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ON THE TOTAL DOMINATION NUMBER OF TOTAL GRAPHS

Abel Cabrera-Martínez

Universidad de Córdoba
Departamento de Matemáticas
Campus de Rabanales, 14071 Córdoba, Spain

e-mail: acmartinez@uco.es

José L. Sánchez

and

José M. Sigarreta Almira

Universidad Autónoma de Guerrero
Facultad de Matemáticas
Carlos E. Adame 54, Col. La Garita
39650 Acapulco, Guerrero, Mexico

e-mail: josesantiesteban@uagro.mx
josemariasigarretaalmira@hotmail.com

Abstract

Let G be a graph with no isolated vertex. A set $D \subseteq V(G)$ is a total dominating set of G if every vertex of G is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of G. In this paper we study the total domination number of total graphs $\mathtt{T}(G)$ of simple graphs G. In particular, we give some relationships that exist between $\gamma_t(\mathtt{T}(G))$ and other domination parameters of G and of some well-known graph operators on G. Finally, we provide closed formulas on $\gamma_t(\mathtt{T}(G))$ for some well-known families of graphs G.

Keywords: total domination, graph operators, total graphs.

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1. Introduction

The theory of domination in graphs is one of the most active research areas within graph theory. This fact can be seen reflected in the more than 5000 papers published on domination and related parameters. Total domination in graphs is the most studied classical variant, with more than 600 published papers. Given a graph G with no isolated vertex, a set $D \subseteq V(G)$ is a *total dominating set* (TDS) of G if $N(v) \cap D \neq \emptyset$ for every vertex v of G. The total domination number of G is defined to be

$$\gamma_t(G) = \min\{|D| : D \text{ is a TDS of } G\}.$$

This parameter was introduced in [10] by Cockayne, Dawes and Hedetniemi. Recent selected results on total domination in graphs can be found in [12, 14]. Among all the papers published on total domination in graphs, at most one tenth are related to graph products and graph operators. In particular, we cite the following works. For instance, the reader is referred to [13, 15] for Cartesian product graphs, [9] for lexicographic product graphs, [11, 22] for direct product graphs, [8] for rooted product graphs, [21] for latin square graphs, [16] for middle graphs, [18] for line graphs $\mathtt{L}(G)$, [23] for graph operators $\mathtt{Q}(G)$, $\mathtt{R}(G)$ and $\mathtt{S}(G)$, and [1] for total graphs $\mathtt{T}(G)$ (where G is a tree).

This last graph operator (total graph $\mathtt{T}(G)$) was introduced by Behzad [2] in 1967. Subsequently, several parameters were studied for this graph operator in different works, including [2, 4, 19, 20, 24, 25]. In this paper we continue with the study of the total domination number of total graphs $\mathtt{T}(G)$. In Subsection 1.1 we introduce some definitions and terminology needed to develop the remaining sections. Section 2 is devoted to obtain some combinatorial results on the total domination number of total graphs. In particular, we give some relationships that exist between $\gamma_t(\mathtt{T}(G))$ and other domination parameters of G and of some well-known graph operators on G. Finally, in Subsection 2.1 we provide closed formulas on $\gamma_t(\mathtt{T}(G))$ for some well-known families of graphs G.

1.1. Definitions and terminology

In order to present our results, we need to introduce some definitions and terminology. Let G be a graph with no isolated vertex of order n with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{v_iv_j : v_i \text{ is adjacent to } v_j\}$, and let $V_E(G) = \{v^{i,j} : v_iv_j \in E(G)\}$ (remark that $v^{i,j} = v^{j,i}$). Given a vertex $v \in V(G)$, $N(v)$ and $N[v]$ represent the *open neighbourhood* and the *closed neighbourhood* of v, respectively. For a set $D \subseteq V(G)$, let $N(D) = \bigcup_{v\in D} N(v)$ and $N[D] = N(D) \cup D$. We denote by $\Delta(G)$ and $\delta(G)$ its *maximum and minimum degrees*, respectively. A *leaf* vertex of G is a vertex of degree one, and a *support* vertex of G is a vertex adjacent to a leaf. The sets of leaves and support vertices will be denoted by $\mathcal{L}(G)$ and $\mathcal{S}(G)$, respectively. As usual, the subgraph of G

induced by $D \subseteq V(G)$ will be denoted by $G[D]$. Moreover, the graph obtained from G by removing all the vertices in $X \subseteq V(G)$ and all the edges incident with a vertex in X will be denoted by $G - X$. Analogously, the graph obtained from G by removing all the edges in $U \subseteq E(G)$ will be denoted by $G - U$.

In [17], Krausz introduced the concept of graph operators of a graph G. Next, we define some graph operators obtained from G.

- The graph $\mathtt{R}(G)$ is the graph obtained from the graph G with vertex set $V(\mathtt{R}(G)) = V(G) \cup V_E(G)$, and in which each vertex $v^{i,j} \in V_E(G)$ is only adjacent to the vertices $v_i, v_j \in V(G)$, i.e., $E(\mathtt{R}(G)) = E(G) \cup \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\}$.
- The *line graph* of the graph G, denoted by $\mathtt{L}(G)$, is the graph whose vertex set is $V(\mathtt{L}(G)) = V_E(G)$, and in which two vertices of $\mathtt{L}(G)$ are adjacent if and only if the corresponding edges in G have a common vertex, i.e., $E(\mathtt{L}(G)) = \{v^{i,j}v^{k,l} : |\{i,j\} \cap \{k,l\}| = 1\}$.
- The graph $\mathtt{Q}(G)$ is the graph obtained from the graph G with vertex set $V(\mathtt{Q}(G)) = V(G) \cup V_E(G)$ and edge set $E(\mathtt{Q}(G)) = E(\mathtt{L}(G)) \cup \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\}$.
- The *total graph* of the graph G, denoted by $\mathtt{T}(G)$, is the graph whose vertex set is $V(\mathtt{T}(G)) = V(G) \cup V_E(G)$, and in which two vertices of $\mathtt{T}(G)$ are adjacent if they are adjacent or incident in G, i.e.,

$$E(\mathtt{T}(G)) = E(G) \cup E(\mathtt{L}(G)) \cup \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in E(G)\}.$$

In Figure 1 we show a graph G, the line graph $\mathtt{L}(G)$, the graphs $\mathtt{R}(G)$ and $\mathtt{Q}(G)$, and the total graph $\mathtt{T}(G)$.

A subset $X \subseteq V(G)$ is an *independent set* of G if the subgraph induced by X has no edges. An independent set $P \subseteq V(G)$ is called a 2-*packing* if $N[x] \cap N[y] = \emptyset$ for every pair of different vertices $x, y \in P$. The *independence number* (respectively, 2-*packing number*) of G, denoted by $\alpha(G)$ (respectively, $\rho(G)$), is the maximum cardinality among all independent sets (respectively, 2-packings) of G. A TDS D of G is a *total outer-independent dominating set* (TOIDS) of G if $V(G) \setminus D$ is an independent set. The total outer-independent domination number of G is defined to be

$$\gamma_{t,oi}(G) = \min\{|D| : D \text{ is a TOIDS of } G\}.$$

This parameter was studied in [5] on graph products and in [7] from combinatorial and complexity point of view. We define a $\gamma_{t,oi}(G)$-*set* as a TOIDS of G of cardinality $\gamma_{t,oi}(G)$. The same agreement will be assumed for optimal parameters associated to other characteristic sets defined in the paper. In this paper, we use the notation K_n, W_n, P_n, C_n and $K_{1,n-1}$ for complete graphs, wheel graphs, path

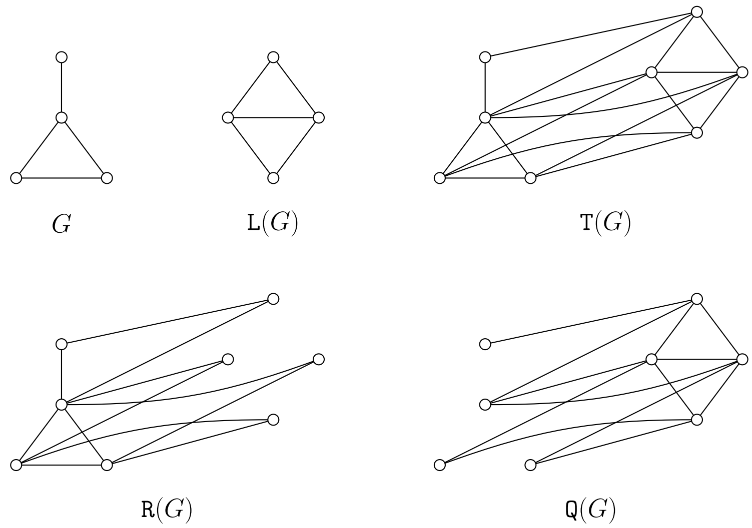

Figure 1. A graph G, and the corresponding graphs $\mathtt{L}(G)$, $\mathtt{R}(G)$, $\mathtt{Q}(G)$ and $\mathtt{T}(G)$.

graphs, cycle graphs and star graphs of order n, respectively. A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph.

2. Combinatorial Results

If G is a disconnected graph with no isolated vertex and $G_1, \ldots, G_r$ $(r \geq 2)$ are the components of G, then

$$\mathtt{T}(G) = \bigcup_{i=1}^{r} \mathtt{T}(G_i).$$

From the result above, we can deduce that every $\gamma_t(\mathtt{T}(G))$-set D satisfies that $D \cap V(\mathtt{T}(G_j))$ is a $\gamma_t(\mathtt{T}(G_j))$-set, for every $j \in \{1, \ldots, r\}$. Therefore, the next remark for the case of disconnected graphs with no isolated vertex is obtained.

Remark 1. If G is a disconnected graph with no isolated vertex and $G_1, \ldots, G_r$ $(r \geq 2)$ are the components of G, then

$$\gamma_t(\mathtt{T}(G)) = \sum_{i=1}^{r} \gamma_t(\mathtt{T}(G_i)).$$

As a consequence of the remark above, throughout this article we will only focus our study on the total graph $\mathtt{T}(G)$ of the nontrivial connected graphs G.

The next results will be two useful tools to provide new bounds for the total domination number of total graphs.

Remark 2. For any nontrivial connected graph G, the following statements hold.

(i) The subgraph of $\mathtt{T}(G)$ induced by $V(G)$ is isomorphic to the graph G.

(ii) The subgraph of $\mathtt{T}(G)$ induced by $V_E(G)$ is isomorphic to the graph $\mathtt{L}(G)$.

(iii) The spanning subgraph $\mathtt{T}(G) - E(\mathtt{L}(G))$ is isomorphic to the graph $\mathtt{R}(G)$.

(iv) The spanning subgraph $\mathtt{T}(G) - E(G)$ is isomorphic to the graph $\mathtt{Q}(G)$.

Lemma 3. *If G is a nontrivial connected graph, then there exists a $\gamma_t(\mathtt{T}(G))$-set D satisfying the next conditions.*

(i) *No vertex in $D \cap V(G)$ is adjacent to a vertex in $D \setminus V(G)$.*

(ii) *$\mathcal{S}(G) \subseteq D$.*

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of G. Let D be a $\gamma_t(\mathtt{T}(G))$-set such that $|D \cap V(G)|$ is maximum. Now, we suppose that there exist $i, j \in \{1, \ldots, n\}$ such that $v_i, v^{i,j} \in D$. Since $N[v^{i,j}] \subseteq N[\{v_i, v_j\}]$, we deduce that $D' = \left(D \setminus \left\{v^{i,j}\right\}\right) \cup \{v_j\}$ is a $\gamma_t(\mathtt{T}(G))$-set and $|D' \cap V(G)| > |D \cap V(G)|$, which is a contradiction. Therefore, D is a $\gamma_t(\mathtt{T}(G))$-set such that no vertex in $D \cap V(G)$ is adjacent to a vertex in $D \setminus V(G)$, which completes the proof of (i). Now, we proceed to prove (ii). If $\mathcal{S}(G) = \emptyset$, then we are done. Assume that $\mathcal{S}(G) \neq \emptyset$ and suppose that there exists $k \in \{1, \ldots, n\}$ such that $v_k \in \mathcal{S}(G) \setminus D$. Let $v_l \in N(v_k) \cap \mathcal{L}(G)$. Since $|N(v_l) \cap V(\mathtt{T}(G))| = 2$, it follows that $v^{k,l} \in D$ (recall that $N[v^{k,l}] \subseteq N[\{v_k, v_l\}] = N[v_k]$). So, $D'' = \left(D \setminus \left\{v^{k,l}\right\}\right) \cup \{v_k\}$ is a $\gamma_t(\mathtt{T}(G))$-set and $|D'' \cap V(G)| > |D \cap V(G)|$, which is a contradiction. Therefore, D is a $\gamma_t(\mathtt{T}(G))$-set such that $\mathcal{S}(G) \subseteq D$, which completes the proof. ∎

The following result provides lower and upper bounds for the total domination number of $\mathtt{T}(G)$ in terms of the total domination numbers of G and $\mathtt{L}(G)$.

Theorem 4. *If G is a connected graph of order at least three, then*

$$\max\{\gamma_t(G), \gamma_t(\mathtt{L}(G))\} \leq \gamma_t(\mathtt{T}(G)) \leq \gamma_t(G) + \gamma_t(\mathtt{L}(G)).$$

Proof. Let D be a $\gamma_t(\mathtt{T}(G))$-set which satisfies Lemma 3. First, we proceed to prove that $\gamma_t(G) \leq \gamma_t(\mathtt{T}(G))$. If $D \subseteq V(G)$, then D is also a TDS of $\mathtt{T}(G)[V(G)]$. So, by Remark 2 we deduce that $\gamma_t(G) = \gamma_t(\mathtt{T}(G)[V(G)]) \leq |D| = \gamma_t(\mathtt{T}(G))$. Hence, from now on we assume that $D \cap V_E(G) \neq \emptyset$. Now, we define a set $W \subseteq V(G) \cap N(D \cap V_E(G))$ of minimum cardinality which satisfies the following conditions.

(a) $W \cap \{v_i, v_j\} \neq \emptyset$ for every vertex $v^{i,j} \in D$.

(b) The subgraph induced by W has no isolated vertex.

Notice that W is well-defined because $N[D \cap V_E(G)]$ satisfies conditions (a) and (b). Now, we observe that the subgraph induced by $D \cap V_E(G)$ does not have isolated vertices by Lemma 3. Then, and by the minimality of $|W|$, we deduce that $|W| \leq |D \cap V_E(G)|$. We claim that $W' = W \cup (D \cap V(G))$ is a TDS of G. By definitions of D and W, it is straightforward that $N(x) \cap W' \neq \emptyset$ for every vertex $x \in W \cup (N[D \cap V(G)] \cap V(G))$. Let $i \in \{1, \ldots, n\}$ such that $v_i \in V(G) \setminus (W \cup N[D \cap V(G)])$. Hence, there exists $j \in \{1, \ldots, n\} \setminus \{i\}$ such that $v^{i,j} \in D$. By definition it follows that $v_i \in N(v_j)$, and by condition (a) it follows that $v_j \in W$, as desired. Therefore, W' is a TDS of G, as required. Thus,

$$\gamma_t(G) \leq |W'| = |W| + |D \cap V(G)| \leq |D \cap V_E(G)| + |D \cap V(G)| = |D| = \gamma_t(\mathtt{T}(G)).$$

Now, we proceed to prove that $\gamma_t(\mathtt{L}(G)) \leq \gamma_t(\mathtt{T}(G))$. If $D \subseteq V_E(G)$, then D is also a TDS of $\mathtt{T}(G)[V_E(G)]$. So, by Remark 2 we deduce that $\gamma_t(\mathtt{L}(G)) = \gamma_t(\mathtt{T}(G)[V_E(G)]) \leq |D| = \gamma_t(\mathtt{T}(G))$. Hence, from now on we assume that $D \cap V(G) \neq \emptyset$.

Let F be a spanning forest of $\mathtt{T}(G)[D \cap V(G)]$. From F, let us define a set $X \subseteq V_E(G) \setminus D$ of minimum cardinality which satisfies the following conditions.

(a') $v^{i,j} \in X$ whenever $v_i v_j \in E(F)$.

(b') If there exists a component $F_{i,j}$ of F isomorphic to P_2 (with vertex set $V(F_{i,j}) = \{v_i, v_j\}$), then $X \cap N(v^{i,j}) \neq \emptyset$.

By the minimality of $|X|$, we deduce that $|X| \leq |V(F)| = |D \cap V(G)|$. Moreover, we notice that the subgraph induced by X has no isolated vertex. We claim that $X' = X \cup (D \cap V_E(G))$ is a TDS of $\mathtt{L}(G)$. By definitions of D and X, it is straightforward that $N(x) \cap X' \neq \emptyset$ for every vertex $x \in X \cup (N[D \cap V_E(G)] \cap V_E(G))$. Let $i, j \in \{1, \ldots, n\}$ such that $v^{i,j} \in V_E(G) \setminus (X \cup N[D \cap V_E(G)])$. Hence, $D \cap \{v_i, v_j\} \neq \emptyset$, and without loss of generality, we suppose that $v_i \in D$. By definitions of D and F, there exists $k \in \{1, \ldots, n\} \setminus \{i\}$ such that $v_k \in N(v_i) \cap D \cap V(F)$. Hence, $v^{i,k} \in N(v^{i,j}) \cap X$, as desired. Therefore, X' is a TDS of $\mathtt{L}(G)$, as required. Thus,

$$\gamma_t(\mathtt{L}(G)) \leq |X'| = |X| + |D \cap V_E(G)| \leq |D \cap V(G)| + |D \cap V_E(G)| = |D| = \gamma_t(\mathtt{T}(G)).$$

Finally, we proceed to prove the upper bound. Let D be a $\gamma_t(\mathtt{T}(G)[V(G)])$-set and D' be a $\gamma_t(\mathtt{T}(G)[V_E(G)])$-set. By Remark 2(i) and (ii) we deduce that $|D| = \gamma_t(G)$ and $|D'| = \gamma_t(\mathtt{L}(G))$. Now, we observe that $D \cup D'$ is a TDS of $\mathtt{T}(G)$. Hence, $\gamma_t(\mathtt{T}(G)) \leq |D \cup D'| = \gamma_t(G) + \gamma_t(\mathtt{L}(G))$, which completes the proof. ■

The lower bound above is achieved for the graph $\mathtt{T}(G)$ given in Figure 1 as $\gamma_t(\mathtt{T}(G)) = \gamma_t(\mathtt{L}(G)) = \gamma_t(G) = 2$. In addition, it is achieved for any star graph and double star graph. After numerous attempts, we have not been able to find examples of graphs that reach the equality in the upper bound given in Theorem 4. In this sense, we propose as a problem to find examples of graphs for which the equality is reached, or to improve this bound.

We continue with an upper bound for the total domination number of $\mathtt{T}(G)$ in terms of the total outer-independent domination numbers of G and $\mathtt{L}(G)$. For this purpose, we shall need to give the following remark.

Remark 5. If G' is a spanning subgraph (with no isolated vertex) of a non-trivial connected graph G, then every TDS of G' is also a TDS of G, and as a consequence,

$$\gamma_t(G) \leq \gamma_t(G').$$

Theorem 6. *The following statements hold for any connected graph G of order $n \geq 3$.*

(i) $\gamma_t(\mathtt{T}(G)) \leq \min\{\gamma_t(\mathtt{Q}(G)), \gamma_t(\mathtt{R}(G))\} \leq \gamma_{t,oi}(G)$.

(ii) $\gamma_t(\mathtt{T}(G)) \leq \gamma_{t,oi}(\mathtt{L}(G)) + |\mathcal{S}(G)|$.

Proof. From Remarks 2 and 5 it follows that $\gamma_t(\mathtt{T}(G)) \leq \min\{\gamma_t(\mathtt{Q}(G)), \gamma_t(\mathtt{R}(G))\}$. In addition, we have that $\gamma_t(\mathtt{R}(G)) = \gamma_{t,oi}(G)$, due to Sigarreta [23]. Therefore, (i) follows.

To conclude the proof, we proceed to prove that $\gamma_t(\mathtt{T}(G)) \leq \gamma_{t,oi}(\mathtt{L}(G)) + |\mathcal{S}(G)|$. Let D be a $\gamma_{t,oi}(\mathtt{T}(G)[V_E(G)])$-set. By Remark 2(ii) we deduce that $|D| = \gamma_{t,oi}(\mathtt{L}(G))$. Now, we claim that $D \cup \mathcal{S}(G)$ is a TDS of $\mathtt{T}(G)$. It is clear that $N(x) \cap (D \cup \mathcal{S}(G)) \neq \emptyset$ for every vertex $x \in V_E(G) \cup \mathcal{L}(G)$. Let $v_i \in V(G) \setminus \mathcal{L}(G)$. As $|N(v_i) \cap V(G)| \geq 2$, let $v_j, v_k \in N(v_i)$. Since $v^{ij} \in N(v^{ik})$, it follows that $D \cap \{v^{ij}, v^{ik}\} \neq \emptyset$, which implies that $N(v_i) \cap D \neq \emptyset$, as required. Hence, $\gamma_t(\mathtt{T}(G)) \leq |D \cup \mathcal{S}(G)| = \gamma_{t,oi}(\mathtt{L}(G)) + |\mathcal{S}(G)|$, as desired. ■

The following result is a direct consequence of Theorems 4 and 6. Observe that the lower bound given in Theorem 4 is achieved for the family of graphs given in the next proposition.

Proposition 7. *The following statements hold for any connected graph G of order $n \geq 3$.*

(i) *If $\gamma_{t,oi}(G) = \gamma_t(G)$, then $\gamma_t(\mathtt{T}(G)) = \gamma_t(\mathtt{R}(G)) = \gamma_t(G)$.*

(ii) *If $\delta(G) \geq 2$ and $\gamma_{t,oi}(\mathtt{L}(G)) = \gamma_t(\mathtt{L}(G))$, then $\gamma_t(\mathtt{T}(G)) = \gamma_t(\mathtt{L}(G))$.*

By Theorem 6, we have that any upper bound for the total outer-independent domination number of G gives us an upper bound for the total domination number of $\mathtt{T}(G)$. In such a sense, the next result provides new upper bounds for $\gamma_{t,oi}(G)$. Recall that $\gamma(G)$ represents the well-known domination number of G, i.e., the minimum cardinality among all dominating sets of G.

Theorem 8. *The following statements hold for any connected graph G of order $n \geq 3$.*

(i) $\gamma_{t,oi}(G) \leq \min\{2n - 2\alpha(G) - \delta(G) - 1, n - \alpha(G) + \gamma(G)\}$.

(ii) *If G is a claw-free graph with $\delta(G) \geq 3$, then $\gamma_{t,oi}(G) = n - \alpha(G)$.*

Proof. First, we proceed to prove (i). The bound $\gamma_{t,oi}(G) \leq 2n-2\alpha(G)-\delta(G)-1$ follows due to Cabrera Martínez et al. [6]. We next prove that $\gamma_{t,oi}(G) \leq n - \alpha(G) + \gamma(G)$. Let I be an $\alpha(G)$-set such that $\mathcal{L}(G) \subseteq I$ and let D be a $\gamma(G)$-set such that $\mathcal{S}(G) \subseteq D$. Now, we define a set $W \subseteq V(G)$ of minimum cardinality which satisfies the following conditions.

(a) $(V(G) \setminus I) \cup D \subseteq W$.

(b) $N(x) \cap W \neq \emptyset$ for every vertex $x \in D$.

We claim that W is a TOIDS of G. As $V(G) \setminus I$ and D are dominating sets, we deduce by (a) that $N(x) \cap W \neq \emptyset$ for every $x \in V(G) \setminus (D \cap W)$. Moreover, if $x \in D \cap W \subseteq D$, then by (b) we have that $N(x) \cap W \neq \emptyset$. Hence, W is a TDS of G. Moreover, it is easy to check that $V(G) \setminus W$ is an independent set because $V(G) \setminus W \subseteq I$. Therefore, W is a TOIDS of G, as desired. Thus, $\gamma_{t,oi}(G) \leq |W|$. Now, by the minimality of $|W|$ we deduce that $|W| \leq |V(G) \setminus I| + |D| = n - \alpha(G) + \gamma(G)$, which completes the proof of (i). Finally, we proceed to prove (ii). From now on, we assume that G is a claw-free graph with $\delta(G) \geq 3$. In order to show that $V(G) \setminus I$ is a TOIDS of G, we only need to prove that the subgraph induced by $V(G) \setminus I$ has no isolated vertex. Let $v \in V(G) \setminus I$ and $v_1, v_2, v_3 \in N(v)$. Since G is claw-free, we deduce that $\{v_1, v_2, v_3\} \nsubseteq I$, which implies that $N(v) \setminus I \neq \emptyset$, as required. Therefore, $V(G) \setminus I$ is a TOIDS of G and so, $\gamma_{t,oi}(G) \leq |V(G) \setminus I| = n - \alpha(G)$. The equality follows by the well-known trivial lower bound $\gamma_{t,oi}(G) \geq n - \alpha(G)$, which completes the proof. ■

The following result is a direct consequence of Theorems 6 and 8.

Theorem 9. *The following statements hold for any connected graph G of order $n \geq 3$.*

(i) $\gamma_t(\mathtt{T}(G)) \leq \min\{2n - 2\alpha(G) - \delta(G) - 1, n - \alpha(G) + \gamma(G)\}$.

(ii) *If G is a claw-free graph with $\delta(G) \geq 3$, then $\gamma_t(\mathtt{T}(G)) \leq n - \alpha(G)$.*

The bounds given in the two previous theorems are tight. For instance, the bounds given in Theorems 8(i) and 9(i) are achieved for the star graph $K_{1,n-1}$ because $\gamma_t(\mathtt{T}(K_{1,n-1})) = |V(K_{1,n-1})| - \alpha(K_{1,n-1}) + \gamma(K_{1,n-1}) = 2$. In addition, the bound given in Theorem 9(ii) is achieved for the complete graph K_4 because $\gamma_t(\mathtt{T}(K_4)) = \gamma_{t,oi}(K_4) = |V(K_4)| - \alpha(K_4) = 3$.

We continue with other bounds for the total domination number of total graphs.

Theorem 10. *If G is a connected graph with diameter* $\operatorname{diam}(G)$ *and order $n \geq 3$, then*

$$2 \leq \gamma_t(\mathtt{T}(G)) \leq n - \rho(G) \leq n - \left\lceil \frac{\operatorname{diam}(G)}{3} \right\rceil.$$

Furthermore,

(i) $\gamma_t(\mathtt{T}(G)) = 2$ *if and only if* $\gamma_{t,oi}(G) = 2$.

(ii) $\gamma_t(\mathtt{T}(G)) = 3$ *if and only if* $\gamma_{t,oi}(G) = 3$.

Proof. The lower bound is straightforward. Now, we proceed to prove the upper bound. Let P be a $\rho(G)$-set such that $P \cap \mathcal{S}(G) = \emptyset$. It is easy to deduce that $V(G) \setminus P$ is a TOIDS of G. Hence, $\gamma_{t,oi}(G) \leq |V(G) \setminus P| = n - \rho(G)$. Now, for any diametrical path $R = v_0v_1 \cdots v_r$ ($r = \mathrm{diam}(G)$), it follows that $P_R = \{v_0, v_3, \ldots, v_{3\lfloor r/3 \rfloor}\}$ is a 2-packing of G. Hence, $\rho(G) \geq |P_R| = \left\lceil \frac{\mathrm{diam}(G)}{3} \right\rceil$. From these two previous inequalities and Theorem 6, we obtain that

$$\gamma_t(\mathtt{T}(G)) \leq \gamma_{t,oi}(G) \leq n - \rho(G) \leq n - \left\lceil \frac{\mathrm{diam}(G)}{3} \right\rceil.$$

Now, we proceed to prove (i). We first suppose that $\gamma_t(\mathtt{T}(G)) = 2$. Let D be a $\gamma_t(\mathtt{T}(G))$-set which satisfies Lemma 3. In this case, either $D \subseteq V_E(G)$ or $D \subseteq V(G)$. If $D \subseteq V_E(G)$, then it is straightforward to see that $|V(G)| = 3$, which implies that $\gamma_{t,oi}(G) = 2$. Now, we suppose that $D \subseteq V(G)$. This implies that D is also a TDS of G. If there exist two adjacent vertices $v_i, v_j \in V(G) \setminus D$, then $N(v^{i,j}) \cap D = \emptyset$, which is a contradiction. Hence, $V(G) \setminus D$ is an independent set, which implies that D is a TOIDS of G. Therefore, $2 \leq \gamma_{t,oi}(G) \leq |D| = 2$, as required. Finally, if $\gamma_{t,oi}(G) = 2$, then by Theorem 6 we have that $2 \leq \gamma_t(\mathtt{T}(G)) \leq 2$, which completes the proof of (i).

Finally, we proceed to prove (ii). We first suppose that $\gamma_t(\mathtt{T}(G)) = 3$. Let D be a $\gamma_t(\mathtt{T}(G))$-set which satisfies Lemma 3. As above, notice that either $D \subseteq V_E(G)$ or $D \subseteq V(G)$. If $D \subseteq V_E(G)$, then $\gamma_{t,oi}(G) \leq 3$ and by (i) we deduce that $\gamma_{t,oi}(G) = 3$, as required. Moreover, if $D \subseteq V(G)$, then proceeding analogously to the previous case, it follows that D is a TOIDS of G. Therefore, $\gamma_{t,oi}(G) \leq |D| = 3$, and by (i) we deduce that $\gamma_{t,oi}(G) = 3$, as required. On the other hand, if $\gamma_{t,oi}(G) = 3$, then by (i) and Theorem 6 it follows that $\gamma_t(\mathtt{T}(G)) = 3$, which completes the proof. ■

The next result provides an upper bound on $\gamma_t(\mathtt{T}(T))$, where T is a tree of order at least three.

Theorem 11. *For any tree T of order $n \geq 3$ and $l(T)$ leaves,*

$$\gamma_t(\mathtt{T}(T)) \leq \frac{3n - l(T) + 2}{4}.$$

Proof. We proceed by induction on the order $n \geq 3$. It is easy to check that $\gamma_t(\mathtt{T}(T)) \leq \frac{3n-l(T)+2}{4}$ for any tree T of order $n \in \{3, 4\}$. These particular cases establish the base cases. Let $n \geq 5$ be an integer and we assume that any tree T' of order $n' < n$ satisfies that $\gamma_t(\mathtt{T}(T')) \leq \frac{3n'-l(T')+2}{4}$. We next proceed to prove that $\gamma_t(\mathtt{T}(T)) \le \frac{3n-l(T)+2}{4}$ for any tree T of order n. For this purpose, let $v_1 \cdots v_d v_{d+1}$ be a diametrical path in T. Notice that $v_d \in \mathcal{S}(T)$ and $v_{d+1} \in \mathcal{L}(T)$. Let us consider the following three cases, which depend on the three schemes given in Figure 2 (Case i works with the scheme (i), with $i \in \{1,2,3\}$).

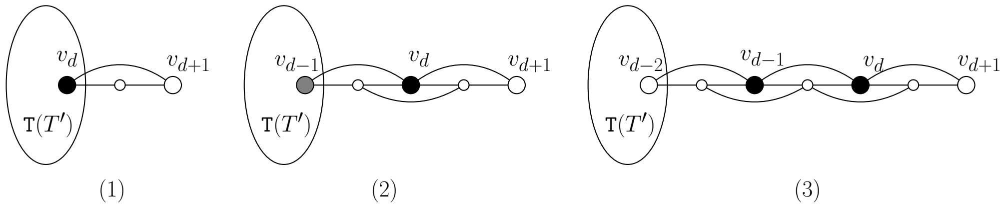

Figure 2. The schemes for the graph $\mathtt{T}(T)$ used in the proof of Theorem 11.

Case 1. $|N(v_d)| \ge 3$. In this case, let $T' = T - \{v_{d+1}\}$. Notice that $l(T') = l(T) - 1$. Let D' be a $\gamma_t(\mathtt{T}(T'))$-set which satisfies Lemma 3. Hence, $|D'| \le \frac{3n'-l(T')+2}{4}$ by induction hypothesis . Also, we have that $v_d \in \mathcal{S}(T')$. By Lemma 3(ii), it follows that $v_d \in D'$, which implies that D' is also a TDS of $\mathtt{T}(T)$. Therefore,

$$\begin{aligned}\gamma_t(\mathtt{T}(T)) \le |D'| &\le \frac{3n' - l(T') + 2}{4}\\ &= \frac{3(n-1) - (l(T) - 1) + 2}{4} \le \frac{3n - l(T) + 2}{4},\end{aligned}$$

as desired. By Case 1, we may henceforth assume in the next cases that $|N(x)| = 2$ for every vertex $x \in \mathcal{S}(T)$.

Case 2. $|N(v_d)| = 2$ and $|N(v_{d-1})| \ge 3$. In this case, let $T' = T - \{v_d, v_{d+1}\}$. Notice that $l(T') \ge l(T) - 1$ and $v_{d-1} \in \mathcal{S}(T') \cup N(\mathcal{S}(T'))$. Among all $\gamma_t(\mathtt{T}(T'))$-sets which satisfy Lemma 3, let D' be a $\gamma_t(\mathtt{T}(T'))$-set such that $|D' \cap \mathcal{L}(T')|$ is minimum. This implies that $v_{d-1} \in D'$. In addition, $|D'| \le \frac{3n'-l(T')+2}{4}$ by induction hypothesis. Now, it is easy to see that $D' \cup \{v_d\}$ is a TDS of $\mathtt{T}(T)$. Therefore,

$$\begin{aligned}\gamma_t(\mathtt{T}(T)) \le |D' \cup \{v_d\}| &\le \frac{3n' - l(T') + 2}{4} + 1\\ &\le \frac{3(n-2) - (l(T) - 1) + 2}{4} + 1 \le \frac{3n - l(T) + 2}{4},\end{aligned}$$

as desired.

Case 3. $|N(v_d)| = |N(v_{d-1})| = 2$. In this case, let $T' = T - \{v_{d-1}, v_d, v_{d+1}\}$. Notice that $l(T') \ge l(T) - 1$. Let D' be a $\gamma_t(\mathtt{T}(T'))$-set which satisfies Lemma 3. Hence, $|D'| \le \frac{3n'-l(T')+2}{4}$ by induction hypothesis. Now, we observe that $D' \cup \{v_{d-1}, v_d\}$ is a TDS of $\mathtt{T}(T)$, which implies that

$$\begin{aligned}\gamma_t(\mathtt{T}(T)) &\le |D' \cup \{v_{d-1}, v_d\}| \le \frac{3n' - l(T') + 2}{4} + 2\\ &\le \frac{3(n-3) - (l(T) - 1) + 2}{4} + 2 = \frac{3n - l(T) + 2}{4},\end{aligned}$$

as desired. Therefore, the proof is complete. ■

2.1. $\gamma_t(\mathtt{T}(G))$ for some specific graphs G

We begin this subsection with the following result given in [1], which states the total domination number of $\mathtt{T}(P_n)$.

Proposition 12. [1] *For any path P_n with $n \ge 2$,*

$$\gamma_t(\mathtt{T}(P_n)) = \begin{cases} \left\lceil \frac{4n}{7} \right\rceil - 1 & \text{if } n \equiv 4 \pmod 7,\\ \left\lceil \frac{4n}{7} \right\rceil & \text{otherwise.} \end{cases}$$

Now, we obtain the total domination number of $\mathtt{T}(C_n)$. For this purpose, we need to state the following three lemmas. In particular, we highlight that Lemmas 13 and 15 lead to the total domination number of $\mathtt{T}(C_n)$.

Lemma 13. *For any cycle C_n with $n \ge 3$,*

$$\gamma_t(\mathtt{T}(C_n)) \le \begin{cases} \left\lceil \frac{4n}{7} \right\rceil + 1 & \text{if } n \equiv 5 \pmod 7,\\ \left\lceil \frac{4n}{7} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. In Figure 3 we show how to construct a $\gamma_t(\mathtt{T}(C_n))$-set for $n \in \{3, \ldots, 9\}$. In this scheme, the set of black-coloured vertices forms a $\gamma_t(\mathtt{T}(C_n))$-set. We now proceed to describe the construction of a TDS D of $\mathtt{T}(C_n)$ for any $n = 7q + r$, where $q \ge 1$ and $0 \le r \le 6$. Let us partition $V(C_n) = \{v_1, \ldots, v_n\}$ into q sets of cardinality 7 and for $r \ge 1$ one additional set of cardinality r, in such a way that the subgraph induced by all these sets are paths. For any $r \in \{0, 3, 4, 5, 6\}$, the restriction of D to each of these q paths of length 7 corresponds to the scheme associated with $\mathtt{T}(C_7)$ in Figure 3, while for the path of length r (if any) we take the scheme associated with $\mathtt{T}(C_r)$. In the cases $r = 1$ or $r = 2$ (with $q \ge 2$), for the first $q-1$ paths of length 7 we take the scheme associated with $\mathtt{T}(C_7)$, and for the path associated with the last 8 or 9 vertices, we take the scheme associated with $\mathtt{T}(C_8)$ or $\mathtt{T}(C_9)$, respectively. From the previous construction, it is easy to deduce that, for $n \equiv 5 \pmod 7$ we have that $\gamma_t(\mathtt{T}(C_n)) \le |D| = 4q+4 = \left\lceil \frac{4n}{7} \right\rceil + 1$, while for $n \not\equiv 5 \pmod 7$ we have that $\gamma_t(\mathtt{T}(C_n)) \le |D| = \left\lceil \frac{4n}{7} \right\rceil$, which completes the proof. ■

Lemma 14. *Let C_n be a cycle with vertex set $V(C_n) = \{v_1, \ldots, v_n\}$ $(n \ge 3)$, where $v_iv_{i+1} \in E(C_n)$ for any $i \in \{1, \ldots, n-1\}$ and $v_1v_n \in E(C_n)$. For any*

$i \in \{1, \ldots, n\}$, let $P_{7,i} = v_i v_{i+1} \cdots v_{i+6}$ be a subgraph of C_n (the subscripts are taken modulo n). If D is a $\gamma_t(\mathtt{T}(C_n))$-set, then

$$\left|D \cap \left(V(\mathtt{T}(P_{7,i})) \cup \left\{v^{i-1,i}\right\}\right)\right| \geq 4.$$

Proof. Let $D' = D \cap \left(V(\mathtt{T}(P_{7,i})) \cup \left\{v^{i,i-1}\right\}\right)$. Notice that $\left|D' \setminus \left\{v_i, v^{i,i-1}\right\}\right| \geq 3$. In addition, if $\left|D' \setminus \left\{v_i, v^{i,i-1}\right\}\right| = 3$, then it is easy to deduce that $\left|D' \cap \left\{v_i, v^{i,i-1}\right\}\right| \geq 1$. Therefore, $|D'| = \left|D' \setminus \left\{v_i, v^{i,i-1}\right\}\right| + \left|D' \cap \left\{v_i, v^{i,i-1}\right\}\right| \geq 4$, which completes the proof. ■

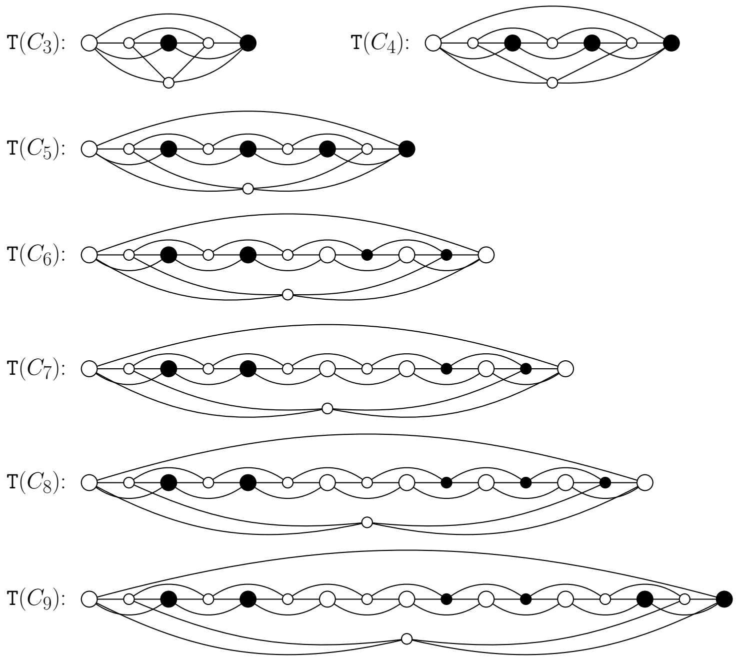

Figure 3. The scheme used in the proof of Lemma 13.

Lemma 15. *For any cycle C_n with $n \geq 3$,*

$$\gamma_t(\mathtt{T}(C_n)) \geq \begin{cases} \left\lceil \frac{4n}{7} \right\rceil + 1 & \text{if } n \equiv 5 \pmod 7, \\ \left\lceil \frac{4n}{7} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let D be a $\gamma_t(\mathtt{T}(C_n))$-set which satisfies Lemma 3. By the upper bound given in Lemma 13 we deduce that there exists $i \in \{1, \ldots, n\}$ such that $v_i, v^{i,i+1} \notin D$ or $v_{i+1}, v^{i,i+1} \notin D$. In any case, we deduce that $\left(\mathtt{T}(C_n) - \left\{v^{i,i+1}\right\}\right) - \{v_i v_{i+1}\} \cong \mathtt{T}(P_n)$. Hence, D is also a TDS of this previous subgraph isomorphic to $\mathtt{T}(P_n)$. Therefore, by this previous fact and Proposition 12, it follows that $\gamma_t(\mathtt{T}(C_n)) = |D| \geq \gamma_t(\mathtt{T}(P_n)) \geq \left\lceil \frac{4n}{7} \right\rceil$ for any integer $n \not\equiv 4, 5 \pmod 7$.

From now on, we assume that $n \equiv 4, 5 \pmod 7$. In Figure 3 we show how to construct a $\gamma_t(\mathtt{T}(C_n))$-set for $n \in \{4, 5\}$, and these values satisfy the lower bound. Let $n = 7q + r$, with $r \in \{4, 5\}$ and $q \geq 1$. Let $V(C_n) = \{v_1, \ldots, v_n\}$, where $v_i v_{i+1} \in E(C_n)$ for any $i \in \{1, \ldots, n-1\}$ and $v_1 v_n \in E(C_n)$. We partition $V(C_n)$ into $X = \{v_1, \ldots, v_r\}$ and $Y = \{v_{r+1}, \ldots, v_n\}$. Let P_r be the subgraph of C_n induced by X. For any $i \in \{1, 8, \ldots, 7q-6\}$, let $P_{7,r+i} = v_{r+i} v_{r+i+1} \cdots v_{r+i+6}$ be a subgraph of C_n. Notice that

$$\begin{aligned} V(\mathtt{T}(C_n)) &= \left(V(\mathtt{T}(P_r)) \cup \left\{v^{1,n}\right\}\right) \\ &\cup \left(\bigcup_{i \in \{1,8,\ldots,7q-6\}} \left(V(\mathtt{T}(P_{7,r+i})) \cup \left\{v^{r+i,r+i-1}\right\}\right) \right). \end{aligned} \tag{1}$$

Let D be a $\gamma_t(\mathtt{T}(C_n))$-set which satisfies Lemma 3. Now, we define the following subsets of D.

$D_r = D \cap \left(V(\mathtt{T}(P_r)) \cup \{v^{1,n}\}\right)$,

$D_q = D \cap \left(\bigcup_{i \in \{1,8,\ldots,7q-6\}} \left(V(\mathtt{T}(P_{7,r+i})) \cup \{v^{r+i,r+i-1}\}\right)\right)$.

By Equality (1) we deduce that $D = D_r \cup D_q$. By Lemma 14 we have that

$$|D_q| \geq 4q = \frac{4(n-r)}{7} = \frac{4n}{7} - \frac{4r}{7}. \tag{2}$$

Now, we analyse the following two possibilities.

Case 1. $r = 4$. In this case, we have that $|D_4| \geq 2$. By Inequality chain (2) we have that $|D_q| \geq 4q$. If $|D_q| \geq 4q + 1$, then we have that $|D| = |D_4| + |D_q| \geq 4q + 3 \geq \left\lceil \frac{4n}{7} \right\rceil$. Otherwise, we have that $|D_q| = 4q$, which implies that $\left|D \cap \left(V(\mathtt{T}(P_{7,4+i})) \cup \left\{v^{4+i,4+i-1}\right\}\right)\right| = 4$ for every $i \in \{1, 8, \ldots, 7q-6\}$. So, $D \cap \left(V(\mathtt{T}(P_{7,4+i})) \cup \left\{v^{4+i,4+i-1}\right\}\right)$ (in each $\mathtt{T}(P_{7,4+i})$) is induced by the set of black-coloured vertices in the scheme of $\mathtt{T}(C_7)$) used in Figure 3. This consequence implies that $|D_4| \geq 3$ and as above, it follows that $|D| = |D_4| + |D_q| \geq 3 + \frac{4n}{7} - \frac{16}{7} \geq \left\lceil \frac{4n}{7} \right\rceil$, as required.

Case 2. $r = 5$. In this case, we have that $|D_5| \geq 2$. By Inequality chain (2) we have that $|D_q| \geq 4q$. First, we suppose that $|D_q| = 4q + 1$ and $|D_5| = 2$. Without loss of generality, we consider that $D_r = \{v_2, v_3\}$. In order to guarantee that $N[x] \cap D \neq \emptyset$ for every $x \in \left\{v^{1,n}, v_5, v^{4,5}\right\}$, it is necessary that $\left|D \cap \left(V(\mathtt{T}(P_{7,6})) \cup \left\{v^{5,6}\right\}\right)\right| \geq 5$ and $\left|D \cap \left(V(\mathtt{T}(P_{7,7q-1})) \cup \left\{v^{7q-2,7q-1}\right\}\right)\right| \geq 5$, which would imply that $|D_q| > 4q + 1$, a contradiction. Now, suppose that $|D_q| = 4q$ and $|D_5| \in \{2, 3\}$. By proceeding analogously to Case 1, we can deduce that $|D_5| \geq 4$ (in order to dominate all vertices in $V(\mathtt{T}(P_5))$), a contradiction. For the rest of the cases it is satisfied that $|D_5| + |D_q| \geq 4q + 4$. This implies that $|D| = |D_5| + |D_q| \geq 4q + 4 \geq \left\lceil \frac{4n}{7} \right\rceil + 1$, as required.

Therefore, the proof is complete. ■

The following result, which is a direct consequence of Lemmas 13 and 15, provides the total domination number of $\mathtt{T}(C_n)$.

Proposition 16. *For any cycle C_n with $n \geq 3$,*

$$\gamma_t(\mathtt{T}(C_n)) = \begin{cases} \left\lceil \frac{4n}{7} \right\rceil + 1 & \text{if } n \equiv 5 \pmod 7, \\ \left\lceil \frac{4n}{7} \right\rceil & \text{otherwise.} \end{cases}$$

Now, we obtain the total domination number of $\mathtt{T}(W_n)$.

Proposition 17. *For any wheel W_n with $n \geq 4$,*

$$\gamma_t(\mathtt{T}(W_n)) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. Let $V(W_n) = \{v_1, v_2, \ldots, v_n\}$. We assume that v_1 is the center of the wheel and that consecutive vertices in $V(W_n) \setminus \{v_1\}$ are adjacent in W_n (in addition, let us assume that $v_2v_n \in E(W_n)$). By Theorem 6 and the fact that $\gamma_{t,oi}(W_n) = \lceil \frac{n+1}{2} \rceil$ (see [6]), we deduce that $\gamma_t(\mathtt{T}(W_n)) \leq \lceil \frac{n+1}{2} \rceil$. We only need to prove that $\gamma_t(\mathtt{T}(W_n)) \geq \lceil \frac{n+1}{2} \rceil$. Let D be a $\gamma_t(\mathtt{T}(W_n))$-set that satisfies the conditions of Lemma 3, has the maximum number of vertices in $V(W_n)$ and, among those satisfying the previous conditions, has the maximum number of vertices in $\{v^{1,2}, v^{1,3}, \ldots, v^{1,n}\}$.

First, we suppose that $D \subseteq V(W_n)$. Since $N(v^{1,i}) \cap V(W_n) = \{v_1, v_i\}$ for every $i \in \{2, \ldots, n\}$ and $|D| \leq \lceil \frac{n+1}{2} \rceil$, it follows that $v_1 \in D$. For every $i \in \{2, \ldots, n\}$, it follows that $N(v^{i,i+1}) \cap D \neq \emptyset$, which implies that $|D \cap \{v_i, v_{i+1}\}| \geq 1$ (we identify $n+1$ with 2). Hence, $|D \cap \{v_2, \ldots, v_n\}| \geq \lceil \frac{n-1}{2} \rceil$, and as a consequence,

$$|D| = |D \cap \{v_2, \ldots, v_n\}| + |\{v_1\}| \geq \left\lceil \frac{n-1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil,$$

as desired. From now on, we consider that $D \cap V_E(W_n) \neq \emptyset$. If $v_1 \in D$, then $v^{1,j} \notin D$ for any $j \in \{2, \ldots, n\}$ by Lemma 3(i), which implies that there exists $i \in \{2, \ldots, n\}$ such that $v^{i,i+1}, v^{i+1,i+2} \in D$. So, $D' = (D \setminus \{v^{i,i+1}, v^{i+1,i+2}\}) \cup \{v_i, v_{i+2}\}$ is a $\gamma_t(\mathtt{T}(W_n))$-set satisfying that $|D' \cap V(G)| > |D \cap V(G)|$, which contradicts the conditions assumed in the proof of Lemma 3. Hence, $v_1 \notin D$. Now, if there exists $i \in \{2, \ldots, n\}$ such that $v^{1,i}, v^{i,i+1} \in D$ (this previous condition is analogous to assuming that $v^{1,i+1}, v^{i,i+1} \in D$), then $D'' = (D \setminus \{v^{i,i+1}\}) \cup \{v^{1,i+1}\}$ is a $\gamma_t(\mathtt{T}(W_n))$-set satisfying that $|D'' \cap \{v^{1,2}, \ldots, v^{1,n}\}| > |D \cap \{v^{1,2}, \ldots, v^{1,n}\}|$, which is a contradiction. From the above, we can deduce that if there exist $i, j \in \{2, \ldots, n\}$ such that $v^{1,i}, v^{j,j+1} \in D$, then $j \notin \{i, i-1\}$. This implication leads to the next complementary cases.

Case 1. $D \cap \{v^{1,2}, \ldots, v^{1,n}\} = \emptyset$. Notice that $|D \cap \{v_i, v^{i-1,i}, v^{i,i+1}\}| \geq 1$ for every $i \in \{2, \ldots, n\}$ because $N(v^{1,i}) \cap D \neq \emptyset$. Moreover, as $N(v_1) \cap D \neq \emptyset$, it follows that $|D \cap \{v_2, \ldots, n\}| \geq 2$ by Lemma 3(i). Therefore,

$$2\gamma_t(\mathtt{T}(W_n)) = 2|D| = \sum_{i \in \{2,\ldots,n\}} \left|D \cap \left\{v_i, v^{i-1,i}, v^{i,i+1}\right\}\right| + |D \cap \{v_2, \ldots, n\}| \geq n + 1.$$

Hence, $\gamma_t(\mathtt{T}(W_n)) \geq \left\lceil \frac{n+1}{2} \right\rceil$, as required.

Case 2. $\left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right| \geq 3$. Let H be the subgraph $\mathtt{T}(W_n) - N\left[D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right]$. We notice that H has no isolated vertex. By Lemma 3(i) and the previous implications, we deduce that $|V(H)| \geq 2(n-1) - 3\left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right|$ and $\Delta(H) \leq 4$. Moreover, we observe that $D' = D \setminus \left\{v^{1,2}, \ldots, v^{1,n}\right\}$ is a $\gamma_t(H)$-set (otherwise, for some $\gamma_t(H)$-set, the set $D_H \cup \left(D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right)$ is a $\gamma_t(\mathtt{T}(W_n))$-set of cardinality less than $|D|$, a contradiction). Therefore, and using the well-known inequality $\gamma_t(H) \geq |V(H)|/\Delta(H)$, we deduce the following.

$$\begin{aligned}
|D| = |D'| + \left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right| &\geq \frac{|V(H)|}{\Delta(H)} + \left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right| \\
&\geq \frac{2(n-1) - 3\left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right|}{4} + \left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right| \\
&= \frac{n-1}{2} + \frac{\left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right|}{4} \geq \frac{n-1}{2} + \frac{3}{4}.
\end{aligned}$$

Hence, $\gamma_t(\mathtt{T}(W_n)) \geq \left\lceil \frac{n-1}{2} + \frac{3}{4} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil$, as required.

Case 3. $\left|D \cap \left\{v^{1,2}, \ldots, v^{1,n}\right\}\right| = 2$. Without loss of generality, we assume that $v^{1,2} \in D$. Let $j \in \{3, \ldots, n\}$ such that $\{v^{1,j}\} = D \cap \left\{v^{1,3}, \ldots, v^{1,n}\right\}$. Now, we consider the next subcases.

Subcase 3.1. $j \in \{3, n\}$. Without loss of generality, we consider that $j = 3$. In this subcase, we proceed analogously to Case 2, considering that $|V(H)| \geq 2(n-1) - 5$. From this previous condition we deduce that $\gamma_t(\mathtt{T}(W_n)) \geq \left\lceil \frac{n+1}{2} \right\rceil$, as required.

Subcase 3.2. $j \in \{4, n-1\}$. Without loss of generality, we consider that $j = 4$. In this subcase, we have that $v_2, v_4, v^{2,3}, v^{3,4} \notin D$ as $v^{1,2}, v^{1,4} \in D$. This implies that $N(v_3) \cap D = \emptyset$ because $v_1 \notin D$, which is a contradiction.

Subcase 3.3. $j \in \{5, \ldots, n-2\}$. In this subcase, we have that $n \geq 8$. Now, let P_{j-3} and P_{n-j} be the subgraphs induced by $\left\{v^{1,3}, \ldots, v^{1,j-1}\right\}$ and $\left\{v^{1,j+1}, \ldots, v^{1,n}\right\}$, respectively. Observe that P_{j-3} and P_{n-j} are paths of $j-3$ and $n-j$ vertices, respectively. With all the above in mind, we deduce that $D \cap V(\mathtt{T}(P_{j-3}))$ and $D \cap V(\mathtt{T}(P_{n-j}))$ are total dominating sets of $\mathtt{T}(P_{j-3})$ and

$\mathtt{T}(P_{n-j})$, respectively. Since $D = (D\cap V(\mathtt{T}(P_{j-3})))\cup(D\cap V(\mathtt{T}(P_{n-j})))\cup\{v^{1,2}, v^{1,j}\}$ and $\gamma_t(\mathtt{T}(P_r)) \geq \frac{4r-2}{7}$ by Proposition 12, we obtain the following.

$$\begin{aligned}|D| &= \left|\{v^{1,2}, v^{1,j}\}\right| + |D\cap V(\mathtt{T}(P_{j-3}))| + |D\cap V(\mathtt{T}(P_{n-j}))|\\ &\geq 2 + \gamma_t(\mathtt{T}(P_{j-3})) + \gamma_t(\mathtt{T}(P_{n-j}))\\ &\geq 2 + \frac{4(j-3)-2}{7} + \frac{4(n-j)-2}{7} = \frac{4n}{7} - \frac{2}{7}.\end{aligned}$$

If $n \geq 11$, then $\frac{4n}{7} - \frac{2}{7} \geq \frac{n+1}{2}$, which implies that $\gamma_t(\mathtt{T}(W_n)) \geq \left\lceil\frac{n+1}{2}\right\rceil$, as required. Finally, in the case in that $n \in \{8, 9, 10\}$, the result can be easily derived by proceeding as before.

From the three cases above, the proof is complete. ∎

Finally, we compute the total domination number of $\mathtt{T}(K_n)$. For this purpose, we shall need the following known results.

Proposition 18. *The following equalities hold for any integer $n \geq 3$.*

(i) [23] $\gamma_t(\mathtt{Q}(K_n)) = \left\lceil\frac{2n}{3}\right\rceil$.

(ii) [18] $\gamma_t(\mathtt{L}(K_n)) = \begin{cases} 2\left\lfloor\frac{n}{3}\right\rfloor & \text{if } n \equiv 0, 1 \pmod 3,\\ 2\left\lfloor\frac{n}{3}\right\rfloor + 1 & \text{if } n \equiv 2 \pmod 3.\end{cases}$

Proposition 19. *For any complete graph K_n with $n \geq 3$,*

$$\gamma_t(\mathtt{T}(K_n)) = \left\lceil\frac{2n}{3}\right\rceil.$$

Proof. By Theorem 6 and Proposition 18(i) we deduce the inequality chain $\gamma_t(\mathtt{T}(K_n)) \leq \gamma_t(\mathtt{Q}(K_n)) = \left\lceil\frac{2n}{3}\right\rceil$. We only need to prove that $\gamma_t(\mathtt{T}(K_n)) \geq \left\lceil\frac{2n}{3}\right\rceil$. Let $V(K_n) = \{v_1, \ldots, v_n\}$ be the vertex set of K_n. Let D be a $\gamma_t(\mathtt{T}(K_n))$-set which satisfies Lemma 3, and without loss of generality we assume that $D \cap V(K_n) = \{v_1, \ldots, v_r\}$. If $r = 0$, then D is a TDS of the spanning subgraph $\mathtt{T}(K_n) - E(K_n)$ (which is isomorphic to the graph $\mathtt{Q}(K_n)$ by Remark 2(iv)). Therefore, $\left\lceil\frac{2n}{3}\right\rceil = \gamma_t(\mathtt{Q}(K_n)) \leq |D| = \gamma_t(\mathtt{T}(K_n))$, as required.

From now on, we suppose that $r > 0$ (by Lemma 3 we deduce that $r \geq 2$). Now, we consider the next sets: $A = \{v_1, \ldots, v_r\} \cup \{v^{i,j} : 1 \leq i < j \leq r\}$, $C = \{v^{i,j} : 1 \leq i \leq r, 1 \leq j \leq n\}$ and $B = V(\mathtt{T}(K_n)) \setminus (A \cup C)$. By Lemma 3 we have that $D \cap C = \emptyset$. This implies that $D \cap B$ is a TDS of the subgraph induced by $V_E(K_n) \setminus C$, which is isomorphic to $\mathtt{L}(K_{n-r})$. This previous fact and Proposition 18(ii) lead to the following inequality chain.

$$\gamma_t(\mathtt{T}(K_n)) = |D| = r + |D \cap B| \geq r + \gamma_t(\mathtt{L}(K_{n-r})) \geq \frac{2n}{3}.$$

From the above, we obtain that $\gamma_t(\mathtt{T}(K_n)) \geq \left\lceil\frac{2n}{3}\right\rceil$, which completes the proof. ∎

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