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DOMINATED PAIR DEGREE SUM CONDITIONS OF SUPEREULERIAN DIGRAPHS

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Abstract

A digraph D is supereulerian if D contains a spanning eulerian subdigraph. In this paper, we propose the following problem: is there an integer t with $0 \leq t \leq n-3$ so that any strong digraph with n vertices satisfying either both $d(u) \geq n-1+t$ and $d(v) \geq n-2-t$ or both $d(u) \geq n-2-t$ and $d(v) \geq n-1+t$, for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$, is supereulerian? We prove the cases when $t = 0, t = n-4$ and $t = n-3$. Moreover, we show that if a strong digraph D with n vertices satisfies $\min\{d^+(u)+d^-(v), d^-(u)+d^+(v)\} \geq n-1$ for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$ of D, then D is supereulerian.

Keywords: supereulerian digraph, spanning eulerian subdigraph, dominated pair degree sum condition.

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1. Introduction

Digraphs considered are loopless and without parallel arcs. We refer the reader to [1] for digraphs for undefined terms and notation. In this paper, we define $[k] = \{1, 2, \ldots, k\}$ for an integer $k > 0$ and use (w, z) to denote an arc oriented from a vertex w to a vertex z and say that w *dominates* z. For any two vertices u, v in a digraph D, if $(u, w), (v, w) \in A(D)$ for some $w \in V(D)$, then we say that $\{u, v\}$ *dominates* w or call the pair $\{u, v\}$ *dominating*; if $(w, u), (w, v) \in A(D)$ for some $w \in V(D)$, then we say that $\{u, v\}$ is *dominated* by w or call the pair $\{u, v\}$ *dominated*. We often write *dipaths* for directed paths, *dicycles* for directed cycles and *ditrails* for directed trails in digraphs. The *length* of a ditrail is the number of its arcs. If a ditrail T starts at w and ends at z, we may call it a (w, z)-ditrail T or $T_{[w,z]}$ and say w is the *initial vertex* of T and z is the *terminal vertex* of T. A (w, z)-ditrail of minimum length in D is called a *shortest* (w, z)*-ditrail* in D. We often write $|D|$ for $|V(D)|$ and use K_n^* to represent the *complete digraph* with n vertices. A digraph D is *semicomplete* if it has no pair of nonadjacent vertices. A digraph D is *strong* if any vertex of a digraph D is reachable from all other vertices of D.

Let $T = v_1v_2 \cdots v_k$ denote a ditrail. For any $1 \leq i \leq j \leq k$, we use $T_{[v_i,v_j]} = v_iv_{i+1} \cdots v_{j-1}v_j$ to denote the *sub-ditrail* of T. Likewise, if $Q = u_1u_2 \cdots u_ku_1$ is a closed ditrail, then for any i, j with $1 \leq i < j \leq k$, $Q_{[u_i,u_j]}$ denotes the sub-ditrail $u_iu_{i+1} \cdots u_{j-1}u_j$. If $T' = w_1w_2 \cdots w_{k'}$ is a ditrail with $v_k = w_1$ and $V(T) \cap V(T') = \{v_k\}$, then we use TT' or $T_{[v_1,v_k]}T'_{[v_k,w_{k'}]}$ to denote the ditrail $v_1v_2 \cdots v_kw_2 \cdots w_{k'}$. If $V(T) \cap V(T') = \emptyset$ and there is a dipath $z_1z_2 \cdots z_t$ with $z_2, \ldots, z_{t-1} \notin V(T) \cup V(T')$ and with $z_1 = v_k$ and $z_t = w_1$, then we use $Tz_1 \cdots z_tT'$ to denote the ditrail $v_1v_2 \cdots v_kz_2 \cdots z_tw_2 \cdots w_{k'}$. In particular, if T is a (v, w)-ditrail of a digraph D and $(u, v), (w, z) \in A(D) - A(T)$, then we use $uvTwz$ to denote the (u, z)-ditrail $D\langle A(T) \cup \{(u, v), (w, z)\}\rangle$. The subdigraphs uvT and Twz are similarly defined.

For a digraph D, $a \in A(D)$ and a subdigraph S of D, we use $D - S$ to denote the subdigraph $D\langle V(D) - V(S)\rangle$, use $D - a$ to denote the subdigraph $D\langle A(D) - a\rangle$, and use $D + a$ to denote the subdigraph $D\langle A(D) + a\rangle$. Let D_1 and D_2 be two digraphs; the *union* $D_1 \cup D_2$ of D_1 and D_2 is a digraph with vertex set $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and arc set $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$. For $S, T \subseteq V(D)$, an (s, t)-dipath P is an (S, T)*-dipath* if $s \in S$, $t \in T$ and $V(P) \cap (S \cup T) = \{s, t\}$. Note that if $S \cap T \neq \emptyset$, then a vertex $s \in S \cap T$ forms an (S, T)-dipath by itself. When S and T are subdigraphs of D, we also talk about an (S, T)-dipath.

Let $d_D^-(s), d_D^+(s), d_D(s) = d_D^-(s) + d_D^+(s), N_D^-(s)$ and $N_D^+(s)$ denote, respectively, the *in-degree, out-degree, degree, in-neighbourhood* and *out-neighbourhood* of a vertex $s \in V(D)$.

In [3], Boesch *et al.* raised the supereulerian problem, which strives to describe graphs that contain spanning eulerian subgraphs. In [10], Pulleyblank showed that deciding whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been many studies on this topic, as revealed in the surveys [4, 5] and [9].

It is natural to try to relate supereulerian graphs to supereulerian digraphs. A digraph D is *supereulerian* if it contains a closed ditrail S with $V(S) = V(D)$, i.e., it has a spanning eulerian subdigraph, and *nonsupereulerian* otherwise. Results on supereulerian digraphs can be found in [2, 6, 7, 8], among others. It is worth pointing out that only a few of degree sum conditions are studied to ensure supereulerianicity in digraphs. In particular, the following have been proved.

Theorem 1 [2]**.** *If a strong digraph D with n vertices satisfies $d^+(u) + d^-(v) \geq n - 1$ for any two vertices u and v with $(u, v) \notin A(D)$, then D is supereulerian.*

Theorem 2 [2]**.** *If a strong digraph D with n vertices satisfies $d(u)+d(v) \geq 2n-3$ for any two nonadjacent vertices u and v, then D is supereulerian.*

It is observed that in Theorems 1 and 2, degree sum conditions on every pairs of nonadjacent vertices are needed to warrant the digraph to be supereulerian. In this article, we will consider a degree sum condition about pairs of dominated (dominating) nonadjacent vertices but no longer on all pairs of nonadjacent vertices. First, we give the following definition.

Definition 3. Given an integer $t \geq 0$, we say a digraph D of order n satisfies the condition C_t if

$$d(u) \geq n - 1 + t, d(v) \geq n - 2 - t \text{ or } d(u) \geq n - 2 - t, d(v) \geq n - 1 + t,$$

for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$ in D.

If u and v are nonadjacent for $u, v \in V(D)$, then $d(u) \leq 2n - 4$ and $d(v) \leq 2n - 4$. Thus $n - 1 + t \leq 2n - 4$ implies $t \leq n - 3$. Then we have $0 \leq t \leq n - 3$ and naturally propose the following problem.

Problem 4. Is there an integer t with $0 \leq t \leq n - 3$ so that any strong digraph with n vertices satisfying the condition C_t is supereulerian?

Problem 4 assumes the existence of nonadjacent vertices. When a strong digraph D contains no nonadjacent vertices, condition C_t is automatically satisfied, and in this case, by Theorem 1.5.3 of [1], D is hamiltonian and so supereulerian. Hence it suffices to settle Problem 4 for strong digraphs which are not semicomplete digraphs. Likewise, when we discuss Theorems 5, 6, 7, 8, we may also assume the digraph under consideration is not a semicomplete digraph.

The purpose of this paper is to prove the cases when $t = 0$, $t = n - 4$ and $t = n - 3$ and show that if a strong digraph D with n vertices satisfies $\min\{d^+(u) + d^-(v), d^-(u) + d^+(v)\} \geq n - 1$ for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$ of D, then D is supereulerian. Moreover, our result, Theorem 8, generalizes Theorem 1. The main results are the following, which are independent of Theorem 2.

Theorem 5. *If a strong digraph D with n vertices satisfies the condition C_0, then D is supereulerian.*

Theorem 6. *If a strong digraph D with n vertices satisfies the condition C_{n-4} for any pair of dominated nonadjacent vertices $\{u, v\}$, then D is supereulerian.*

Theorem 7. *If a strong digraph D with n vertices satisfies the condition C_{n-3} for any pair of dominated nonadjacent vertices $\{u, v\}$, then D is supereulerian.*

Theorem 8. *If a strong digraph D with n vertices satisfies, for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$, $\min\{d^+(u)+d^-(v), d^-(u)+d^+(v)\} \geq n - 1$, then D is supereulerian.*

In Section 2, we give the proofs of Theorems 5–8 and shall display examples of nonsupereulerian digraphs to demonstrate the sharpness of our results in some sense. The last section is devoted to some concluding remarks.

2. Main Results

The following lemmas will be useful.

Lemma 9. *Let D be a digraph, $S = u_1u_2 \cdots u_s$ and $T = v_1v_2 \cdots v_t$ be two arc disjoint ditrails of D. If D does not contain a (u_1, u_s)-ditrail with vertex set $V(S) \cup V(T)$, then $d^-_S(v_1) + d^+_S(v_t) \leq |S|$.*

Proof. As D does not contain a (u_1, u_s)-ditrail with vertex set $V(S) \cup V(T)$, we have $|\{(u_i, v_1), (v_t, u_i)\} \cap A(D)| \leq 1$ for any $u_i \in V(S)$. Accordingly, we obtain $d^-_S(v_1) + d^+_S(v_t) \leq |S|$ as required. ■

Corollary 10. *Let D be a digraph, $S = u_1u_2 \cdots u_s$ be a ditrail in D and $x \in V(D) - V(S)$. If D does not contain a (u_1, u_s)-ditrail with vertex set $V(S) \cup \{x\}$, then $d_S(x) \leq |S|$.*

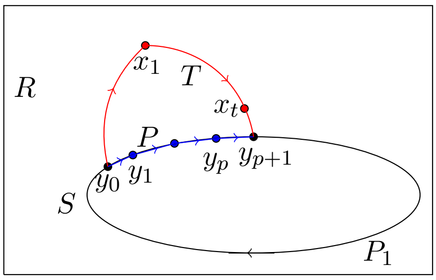
Throughout the proofs of Theorems 5–8, we let D denote a strong nonsupereulerian digraph with n vertices, and let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be the collection of closed ditrails such that $|V(S_1)| = \cdots = |V(S_k)|$ is maximized in D (possibly

$k = 1$). Let $S = y_0y_1 \cdots y_py_{p+1} \cdots y_my_0$ be the closed ditrail such that $|A(S)|$ is maximized in $\mathcal{S}$. Thus

(1) $\qquad |V(S)|$ is maximized in D and $|A(S)|$ is maximized in $\mathcal{S}$.

Let $|V(S)| = s$. As D is not supereulerian, $1 < s < n$. Since D is strong, there exists an (S, S)-dipath T with $|T| \geq 3$.

Throughout the proofs of Theorems 5–7, for an integer $l \geq 1$ and $i \in [l]$, let T_i be an (S, S)-dipath with $|T_i| \geq 3$ such that $|V(P_i)|$ of the ditrail P_i is minimum in S, where $|V(P_1)| = \cdots = |V(P_l)|$ and P_i is a shortest (u_i, v_i)-ditrail which travels along S from u_i to v_i such that the initial vertex u_i of P_i is the initial vertex of T_i and the terminal vertex v_i of P_i is the terminal vertex of T_i. Choose an (S, S)-dipath T with $|T| \geq 3$ in $\{T_1, \ldots, T_l\}$ such that $|A(P)|$ of the ditrail P is minimum in $\{P_1, \ldots, P_l\}$. Thus $|V(P)|$ is minimum in S and $|A(P)|$ is minimum in $\{P_1, \ldots, P_l\}$. Without loss of generality, write $T = y_0x_1x_2 \cdots x_ty_{p+1}$. Let $W = \{y_1, y_2, \ldots, y_p\}$ be the set of internal vertices of P, P' be the ditrail which travels along S from y_{p+1} to y_0, r be the maximum integer $1 \leq i \leq p$ such that D contains a (y_{p+1}, y_0)-ditrail P_1 with vertex set $V(P_1) = V(P') \cup \{y_0, y_1, \ldots, y_{r-1}\}$ and $R = D - S$. Then $|P'| = s - p + c'$, and $|P_1| = s - p + c$, where $c' = |W \cap P'|$ and $c = |W \cap P_1|$.



A nonsupereulerian strong digraph D

Figure 1. The illustration for the proofs of Theorems 5–7.

***Proof of Theorem* 5.** By (1), we have $y_0 \neq y_{p+1}$, $(y_0, y_{p+1}) \notin A(S)$ and $p \geq 1$. This together with the fact that P is a (y_0, y_{p+1})-ditrail implies $d_P^+(y_0) - d_P^-(y_0) = 1$ and $d_P^-(y_{p+1}) - d_P^+(y_{p+1}) = 1$. If for an integer $k \geq 1$, $d_P^+(y_0) = k + 1 \geq 2$, then $d_P^-(y_0) = k$. Hence we can denote the ditrail $P = y_0 \cdots y_0^1 \cdots y_0^2 \cdots y_0^k \cdots y_{p+1}$, where $y_0 = y_0^1 = y_0^2 = \cdots = y_0^k$. For any $h \in [k]$, $P_{[y_0^h, y_{p+1}]} = y_0^h \cdots y_{p+1}$ is also a (y_0, y_{p+1})-ditrail which travels along S from y_0 to y_{p+1}. If $\left|V\left(P_{[y_0^h, y_{p+1}]}\right)\right| = |V(P)|$, then $\left|A\left(P_{[y_0^h, y_{p+1}]}\right)\right| < |A(P)|$, contrary to the choice of P. If

$\left|V\left(P_{[y_0^h, y_{p+1}]}\right)\right| \neq |V(P)|$, then $\left|V\left(P_{[y_0^h, y_{p+1}]}\right)\right| < |V(P)|$ and $\left|A\left(P_{[y_0^h, y_{p+1}]}\right)\right| < |A(P)|$, contrary to the choice of P. Hence $k = 0$, $d_P^+(y_0) = 1$ and $d_P^-(y_0) = 0$. By similar arguments, we can get that $d_P^-(y_{p+1}) = 1$ and $d_P^+(y_{p+1}) = 0$. Therefore,

$$d_P^+(y_0) = d_P^-(y_{p+1}) = 1 \text{ and } d_P^-(y_0) = d_P^+(y_{p+1}) = 0. \tag{2}$$

First for any $i \in [t]$ we have $d_W(x_i) = 0$ by the choice of T and (1). If $d_W^+(x_i) > 0$ or $d_W^-(x_i) > 0$, then without loss of generality, we may assume that $d_W^+(x_i) > 0$, and so there exists a vertex $y_j \in W$ ($j \in [p]$) such that $(x_i, y_j) \in A(D)$. If $2 \leq j \leq p$, then we can get another (S, S)-dipath T' with the initial vertex y_0 and the terminal vertex y_j such that the length of (y_0, y_j)-ditrail P' in S is less then the length of P in S, contrary to the choice of T above. If $j = 1$, then $S \cup T_{[y_0, x_i]} + (x_i, y_1) - (y_0, y_1)$ is a closed ditrail with $\left|S \cup T_{[y_0, x_i]} + (x_i, y_1) - (y_0, y_1)\right| > |S|$, contrary to (1). Therefore $d_W^+(x_i) = 0$. The proof for $d_W^-(x_i) = 0$ is similar. In particular, x_i and y_j are nonadjacent, for $i \in [t]$ and $j \in [p]$.

By the definition of P', $x_i \notin V(P')$. If for some $x_i \in V(T)$, D contains a (y_{p+1}, y_0)-ditrail S' with vertex set $V(P') \cup \{x_i\}$, then $S' \cup P$ is a closed ditrail with $|S' \cup P| > |S|$, contrary to (1). Thus D does not have a (y_{p+1}, y_0)-ditrail with vertex set $V(P') \cup \{x_i\}$, for any $x_i \in V(T)$. By Corollary 10 and $d_W(x_i) = 0$, we can deduce that

$$d_S(x_i) = d_{P'}(x_i) = d_{P'-W}(x_i) \leq |P'| - c' = s - p. \tag{3}$$

Obviously,

$$d_{W-P_1}(y_j) \leq 2(p - c - 1). \tag{4}$$

Furthermore, by the choice of T and (1), there is no vertex $z \in R$ satisfying $\{(y_j, z), (z, x_i)\} \subseteq A(D)$ or $\{(x_i, z), (z, y_j)\} \subseteq A(D)$, for $i \in [t]$ and $j \in [p]$. Accordingly,

$$d_R(x_i) + d_R(y_j) \leq 2(n - s - 1). \tag{5}$$

Let y_k be any vertex in W such that $(y_0, y_k) \in A(D)$. Combining (3)–(5) with the fact that the pair of nonadjacent vertices $\{x_1, y_k\}$ is dominated by y_0 and the pair of nonadjacent vertices $\{x_t, y_p\}$ dominates y_{p+1}, we get

$$2n - 3 \leq d(x_1) + d(y_k) \leq d_{P_1}(y_k) + 2n - |P_1| - 4 - c$$

and

$$2n - 3 \leq d(x_t) + d(y_p) \leq d_{P_1}(y_p) + 2n - |P_1| - 4 - c.$$

Accordingly,

$$d_{P_1}(y_k) \geq |P_1| + 1 + c \text{ and } d_{P_1}(y_p) \geq |P_1| + 1 + c. \tag{6}$$

Now we consider two cases in the following.

Case 1. $c = |W \cap P_1| = 0$. In this case, we have $W \cap P_1 = \emptyset$ and $S = P + P_1$. By (1), D does not have a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup V\big(P_{[y_1,y_p]}\big)$. Then by Lemma 9, we get

$$d^-_{P_1}(y_1) + d^+_{P_1}(y_p) \le |P_1| = s - p. \tag{7}$$

By symmetry, assume that $y_1 = y_k$. If $y_1 = y_p$, then $d_{P_1}(y_1) \le |P_1|$, contrary to (6). Thus $y_1 \ne y_p$. By (6), $d_{P_1}(y_1) \ge |P_1|+1+c$. There must exist vertices $y_a, y_c \in V(P_1)$ such that $\{(y_a, y_1), (y_1, y_a), (y_p, y_c), (y_c, y_p)\} \subseteq A(D)$. Since $W \cap P_1 = \emptyset$ and $S = P + P_1$, we have $y_1, y_p \in W, y_1, y_p \notin V(P_1), y_a, y_c \in V(P_1)$ and $y_a, y_c \notin W$. Then $(y_a, y_1), (y_1, y_a), (y_p, y_c), (y_c, y_p) \notin A(P_1)$.

By (2), we have $d^-_P(y_{p+1}) = |\{(y_p, y_{p+1})\} \cap A(D)| = 1$ and $d^+_P(y_{p+1}) = 0$. If $y_a = y_{p+1}$, then as $y_p \ne y_1$, we have $(y_{p+1}, y_1), (y_1, y_{p+1}) \notin A(P)$. Therefore $(y_a, y_1), (y_1, y_a) \notin A(S)$. But then we can get a closed ditrail $S' = S \cup \{(y_a, y_1), (y_1, y_a)\}$ with $|A(S')| > |A(S)|$, contrary to (1). Thus $y_a \ne y_{p+1}$. Similarly, we can get that $y_c \ne y_0$.

If $y_a \ne y_0$, then $(y_1, y_a), (y_a, y_1) \notin A(P)$. Therefore $(y_a, y_1), (y_1, y_a) \notin A(S)$. But then we can get a closed ditrail $S' = S \cup \{(y_a, y_1), (y_1, y_a)\}$ with $|A(S')| > |A(S)|$, contrary to (1). Thus $y_a = y_0$. Similarly, we can get that $y_c = y_{p+1}$. Then $\{(y_0, y_1), (y_1, y_0), (y_p, y_{p+1}), (y_{p+1}, y_p)\} \subseteq A(D) - A(P_1)$.

By (1) with (6), for any $y_i \in V(P_1) - y_0$ and $y_j \in V(P_1) - y_{p+1}$, we have $|\{(y_i, y_1), (y_1, y_i) \cap A(D)\}| = 1$ and $|\{(y_j, y_p), (y_p, y_j)\} \cap A(D)| = 1$. By (2), $(y_1, y_0), (y_{p+1}, y_p) \notin A(P)$. Then $(y_1, y_0), (y_{p+1}, y_p) \notin A(S)$. If $(y_m, y_1) \in A(D)$, note that $(y_m, y_1) \notin A(S)$, then we can get a closed ditrail $S' = S + (y_m, y_1) + (y_1, y_0) - (y_m, y_0)$ with $|A(S')| > |A(S)|$, contrary to (1). Thus $(y_m, y_1) \notin A(D)$ and $(y_1, y_m) \in A(D)$. Continuing this process, we finally conclude that for any $y_i \in V(P_1) - y_0$, $(y_i, y_1) \notin A(D)$ and $(y_1, y_i) \in A(D)$. Similarly, we can get that for any $y_j \in V(P_1) - y_{p+1}$, $(y_p, y_j) \notin A(D)$ and $(y_j, y_p) \in A(D)$. In particular, $(y_{p+1}, y_1) \notin A(D)$.

Now we have $d^+_{P_1}(y_1) = |P_1| = d^-_{P_1}(y_p)$ and $d^-_{P_1}(y_1) = 1 = d^+_{P_1}(y_p)$. Combining (3) with the fact that $d_R(x_1) + d_R(y_1) \le 2(n - s - 1)$ and the assumption of the theorem, note that the pair of nonadjacent vertices $\{x_1, y_1\}$ is dominated by y_0, we obtain $d_S(y_1) \ge s+p-1$. Since $d_{P_1}(y_1) = |P_1|+1 = s-p+1$, $d_W(y_1) \ge 2(p-1)$. That is, for any $y_j \in W$, $(y_j, y_1), (y_1, y_j) \in A(D)$. But then we can get a closed ditrail $S' = T \cup P_1 \cup \{(y_j, y_1), (y_1, y_j)\} \cup \{(y_0, y_1), (y_1, y_0)\}$, for every $y_j \in W$, with $|S'| > |S|$, contrary to (1).

Case 2. $c = |W \cap P_1| \ge 1$. From (6) and Corollary 10 it follows that D contains a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{y_k\}$. Note that $V(P_1) = V(P') \cup \{y_0, y_1, \ldots, y_{r-1}\}$. By (1), $r - 1 \le p - 1$. Then there exists an integer r with $2 \le r \le p$ such that D contains a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup$

$\{y_{r-1}\}$ but does not contain a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{y_r\}$. In particular, $y_{r-1} \in V(P_1)$. Therefore D does not contain a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup V(S_{[y_{r-1}, y_r]})$.

By Corollary 10 and the fact that D does not contain a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{y_r\}$, $d_{P_1}(y_r) \leq |P_1| = s - p + c$. By (6), $y_r \neq y_k$, that is $(y_0, y_r) \notin A(D)$. From this with (3)–(5), we obtain $d(y_r) + d(x_1) \leq 2n - 5$. By assumption, note that $\{x_1, y_1\}$ is a pair of dominated nonadjacent vertices, we have $d(x_1) \geq n - 2$. Therefore, we get

$$d(y_r) \leq n - 3. \tag{8}$$

Note that $y_{r-1} \in V(P_1)$ and $y_r \notin V(P_1)$. Let $P_1 = y_{p+1} y_{a-d} y_{a-d+1} \cdots y_{a-2} y_{r-1}$ $y_a y_{a+1} \cdots y_{a+l} y_0$.

Therefore by (8) and the assumption of the theorem, as the pair $\{y_r, y_a\}$ is dominated by y_{r-1}, we have that y_r and y_a are adjacent. If $(y_r, y_a) \in A(D)$, then D contains a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup V(S_{[y_{r-1}, y_r]})$, a contradiction. So assume that $(y_a, y_r) \in A(D)$. Then the pair $\{y_{a+1}, y_r\}$ is dominated by y_a, we similarly conclude that $(y_{a+1}, y_r) \in A(D)$. Continuing this process, we can deduce that $(y_0, y_r) \in A(D)$, which contradicts the conclusion above $(y_0, y_r) \notin A(D)$. This proves Theorem 5. ■

***Proof of Theorem* 6.** By similar arguments as in the proof of Theorem 5, we obtain that $d_W(x_i) = 0$ for any $i \in [t]$ and $d_{P_1}(y_1) \geq |P_1| + 1$. From this and Corollary 10 it follows that D contains a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{y_1\}$. Since D does not contain a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{y_r\}$, $|W| \geq 2$. This together with $d_W(x_1) = 0$ implies

$$d(x_1) \leq 2(n-3) = 2n - 6.$$

Since $\{y_1, x_1\}$ is a pair of dominated nonadjacent vertices and by assumption, we get that $d(y_1) \geq 2n - 5$ and $d(x_1) \geq 2$. By the choice of T, y_1 and any vertex of $T_{[x_1, x_t]}$ are not adjacent. This together with $d(y_1) \geq 2n - 5$, we obtain $t = 1$. As we known the pair of nonadjacent vertices $\{y_1, x_1\}$ satisfying $d(y_1) \leq 2n - 4$ and $d(x_1) \leq 2n - 4$. Then we have $2n - 5 \leq d(y_1) \leq 2n - 4$. If $d(y_1) = 2n - 4$, then we get $(z, y_1), (y_1, z) \in A(D)$ for any $z \in V(D) - \{y_1, x_1\}$. Thus, for all $j \in [m]$ and $j \notin \{0, 1, p+1\}$, $S' = y_0 x_1 y_{p+1} y_1 y_0 \cup \{(y_1, y_j), (y_j, y_1)\}$ is a closed ditrail with $|S'| > |S|$, contrary to (1). So assume that $d(y_1) = 2n - 5$. We can assume that $d(y_p) = 2n - 5$ by similar arguments as above.

If $\{(y_1, y_j), (y_j, y_1)\} \subseteq A(D)$ for any $j \in [m]$, then, for all $i \in [m]$ and $i \notin \{0, 1, p+1\}$, $S' = y_0 x_1 y_{p+1} y_1 y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with $|S'| > |S|$, a contradiction. Thus, we consider two cases in the following.

Case 1. $(y_1, y_j) \notin A(D)$ for some $j \in [m]$. In this case, $(y_j, y_1) \in A(D)$ and $\{(y_i, y_1), (y_1, y_i)\} \subseteq A(D)$, where $i \in [m]$ and $i \notin \{1, j\}$. If $j = 0$, then, for all $i \in [m]$ and $i \notin \{0, 1, p+1, m\}$, $S' = y_0x_1y_{p+1}y_1y_my_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with $|S'| > |S|$, a contradiction. Thus $j \neq 0$. If $j < p+1$, then, for all $i \in [m]$ and $i \notin \{0, 1, \ldots, j, p+1\}$, $S' = y_0x_1y_{p+1}S_{[y_1,y_j]}y_1y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with $|S'| > |S|$, contrary to (1). So $j \geq p+1$. Then $S' = y_0x_1y_{p+1}S_{[y_{p+1},y_j]}y_1y_2y_1y_3 \cdots y_1y_py_1y_{j+1}y_1y_{j+2} \cdots y_1y_my_1y_0$ is a closed ditrail with $|S'| > |S|$, contrary to (1).

Case 2. $(y_j, y_1) \notin A(D)$ for some $j \in [m]$. In this case, $(y_1, y_j) \in A(D)$ and $\{(y_i, y_1), (y_1, y_i)\} \subseteq A(D)$, where $i \in [m]$ and $i \notin \{1, j\}$. If $j = p$, that is $(y_p, y_1) \notin A(D)$, then by $d(y_p) = 2n-5$, we have $\{(y_p, y_{p+1}), (y_{p+1}, y_p)\} \subseteq A(D)$. Then $S' = y_0x_1y_{p+1}y_py_{p+1}y_1y_0y_1y_2 \cdots y_1y_{p-1}y_1y_{p+2}y_1y_{p+3} \cdots y_1y_my_1y_0$ is a closed ditrail with $|S'| > |S|$, contrary to (1). If $j = p+1$, then, for all $i \in [m]$ and $i \notin \{0, 1, p+1, p+2\}$, $S' = y_0x_1y_{p+1}y_{p+2}y_1y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with $|S'| > |S|$, contrary to (1). If $j \leq p-1$, then, for all $i \in [m]$ and $i \notin \{0, 1, \ldots, j+1, p+1\}$, $S' = y_0x_1y_{p+1}S_{[y_1,y_{j+1}]}y_1y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with $|S'| > |S|$, contrary to (1). Thus $j \geq p+2$, then $S' = y_0x_1y_{p+1}y_1y_jS_{[y_j,y_0]}y_1y_2y_1y_3 \cdots y_1y_py_1$ is a closed ditrail with $|S'| > |S|$, contrary to (1). This proves Theorem 6. ■

***Proof of Theorem* 7.** By the same arguments as in the proof of Theorem 6, we obtain

$$d(x_1) \leq 2(n-3) = 2n-6.$$

By assumption and since $\{y_1, x_1\}$ is a pair of dominated nonadjacent vertices, we see that $d(y_1) \geq 2n-4$ and $d(x_1) \geq 1$. This implies that $(z, y_1), (y_1, z) \in A(D)$ for any $z \in V(D) - \{y_1, x_1\}$. By the choice of T, y_1 and any vertex of $T_{[x_1,x_t]}$ are not adjacent. This together with $d(y_1) \geq 2n-4$, we obtain $t = 1$. Then, for all $j \in [m]$ and $j \notin \{0, 1, p+1\}$, $S' = y_0x_1y_{p+1}y_1y_0 \cup \{(y_1, y_j), (y_j, y_1)\}$ is a closed ditrail with $|S'| > |S|$, contrary to (1). This proves Theorem 7. ■

***Proof of Theorem* 8.** First, we show that D contains an (S, S)-dipath T with $V(T) = 3$.

Since D is strong, there exists a vertex $r \in V(R)$ and a vertex $y_i \in V(S)$ such that $(y_i, r) \in A(D)$, and a dipath P from r to a vertex $y_j \in V(S)$ such that $V(P) \cap V(S) = \{y_j\}$, where $i, j \in \{0, 1, \ldots, m\}$. If $(y_a, r) \in A(D)$ for all $a \in \{0, 1, \ldots, m\}$, then $S' = S \cup P + (y_j, r)$ is a closed ditrail with $|S'| > |S|$, contrary to the maximality of S. So there exists a vertex $y_a \in V(S)$ such that $(y_a, r) \notin A(D)$. Using this together with the fact that there exists a vertex $y_i \in V(S)$ such that $(y_i, r) \in A(D)$ we can conclude, without loss of generality, that $(y_i, r) \in A(D)$ but $(y_{i+1}, r) \notin A(D)$. If $(r, y_{i+1}) \in A(D)$, then $S' = S + (y_i, r) + (r, y_{i+1}) - (y_i, y_{i+1})$ is a closed ditrail with $|S'| > |S|$, contrary to the maximality of S. Therefore r and y_{i+1} are nonadjacent. By assumption, as the pair $\{r, y_{i+1}\}$ is dominated by y_i, we get $d^+(r) + d^-(y_{i+1}) \geq n-1$. Then we can obtain that there exists a vertex $z \in V(D) - \{r, y_{i+1}\}$ such that $\{(r, z), (z, y_{i+1})\} \subseteq A(D)$. By (1), we have that $z \in V(S)$. Then, $T = y_i r z$ is an (S, S)-dipath with three vertices.

Now choose an (S, S)-dipath T with $V(T) = 3$ such that the length of the ditrail P is minimum in S, where P is a shortest (u, v)-ditrail which travels along S from u to v such that the initial vertex u of P is the initial vertex of T and the terminal vertex v of P is the terminal vertex of T. Without loss of generality, write $T = y_0 x_1 y_{p+1}$, that is $u = y_0$ and $v = y_{p+1}$. Let $W = \{y_1, y_2, \ldots, y_p\}$ be the set of internal vertices of P, P_1 be a longest ditrail from y_{p+1} to y_0 in S. Then $|P_1| = s - p + c$, where $c = |W \cap P_1|$.

By the choice of T and (1), we have

$$d_W(x_1) = 0. \tag{9}$$

If D contains a (y_{p+1}, y_0)-ditrail P' with vertex set $V(P_1) \cup V(P_{[y_1,y_p]})$, then $S' = P' \cup T$ is a closed ditrail with $|S'| > |S|$, contrary to the maximality of S. Thus D does not have a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup V\big(P_{[y_1,y_p]}\big)$. Then by Lemma 9, we have

$$d^+_{P_1}(y_p) + d^-_{P_1}(y_1) \le |P_1|. \tag{10}$$

From (1) and Corollary 10, we have that D does not have a (y_{p+1}, y_0)-ditrail with vertex set $V(P_1) \cup \{x_1\}$ and

$$d_{P_1}(x_1) \le |P_1|. \tag{11}$$

By (1), there is no vertex $u \in R$ satisfying $\{(y_p, u), (u, x_1)\} \subseteq A(D)$ or $\{(x_1, u), (u, y_1)\} \subseteq A(D)$. Then

$$d^-_R(x_1) + d^+_R(y_p) + d^+_R(x_1) + d^-_R(y_1) \le 2(n - s - 1). \tag{12}$$

It is obvious that

$$d^+_{W-P_1}(y_p) + d^-_{W-P_1}(y_1) \le 2(p - c - 1). \tag{13}$$

By adding (9)–(13), note that $\{x_1, y_1\}$ is a pair of dominated nonadjacent vertices and $\{x_1, y_p\}$ is a pair of dominating nonadjacent vertices, we get

$$2n - 2 \le d^+(x_1) + d^-(y_1) + d^+(y_p) + d^-(x_1) \le 2n - 4,$$

a contradiction. This proves Theorem 8. ■

It is obvious that Theorems 6 and 7 are best possible in some sense, since the condition of Theorem 6 or 7 cannot be weakened by more than a constant, otherwise, it is not strong and impossible to be supereulerian. Now we demonstrate an example with the condition $d^+(u) + d^-(v) = d^+(v) + d^-(u) = n - 2$ for a pair of dominated (dominating) nonadjacent vertices $\{u, v\}$ which does not necessarily imply being supereulerian. Theorems 5 and 8 are thus best possible.

Example 11. We construct a strong digraph D with $V(D) = \{u, v, w, w'\} \cup V(K^*_{n-4})$ and the arcs of D are shown in (i) and (ii) below. (See Figure 2.)

(i) $w' \cup K^*_{n-4}$ is a complete digraph.

(ii) $(w', w) \in A(D)$, $N^+(w) = \{u, v\} \cup V(K^*_{n-4})$ and $N^+(u) = \{w'\} \cup V(K^*_{n-4}) = N^+(v)$.

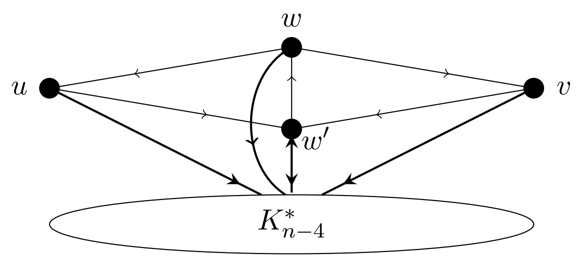


Figure 2. The strong digraph D.

It is not difficult to show that the digraph D of Figure 2 is nonsupereulerian. In fact, as $d^-_D(u) = d^-_D(v) = 1$ in D, any spanning eulerian subdigraph S (if it exists) of D has to contain the arcs $(w, u), (w, v)$, that is $d^+_S(w) = d^-_S(w) \geq 2$ in any spanning eulerian subdigraph S (if it exists) of D. And $d^-_D(w) \geq d^-_S(w)$. However $d^-_D(w) = 1$, so such a spanning eulerian subdigraph does not exist. Consequently, it is obvious that $d^+_D(u) + d^-_D(v) = d^+_D(v) + d^-_D(u) = n - 2$ and $\{u, v\}$ is the only pair of dominated or dominating nonadjacent vertices. Therefore, Example 11 demonstrates that there are infinitely many nonsupereulerian digraphs satisfying $d^+_D(u) + d^-_D(v) = d^+_D(v) + d^-_D(u) = n - 2$ for a pair of dominated or dominating nonadjacent vertices $\{u, v\}$. Thus conditions of Theorems 5 and 8 cannot be weakened by more than a constant. So the result of Theorems 5 and 8 are sharp.

3. Concluding Remarks

The remaining case of Problem 4 is $1 \leq t \leq n - 5$.

Conjecture 12. *There exists an integer t with $1 \leq t \leq n - 5$ so that any strong digraph with n vertices satisfying the condition C_t is supereulerian.*

We believe this can be generalized to the following. If the result is ture, then it, together with the fact that any strong semicomplete digraph is supereulerian, can be seen as a generalization of Theorem 2.

Conjecture 13. *If a strong digraph with n vertices satisfies $d(u)+d(v) \geq 2n-3$ for any pair of dominated or dominating nonadjacent vertices $\{u, v\}$, then it is supereulerian.*

By Theorems 6 and 7, we propose the following.

Conjecture 14. *There exists an integer t with $0 \leq t \leq n-5$ so that any strong digraph with n vertices satisfying the condition C_t, for any pair of dominated nonadjacent vertices $\{u, v\}$, is supereulerian.*

Conjecture 15. *If a strong digraph with n vertices satisfies $d(u)+d(v) \geq 2n-3$ for any pair of dominated nonadjacent vertices $\{u, v\}$, then it is supereulerian.*

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