Discussiones Mathematicae
Graph Theory 44 (2024) 893–911
https://doi.org/10.7151/dmgt.2477

THE m-BIPARTITE RAMSEY NUMBER $BR_m(H_1, H_2)$

Yaser Rowshan

Department of Mathematics
Institute for Advanced Studies in Basic Sciences (IASBS)
Zanjan 66731-45137, Iran

e-mail: y.rowshan@iasbs.ac.ir, y.rowshan.math@gmail.co

Abstract

In a (G^1, G^2) coloring of a graph G, every edge of G is in G^1 or G^2. For two bipartite graphs H_1 and H_2, the bipartite Ramsey number $BR(H_1, H_2)$ is the least integer $b \geq 1$, such that for every (G^1, G^2) coloring of the complete bipartite graph $K_{b,b}$, results in either $H_1 \subseteq G^1$ or $H_2 \subseteq G^2$. As another view, for bipartite graphs H_1 and H_2 and a positive integer m, the m-bipartite Ramsey number $BR_m(H_1, H_2)$ of H_1 and H_2 is the least integer n ($n \geq m$) such that every subgraph G of $K_{m,n}$ results in $H_1 \subseteq G$ or $H_2 \subseteq \overline{G}$. The size of m-bipartite Ramsey number $BR_m(K_{2,2}, K_{2,2})$, the size of m-bipartite Ramsey number $BR_m(K_{2,2}, K_{3,3})$ and the size of m-bipartite Ramsey number $BR_m(K_{3,3}, K_{3,3})$ have been computed in several articles up to now. In this paper we determine the exact value of $BR_m(K_{2,2}, K_{4,4})$ for each $m \geq 2$.

Keywords: Ramsey numbers, bipartite Ramsey numbers, complete graphs, m-bipartite Ramsey number.

2020 Mathematics Subject Classification: 05D10, 05C55.

1. Introduction

In a (G^1, G^2) coloring of a graph G, every edge of G is in G^1 or G^2. For two graphs G and H, the Ramsey number $R(G, H)$ is the smallest positive integer n such that for every (G^1, G^2) coloring of the complete graph K_n results in either $G \subseteq G^1$ or $H \subseteq G^2$. Frank Harary always liked this notation! All such Ramsey numbers $R(G, H)$ exist as well, and if G has order n and H has order m, then $R(G, H) \leq R(K_n, K_m)$. In [1], Beineke and Schwenk introduced a bipartite version of Ramsey numbers. For given bipartite graphs H_1 and H_2, the bipartite Ramsey number $BR(H_1, H_2)$ is defined as the smallest positive integer b such that any subgraph G of $K_{b,b}$ results in either an $H_1 \subseteq G$ or $H_2 \subseteq \overline{G}$. One can refer to [3, 4, 8–14, 16–19, 21, 22], and their references for further studies.

Suppose that (G^1, G^2) be any 2-edges coloring of $K_{m,n}$ when $m \neq n$, also let H_1 and H_2 be two bipartite graphs. For each $m \geq 1$, the m-bipartite Ramsey number $BR_m(H_1, H_2)$ of H_1 and H_2 is the least integer $n (n \geq m)$, so that any (G^1, G^2) coloring of $K_{m,n}$ results in either a $H_1 \subseteq G^1$ or $H_2 \subseteq G^2$. Bi, Chartrand and Zhang in [2] evaluate $BR_m(K_{2,2}, K_{3,3})$ for each $m \geq 2$. Recently Rowshan and Gholami, by another simple proof, evaluated $BR_m(K_{2,2}, K_{3,3})$ in [20]. Bi, Chartrand and Zhang in [5] evaluate $BR_m(K_{3,3}, K_{3,3})$.

In this article we determine the exact value of $BR_m(K_{2,2}, K_{4,4})$ for each $m \geq 2$. In particular we prove the following theorem.

Theorem 1. *Let $m \geq 2$ be a positive integer. Then*

$$BR_m(K_{2,2}, K_{4,4}) = \begin{cases} \text{does not exist}, & if \quad m \in \{2, 3, 4\}, \\ 26, & if \quad m = 5, \\ 22, & if \quad m \in \{6, 7\}, \\ 16, & if \quad m = 8, \\ 14, & if \quad m \in \{9, 10, \ldots, 13\}, \\ m, & if \quad m \geq 14. \end{cases}$$

There is a familiar problem corresponding to the Ramsey number $R(K_3, K_3)$, which is stated as follows: What is the last number of people who must be present at a meeting, where every two people are either acquaintances or strangers, so that there are three among them who are either mutual acquaintances or mutual strangers? Since $R(K_3, K_3) = 6$, the answer to this question is 6. On the other hand, for a gathering of people, five of whom are men, what is the smallest number of women who must also be present at the meeting so that there are four among them, two men and two women, where each man is an acquaintance of each woman, or there are eight among them, four men and four women, where each man is a stranger of each woman. By Theorem 1, as $BR_5(K_{2,2}, K_{4,4}) = 26$, the required number of women to be present is 26.

2. Preparations

Suppose that $G = (V(G), E(G))$ is a graph. The degree of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$. For each $v \in V(G)$, $N_G(v) = \{u \in V(G) : vu \in E(G)\}$. The maximum and minimum degrees of $V(G)$ are defined by $\Delta(G)$ and $\delta(G)$, respectively. Let $G[X, Y]$ (or in short $[X, Y]$), be a bipartite graph with bipartition $X \cup Y$. Suppose that $E(G[X', Y'])$, in short $E[X', Y']$, denotes the edge set of

$G[X', Y']$. We use $\Delta(G_X)$ to denote the maximum degree of vertices in part X of G. Assume that G is a graph of size n, the complement of a graph G is denoted by $\overline{G}$, where $V(\overline{G}) = V(K_n)$ and $E(\overline{G}) = E(K_n) \setminus E(G)$. Now, assume that G is a subgraph of complete bipartite graph $K_{m,n}$, the complement of a graph G is denoted by $\overline{G}$, where $V(\overline{G}) = V(K_{m,n})$ and $E(\overline{G}) = E(K_{m,n}) \setminus E(G)$.

Let (G^1, G^2) be a 2-coloring of a graph G. Then every edge of G is in G^1 or G^2. For given graphs G, H_1, and H_2, we say G is 2-colorable to (H_1, H_2) if there exists a 2-edge decomposition of G, say (G^1, G^2), where $H_i \nsubseteq G^i$ for each $i = 1, 2$. We use $G \nrightarrow (H_1, H_2)$ to show that G is 2-colorable to (H_1, H_2).

Definition. The Zarankiewicz number $z((m_1, m_2), K_{n_1,n_2})$ is defined to be the maximum number of edges in any subgraph G of the complete bipartite graph K_{m_1,m_2}, such that G does not contain K_{n_1,n_2}.

Lemma 2 [7]. *The following results on $z((m, n), K_{t,t})$ are true.*

- $z((7, 14), K_{2,2}) \leq 31$.
- $z((7, 16), K_{2,2}) \leq 34$.
- $z((8, 14), K_{2,2}) \leq 35$.
- $z((8, 16), K_{2,2}) \leq 38$.
- $z((8, 16), K_{4,4}) \leq 90$.
- $z((9, 14), K_{2,2}) \leq 39$.
- $z((9, 14), K_{4,4}) \leq 88$.
- $z((10, 14), K_{2,2}) \leq 42$.
- $z((10, 14), K_{4,4}) \leq 97$.

Proof. By using the bounds in Table 5 of [7] and Table C.0 of [6], the proposition holds. ∎

Hattingh and Henning in [15] prove the next theorem.

Theorem 3 [15]. *$BR(K_{2,2}, K_{4,4}) = 14$.*

Lemma 4. *Assume that G is a subgraph of $K_{|X|,|Y|}$, where $|X| = m \geq 5$, and $|Y| = n \geq 8$. If $\Delta(G_X) \geq 8$, then either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $\Delta(G_X) = 8$ and $N_G(x) = Y'$, where $|Y'| = 8$. Suppose that $K_{2,2} \nsubseteq G$, hence $|N_G(x') \cap Y'| \leq 1$ for any $x' \in X \setminus \{x\}$. So, since $|X| \geq 5$ and $|Y'| = 8$, it is easy to check that $K_{4,4} \subseteq \overline{G}[X \setminus \{x\}, Y']$. ∎

Lemma 5. *Let G be a subgraph of $K_{|X|,|Y|}$, where $|X| = m \geq 9$, and $|Y| = n \geq 9$. If $\Delta(G_X) \geq 7$, then either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $N_G(x) = Y'$, where $|Y'| = 7$. Suppose that $K_{2,2} \nsubseteq G$, so $|N_G(x') \cap Y'| \leq 1$ for each $x' \in X \setminus \{x\}$. Since $|X| \geq 9$ and $|Y'| = 7$, if $|N_G(x') \cap Y'| = 0$ for some $x' \in X \setminus \{x\}$, then one can check that $K_{4,4} \subseteq \overline{G}[X \setminus \{x\}, Y']$. So suppose that $|N_G(x') \cap Y'| = 1$ for each $x' \in X \setminus \{x\}$. Therefore as $|X| \geq 9$ and $|Y'| = 7$ by pigeon-hole principle, it is easy to say that there exist two vertices of $X \setminus \{x\}$, say x', x'', such that $N_G(x') \cap Y' = N_G(x'') \cap Y'$. Let $N_G(x') \cap Y' = N_G(x'') \cap Y' = \{y'\}$. Hence one can check that $K_{4,4} \subseteq \overline{G}[X \setminus \{x\}, Y' \setminus \{y'\}]$. Hence lemma holds. ■

3. Proof of the Main Results

In this section, we prove Theorem 1. To simplify the comprehension, let us split the proof of Theorem 1 into small parts. We begin with the following result.

Theorem 6. *$BR_5(K_{2,2}, K_{4,4}) = 26$.*

Proof. Suppose that $(X = \{x_1, x_2, \ldots, x_5\}, Y = \{y_1, y_2, \ldots, y_{25}\})$ are the partition sets of $K = K_{5,25}$ and G is a subgraph of K such that for each $x \in X$, $N_G(x)$ is defined as follows.

- **(A1):** $N_G(x_1) = \{y_1, y_2, \ldots, y_7\}$.
- **(A2):** $N_G(x_2) = \{y_1, y_8, y_9, \ldots, y_{13}\}$.
- **(A3):** $N_G(x_3) = \{y_2, y_8, y_{14}, y_{15}, \ldots, y_{18}\}$.
- **(A4):** $N_G(x_4) = \{y_3, y_9, y_{14}, y_{19}, \ldots, y_{22}\}$.
- **(A5):** $N_G(x_5) = \{y_4, y_{10}, y_{15}, y_{19}, y_{23}, y_{24}, y_{25}\}$.

Now, by (Ai) and (Aj), it can be said that $|N_G(x_i) \cap N_G(x_j)| = 1$ for each $i, j \in \{1, 2, \ldots, 5\}$, which means that $K_{2,2} \nsubseteq G$. Also, for each $i \in \{1, 2, \ldots, 5\}$, it is easy to check that $\left|\bigcup_{j=1, j\neq i}^{j=5} N_G(x_j)\right| = 22$. Therefore, $K_{4,4} \nsubseteq \overline{G}[X \setminus \{x_i\}, Y]$ for each $x_i \in X$, which means that $BR_5(K_{2,2}, K_{4,4}) \geq 26$.

Now assume that $(X = \{x_1, x_2, \ldots, x_5\}$, $Y = \{y_1, \ldots, y_{26}\})$ are the partition sets of $K_{5,26}$ and let G be a subgraph of $K_{5,26}$ such that $K_{2,2} \nsubseteq G$. Consider $\Delta = \Delta(G_X)$. Since $K_{2,2} \nsubseteq G$, if $\Delta \geq 8$, then by Lemma 4, the proof is complete. Also, since $|Y| = 26$, if $\Delta \leq 5$, then it is easy to check that $K_{4,4} \subseteq \overline{G}$. Therefore let $\Delta \in \{6, 7\}$. Now we verify the next claim.

Claim 7. *If $\Delta = 7$, then we have $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $\Delta = \deg_G(x_1)$ and $N_G(x_1) = Y_1 = \{y_1, \ldots, y_7\}$. Since $K_{2,2} \nsubseteq G$ we have $|N_G(x_i) \cap Y_1| \leq 1$. Also as $|Y_1| = 7$, we have $|N_G(x_i) \cap Y_1| = 1$ and $N_G(x_i) \cap Y_1 \neq N_G(x_j) \cap Y_1$ for any $i, j \in \{2, 3, 4, 5\}$. Otherwise in any case one can checked that $K_{4,4} \subseteq \overline{G}[X, Y_1]$. Suppose that there exists a vertex of $X \setminus \{x_1\}$, so that $|N_G(x)| = 6$, and without loss of generality, suppose that $x = x_2$ and $N_G(x_2) = Y_2 = \{y_1, y_8, \ldots, y_{12}\}$. Therefore, since $K_{2,2} \nsubseteq G$, we have $|N_G(x_i) \cap (Y_2 \setminus \{y_1\})| \leq 1$, also one can checked that $|N_G(x_i) \cap (Y_2 \setminus \{y_1\})| = 1$ for at least two $i \in \{3, 4, 5\}$, otherwise $K_{4,4} \subseteq \overline{G}[X \setminus \{x_2\}, Y_2 \setminus \{y_1\}]$. Without loss of generality, assume that $|N_G(x_i) \cap (Y_2 \setminus \{y_1\})| = 1$ for $i = 3, 4$. So, as $\Delta = 7$, $|N_G(x_i) \cap (Y_1 \cup Y_2)| = 2$, one can say that $|N_G(x_i) \cap Y \setminus (Y_1 \cup Y_2)| \leq 5$, therefore $\left|\bigcup_{j=1}^{j=4} N_G(x_j)\right| \leq 22$, which means that $K_{4,4} \subseteq \overline{G}$.

Now let $|N_G(x)| = 7$ for each $x \in X$. Suppose that $N_G(x_i) = Y_i$ for each $i \in \{1, 2, \ldots, 5\}$. Hence, since $K_{2,2} \nsubseteq G$, one can say that $|N_G(x_i) \cap Y_j| = 1$ and $N_G(x_i) \cap Y_j \neq N_G(x_l) \cap Y_j$ for each $i, l, j \in \{1, 2, 3, 4, 5\}$ where $i \neq j \neq l \neq i$, otherwise $K_{4,4} \subseteq \overline{G}[X, Y]$. Therefore, it is easy to say that $\left|\bigcup_{j=1, j\neq i}^{j=5} N_G(x_j)\right| = 22$. Hence as $|Y| = 26$, then $K_{4,4} \subseteq \overline{G}[X \setminus \{x\}, Y]$ for each $x \in X$. So, the claim holds. □

Hence, we may assume that $\Delta = 6$. Without loss of generality, let $N_G(x_1) = Y_1$. Since $K_{2,2} \nsubseteq G$, thus $|N_G(x_i) \cap Y_1| \leq 1$ for each $i \in \{2, 3, 4, 5\}$. Also, we may suppose that $|N_G(x_i) \cap Y_1| = 1$ for at least three vertices of $X \setminus \{x_1\}$, otherwise $K_{4,4} \subseteq \overline{G}[X, Y_1]$. Now, without loss of generality, let $X_1 = \{x \in X \setminus \{x_1\}, |N_G(x_i) \cap Y_1| = 1\}$. As $|Y| = 26$, $|X_1| \geq 3$, and $\Delta = 6$, one can check that $\left|\bigcup_{x \in X_1 \cup \{x_1\}} N_G(x)\right| \leq 21$, which means that $K_{4,4} \subseteq \overline{G}$. Hence we have $BR_5(K_{2,2}, K_{4,4}) = 26$. ■

In the following theorem, we compute the size of $BR_m(K_{2,2}, K_{4,4})$ for $m = 6, 7$.

Theorem 8. $BR_6(K_{2,2}, K_{4,4}) = BR_7(K_{2,2}, K_{4,4}) = 22$.

Proof. It suffices to show the following.

- **(I):** $K_{7,21} \nrightarrow (K_{2,2}, K_{4,4})$.
- **(II):** $BR_6(K_{2,2}, K_{4,4}) \leq 22$.

We begin with (I). Let $(X = \{x_1, \ldots, x_7\}, Y = \{y_1, \ldots, y_{21}\})$ be the partition sets of $K = K_{7,21}$ and let G be a subgraph of K such that for each $x \in X$ we define $N_G(x)$ as follows.

- **(D1):** $N_G(x_1) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$.
- **(D2):** $N_G(x_2) = \{y_1, y_7, y_8, y_9, y_{10}, y_{11}\}$.
- **(D3):** $N_G(x_3) = \{y_2, y_7, y_{12}, y_{13}, y_{14}, y_{15}\}$.
- **(D4):** $N_G(x_4) = \{y_3, y_8, y_{12}, y_{16}, y_{17}, y_{18}\}$.
- **(D5):** $N_G(x_5) = \{y_4, y_9, y_{13}, y_{16}, y_{19}, y_{20}\}$.
- **(D6):** $N_G(x_6) = \{y_5, y_{10}, y_{14}, y_{17}, y_{19}, y_{21}\}$.
- **(D7):** $N_G(x_7) = \{y_6, y_{11}, y_{15}, y_{18}, y_{20}, y_{21}\}$.

Now, for each $i, j \in \{1, 2, \ldots, 7\}$ by (Di) and (Dj) it can be said as follows.

- **(E1):** $|N_G(x_i) \cap N_G(x_j)| = 1$ for each $i, j \in \{1, 2, \ldots, 7\}$.
- **(E2):** $\left|\bigcup_{i=1}^{i=4} N_G(x_{j_i})\right| = 18$ for each $j_1, j_2, j_3, j_4 \in \{1, \ldots, 7\}$.

By (E1), one can say that $K_{2,2} \nsubseteq G$. Also by (E2), it is easy to say that $K_{4,4} \nsubseteq \overline{G}$ which means that $K_{7,21} \nrightarrow (K_{2,2}, K_{4,4})$, that is the part (I) is correct.

Now we show that (II) is established, that is, we show that for any subgraph of $K_{6,22}$, say G, either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$. Let $(X = \{x_1, x_2, \ldots, x_6\}, Y = \{y_1, y_2, \ldots, y_{22}\})$ be the partition sets of $K = K_{6,22}$ and G be a subgraph of K, where $K_{2,2} \nsubseteq G$. We show that $K_{4,4} \subseteq \overline{G}$. Consider $\Delta = \Delta(G_X)$. Since $K_{2,2} \nsubseteq G$, by Lemma 4, $\Delta \leq 7$. Now we verify the following claim.

Claim 9. *If $\Delta = 7$, then $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $|N_G(x_1) = Y_1| = 7$. Since $K_{2,2} \nsubseteq G$, for each $i, j \in \{2, \ldots, 6\}$, we have $|N_G(x_i) \cap Y_1| \leq 1$ and $N_G(x_i) \cap Y_1 \neq N_G(x_j) \cap Y_1$, otherwise since $|Y_1| = 7$, it is easy to check that $K_{4,4} \subseteq \overline{G}[X, Y_1]$. Therefore as $|X| = 6$ and $|Y_1| = 7$, for each $x \neq x_1$ it can be said that $K_{4,3} \subseteq \overline{G}[X \setminus \{x_1, x\}, Y_1]$. So, as $|Y| = 22$, if there exists a vertex of $Y \setminus Y_1$ so that $|N_{\overline{G}}(y) \cap (X \setminus \{x_1\})| \geq 4$, then $K_{4,4} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1 \cup \{y\}]$. Hence assume that $|N_G(y) \cap (X \setminus \{x_1\})| \geq 2$ for each $y \in Y \setminus Y_1$, that is $|E(G[X, Y \setminus Y_1])| \geq 30$. Therefore by pigeon-hole principle, one can check that there exists at least one vertex of $X \setminus \{x_1\}$, say x_2, such that $|N_G(x_2) \cap (Y \setminus Y_1)| \geq 5$. Suppose that $N_G(x_2) \cap (Y \setminus Y_1) = Y_2$. So as $K_{2,2} \nsubseteq G$, $|N_G(x_i) \cap Y_2| \leq 1$ for each $i \in \{3, 4, 5, 6\}$. Therefore as $|Y_2| \geq 5$, we have $K_{4,1} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_2]$. Hence as $K_{4,3} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1]$, it can be said that, $K_{4,4} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup Y_2]$. Hence the claim holds. □

Therefore, by Claim 9 assume that $\Delta \leq 6$. Let $\Delta = 5$ and without loss of generality, suppose that $\Delta = |N_G(x_1)|$. Since $K_{2,2} \nsubseteq G$, thus $|N_G(x_i) \cap N_G(x_1)| = 1$ for at least three vertices of $X \setminus \{x_1\}$, say x_2, x_3, x_4, otherwise $K_{4,4} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1]$. Hence, as $\Delta = 5$, one can check that $\left|\bigcup_{i=1}^{i=4} Y_i\right| \leq 17$, where $Y_i = N_G(x_i)$. So, since $|Y| = 22$, we have $K_{4,4} \subseteq \overline{G}$.

So let $\Delta = 6$. Without loss of generality, let $\Delta = \deg_G(x_1)$ and $N_G(x_1) = Y_1$. Since $K_{2,2} \nsubseteq G$, $|N_G(x_i) \cap Y_1| \leq 1$ for each $i \in \{2, 3, 4, 5, 6\}$. Now we verify the following claim.

Claim 10. *$|N_G(x_i) \cap Y_1| = 1$ and $N_G(x_i) \cap Y_1 \neq N_G(x_j) \cap Y_1$ for each $i, j \in \{2, 3, 4, 5, 6\}$.*

Proof. By contradiction, without loss of generality, assume that $|N_G(x_2) \cap Y_1| = 0$. Therefore, $|N_G(x_i) \cap Y_1| = 1$ and $N_G(x_i) \cap Y_i \neq N_G(x_j) \cap Y_i$ for each $i, j \in \{3, 4, 5, 6\}$, otherwise $K_{4,4} \subseteq \overline{G}[X, Y_1]$. Now as $|X| = |Y_1| = 6$, it can be said that $K_{4,2} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1]$. If $|N_G(x_2)| = 6$, then $K_{4,2} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, N_G(x_2)]$, hence $K_{4,4} \subseteq \overline{G}[X \setminus \{x_1, x_2\}, Y_1 \cup N_G(x_2)]$. So suppose that $|N_G(x_2)| \leq 5$. If $|N_G(x_2)| \leq 3$, then as $|Y| = 22$, it is clear that $K_{4,4} \subseteq \overline{G}$. Hence we may suppose that $4 \leq |N_G(x_2)| \leq 5$. If $|N_G(x_2)| = 4$, then there exist at least two vertices of $X \setminus \{x_1, x_2\}$, say x', x'', such that $|N_G(x_2) \cap N_G(x)| = 1$ for each $x \in \{x', x''\}$, otherwise $K_{4,4} \subseteq \overline{G}[X \setminus \{x_2\}, N_G(x_2)]$. Without loss of generality, assume that $x' = x_3, x'' = x_4$. Hence as $\Delta = 6$, $|N_G(x_2)| = 4$, and $|N_G(x_i) \cap N_G(x_j)| = 1$ for each $i \in \{1, 2\}$ and $j \in \{3, 4\}$, one can check that $\left|\bigcup_{j=1}^{j=4} N_G(x_j)\right| \leq 18$, which means that $K_{4,4} \subseteq \overline{G}$. So suppose that $|N_G(x_2)| = 5$, hence for at least three vertices of $X \setminus \{x_1, x_2\}$, say $\{x_3, x_4, x_5\}$, we have $|N_G(x_2) \cap N_G(x_i)| = 1$, otherwise $K_{4,4} \subseteq \overline{G}$. If $|N_G(x_i)| \leq 5$ for at least one vertex of $\{x_3, x_4, x_5\}$, then as $\Delta = 6$, $|N_G(x_2)| = 5$, and $|N_G(x_i) \cap N_G(x_j)| = 1$ for each $i \in \{1, 2\}$ and each $j \in \{3, 4\}$, one can say that $\left|\bigcup_{j=1}^{j=4} N_G(x_j)\right| \leq 18$, which means that $K_{4,4} \subseteq \overline{G}$. So suppose that $|N_G(x_i)| = 6$, for each $x \in \{x_3, x_4, x_5\}$. Consider $N_G(x_3)$, hence there is at least one vertex of $\{x_4, x_5\}$, say x, such that $|N_G(x_3) \cap N_G(x_2) \cap (Y \setminus (Y_1 \cup Y_2))| = 1$ where $Y_i = N_G(y_i)$, otherwise $K_{4,4} \subseteq \overline{G}$. Therefore $|N_G(x_3) \cap (Y \setminus (Y_1 \cup Y_2 \cup Y_3))| = 3$, which means that $\left|\bigcup_{j=1}^{j=4} N_G(x_j)\right| = 18$. Therefore, it is easy to say that $K_{4,4} \subseteq \overline{G}[X \setminus \{x_5, x_6\}, (Y \setminus (Y_1 \cup Y_2 \cup Y_3 \cup Y_4))]$. For the case that there exist $i, j \in \{2, 3, \ldots, 6\}$ such that $N_G(x_i) \cap Y_1 = N_G(x_j) \cap Y_1$ the proof is the same. □

Hence by Claim 10, without loss of generality, we may suppose that $Y_1 = \{y_1, \ldots, y_6\}$ and let $x_i y_{i-1} \in E(G)$ for $i = 2, \ldots, 6$. Now we verify the following claim.

Claim 11. $|N_G(x_i)| = 6 = \Delta$ *for each* $i \in \{2, 3, 4, 5, 6\}$.

Proof. By contradiction, let $|N_G(x_2)| \leq 5$, that is $|N_G(x_2) \cap (Y \setminus Y_1)| \leq 4$. Without loss of generality, assume that $N_G(x_2) \cap (Y \setminus Y_1) = Y_2$. Since $|X| = 6$, one can say that there is at least two vertices of $X \setminus \{x_1, x_2\}$, say x_3, x_4, such that $|N_G(x_i) \cap Y_2| = 1$, for $i = 2, 3$, otherwise $K_{4,4} \subseteq \overline{G}[X \setminus \{x_2\}, Y_2]$. Hence as $\Delta = 6$, one can check that $\left|\bigcup_{i=1}^{i=4} Y_i\right| \leq 18$, where $Y_i = N_G(x_i)$. So as and $|Y| = 22$, we have $K_{4,4} \subseteq \overline{G}\left[\{x_1, x_2, x_3, x_4\}, Y \setminus \bigcup_{i=1}^{i=4} Y_i\right]$. □

Therefore by Claims 10 and 11, it can be said that $\left|\bigcup_{i=1}^{i=6} Y_i\right| = 21$, that is there exists one vertex of Y, say y_{22}, such that $K_{6,1} \subseteq \overline{G}[X, \{y_{22}\}]$. Suppose that $Y_i = N_G(x_i)$ for $i = 1, 2$, and without loss of generality, assume that $Y_1 = \{y_1, \ldots, y_6\}$ and $Y_2 = \{y_1, y_7, \ldots y_{11}\}$. Hence by Claims 10 and 11, it is easy to say that there exists one vertex of $Y_1 \setminus \{y_1\}$, say y', and one vertex of $Y_2 \setminus \{y_1\}$ say y'' such that $K_{4,2} \subseteq \overline{G}[\{x_3, \ldots, x_6\}, \{y', y''\}]$. Hence, one can check that $K_{4,4} \subseteq \overline{G}[\{x_3, \ldots, x_6\}, \{y_1, y', y'', y_{22}\}]$.

Thus, for any subgraph of $K_{6,22}$, say G, either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$. Hence, $BR_6(K_{2,2}, K_{4,4}) \leq 22$ and so by (I), $BR_6(K_{2,2}, K_{4,4}) = 22$. This also implies that for any subgraph of $K_{7,22}$, say G, either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$. Therefore, by (I), $BR_7(K_{2,2}, K_{4,4}) = 22$. Hence the theorem holds. ■

Let G be a subgraph of $K_{|X|,|Y|} = K_{m,n}$ where $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ are the partition sets of $K_{|X|,|Y|}$. For any subgraph G of $K_{m,n}$, suppose that $A[G] = [a_{ij}]$ is an $m \times n$ matrix, where for each $i \in [m]$ and each $j \in [n]$, $a_{ij} = 1$ if the edges $x_iy_j \in E(G)$, and $a_{ij} = 0$ if the edges $x_iy_j \in E(\overline{G})$. The matrix $A[G] = [a_{ij}]$ represents a 2-colored $K_{m,n}$. In the next two theorems, we find the value of $BR_8(K_{2,2}, K_{4,4})$, by considering a particular 2-colored of $K_{8,15}$ and a 2-colored of $K_{8,16}$.

Theorem 12. $K_{8,15} \nrightarrow (K_{2,2}, K_{4,4})$.

Proof. Let G be a subgraph of $K_{8,15}$ such that $A[G]$ is shown in the following matrix

$$A[G] = A_{8\times 15} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Set $X_1 = \{x_1, x_2, x_3, x_4\}$ and $X_2 = X \setminus X_1$. Therefore by matrix $A[G]$, it can be said that the following items are true.

- **(P1):** $|N_G(x_i) \cap N_G(x_j)| = 1$ for each $i, j \in \{1, \dots, 8\}$.
- **(P2):** $|N_G(x_i)| = 4$ for each $x \in X_1$.
- **(P3):** $|N_G(x_i)| = 5$ for each $x \in X_2$.
- **(P4):** $\left|\bigcup_{i=1}^{i=4} N_G(x_i)\right| = 13$.
- **(P5):** $\left|\bigcup_{i=5}^{i=8} N_G(x_i)\right| = 14$.

Therefore by (P1), we have G is $K_{2,2}$-free ($K_{2,2} \nsubseteq G$), also by (P4) and (P6), for $i = 1, 2$, it can be said that $K_{4,4} \nsubseteq \overline{G}[X_i, Y]$. Also by $A[G]$, it can be said that the following items are true.

- **(M1):** $|N_G(x_i) \cup N_G(x_j)| = 7$ for each $x_i, x_j \in X_1$.
- **(M2):** $|N_G(x_i) \cup N_G(x_j) \cup N_G(x_l)| = 10$ for each $x_i, x_j, x_l \in X_1$.

- **(M3):** $|N_G(x_i) \cup N_G(x_j)| = 9$ for each $x_i, x_j \in X_2$.
- **(M4):** $|N_G(x_i) \cup N_G(x_j) \cup N_G(x_l)| = 12$ for each $x_i, x_j, x_l \in X_2$.

Now we verify the following claim.

Claim 13. *$\overline{G}$ is $K_{4,4}$-free.*

Proof. By contradiction, suppose that K is a copy of $K_{4,4}$ in $\overline{G}$, also assume that $V(K) \cap X = \{w_1, w_2, w_3, w_4\} = W$ and $V(K) \cap Y = \{w'_1, w'_2, w'_3, w'_4\} = W'$. Since $K_{4,4} \subseteq \overline{G}[W, W']$, by (P4) and (M4), one can say that $1 \leq |W \cap X_2| \leq 2$. Assume that $|W \cap X_i| = 2$. Without loss of generality, let $w_1, w_2 \in X_1$ and $w_3, w_4 \in X_2$. Hence, by (M1), we have $|N_G(w_1) \cup N_G(w_2)| = 7$. Now consider w_3, w_4, as $|N_G(w_i) \cap N_G(w_j)| = 1$ for each $i, j \in \{1, \ldots, 8\}$, and $y_1 \notin N_G(x)$ for each $x \in X_2$ one can say that $|N_G(w_1) \cup N_G(w_2) \cup N_G(w_i)| = 10$ for each $i \in \{3, 4\}$. Also, as $|N_G(w_i) \cap N_G(w_j)| = 1$, it is easy to check that $|N_G(w_1) \cup N_G(w_2) \cup N_G(w_3) \cup N_G(w_4)| \geq 12$, which means that $K_{4,4} \nsubseteq \overline{G}[W, Y]$, a contradiction. So assume that $|W \cap X_1| = 3$, and without loss of generality, let $w_4 \in X_2$. Hence, by (P1), (P3), and (M2), one can say that $|N_G(w_1) \cup N_G(w_2) \cup N_G(w_3) \cup N_G(w_4)| = 12$, which means that $K_{4,4} \nsubseteq \overline{G}[W, Y]$, a contradiction again. Hence the claim holds. □

Therefore by (P1) and by Claim 13, we have the proof of the theorem is complete. ■

To prove the next theorem, we need to establish the following lemma.

Lemma 14 [7]**.** *The following results on $z((m, n), K_{t,t})$ are true.*

- $z((5, 6), K_{2,2}) \leq 14$.
- $z((6, 9), K_{2,2}) \leq 21$.
- $z((6, 12), K_{2,2}) \leq 25$.
- $z((7, 9), K_{2,2}) \leq 24$.
- $z((7, 12), K_{2,2}) \leq 28$.
- $z((7, 16), K_{2,2}) \leq 34$.

Proof. By using the bounds in Table 5 of [7] and Table C.4 of [6], the proposition holds. ■

Theorem 15. $BR_8(K_{2,2}, K_{4,4}) = 16$.

Proof. Let G be any subgraph of $K_{8,16}$. Since $|E(K_{8,16})| = 128$, we may assume that $|E(G)| = 38$ and $|E(\overline{G})| = 90$, otherwise by Lemma 2 as $z((8, 16), K_{2,2}) \leq 38$ and $z((8, 16), K_{4,4}) \leq 90$, it can be said that either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$, that is the proof is complete. Now let $|E(G)| = 38$, $|E(\overline{G})| = 90$ and without loss of generality, assume that $K_{2,2} \nsubseteq G$. Since $z((7,16),K_{2,2}) \leq 34$, if there exists a vertex of G, say x, such that $|N_G(x)| \leq 3$, then it is easy to say that $|E(G \setminus \{x\})| \geq 35$. Therefore, we have $K_{2,2} \subseteq G$, a contradiction. Hence, $\delta(G) = 4$. Define X' as follows

$$X' = \{x \in X : \deg_G(x) = 4\}.$$

Since $|E(G)| = 38$ and $|X| = 8$, it is clear that $|X'| \geq 2$. By considering X', we verify the following claim.

Claim 16. *If there exist two members of X', say x, x', such that $|N_G(x) \cap N_G(x')| = 1$, then $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $x = x_1, x' = x_2$, $N_G(x_1) = Y_1 = \{y_1, y_2, y_3, y_4\}$ and $N_G(x_2) = Y_2 = \{y_1, y_5, y_6, y_7\}$. Now set $Y' = Y_1 \cup Y_2$. Hence we have the following fact.

- **(F1):** There is at least one member of $X_1 = X \setminus \{x_1, x_2\}$, say x, such that $|N_G(x) \cap Y'| = 2$.

To prove (F1), by contradiction, let $|N_G(x) \cap Y'| \leq 1$ for each $x \in X_1$. Set $X_1 = X \setminus \{x_1, x_2\}$ and $Y' = Y \setminus (Y_1 \cup Y_2)$, therefore we have $|E(G[X_1, Y'])| \geq 25$. Hence by Lemma 14, as $z((6,9),K_{2,2}) \leq 24$, we have $K_{2,2} \subseteq G$, a contradiction. Hence the fact (F1) is correct.

Therefore, by (F1) without loss of generality, let $|N_G(x_3) \cap Y'| = 2$ and $Y_3 = \{y_2, y_5, y_8, y_9\} \subseteq N_G(x_3)$. If $x_3 \in X'$, then by an argument similar to the proof of (F1), one can check that there exists at least one member of $X \setminus \{x_1, x_2, x_3\}$, say x, such that $|N_G(x)| \leq 5$ and $|N_G(x) \cap (Y' \cup Y_3)| \geq 2$. Therefore, $|N_G(x) \cap Y \setminus (Y' \cup Y_3)| \leq 3$, hence it is easy to say that $K_{4,4} \subseteq \overline{G}[\{x_1, x_2, x_3, x\}, Y \setminus (Y' \cup Y_3)]$. So, suppose that $|N_G(x_3)| \geq 5$. Now we verify the following two cases.

Case 1. $|N_G(x_3)| = 5$. In this case, we have the following fact.

- **(F2):** There exists one member of $X \setminus \{x_1, x_2, x_3\}$ say x, such that $|N_G(x)| = 5$ and $|N_G(x) \cap (Y' \cup Y_3)| = 3$.

To prove (F2), by contradiction, let $|N_G(x) \cap (Y' \cup Y_3)| \leq 2$ for each $X \setminus \{x_1, x_2, x_3\}$. Set $X_1 = X \setminus \{x_1, x_2, x_3\}$ and $Y'' = Y \setminus (Y' \cup Y_3)$, therefore $|E(G[X_1, Y''])| \geq 15$. Hence by Lemma 14, as $z((5,6),K_{2,2}) \leq 14$, we have $K_{2,2} \subseteq G$, a contradiction. Hence the fact (F2) is true.

Therefore, by (F2) without loss of generality, let $|N_G(x_4) \cap (Y' \cup Y_3)| = 3$ also assume that $Y_4 = \{y', y'', y'''\} \subseteq N_G(x_4) \cap (Y' \cup Y_3)$. If $|N_G(x_4)| \leq 5$, then it is easy to say that $K_{4,4} \subseteq \overline{G}[\{x_1, x_2, x_3, x_4\}, Y \setminus (Y' \cup Y_3 \cup N_G(x_4))]$. So suppose that $|N_G(x_4)| = 6$ and assume that $Y_4 = \{y', y'', y''', y_{11}, y_{12}, y_{13}\} \subseteq N_G(x_4)$. Now set $X'' = \{x_5, x_6, x_7, x_8\}$ and $Y''' = \{y_{14}, y_{15}, y_{16}\}$, therefore one can check that $|N_G(x) \cap Y'''| \geq 2$ for each $x \in X''$. Otherwise, if there is a vertex of X'', say x'', such that $|N_G(x'') \cap Y'''| \leq 1$, then it is easy to say that either $|N_G(x'')| = 5$ and $|N_G(x'') \cap (Y' \cup Y_3)| = 3$ or $|N_G(x'')| = 4$ and $|N_G(x'') \cap (Y' \cup Y_3)| = 2$, in any case the fact (F2) is true by setting $x = x''$. Therefore as $|X_2| = 4$, $|Y'''| = 3$ and $|N_G(x) \cap Y'''| \geq 2$ for each $x \in X''$, it is easy to checked that $K_{2,2} \subseteq G[X'', Y''']$, a contradiction.

Case 2. $|N_G(x_3)| = 6$. Without loss of generality, let $N_G(x_3) = Y_3 = \{y_2, y_5, y_8, y_9, y_{10}, y_{11}\}$. Now set $X'' = \{x_4, x_5, x_6, x_7, x_8\}$ and $Y''' = \{y_{12}, \ldots, y_{16}\}$. If there exists a vertex of X'', say x, such that $|N_G(x) \cap Y'''| \leq 2$, then it is easy to say that either $|N_G(x)| = 5$ and $|N_G(x) \cap Y'| = 2$ or $|N_G(x)| = 4$ and $|N_G(x) \cap Y'| \geq 1$, in any case the proof is complete by Case 1. Therefore, let $|N_G(x) \cap Y'''| \geq 3$ for at lest three vertices of X''. Now as $|Y'''| = 5$, its easy to say that $K_{2,2} \subseteq G[X'', Y''']$, a contradiction.

Therefore by Cases 1 and 2, the proof of the claim is complete. □

Now by Claim 16, we have the following claim.

Claim 17. $\Delta = 5$.

Proof. By contradiction, suppose that $\Delta = 6$, therefore $|X'| \geq 3$. Without loss of generality, let $|N_G(x_1) = Y_1| = \Delta$ and $x_2, x_3, x_4 \in X'$. Since $K_{2,2} \nsubseteq G$, $|N_G(x) \cap Y_1| \leq 1$ for each $x \in X \setminus \{x_1\}$. Also by Claim 16, we have $|N_G(x) \cap N_G(x')| = 0$ for each $x, x' \in X'$. So, without loss of generality, let $N_G(x_2) = Y_2 = \{y_1, y_2, y_3, y_4\}$, $N_G(x_3) = Y_3 = \{y_5, y_6, y_7, y_8\}$ and $N_G(x_4) = Y_4 = \{y_9, y_{10}, y_{11}, y_{12}\}$. Set $W = Y \setminus (Y_1 \cup Y_2 \cup Y_3) = \{y_{13}, y_{14}, y_{15}, y_{16}\}$. As $N_G(x_1) = Y_1$ and $K_{2,2} \nsubseteq G$, we have $|Y_1 \cap W| \geq 3$. Without loss of generality, let $W' = \{y_{13}, y_{14}, y_{15}\} \subseteq Y_1 \cap W$. Also as $|N_G(x) \cap W| \geq 2$ for each $x \in X \setminus \{x_1, \ldots, x_4\}$, then it is easy to check that $K_{2,2} \subseteq G[\{x_5, x_6, \ldots, x_8\}, W]$, a contradiction. Hence the claim holds. □

Now by Claim 17, and as $|E(G)| = 38$, we have $|X'| = 2$. Without loss of generality, let $X' = \{x_1, x_2\}$. Also by Claim 16, without loss of generality, suppose that $N_G(x_1) = Y_1 = \{y_1, y_2, y_3, y_4\}$, $N_G(x_2) = Y_2 = \{y_5, y_6, y_7, y_8\}$. Set $X_1 = X \setminus \{x_1, x_2\}$. Now we verify the next claim.

Claim 18. *For each $x \in X_1$ and $i = 1, 2$, we have $|N_G(x) \cap Y_i| = 1$.*

Proof. By contradiction, without loss of generality, suppose that $|N_G(x_3) \cap Y_1| = 0$. Since $|X| = 8$, and $|N_G(x)| = 5$ for each $x \in X_1$, either if $|N_G(x_3) \cap Y_2| = 0$ or there exists a vertex of $X_1 \setminus \{x_3\}$, say x', such that $|N_G(x') \cap Y_1| = 0$, then it is easy to say that $K_{2,2} \subseteq G[X_1, Y \setminus (Y_1 \cup Y_2)]$, a contradiction. So let $|N_G(x_3) \cap Y_1| = 0$, $|N_G(x_3) \cap Y_2| = 1$ and without loss of generality, let $N_G(x_3) = Y_3 = \{y_5, y_9, y_{10}, y_{11}, y_{12}\}$. Set $X' = X \setminus \{x_1\}$ and $Y' = Y \setminus Y_1$. Since $K_{2,2} \nsubseteq G$, we have $|N_G(x) \cap Y_1| \leq 1$ for each $x \in X' \setminus \{x_3\}$ and as $|N_G(x_2) \cap Y_1| = 0$, we have $|E(G[X, Y_1])| \leq 9$. Therefore as $|N_G(x_1) = Y_1| = 4$, we have $|E(G[X', Y'])| \geq 29$. Hence by Lemma 14 as $z((7, 12), K_{2,2}) \leq 28$, we have $K_{2,2} \subseteq G$, a contradiction. Hence the claim holds. □

Therefore by Claim 18, we have $|N_G(x) \cap Y_i| = 1$ for each $x \in X_1$ and each $i = 1, 2$. If there is a vertex of $Y_1 \cup Y_2$, say y, so that $|N_G(y) \cap X_1| \geq 3$, then as $|Y \setminus (Y_1 \cup Y_2)| = 8$ and $|N_G(x) \cap Y \setminus (Y_1 \cup Y_2)| = 3$, by pigeon-hole principle, one can check that $K_{2,2} \subseteq G$, a contradiction. Therefore we may suppose that $|N_G(y) \cap X_1| \leq 2$ for each $y \in Y_1 \cup Y_2$. Therefore as $|X_1| = 6$ and $|Y_1| = 4$ there exist two member of Y_1, say y, y', such that $|N_G(y) \cap X_1| = |N_G(y') \cap X_1| = 1$. Without loss of generality, let $y = y_1$ and $y' = y_2$. By symmetry, as $|X_1| = 6$ and $|Y_2| = 4$ there exist two member of Y_2, say y'', y''', such that $|N_G(y'') \cap X_1| = |N_G(y''') \cap X_1| = 1$. Without loss of generality, let $y'' = y_5$ and $y''' = y_6$. Now set $X' = X \setminus \{x_1, x_2\}$ and $Y' = Y \setminus \{y_1, y_2, y_5, y_6\}$. Since $|N_G(x_i)| = 4$ for each $i = 1, 2$, $y_1, y_2 \in N_G(x_1)$, $y_5, y_6 \in N_G(x_2)$ and as $|N_G(y) \cap X_1| = 1$ for each $y \in \{y_1, y_2, y_5, y_6\}$, one can say that $|E(G[X, \{y_1, y_2, y_5, y_6\}])| = 8$ and $|E(G[\{x_1, x_2\}, Y'])| = 4$, therefore as $|E(G)| = 38$ one can said that $|E(G[X', Y']| = 26$. Hence by Lemma 14, as $z((6, 12), K_{2,2}) \leq 25$, we have $K_{2,2} \subseteq G$, a contradiction.

Hence we have the assumption that $|E(G)| = 38$ and $|E(\overline{G})| = 90$ does not hold. Therefore by Lemma 2, as $z((8, 16), K_{2,2}) \leq 38$ and $z((8, 16), K_{4,4}) \leq 90$, it can be said that either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$, that is $BR_8(K_{2,2}, K_{4,4}) \leq 16$. Therefore, by Theorem 12, we have $BR_8(K_{2,2}, K_{4,4}) = 16$, and the proof is complete. ■

In the following theorem, we find the value of $BR_9(K_{2,2}, K_{4,4})$.

Theorem 19. *$BR_9(K_{2,2}, K_{4,4}) = 14$.*

Proof. Consider $K = K_{9,14}$, let $(X = \{x_1, \ldots, x_9\}, Y = \{y_1, y_2, \ldots, y_{14}\})$ be the partition sets of K, and let G be a subgraph of K such that $K_{2,2} \nsubseteq G$. Consider $\Delta = \Delta(G_X)$. By Lemma 5, $\Delta \leq 6$. Now, we verify the following claim.

Claim 20. *If $\Delta = 6$, then $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $|N_G(x_1) = Y_1| = 6$. Since $K_{2,2} \nsubseteq G$, $|N_G(x) \cap Y_1| \leq 1$ for each $x \in X \setminus \{x_1\}$. Therefore, as $|X| = 9$ and $|Y_1| = 6$, if there is a vertex of $X \setminus \{x_1\}$, say x, or there is a vertex of Y_1, say y, such that either $|N_G(x) \cap Y_1| = 0$ or $|N_G(y) \cap (X \setminus \{x_1\})| \geq 3$, then by pigeon-hole principle, one can check that $K_{4,4} \subseteq \overline{G}[X \setminus \{x_1\}, Y_1]$. Hence, let $|N_G(x) \cap Y_1| = 1$ for each $x \in X \setminus \{x_1\}$ and $|N_G(y) \cap (X \setminus \{x_1\})| \leq 2$ for each $y \in Y_1$. Now, as $|X| = 9$ and $|Y_1| = 6$, there exist two member of Y_1, say y_1, y_2, so that $|N_G(y_i) \cap (X \setminus \{x_1\})| = 2$. Without loss of generality, let $N_G(y_1) \cap (X \setminus \{x_1\}) = \{x_2, x_3\}$ and $N_G(y_2) \cap (X \setminus \{x_1\}) = \{x_4, x_5\}$. Hence, $K_{4,4} \subseteq \overline{G}[\{x_2, x_3, x_4, x_5\}, Y_1 \setminus \{y_1, y_2\}]$, so the claim holds. □

Since $|E(K_{9,14})| = 126$, we have $38 \leq |E(G)| \leq 39$ and $87 \leq |E(\overline{G})| \leq 88$. Otherwise by Lemma 2, as $z((9, 14), K_{2,2}) \leq 39$ and $z((9, 14), K_{4,4}) \leq 88$, it can be said that either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$. Now we define A as follows

$$A = \{x \in X : \deg_G(x) = 5\}.$$

By Claim 20 and since $K_{2,2} \nsubseteq G$ and $38 \leq |E(G)| \leq 39$, it is easy to say that there exist at least two members of X, say x, x', such that $\Delta = \deg_G(x) = \deg_G(x') = 5$. So $|A| \geq 2$. Now we verify the following two claims.

Claim 21. *We have* $|A| \leq 3$.

Proof. By contradiction we may assume that $|A| \geq 4$. Now consider $|E(G)|$. Since $38 \leq |E(G)| \leq 39$, if $|E(G)| = 38$, then as $|X| = 9$ one can said that there exist two vertices of $X \setminus A$ say x', x'' such that for each $x \in \{x', x''\}$, $|N_G(x_i)| \leq 3$. Therefore $|E(G)| - |N_G(x') \cup N_G(x'')| \geq 32$. Hence by Lemma 2, as $z((7, 14), K_{2,2}) \leq 31$ we have $K_{2,2} \subseteq G[X \setminus \{x_5, x_6\}, Y]$, a contradiction. So suppose that $|E(G)| = 39$, then as $|X| = 9$ there is one vertex of $X \setminus A$ say x such that for $|N_G(x)| \leq 3$. Therefore we have $|E(G)| - |N_G(x)| \geq 36$. Hence by Lemma 2, as $z((8, 14), K_{2,2}) \leq 35$ we have $K_{2,2} \subseteq G[X \setminus \{x\}, Y]$, a contradiction again. Hence the claim holds. □

Claim 22. *For each* $x, x' \in A$, $|N_G(x) \cap N_G(x')| = 1$.

Proof. By contradiction, without loss of generality, let $x_1, x_2 \in A$ and $|N_G(x_1) \cap N_G(x_2)| = 0$, $Y_1 = N_G(x_1) = \{y_1, \ldots, y_5\}$ and $N_G(x_2) = Y_2 = \{y_6, \ldots, y_{10}\}$. By Claim 21, assume that $|A| = 3$ and without loss of generality, let $x_3 \in A$, $N_G(x_3) = Y_3$. As $K_{2,2} \nsubseteq G$, one can say that $|N_G(x_3) \cap Y \setminus (Y_1 \cup Y_2)| \geq 3$. Therefore, since $|X| = 9$, $|A| = 3$, and $|E(G)| \leq 39$, we have $|N_G(x)| = 4$ for at least five vertices of $X \setminus A$. Now, since $K_{2,2} \nsubseteq G$, one can say that $|N_G(x) \cap Y \setminus (Y_1 \cup Y_2)| = 2$ for this vertices. Hence, as $|N_G(x_3) \cap Y \setminus (Y_1 \cup Y_2)| \geq 3$, it is easy to check that $K_{2,2} \subseteq G[X \setminus \{x_1, x_2\}, Y \setminus (Y_1, Y_2)]$, a contradiction. So let $|A| = 2$. Therefore $|E(G)| = 38$, and $|N_G(x)| = 4$ for each $x \in X \setminus A$ and $|N_G(x) \cap Y \setminus (Y_1 \cup Y_2)| \geq 2$. Hence as $|Y \setminus (Y_1 \cup Y_2)| = 4$ and $|X \setminus A| = 7$, by pigeon-hole principle, it is easy to say that $K_{2,2} \subseteq G[X \setminus \{x_1, x_2\}, Y \setminus (Y_1, Y_2)]$, a contradiction again. So the claim holds. □

Now, without loss of generality, let $x_1, x_2 \in A$ and let $Y_1 = \{y_1, \ldots, y_5\}$, $Y_2 = \{y_1, y_6 \ldots, y_9\}$, where $Y_i = N_G(x_i)$ for $i = 1, 2$. Define A' as follows

$$A' = \{x \in X \setminus A : \deg_G(x) = 4\}.$$

By Claim 21, as $38 \leq |E(G)| \leq 39$, we have $|A'| \geq 5$. Now we verify the next two claims.

Claim 23. *If* $|A| = 3$, *then* $y_1 \notin N_G(x)$ *for each* $x \in A \setminus \{x_1, x_2\}$.

Proof. Without loss of generality, suppose that $x_3 \in A$ and by contradiction let $y_1 \in N_G(x_3)$. Hence $\left|\bigcup_{i=1}^{i=3} Y_i\right| = 13$. Assume that $\{y_{14}\} = Y \setminus \bigcup_{i=1}^{i=3} Y_i$. So as $K_{2,2} \nsubseteq G$, $y_{14} \in N_G(x)$ and $y_1 \notin N_G(x)$ for each $x \in A'$. Also $|N_G(x) \cap Y_i \setminus \{y_1\}| = 1$ for each $x \in A'$ and $i = 1, 2$. Now, as $|Y_1 \setminus \{y_1\}| = 4$ and $|A'| \geq 5$, it is clear that $K_{2,2} \subseteq G[A', \{y, y_{14}\}]$ for some $y \in Y_1$, a contradiction. □

Claim 24. *If* $|A| = 3$*, then* $K_{4,4} \subseteq \overline{G}$.

Proof. Without loss of generality, let $A = \{x_1, x_2, x_3\}$. Also by Claims 23 and 22, without loss of generality, let $N_G(x_3) = Y_3 = \{y_2, y_6, y_{10}, y_{11}, y_{12}\}$. Therefore $\left|\bigcup_{i=1}^{i=3} Y_i\right| = 12$. Since $|A'| \geq 5$, without loss of generality, we may suppose that $\{x_4, x_5, x_6, x_7, x_8\} \subseteq A'$. Also one can assume that $|N_G(x) \cap Y'''| = 0$ for each $x \in A'$, where $Y''' = \{y_1, y_2, y_6\}$. Otherwise, let $y_1 \in N_G(x_4)$ (for other case the proof is the same). So, as $K_{2,2} \nsubseteq G$, we have $|N_G(x_4) \cap (Y_3 \setminus \{y_2, y_6\})| = 1$ that is $\{y_{13}, y_{14}\} \subseteq N_G(x_4)$. Now without loss of generality, assume that $N_G(x_4) = \{y_1, y_{10}, y_{13}, y_{14}\}$. Therefore, it can be said that $|N_G(x) \cap \{y_1, y_2, y_6, y_{10}\}| = 0$ for each $x \in A' \setminus \{x_4\}$, otherwise $K_{2,2} \subseteq G[\{x_4, x\}, \{y, y'\}]$ for some $y \in \{y_1, y_2, y_6, y_{10}\}$ and $y' \in \{y_{13}, y_{14}\}$, a contradiction. Therefore, $K_{4,4} \subseteq \overline{G}[A' \setminus \{x_4\}, \{y_1, y_2, y_6, y_{10}\}]$.

So, let $|N_G(x) \cap Y'''| = 0$ for each $x \in A'$. If there is a vertex of A', say x, such that $|N_G(x) \cap \{y_{13}, y_{14}\}| = 2$, then the proof is the same. Hence we may assume that $|N_G(x) \cap \{y_{13}, y_{14}\}| = 1$ for each $x \in A'$. Also as $K_{2,2} \nsubseteq G$ for each $x \in A'$, we have $|N_G(x) \cap \{y_3, y_4, y_5\}| = |N_G(x) \cap \{y_7, y_8, y_9\} = |N_G(x) \cap \{y_{10}, y_{11}, y_{12}\} = |N_G(x) \cap \{y_{13}, y_{14}\} = 1$. Therefore, as $|A'| \geq 5$, by pigeon-hole principle, there exists at least one member of $\{y_{13}, y_{14}\}$, say y_{13}, such that $|N_G(y_{13}) \cap A'| \geq 3$. As $|N_G(x) \cap \{y_3, y_4, y_5\}| = 1$, if $|N_G(y_{13}) \cap A'| \geq 4$, then it is easy to check that $K_{2,2} \subseteq G[N_G(y_{13}) \cap A', \{y, y_{13}\}]$ for some $y \in \{y_3, y_4, y_5\}$, a contradiction. Hence, without loss of generality, let $\{x_4, x_5, x_6\} = N_G(y_{13}) \cap A'$, that is $x_7, x_8 \in N_G(y_{14})$.

Now, without loss of generality, assume that $N_G(x_4) = Y_4 = \{y_3, y_7, y_{10}, y_{13}\}$, $N_G(x_5) = Y_5 = \{y_4, y_8, y_{11}, y_{13}\}$, and $N_G(x_6) = Y_6 = \{y_5, y_9, y_{12}, y_{13}\}$. Consider $D = \{y_5, y_9, y_{12}\}$, for $i = 7, 8$ if there exists a vertex of D, say y, such that $yx_i \in E(G)$, then $K_{2,2} \subseteq G[\{x_7, x_8\}, \{y, y_{14}\}]$, a contradiction. Also there is a vertex of D, say y, so that $yx_7, yx_8 \in E(\overline{G})$, otherwise $K_{2,2} \subseteq G[\{x_6, x_i\}, D]$ for some $i \in \{7, 8\}$, a contradiction again. Therefore without loss of generality, let $x_7y_5, x_8y_5 \in E(\overline{G})$, hence $K_{4,4} \subseteq \overline{G}[\{x_4, x_5, x_7, x_8\}, Y''' \cup \{y_5\}]$. So the claim holds. □

Now by Claim 24, assume that $A = \{x_1, x_2\}$, therefore $A' = X \setminus \{x_1, x_2\}$. For $i = 1, 2$ set $Y'_i = Y_i \setminus \{y_1\}$ and set $Y' = \{y_{10}, \ldots, y_{14}\}$. Now we verify two claims as follows.

Claim 25. *For each* $y \in Y'_1 \cup Y'_2$*, we have* $|N_G(y) \cap A'| \leq 2$.

Proof. By contradiction, let $|N_G(y) \cap A'| \geq 3$ for at least one $y \in Y_1' \cup Y_2'$ and let $X' = N_G(y) \cap A'$. As $|N_G(x) \cap Y'| \geq 2$ and $|X'| \geq 3$, it can be said that $|N_G(y') \cap X'| \geq 2$ for one $y' \in Y'$, which means that $K_{2,2} \subseteq G[X', \{y, y'\}]$, a contradiction. □

By an argument similar to the proof of Claim 25, we can say that the following claim is established.

Claim 26. $|N_G(y_1) \cap A'| \leq 1$.

Now by Claim 26, we verify two cases as follows.

Case 1. $|N_G(y_1) \cap A'| = 0$. In this case, we verify the following claim.

Claim 27. *If there exists a vertex of A', say x, such that $|N_G(x) \cap Y'| = 3$, then $K_{4,4} \subseteq \overline{G}$.*

Proof. Without loss of generality, let $|N_G(x_3) \cap Y'| = 3$. Therefore as $|N_G(x_3)| = 4$, one can say that $|N_G(x_3) \cap Y_i'| = 0$ for one $i \in \{1, 2\}$. Without loss of generality, assume that $|N_G(x_3) \cap Y_1'| = 0$. Hence, as $|A' \setminus \{x_3\}| = 6$, $|Y_1'| = 4$, and $|N_G(x) \cap Y_1'| \leq 1$ for each $x \in A' \setminus \{x_3\}$, then one can say that there exist two vertices of Y_1', say y_1', y_2', such that $|N_G(y) \cap A'| = 1$ for each $y \in \{y_1', y_2'\}$. Also as $|A'| = 7$, $|Y_2'| = 4$, and $|N_G(x) \cap Y_2'| \leq 1$ for each $x \in A' \setminus \{x_3\}$, thus there exists one vertex of Y_2', say y_3', such that $|N_G(y_3') \cap A'| = 1$. Set $W = \{y_1, y_1', y_2', y_3'\}$. Since $|N_G(y_1) \cap A'| = 0$, we have $\left|\bigcup_{y \in W}(N_G(y) \cap A')\right| \leq 3$, so since $|A'| = 7$, we have $K_{4,4} \subseteq \overline{G}[A', W]$. □

Therefore by Claim 27, we may suppose that $|N_G(x) \cap Y_i'| = 1$ for each $x \in A'$ and each $i \in \{1, 2\}$. Hence by Claim 25, and using the fact that $|A'| = 7$, $|Y_i| = 4$, it can be said that there exist one vertex of Y_1', say y_1', and one vertex of Y_2', say y_2', such that $|N_G(y) \cap A'| = 1$ for each $y \in \{y_1', y_2'\}$. Also as $|N_G(x) \cap Y_i'| = 1$ for each $x \in X'$ and each $i \in \{1, 2\}$, we have $|N_G(x) \cap Y'| = 2$ for each $x \in X'$. Hence by Claim 25, using the fact that $|A'| = 7$ and $|Y'| = 5$, it can be said that there exists one vertex of Y', say y_3', such that $|N_G(y) \cap A'| = 2$. Now set $W = \{y_1, y_1', y_2', y_3'\}$. We note that $N_G(y_1) = \{x_1, x_2\}$. Hence assume that $N_G(y_1') = \{x_1, x_1'\}$, $N_G(y_2') = \{x_2, x_2'\}$ also let $N_G(y_3') = \{x_3', x_4'\}$, where $x_i' \in A'$. If there exist $i, j \in \{1, 2, 3, 4\}$ such that $x_i' = x_j'$, then it can be said that $\left|\bigcup_{y \in W}(N_G(y))\right| \leq 5$, which means that $K_{4,4} \subseteq \overline{G}[X, W]$. So suppose that $x_i' \neq x_j'$ for each $i, j \in \{1, 2, 3, 4\}$. Without loss of generality, assume that $x_1' = x_3, x_2' = x_4, x_3' = x_5$ and $x_4' = x_6$. Now consider $X'' = \{x_5, \ldots, x_9\}$. If there exists a vertex of $Y \setminus W$, say y, such that $|N_G(y) \cap X''| \leq 1$, then it is easy to say that $K_{4,4} \subseteq \overline{G}[X'', \{y_1, y_1', y_2', y\}]$. So suppose that $|N_G(y) \cap X''| \geq 2$ for each $Y \setminus W$. Therefore as $|Y \setminus W| = 10$, $N_G(y_3') = \{x_5, x_6\}$, and $|X''| = 5$, one can say that $|E(G[X'', Y])| \geq 22$, that is there exist at least two vertices of X'' with degree at least 5, which is impossible. Therefore, the proof of Case 1 is complete.

Case 2. $|N_G(y_1) \cap A'| = 1$. Without loss of generality, we may assume that $N_G(y_1) = \{x_1, x_2, x_3\}$ also let $N_G(x_3) = Y_3 = \{y_1, y_{10}, y_{11}, y_{12}\}$. Since $|N_G(x) \cap Y_i'| \leq 1$ for each $x \in A'$ and each $i \in \{1, 2\}$, we have $|N_G(x) \cap Y_4'| \geq 1$ for each $x \in A' \setminus \{x_3\}$, where $Y_4' = \{y_{13}, y_{14}\}$. Set $Y_i' = Y_i \setminus \{y_1\}$ for $i = 1, 2$ and $Y_3' = \{y_{10}, y_{11}, y_{12}\}$. Hence we have the following claim.

Claim 28. *We have*

- **(P1):** $|N_G(y) \cap (X \setminus \{x_1, x_2, x_3\})| = 3$ *for each* $y \in Y_4' = \{y_{13}, y_{14}\}$.
- **(P2):** $|N_G(x) \cap Y_i'| = 1$ *for each* $x \in A' \setminus \{x_3\}$ *and each* $i \in \{1, 2, 3\}$.

Proof. To prove (P1), by contradiction, without loss of generality, let $|N_G(y_{13}) \cap (X \setminus \{x_1, x_2, x_3\})| \geq 4$. Without loss of generality, assume that $A'' = \{x_4, x_5, x_6, x_7\} \subseteq N_G(y_{13})$. If $|N_G(x) \cap \{y_{14}\}| = 0$, then one say that $|N_G(x) \cap Y_3'| = 1$ for each $x \in A''$, and since $|A''| = 4$ and $|Y_3'| = 3$, it is easy to say that $K_{2,2} \subseteq G[A'', \{y, y_{13}\}]$ for some $y \in Y_3'$. So suppose that $\{y_{14}\} \in N_G(x)$ for one $x \in A''$. Without loss of generality, assume that $y_{14} \in N_G(x_4)$. If $|N_G(x_4) \cap Y_3'| = 1$, then the proof is the same. So suppose that $|N_G(x_4) \cap Y_3'| = 0$, and without loss of generality, assume that $N_G(x_4) = \{y_2, y_6, y_{13}, y_{14}\}$. Now set $B = X \setminus \{x_1, x_2, x_3, x_4\}$ and $B' = \{y_1, y_2, y_6\}$. As $|N_G(y_1) \cap A'| = 1$, and for each $x \in X \setminus \{x_1, x_2\}$, $|N_G(x) \cap Y_4'| \geq 1$, therefore we have $K_{5,3} \cong [B, B'] \subseteq \overline{G}$. Hence, if there exists a vertex of $Y \setminus B'$, say y, such that $|N_{\overline{G}}(y) \cap B| \geq 4$, then $K_{4,4} \cong [B, B' \cup \{y\}] \subseteq \overline{G}$. So let $|N_{\overline{G}}(y) \cap B| \leq 3$, that is $|N_G(y) \cap B| \geq 2$ for each $y \in Y \setminus B'$. Now as $|Y \setminus B'| = 11$, one can said that $|E(G[B, Y \setminus B'])| \geq 22$. Therefore as $|B| = 5$, one can say that there exists at least two vertices of B, say x, x', such that $|N_G(x)| = |N_G(x')| = 5$, a contradiction to $|A = \{x_1, x_2\}| = 2$.

To prove (P2), if for one $x \in A' \setminus \{x_3\}$ and one $i \in \{1, 2, 3\}$, $|N_G(x) \cap Y_i'| = 0$, then $|N_G(x) \cap Y_4'| = 2$. Hence as $|A' \setminus \{x_3\}| = 6$, and $|N_G(x) \cap Y_4'| \geq 1$ for each $x \in A' \setminus \{x_3\}$, then it can be checked that $|N_G(y) \cap (X \setminus \{x_1, x_2, x_3\})| \geq 4$ for one $y \in Y_4'$, and the proof is complete by part (P1). □

Now by Claim 28, without loss of generality, suppose that $R = N_G(y_{13}) \cap (X \setminus \{x_1, x_2, x_3\}) = \{x_4, x_5, x_6\}$ and $R' = N_G(y_{14}) \cap X \setminus \{x_1, x_2, x_3\} = \{x_7, x_8, x_9\}$. Hence by (P1) and (P2) and without loss of generality, we can suppose the following.

- **(P3):** $N_G(x_4) = \{y_2, y_6, y_{10}, y_{13}\}$.
- **(P4):** $N_G(x_5) = \{y_3, y_7, y_{11}, y_{13}\}$.
- **(P5):** $N_G(x_6) = \{y_4, y_8, y_{12}, y_{13}\}$.

Now consider R', by (P2), without loss of generality, let $y_{10} \in N_G(x_7)$, $y_{11} \in N_G(x_8)$, and $y_{12} \in N_G(x_9)$. As $|R'| = 3$ and $|Y_1'| = |Y_2'| = 4$ and by (P2), it can be said that the following properties are established.

- **(P6):** There exists a vertex of R', say x, so that $|N_G(x) \cap (Y_1' \setminus \{y_5\})| = |N_G(x) \cap (Y_2' \setminus \{y_9\})| = 1$.

Therefore, by (P6) without loss of generality, we may suppose that $x = x_7$, $N_G(x_7) \cap (Y_1' \setminus \{y_5\}) = \{y'\}$ and $N_G(x_7) \cap (Y_2' \setminus \{y_9\}) = \{y''\}$. Since $y_{10} \in N_G(x_7)$, it can be said that $y' \neq y_2$ and $y'' \neq y_6$, otherwise either $K_{2,2} \subseteq G[\{x_4, x_7\}, \{y_2, y_{10}\}]$ or $K_{2,2} \subseteq G[\{x_4, x_7\}, \{y_6, y_{10}\}]$, a contradiction. Hence without loss of generality, assume that $y' = y_3$. As $y_3, y_7 \in N_G(x_5)$, one can say that $y'' = y_8$. Therefore we have the following.

- **(P7):** $N_G(x_7) = \{y_3, y_8, y_{10}, y_{13}\}$.

Now by considering (P4), (P5), (P6), and (P7) and since $N_G(x_3) \cap (Y_1' \cup Y_2') = \emptyset$, one can say that $K_{4,4} \subseteq \overline{G}[\{x_3, x_5, x_6, x_7\}, \{y_2, y_5, y_6, y_9\}]$. Which means that the proof of Case 2 is complete.

Therefore by Case 1 and Case 2, the proof is complete. ■

In the following theorem, we find the values of $BR_m(K_{2,2}, K_{4,4})$ for each $m \in \{10, \ldots, 13\}$.

Theorem 29. $BR_m(K_{2,2}, K_{4,4}) = 14$ *for each* $m \in \{10, 11, 12, 13\}$.

Proof. By Theorem 3 as $BR(K_{2,2}, K_{4,4}) = 14$, it is sufficient to show that $BR_{10}(K_{2,2}, K_{4,4}) = 14$. Suppose that G is a subgraph of $K_{10,14}$, hence as $|E(K_{10,14})| = 140$, then either $|E(G)| \geq 43$ or $|E(\overline{G})| \geq 98$. Therefore by Lemma 2, as $z((10, 14), K_{2,2}) \leq 42$ and $z((10, 14), K_{4,4}) \leq 97$, it can be said that either $K_{2,2} \subseteq G$ or $K_{4,4} \subseteq \overline{G}$. Therefore, $BR_m(K_{2,2}, K_{4,4}) = 14$, for each $m \in \{10, 11, 12, 13\}$. ■

***Proof of Theorem* 1.** For $m = 2, 3, 4$, it is easy to say that $BR_m(K_{2,2}, K_{4,4})$ does not exist. Now, combining Theorems 3, 6, 8, 12, 15, 19, and 29, we conclude that the proof of Theorem 1 is complete. ■

References

[1] L.W. Beineke and A.J. Schwenk, *On a bipartite form of the Ramsey problem,* in: Proc. Fifth British Combinatorial Conference, Aberdeen, 1975, Util. Math. **15** (1976) 17–22.

[2] Z. Bi, G. Chartrand and P. Zhang, *Another view of bipartite Ramsey numbers*, Discuss. Math. Graph Theory **38** (2018) 587–605.
https://doi.org/10.7151/dmgt.2034

[3] M. Bucić, S. Letzter and B. Sudakov, *Multicolour bipartite Ramsey number of paths*, Electron. J. Combin. **26** (2019) #P3.60.
https://doi.org/10.37236/8458

[4] M. Bucić, S. Letzter and B. Sudakov, 3-*color bipartite Ramsey number of cycles and paths*, J. Graph Theory **92** (2019) 445–459.
https://doi.org/10.1002/jgt.22463

[5] G. Chartrand and P. Zhang, *New directions in Ramsey theory*, Discrete Math. Lett. **6** (2021) 84–96.
https://doi.org/10.47443/dml.2021.s110

[6] A.F. Collins, Bipartite Ramsey Numbers and Zarankiewicz Numbers, M.Sc. Thesis (Rochester Institute of Technology, 2015).

[7] A.F. Collins, A.W.N. Riasanovsky, J.C. Wallace and S.P. Radziszowski, *Zarankiewicz numbers and bipartite Ramsey numbers*, J. Algorithms Comput. **47** (2016) 63–78.
https://doi.org/10.22059/jac.2016.7943

[8] D. Day, W. Goddard, M.A. Henning, and H.C. Swart, *Multipartite Ramsey numbers*, Ars Combin. **58** (2001) 23–32.

[9] J. Dybizbański, T. Dzido and S. Radziszowski, *On some Zarankiewicz numbers and bipartite Ramsey Numbers for quadrilateral*, Ars Combin. **119** (2016) 275–287.

[10] R.J. Faudree and R.H. Schelp, *Path-path Ramsey-type numbers for the complete bipartite graph*, J. Combin. Theory Ser. B **19** (1975) 161–173.
https://doi.org/10.1016/0095-8956(75)90081-7

[11] M. Gholami and Y. Rowshan, *The bipartite Ramsey numbers $BR(C_8, C_{2n})$* (2021). arXiv:2108.02630

[12] W. Goddard, M.A. Henning and O.R. Oellermann, *Bipartite Ramsey numbers and Zarankiewicz numbers*, Discrete Math. **219** (2000) 85–95.
https://doi.org/10.1016/S0012-365X(99)00370-2

[13] A. Gyárfás and J. Lehel, *A Ramsey-type problem in directed and bipartite graphs*, Period. Math. Hungar. **3** (1973) 299–304.
https://doi.org/10.1007/BF02018597

[14] I. Hatala, T. Héger, and S. Mattheus, *New values for the bipartite Ramsey number of the four-cycle versus stars*, Discrete Math. **344** (2021) 112320.
https://doi.org/10.1016/j.disc.2021.112320

[15] J. Hattingh and M. Henning, *Bipartite Ramsey theory*, Util. Math. **53** (1998) 217–230.

[16] J.H. Hattingh and M.A. Henning, *Star-path bipartite Ramsey numbers*, Discrete Math. **185** (1998) 255–258.
https://doi.org/10.1016/S0012-365X(97)00205-7

[17] R. Lakshmi and D. Sindhu, *Three-colour bipartite Ramsey number $R_b(G_1, G_2, P_3)$*, Electron. J. Graph Theory Appl. (EJGTA) **8** (2020) 195–204.
https://doi.org/10.5614/ejgta.2020.8.1.14

[18] V. Longani, *Some bipartite Ramsey numbers*, Southeast Asian Bull. Math. **26** (2003) 583–592.
https://doi.org/10.1007/s100120200062

[19] G. Raeisi, *Star-path and star-stripe bipartite Ramsey numbers in multicoloring*, Trans. Comb. **4** (2015) 37–42.
https://doi.org/https://doi.org/10.22108/toc.2015.7275

[20] Y. Rowshan and M. Gholami, *Another view of bipartite Ramsey numbers*, (2022).
https://doi.org/10.48550/arXiv.2201.12844

[21] Y. Rowshan, M. Gholami and S. Shateyi, *A proof of a conjecture on bipartite Ramsey numbers* $B(2,2,3)$, Mathematics **10** (2022) 701.
https://doi.org/10.3390/math10050701

[22] Y. Rowshan, M. Gholami and S. Shateyi, *The size, multipartite Ramsey numbers for* nK_2 *versus path–path and cycle*, Mathematics **9** (2021) 764.
https://doi.org/10.3390/math9070764

Received 10 February 2022
Revised 17 October 2022
Accepted 18 October 2022
Available online 26 November 2022