denote the Cartesian product of k copies of G.

Theorem 3.2. *Let G be a strong digraph with order $n \geq 2$. For any integer $k \geq 2$, we have $\lambda(G^k) = k\delta^0(G)$.*

Proof. Our proofs are divided into two cases, according to whether $\lambda(G) = 1$.

Case 1. $\lambda(G) = 1$. The case that $n = 2$ is trivial, and we just consider the case that $n \geq 3$. Now G contains a cut arc e. Let $G' = G - e$. The strong component digraph $SC(G')$ of G' is acyclic and hence has an acyclic ordering. Let $G_1, G_2, \ldots, G_t$ be the components of G' labelled in such an acyclic ordering. Observe that e must be an arc of G from G_t to G_1. Without loss of generality, we assume that the digraph G has as many arcs as possible, that is, $G_{i_1} \Rightarrow G_{i_2}$ whenever $1 \leq i_1 < i_2 \leq t$, and each G_i is a complete digraph (Note that in this case, G has the maximum minimum semi-degree.)

Subcase 1.1. $|G_1| = \min\{|G_i| \mid i \in [t]\}$ or $|G_t| = \min\{|G_i| \mid i \in [t]\}$. Clearly, $\delta^0(G) \leq n/t \leq n/2$.

Subcase 1.2. $|G_{i'}| = \min\{|G_i| \mid i \in [t]\}$ for some $1 < i' < t$. Without loss of generality, let $|G_t| \geq |G_1| \geq |G_{i'}|$. Furthermore, by the assumption that $G_1 \Rightarrow G_{i'} \Rightarrow G_t$, we have $\min\{d^+(v), d^-(v)\} \geq \min\{|G_1|, |G_t|\}$ for any $v \in V(G_{i'})$, hence

$$\delta^0(G) = \min\{|G_1| - 1 + 1, |G_t| - 1 + 1\} = \min\{|G_1|, |G_t|\} = |G_1|$$

(since G_1 and G_t are both complete digraphs, and there is exactly one arc e from G_t to G_1), which means that $\delta^0(G) \leq n/2$.

In both subcases, we have $\delta^0 \leq n/2$. By Theorem 3.1,

$$\lambda(G\Box G) = \min\left\{\lambda(G)\, n,\, 2\delta^0(G)\right\} = 2\delta^0(G).$$

Now let $k \geq 3$ and we prove the result by induction on k. Assume that $\lambda(G^{k-1}) = (k-1)\,\delta^0(G)$. We have

$$\begin{aligned}\lambda\big(G\Box G^{k-1}\big) &= \min\left\{\lambda(G)\, n^{k-1},\, \lambda(G^{k-1})n,\, \delta^0(G) + \delta^0\big(G^{k-1}\big)\right\}\\ &= \min\left\{n^{k-1},\, (k-1)\delta^+(G)\, n, (k-1)\delta^-(G)\, n,\, k\delta^0(G)\right\}.\end{aligned}$$

Clearly, $k\delta^0(G) \leq (k-1)\,\delta^0(G)\, n$ (since $n \geq 2$). If $G = \overleftrightarrow{K}_2$, then $k\delta^0(G) = k \leq 2^{k-1} = |G^{k-1}|$. If $n \geq 3$, then $k\delta^0(G) \leq kn \leq |G^{k-1}| = n^{k-1}$. Therefore, $\lambda(G^k) = k\delta^0(G)$ when $\lambda(G) = 1$.

Case 2. $\lambda(G) \geq 2$. In this case, we clearly have $2\delta^0(G) \leq \lambda(G)\, n$. When $k = 2$, by Theorem 3.1, we have $\lambda(G^2) = 2\delta^0(G)$. When $k \geq 3$, by induction, $\lambda(G^{k-1}) = (k-1)\,\delta^0(G)$, and

$$\lambda\big(G\Box G^{n-1}\big) = \min\left\{\lambda(G)\, n^{k-1},\, (k-1)\delta^0(G)\, n,\, k\delta^0(G)\right\}.$$

Clearly, $k\delta^0(G) \leq (k-1)\,\delta^0(G)\, n$, and $k\delta^0(G) \leq k(n-1) \leq kn \leq 2n^{k-1} \leq \lambda(G)\, n^{k-1}$. Therefore, $\lambda(G^k) = k\delta^0(G)$ when $\lambda(G) \geq 2$. ■

By Theorem 3.2 and the fact that $\lambda_k(D) \leq \lambda(D)$ for any digraph D [11], we have the following sharp upper bound for $\lambda_2(G\Box G)$. To see its sharpness, we just consider the following example: $\lambda_2(\overleftrightarrow{K}_n \Box \overleftrightarrow{K}_n) = 2n - 2 = 2\delta^0(\overleftrightarrow{K}_n)$.

Corollary 1. *Let G be a strong digraph on at least two vertices. We have*

$$\lambda_2(G\Box G) \leq 2\delta^0(G).$$

Moreover, this bound is sharp.

4. General Bounds

By Theorem 3.1 and the fact that $\lambda_k(D) \leq \lambda(D)$ for any digraph D [11], we get the following sharp bound for $\lambda_2(G\Box H)$ given in Theorem 4.1, where the sharpness is proved by Theorems 5.5.

Theorem 4.1. *Let G and H be two strong digraphs, each with at least two vertices. Then*

$$\lambda_2(G\Box H) \leq \min\left\{\lambda(G)|H|, \lambda(H)|G|, \delta^+(G)+\delta^+(H), \delta^-(G)+\delta^-(H)\right\}.$$

Moreover, this bound is sharp.

Now we will provide a lower bound for $\lambda_2(G\Box H)$ for strong digraphs G and H.

Theorem 4.2. *Let G and H be two strong digraphs. We have*

$$\lambda_2(G\Box H) \geq \lambda_2(G) + \lambda_2(H) - 1.$$

Proof. It suffices to show that there are at least $\lambda_2(G)+\lambda_2(H)-1$ pairwise arc-disjoint S-strong subgraphs for any $S \subseteq V(G\Box H)$ with $|S| = 2$. Let $S = \{x, y\}$ and $S = \{(u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2})\}$. Consider the following two cases.

Case 1. x and y are in the same $H(u_i)$ or $G(v_j)$ for some $1 \leq i \leq n, 1 \leq j \leq m$. We will prove that, in this case, $\lambda_2(G\Box H) \geq \lambda_2(G) + \lambda_2(H)$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{1,2}$. We know there are at least $\lambda_2(H)$ pairwise arc-disjoint S-strong subgraphs in the subgraph $H(u_1)$, and so it suffices to find the remaining $\lambda_2(G)$ S-strong subgraphs in $G\Box H$.

We know there are at least $\lambda_2(G)$ pairwise arc-disjoint $\{x, u_{2,1}\}$-strong subgraphs, say $D_i(v_1)$ ($i \in [\lambda_2(G)]$), in $G(v_1)$. For each $i \in [\lambda_2(G)]$, we can choose an out-neighbor, say $u_{t_i,1}$ ($i \in [\lambda_2(G)]$), of x in $D_i(v_1)$ such that these out-neighbors are distinct. Then in $H(u_{t_i})$, we know there are $\lambda_2(H)$ pairwise arc-disjoint $\{u_{t_i,1}, u_{t_i,2}\}$-strong subgraphs, we choose one such strong subgraph, say $D(H(u_{t_i}))$. For each $i \in [\lambda_2(G)]$, let $D_i(v_2)$ be the $\{u_{t_i,2}, y\}$-strong subgraph corresponding to $D_i(v_1)$ in $G(v_2)$. We now construct the remaining $\lambda_2(G)$ S-strong subgraphs by letting $D_i = D_i(v_1) \cup D(H(u_{t_i})) \cup D_i(v_2)$ for each $i \in [\lambda_2(G)]$. Combining the former $\lambda_2(H)$ pairwise arc-disjoint S-strong subgraphs with the $\lambda_2(G)$ S-strong subgraphs, we can obtain $\lambda_2(G) + \lambda_2(H)$ strong subgraphs. Observe all these strong subgraphs are pairwise arc-disjoint.

Case 2. x and y belong to distinct $H(u_i)$ and $G(v_j)$. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$.

There are at least $\lambda_2(G)$ pairwise arc-disjoint $\{x, u_{2,1}\}$-strong subgraphs, say $A_i(v_1)$ ($i \in [\lambda_2(G)]$), in $G(v_1)$. For each $i \in [\lambda_2(G)]$, we can choose an out-neighbor, say $u_{t_i,1}$ ($i \in [\lambda_2(G)]$), of x in $A_i(v_1)$ such that these out-neighbors are distinct. Then in $H(u_{t_i})$, we know that there are $\lambda_2(H)$ pairwise arc-disjoint $\{u_{t_i,1}, u_{t_i,2}\}$-strong subgraphs; we choose one such strong subgraph, say $A(H(u_{t_i}))$. For each $i \in [\lambda_2(G)]$, let $A_i(v_2)$ be the $\{u_{t_i,2}, y\}$-strong subgraph corresponding to $A_i(v_1)$ in $G(v_2)$. We now construct the $\lambda_2(G)$ S-strong subgraphs by letting $A_i = A_i(v_1) \cup A(H(u_{t_i})) \cup A_i(v_2)$ for each $i \in [\lambda_2(G)]$.

Similarly, there are at least $\lambda_2(H)$ pairwise arc-disjoint $\{x, u_{1,2}\}$-strong subgraphs, say $B_j(u_1)$ ($j \in [\lambda_2(H)]$), in $H(u_1)$. For each $j \in [\lambda_2(H)]$, we can choose an out-neighbor, say u_{1,t'_j} ($j \in [\lambda_2(H)]$), of x in $B_j(u_1)$ such that these out-neighbors are distinct. Then in $G(v_{t'_j})$, we know there are $\lambda_2(G)$ pairwise arc-disjoint $\{u_{1,t'_j}, u_{2,t'_j}\}$-strong subgraphs, we choose one such strong subgraph, say $B(G(v_{t'_j}))$. For each $j \in [\lambda_2(H)]$, let $B_j(u_2)$ be the $\{u_{2,t'_j}, y\}$-strong subgraph corresponding to $B_j(u_1)$ in $H(u_2)$. We now construct the other $\lambda_2(H)$ S-strong subgraphs by letting $B_j = B_j(u_1) \cup B(G(v_{t'_j})) \cup B_j(u_2)$ for each $j \in [\lambda_2(H)]$.

Subcase 2.1. $t_i \neq 2$ for any $i \in [\lambda_2(G)]$ and $t'_j \neq 2$ for any $j \in [\lambda_2(H)]$, that is, $u_{2,1}$ was not chosen as an out-neighbor of $u_{1,1}$ in $G(v_1)$ and $u_{1,2}$ was not chosen as an out-neighbor of $u_{1,1}$ in $H(u_1)$. We can check the above $\lambda_2(G) + \lambda_2(H)$ strong subgraphs are pairwise arc-disjoint.

Subcase 2.2. $t_i = 2$ for some $i \in [\lambda_2(G)]$ or $t'_j = 2$ for some $j \in [\lambda_2(H)]$, that is, $u_{2,1}$ was chosen as an out-neighbor of $u_{1,1}$ in $G(v_1)$ or $u_{1,2}$ was chosen as an out-neighbor of $u_{1,1}$ in $H(u_1)$. We also assume that not both conditions are fulfilled. Without loss of generality, we may assume that $t_i = 2$ and $t'_j \neq 2$, that is, $u_{2,1}$ was chosen as an out-neighbor of $u_{1,1}$ in $G(v_1)$ and $u_{1,2}$ was not chosen as an out-neighbor of $u_{1,1}$ in $H(u_1)$. Since $A\left(H\left(u_{t_1}\right)\right)$ is the only digraph that is potentially not disjoint with a digraph $B_j\left(u_2\right)$ we find that there are at least $\lambda_2(G) + \lambda_2(H) - 1$ pairwise disjoint $\{x, y\}$-strong subgraphs A_i, B_j for $i \in \left[\lambda_2\left(G\right)\right], j \in \left[\lambda_2\left(H\right)\right]$. Otherwise, we can check the above $\lambda_2(G) + \lambda_2(H)$ strong subgraphs are pairwise arc-disjoint and get the desired S-strong subgraphs.

Subcase 2.3. $t_i = 2$ for some $i \in [\lambda_2(G)]$ and $t'_i = 2$ for some $j \in [\lambda_2(H)]$, we may assume that $t_1 = 2$ and $t'_1 = 2$, and replace A_1, B_1 by $\overline{A}_1$, $\overline{B}_1$, respectively as follows: let $\overline{A}_1 = A_1(v_1) \cup B(H(u_{t_1}))$ and $\overline{B}_1 = B_1(u_1) \cup A_1(v_2)$. We can check that the current $\lambda_2(G) + \lambda_2(H)$ strong subgraphs are pairwise arc-disjoint.

Hence, the bound holds. This completes the proof. ■

In order to improve upper bound for $\lambda_2\left(G \Box H\right)$, we add an additional condition and prove that this lower bound is tight.

Theorem 4.3. *Let G and H be two strong digraphs such that $\delta^+\left(H\right) > \lambda_2\left(H\right)$. We have*

$$\lambda_2(G \Box H) \geq \lambda_2(G) + \lambda_2(H). \tag{6}$$

Moreover, this bound is sharp.

Proof. It suffices to show that there are at least $\lambda_2(G) + \lambda_2(H)$ pairwise arc-disjoint S-strong subgraphs for any $S = \{x, y\} \subseteq V(G\Box H)$. We consider the following two cases.

Case 1. x and y are in the same $H(u_i)$ or $G(v_j)$ ($i \in [n]$, $j \in [m]$) as shown in Figure 2. This proof is similar to Case 1 of Theorem 4.2, so we omit the details.

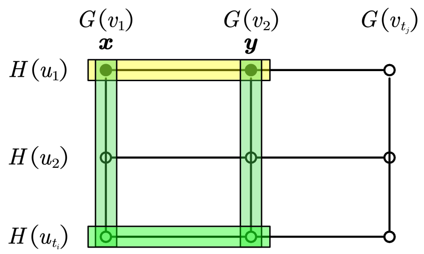

Figure 2. x and y are in the same $H(u_i)$ or $G(v_j)$.

Case 2. x and y belong to distinct $H(u_i)$ and $G(v_j)$ ($i \in [n], j \in [m]$) (as shown the figure in Figure 3). Without loss of generality, assume that $x = u_{1,1}$, $y = u_{2,2}$.

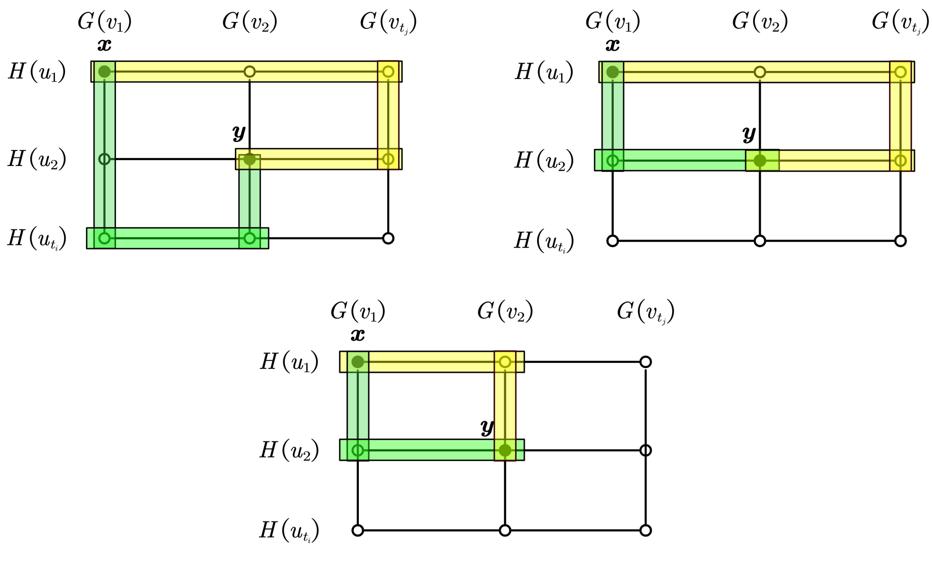

Figure 3. x and y belong to distinct $H(u_i)$ and $G(v_j)$.

Subcase 2.1. $t_i \neq 2$ ($i \in [\lambda_2(G)]$) and $t'_j \neq 2$ ($j \in [\lambda_2(H)]$) (as shown the left figure in Figure 3), this proof is similar to that of Subcase 2.1 of Theorem 4.2 and we omit the details.

Subcase 2.2. $t_i = 2$ for some $i \in [\lambda_2(G)]$ or $t'_j = 2$ for some $j \in [\lambda_2(H)]$ (as shown the middle figure in Figure 3), that is, $u_{2,1} \in N_G^+(x)$ in $G(v_1)$ or $u_{1,2} \in N_H^+(x)$ in $H(u_1)$. Without loss of generality, assume that $u_{2,1} \in N_G^+(x)$ in $G(v_1)$ and $u_{1,2} \notin N_H^+(x)$ in $H(u_1)$. Since $\delta^+(H) > \lambda_2(H)$, that is, $A(A_i) \cap A(A_j) = \emptyset$, the above $\lambda_2(G) + \lambda_2(H)$ strong subgraphs, A_i, B_j $(i \in [\lambda_2(G)], j \in [\lambda_2(H)])$, are desired pairwise arc-disjoint strong subgraphs. It can be checked that the current $\lambda_2(G) + \lambda_2(H)$ strong subgraphs are pairwise arc-disjoint.

Subcase 2.3. $t_i = 2$ for some $i \in [\lambda_2(G)]$ and $t'_j = 2$ for some $j \in [\lambda_2(H)]$ (as shown the right figure in Figure 3), this proof is similar to that of Subcase 2.3 of Theorem 4.2 and we omit the details.

By Case 1 and Case 2, the lower bound (6) holds. To prove its sharpness, we need the following example: Let $G = \overleftrightarrow{K}_2$, and let H be a symmetric digraph whose underlying graph H_1 is defined as follows: $V(H_1) = A \cup B$, both A and B induce a clique of H_1 and there is an edge between v_3 and v_4, where $A = \{v_i \mid 1 \leq i \leq 3\}$, $B = \{v_i \mid 3 \leq i \leq 6\}$. Observe that $\delta^+(H) > \lambda_2(H)$, $\lambda_2(G) = \lambda_2(H) = 1$. We will prove that $\lambda_2(G \Box H) = 2$. As $\lambda_2(G \Box H) \leq \delta^0(G \Box H) = 2$, it suffices to show that for any $S = \{x, y\} \subseteq V(G \Box H)$, there are at least two pairwise arc-disjoint S-strong subgraphs in $G \Box H$.

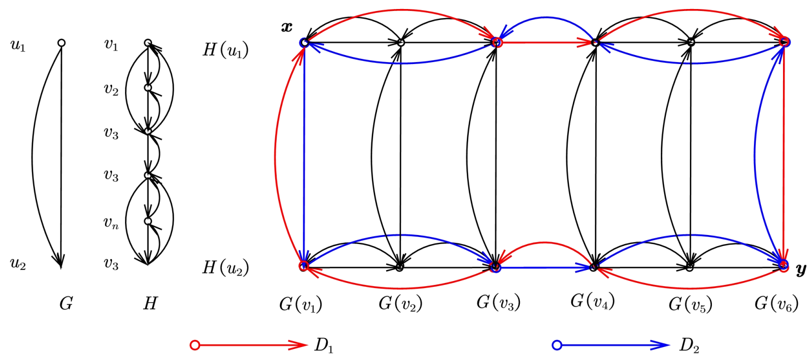

Figure 4. Digraphs G and H and their Cartesian product.

We will consider only the case when x, y are neither in the same G nor in the same H, as the arguments for the remaining cases are similar. Without loss of generality, let $x = u_{1,1}$, $y = u_{2,6}$. We obtain two pairwise arc-disjoint S-strong subgraphs in $G \Box H$, say D_1, D_2 (as shown in Figure 4) such that

$V(D_1) = \{x, y, u_{2,1}, u_{1,3}, u_{1,4}, u_{1,6}, u_{2,3}, u_{2,4}\}$ and
$A(D_1) = \{xu_{1,3}, u_{1,3}u_{1,4}, u_{1,4}u_{1,6}, u_{1,6}y, yu_{2,4}, u_{2,4}u_{2,3}, u_{2,3}u_{2,1}, u_{2,1}x\}$.
$V(D_2) = \{x, y, u_{2,1}, u_{1,3}, u_{1,4}, u_{1,6}, u_{2,3}, u_{2,4}\}$ and
$A(D_2) = \{xu_{2,1}, u_{2,1}u_{1,4}, u_{1,4}u_{1,6}, u_{1,6}y, yu_{2,4}, u_{2,4}u_{2,3}, u_{2,3}u_{2,1}, u_{2,1}x\}$.

Hence, the bound holds and is sharp. This completes the proof. ■

5. Exact Values for Digraph Classes

In this section, we will obtain exact values for the strong subgraph 2-arc-connectivity of Cartesian product of two digraphs belonging to some digraph classes.

Proposition 5.1. *We have* $\lambda_2(\overrightarrow{C}_n \Box \overrightarrow{C}_m) = 2$.

Proof. Let $S = \{x, y\}$, we will consider only the case when x, y are neither in the same $\overrightarrow{C}_n(u_i)$ nor in the same $\overrightarrow{C}_m(v_j)$ for some $1 \le i \le n$, $1 \le j \le m$, since the arguments for remaining cases are similar. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$. We can get two pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \Box \overrightarrow{C}_m$, say D_1 and D_2 (as shown in Figure 5) such that

$V(D_1) = \{x, y, u_{1,2}, \ldots, u_{2,m-1}, u_{2,m}, \ldots, u_{n,m}, u_{1,m}\}$ and
$A(D_1) = \{xu_{1,2}, u_{1,2}y, \ldots, u_{2,m-1}u_{2,m}, \ldots, u_{n-1,m}u_{n,m}, u_{n,m}u_{1,m}, u_{1,m}x\}$.
$V(D_2) = \{x, y, u_{2,1}, \ldots, u_{n-1,2}, \ldots, u_{n-1,m-1}, u_{n-1,m}, u_{n-1,1}\}$ and
$A(D_2) = \{xu_{2,1}, u_{2,1}y, \ldots, u_{n-2,2}u_{n-1,2}, \ldots, u_{n-1,m-1}u_{n-1,m}, u_{n-1,m}u_{n-1,1},$
$u_{n-1,1}u_{n,1}, u_{n,1}x\}$.

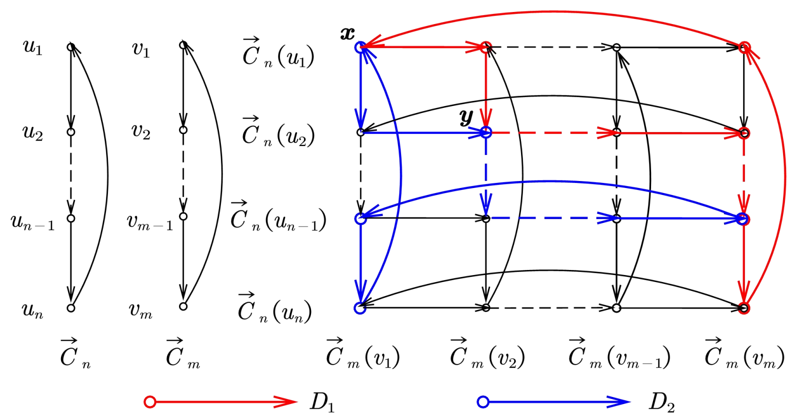

Figure 5. $\overrightarrow{C}_n$, $\overrightarrow{C}_m$ and their Cartesian product.

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \ge \lambda_2(\overrightarrow{C}_n \Box \overrightarrow{C}_m) \ge 2$. This completes the proof. ■

Proposition 5.2. We have $\lambda_2(\overrightarrow{C}_n \Box \overleftrightarrow{C}_m) = 3$.

Proof. Let $S = \{x, y\}$, we will consider only the case when x, y are neither in the same $\overrightarrow{C}_n(u_i)$ nor in the same $\overleftrightarrow{C}_m(v_j)$ for some $1 \le i \le n$, $1 \le j \le m$, since

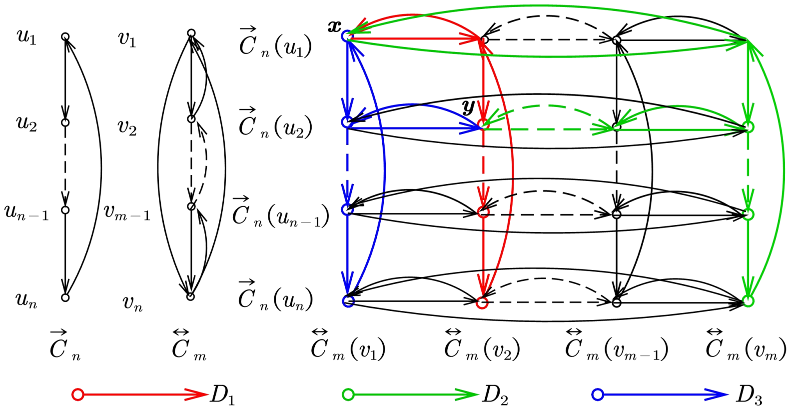

Figure 6. $\overrightarrow{C}_n$, $\overleftrightarrow{C}_m$ and their Cartesian product.

the arguments for remaining cases are similar. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$. We can get three pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \Box \overleftrightarrow{C}_m$, say D_1, D_2 and D_3 (as shown in Figure 6) such that

$V(D_1) = \{x, y, \ldots, u_{n-1,2}, u_{n,2}, u_{1,2}\}$ and
$A(D_1) = \{xu_{1,2}, u_{1,2}y, \ldots, u_{n-1,2}u_{n,2}, u_{n,2}u_{1,2}, u_{1,2}x\}$.
$V(D_2) = \{x, y, u_{1,m}, u_{2,m}, u_{2,m-1}, \ldots, u_{n-1,m-1}, u_{n,m}\}$ and
$A(D_2) = \{xu_{1,m}, u_{1,m}u_{2,m}, u_{2,m}u_{2,m-1}, \ldots, u_{2,3}y, yu_{2,3}, \ldots, u_{2,m-1}u_{2,m}, \ldots,$
$u_{n-1,m}u_{n,m}, u_{n,m}u_{1,m}, u_{1,m}x\}$.
$V(D_3) = \{x, y, u_{2,1}, \ldots, u_{n-1,1}, u_{n,1}\}$ and
$A(D_3) = \{xu_{2,1}, u_{2,1}y, yu_{2,1}, \ldots, u_{n-1,1}u_{n,1}, u_{n,1}x\}$.

Then we have $3 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_2(\overrightarrow{C}_n \Box \overleftrightarrow{C}_m) \geq 3$. This completes the proof. ■

Proposition 5.3. We have $\lambda_2(\overrightarrow{C}_n \Box \overleftrightarrow{T}_m) = 2$.

Proof. Let $S = \{x, y\}$, we will consider only the case when x, y are neither in the same $\overrightarrow{C}_n(u_i)$ nor in the same $\overleftrightarrow{T}_m(v_j)$ for some $1 \leq i \leq n$, $1 \leq j \leq m$, as the arguments for the remaining cases are similar. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$. We can get two pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \Box \overleftrightarrow{T}_m$, say D_1 and D_2 (as shown in Figure 7) such that

$V(D_1) = \{x, y, \ldots, u_{n-1,2}, u_{n,2}, u_{1,2}, u_{2,1}\}$ and
$A(D_1) = \{xu_{2,1}, u_{2,1}y, \ldots, u_{n-1,2}u_{n,2}, u_{n,2}u_{1,2}, u_{1,2}x\}$.
$V(D_2) = \{x, y, u_{1,2}, u_{2,1}, \ldots, u_{n-1,1}, u_{n,1}\}$ and

$A(D_2) = \{xu_{1,2}, u_{1,2}y, yu_{2,1}, \ldots, u_{n-1,1}u_{n,1}, u_{n,1}x\}$.

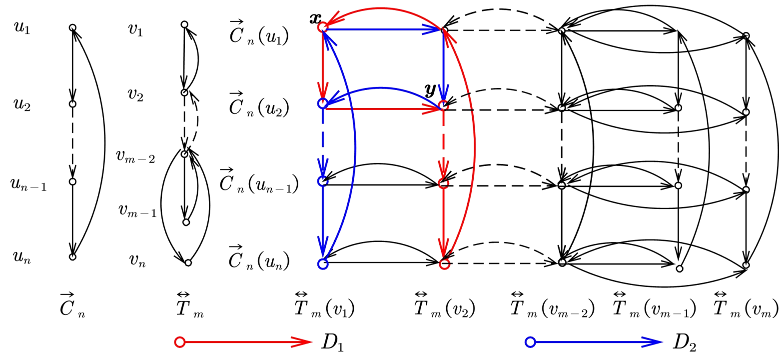

Figure 7. $\overrightarrow{C}_n$, $\overleftrightarrow{T}_m$ and their Cartesian product.

Then we have $2 = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_2(\overrightarrow{C}_n \square \overleftrightarrow{T}_m) \geq 2$. This completes the proof. ■

Proposition 5.4. We have $\lambda_2(\overrightarrow{C}_n \square \overleftrightarrow{K}_m) = m$.

Proof. Let $S = \{x, y\}$, we will consider only the case when x, y are neither in the same $\overrightarrow{C}_n(u_i)$ nor in the same $\overleftrightarrow{K}_m(v_j)$ for some $1 \leq i \leq n$, $1 \leq j \leq m$, as the arguments for the remaining cases are similar. Without loss of generality, we may assume that $x = u_{1,1}$, $y = u_{2,2}$.

We first show that $\lambda_2(\overrightarrow{C}_n \square \overleftrightarrow{K}_2) = 2$. When $m = 2$, we can get two pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \square \overleftrightarrow{K}_2$, say D_1 and D_2 (as shown in Figure 8) satisfying:

$V(D_1) = \{x, y, u_{1,2}, u_{2,1}, \ldots, u_{n-1,1}, u_{n,1}\}$ and
$A(D_1) = \{xu_{1,2}, u_{1,2}y, yu_{2,1}, \ldots, u_{n-1,1}u_{n,1}, u_{n,1}x\}$.
$V(D_2) = \{x, y, u_{1,2}, u_{2,1}, \ldots, u_{n-1,1}, u_{n,1}\}$ and
$A(D_2) = \{xu_{2,1}, u_{2,1}y, \ldots, u_{n-1,2}u_{n,2}, u_{n,2}u_{1,2}, u_{1,2}x\}$.

The propositon is now proved by induction on m. Suppose that when $m = k$, we have $\lambda_2(\overrightarrow{C}_n \square \overleftrightarrow{K}_k) = k$. We shall show that $\lambda_2(\overrightarrow{C}_n \square \overleftrightarrow{K}_{k+1}) = k + 1$ when $m = k + 1$. Since we can get k pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \square \overleftrightarrow{K}_k$, say $D_1, D_2, \ldots, D_k$. When $m = k+1$, that is, the degree of each vertex increases by 2 in $\overleftrightarrow{K}_k$, we can get $k+1$ pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \square \overleftrightarrow{K}_{k+1}$, say $D_1, D_2, \ldots, D_k, D_{k+1}$. By the symmetry of the complete digraph, the same conclusion is drawn in the two cases where x, y belong to $\overleftrightarrow{K}_{k+1}$, and x,

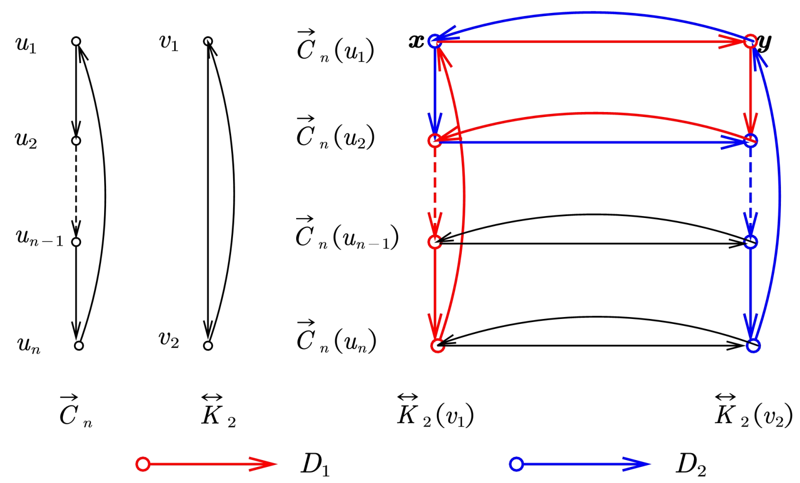

Figure 8. $\overrightarrow{C}_n$, $\overleftrightarrow{K}_2$ and their Cartesian product.

y belong to $\overleftrightarrow{K}_k$ and $\overleftrightarrow{K}_{k+1}$, respectively. From the above argument, the original proposition holds for any positive integer, we can get m pairwise arc-disjoint S-strong subgraphs in $\overrightarrow{C}_n \Box \overleftrightarrow{K}_m$, say $D_1, D_2, \ldots, D_j(2 < j \leq m), \ldots, D_{m-1}, D_m$ (as shown in Figure 9) such that

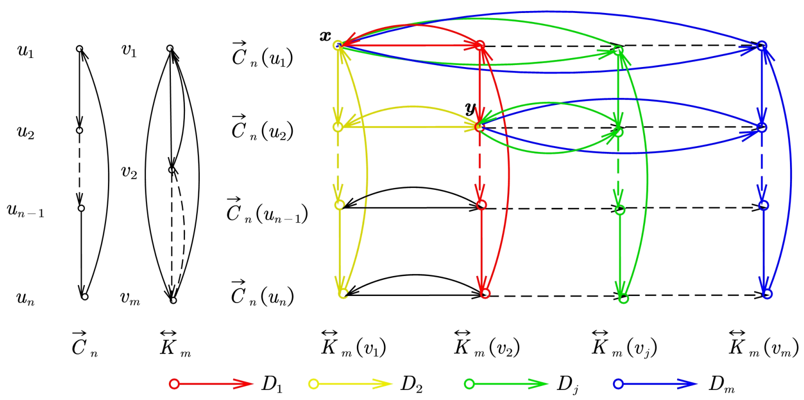

Figure 9. $\overrightarrow{C}_n$, $\overleftrightarrow{K}_m$ and their Cartesian product.

$V(D_1) = \{x, y, u_{1,2}, \ldots, u_{n-1,2}, u_{n,2}\}$ and
$A(D_1) = \{xu_{1,2}, u_{1,2}y, \ldots, u_{n-1,2}u_{n,2}, u_{n,2}u_{1,2}, u_{1,2}x\}$.
$V(D_2) = \{x, y, u_{2,1}, \ldots, u_{n-1,2}, u_{n,2}\}$ and

$A(D_2) = \{xu_{2,1}, u_{2,1}y, yu_{2,1}, \ldots, u_{n-1,1}u_{n,1}, u_{n,1}x\}$.
$V(D_j) = \{x, y, u_{1,2}, u_{2,j}, \ldots, u_{n-1,j}, u_{n,j}\}$ and
$A(D_j) = \{xu_{1,j}, u_{1,j}u_{2,j}, u_{2,j}y, yu_{2,j}, \ldots, u_{n-1,j}u_{n,j}, u_{n,j}u_{1,j}, u_{1,j}x\}$.

Then we have $m = \min\{\delta^+(D), \delta^-(D)\} \geq \lambda_2(\overrightarrow{C}_n \Box \overleftrightarrow{K}_m) \geq m$. This completes the proof. ■

Since $\lambda_2(\overleftrightarrow{Q}) = \lambda(Q)$ for any undirected graph Q, using Cartesian product definition, we have

$$\lambda_2(\overleftrightarrow{G} \Box \overleftrightarrow{H}) = \lambda(G \Box H) \tag{7}$$

for undirected graphs G and H.

Propositions 5.1–5.4 and formulas (7) and (1) imply the following theorem. Indeed, entries in the first row and columns of Table 1 follow from Propositions 5.1–5.4 and all other entries can be easily computed using (7) and (1).

Theorem 5.5. *The following table for the strong subgraph* 2*-arc-connectivity of Cartesion products of some digraph classes holds.*

	$\overrightarrow{C}_m$	$\overleftrightarrow{C}_m$	$\overleftrightarrow{T}_m$	$\overleftrightarrow{K}_m$
$\overrightarrow{C}_n$	2	3	2	m
$\overleftrightarrow{C}_n$	3	4	3	$m+1$
$\overleftrightarrow{T}_n$	2	3	2	m
$\overleftrightarrow{K}_n$	n	$n+1$	n	$n+m-2$

Table 1. Exact values of λ_2 for Cartesian products of some digraph classes.

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