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# DOMINATED PAIR DEGREE SUM CONDITIONS OF SUPEREULERIAN DIGRAPHS

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### Abstract

A digraph D is superculerian if D contains a spanning culerian subdigraph. In this paper, we propose the following problem: is there an integer t with  $0 \le t \le n-3$  so that any strong digraph with n vertices satisfying either both  $d(u) \ge n-1+t$  and  $d(v) \ge n-2-t$  or both  $d(u) \ge n-2-t$ and  $d(v) \ge n-1+t$ , for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$ , is superculerian? We prove the cases when t = 0, t = n-4and t = n-3. Moreover, we show that if a strong digraph D with n vertices satisfies  $\min\{d^+(u)+d^-(v), d^-(u)+d^+(v)\} \ge n-1$  for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$  of D, then D is superculerian.

**Keywords:** supereulerian digraph, spanning eulerian subdigraph, dominated pair degree sum condition.

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#### 1. INTRODUCTION

Digraphs considered are loopless and without parallel arcs. We refer the reader to [1] for digraphs for undefined terms and notation. In this paper, we define  $[k] = \{1, 2, \dots, k\}$  for an integer k > 0 and use (w, z) to denote an arc oriented from a vertex w to a vertex z and say that w dominates z. For any two vertices u, v in a digraph D, if  $(u, w), (v, w) \in A(D)$  for some  $w \in V(D)$ , then we say that  $\{u, v\}$  dominates w or call the pair  $\{u, v\}$  dominating; if  $(w, u), (w, v) \in A(D)$  for some  $w \in V(D)$ , then we say that  $\{u, v\}$  is dominated by w or call the pair  $\{u, v\}$ *dominated.* We often write *dipaths* for directed paths, *dicycles* for directed cycles and *ditrails* for directed trails in digraphs. The *length* of a ditrail is the number of its arcs. If a ditrail T starts at w and ends at z, we may call it a (w, z)-ditrail T or  $T_{[w,z]}$  and say w is the *initial vertex* of T and z is the *terminal vertex* of T. A (w, z)-ditrail of minimum length in D is called a *shortest* (w, z)-ditrail in D. We often write |D| for |V(D)| and use  $K_n^*$  to represent the *complete digraph* with *n* vertices. A digraph *D* is *semicomplete* if it has no pair of nonadjacent vertices. A digraph D is strong if any vertex of a digraph D is reachable from all other vertices of D.

Let  $T = v_1 v_2 \cdots v_k$  denote a ditrail. For any  $1 \leq i \leq j \leq k$ , we use  $T_{[v_i,v_j]} = v_i v_{i+1} \cdots v_{j-1} v_j$  to denote the *sub-ditrail* of T. Likewise, if  $Q = u_1 u_2 \cdots u_k u_1$  is a closed ditrail, then for any i, j with  $1 \leq i < j \leq k$ ,  $Q_{[u_i,u_j]}$  denotes the sub-ditrail  $u_i u_{i+1} \cdots u_{j-1} u_j$ . If  $T' = w_1 w_2 \cdots w_{k'}$  is a ditrail with  $v_k = w_1$  and  $V(T) \cap V(T') = \{v_k\}$ , then we use TT' or  $T_{[v_1,v_k]}T'_{[v_k,w_{k'}]}$  to denote the ditrail  $v_1 v_2 \cdots v_k w_2 \cdots w_{k'}$ . If  $V(T) \cap V(T') = \emptyset$  and there is a dipath  $z_1 z_2 \cdots z_t$  with  $z_2, \ldots, z_{t-1} \notin V(T) \cup V(T')$  and with  $z_1 = v_k$  and  $z_t = w_1$ , then we use  $Tz_1 \cdots z_t T'$  to denote the ditrail  $v_1 v_2 \cdots v_k z_2 \cdots v_k z_2 \cdots z_t w_2 \cdots w_{k'}$ . In particular, if T is a (v, w)-ditrail of a digraph D and  $(u, v), (w, z) \in A(D) - A(T)$ , then we use uvTwz to denote the (u, z)-ditrail  $D\langle A(T) \cup \{(u, v), (w, z)\}\rangle$ . The subdigraphs uvT and Twz are similarly defined.

For a digraph D,  $a \in A(D)$  and a subdigraph S of D, we use D - S to denote the subdigraph  $D\langle V(D) - V(S) \rangle$ , use D - a to denote the subdigraph  $D\langle A(D) - a \rangle$ , and use D + a to denote the subdigraph  $D\langle A(D) + a \rangle$ . Let  $D_1$  and  $D_2$  be two digraphs; the union  $D_1 \cup D_2$  of  $D_1$  and  $D_2$  is a digraph with vertex set  $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$  and arc set  $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$ . For  $S, T \subseteq V(D)$ , an (s, t)-dipath P is an (S, T)-dipath if  $s \in S$ ,  $t \in T$  and  $V(P) \cap (S \cup T) = \{s, t\}$ . Note that if  $S \cap T \neq \emptyset$ , then a vertex  $s \in S \cap T$  forms an (S, T)-dipath by itself. When S and T are subdigraphs of D, we also talk about an (S, T)-dipath.

Let  $d_D^-(s), d_D^+(s), d_D(s) = d_D^-(s) + d_D^+(s), N_D^-(s)$  and  $N_D^+(s)$  denote, respectively, the *in-degree, out-degree, degree, in-neighbourhood* and *out-neighbourhood* of a vertex  $s \in V(D)$ .

In [3], Boesch *et al.* raised the supereulerian problem, which strives to describe graphs that contain spanning eulerian subgraphs. In [10], Pulleyblank showed that deciding whether a graph is supereulerian, even within planar graphs, is NP-complete. There have been many studies on this topic, as revealed in the surveys [4, 5] and [9].

It is natural to try to relate supereulerian graphs to supereulerian digraphs. A digraph D is supereulerian if it contains a closed ditrail S with V(S) = V(D), i.e., it has a spanning eulerian subdigraph, and nonsupereulerian otherwise. Results on supereulerian digraphs can be found in [2, 6, 7, 8], among others. It is worth pointing out that only a few of degree sum conditions are studied to ensure supereulerianicity in digraphs. In particular, the following have been proved.

**Theorem 1** [2]. If a strong digraph D with n vertices satisfies  $d^+(u) + d^-(v) \ge n-1$  for any two vertices u and v with  $(u,v) \notin A(D)$ , then D is superculerian.

**Theorem 2** [2]. If a strong digraph D with n vertices satisfies  $d(u)+d(v) \ge 2n-3$  for any two nonadjacent vertices u and v, then D is supereulerian.

It is observed that in Theorems 1 and 2, degree sum conditions on every pairs of nonadjacent vertices are needed to warrant the digraph to be supereulerian. In this article, we will consider a degree sum condition about pairs of dominated (dominating) nonadjacent vertices but no longer on all pairs of nonadjacent vertices. First, we give the following definition.

**Definition 3.** Given an integer  $t \ge 0$ , we say a digraph D of order n satisfies the condition  $C_t$  if

$$d(u) \ge n - 1 + t, d(v) \ge n - 2 - t \text{ or } d(u) \ge n - 2 - t, d(v) \ge n - 1 + t,$$

for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$  in D.

If u and v are nonadjacent for  $u, v \in V(D)$ , then  $d(u) \leq 2n - 4$  and  $d(v) \leq 2n - 4$ . Thus  $n - 1 + t \leq 2n - 4$  implies  $t \leq n - 3$ . Then we have  $0 \leq t \leq n - 3$  and naturally propose the following problem.

**Problem 4.** Is there an integer t with  $0 \le t \le n-3$  so that any strong digraph with n vertices satisfying the condition  $C_t$  is supercularian?

Problem 4 assumes the existence of nonadjacent vertices. When a strong digraph D contains no nonadjacent vertices, condition  $C_t$  is automatically satisfied, and in this case, by Theorem 1.5.3 of [1], D is hamiltonian and so superculerian. Hence it suffices to settle Problem 4 for strong digraphs which are not semicomplete digraphs. Likewise, when we discuss Theorems 5, 6, 7, 8, we may also assume the digraph under consideration is not a semicomplete digraph. The purpose of this paper is to prove the cases when t = 0, t = n - 4and t = n - 3 and show that if a strong digraph D with n vertices satisfies  $\min\{d^+(u) + d^-(v), d^-(u) + d^+(v)\} \ge n - 1$  for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$  of D, then D is supercularian. Moreover, our result, Theorem 8, generalizes Theorem 1. The main results are the following, which are independent of Theorem 2.

**Theorem 5.** If a strong digraph D with n vertices satisfies the condition  $C_0$ , then D is supereulerian.

**Theorem 6.** If a strong digraph D with n vertices satisfies the condition  $C_{n-4}$  for any pair of dominated nonadjacent vertices  $\{u, v\}$ , then D is supereulerian.

**Theorem 7.** If a strong digraph D with n vertices satisfies the condition  $C_{n-3}$  for any pair of dominated nonadjacent vertices  $\{u, v\}$ , then D is supereulerian.

**Theorem 8.** If a strong digraph D with n vertices satisfies, for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$ ,  $\min\{d^+(u)+d^-(v), d^-(u)+d^+(v)\} \ge n-1$ , then D is superculerian.

In Section 2, we give the proofs of Theorems 5–8 and shall display examples of nonsupereulerian digraphs to demonstrate the sharpness of our results in some sense. The last section is devoted to some concluding remarks.

## 2. Main Results

The following lemmas will be useful.

**Lemma 9.** Let D be a digraph,  $S = u_1 u_2 \cdots u_s$  and  $T = v_1 v_2 \cdots v_t$  be two arc disjoint ditrails of D. If D does not contain a  $(u_1, u_s)$ -ditrail with vertex set  $V(S) \cup V(T)$ , then  $d_S^-(v_1) + d_S^+(v_t) \leq |S|$ .

**Proof.** As D does not contain a  $(u_1, u_s)$ -ditrail with vertex set  $V(S) \cup V(T)$ , we have  $|\{(u_i, v_1), (v_t, u_i)\} \cap A(D)| \leq 1$  for any  $u_i \in V(S)$ . Accordingly, we obtain  $d_S^-(v_1) + d_S^+(v_t) \leq |S|$  as required.

**Corollary 10.** Let D be a digraph,  $S = u_1 u_2 \cdots u_s$  be a ditrail in D and  $x \in V(D) - V(S)$ . If D does not contain a  $(u_1, u_s)$ -ditrail with vertex set  $V(S) \cup \{x\}$ , then  $d_S(x) \leq |S|$ .

Throughout the proofs of Theorems 5–8, we let D denote a strong nonsupereulerian digraph with n vertices, and let  $S = \{S_1, \ldots, S_k\}$  be the collection of closed ditrails such that  $|V(S_1)| = \cdots = |V(S_k)|$  is maximized in D (possibly

k = 1). Let  $S = y_0 y_1 \cdots y_p y_{p+1} \cdots y_m y_0$  be the closed ditrail such that |A(S)| is maximized in S. Thus

(1) |V(S)| is maximized in D and |A(S)| is maximized in S.

Let |V(S)| = s. As D is not supercularian, 1 < s < n. Since D is strong, there exists an (S, S)-dipath T with  $|T| \ge 3$ .

Throughout the proofs of Theorems 5–7, for an integer  $l \geq 1$  and  $i \in [l]$ , let  $T_i$  be an (S, S)-dipath with  $|T_i| \geq 3$  such that  $|V(P_i)|$  of the ditrail  $P_i$  is minimum in S, where  $|V(P_1)| = \cdots = |V(P_l)|$  and  $P_i$  is a shortest  $(u_i, v_i)$ -ditrail which travels along S from  $u_i$  to  $v_i$  such that the initial vertex  $u_i$  of  $P_i$  is the initial vertex of  $T_i$  and the terminal vertex  $v_i$  of  $P_i$  is the terminal vertex of  $T_i$ . Choose an (S, S)-dipath T with  $|T| \geq 3$  in  $\{T_1, \ldots, T_l\}$  such that |A(P)| of the ditrail P is minimum in  $\{P_1, \ldots, P_l\}$ . Thus |V(P)| is minimum in S and |A(P)| is minimum in  $\{P_1, \ldots, P_l\}$ . Without loss of generality, write  $T = y_0 x_1 x_2 \cdots x_t y_{p+1}$ . Let  $W = \{y_1, y_2, \ldots, y_p\}$  be the set of internal vertices of P, P' be the ditrail which travels along S from  $y_{p+1}$  to  $y_0, r$  be the maximum integer  $1 \leq i \leq p$  such that D contains a  $(y_{p+1}, y_0)$ -ditrail  $P_1$  with vertex set  $V(P_1) = V(P') \cup \{y_0, y_1, \ldots, y_{r-1}\}$  and R = D - S. Then |P'| = s - p + c', and  $|P_1| = s - p + c$ , where  $c' = |W \cap P'|$  and  $c = |W \cap P_1|$ .



A nonsuperculerian strong digraph D

Figure 1. The illustration for the proofs of Theorems 5–7.

**Proof of Theorem 5.** By (1), we have  $y_0 \neq y_{p+1}$ ,  $(y_0, y_{p+1}) \notin A(S)$  and  $p \ge 1$ . This together with the fact that P is a  $(y_0, y_{p+1})$ -ditrail implies  $d_P^+(y_0) - d_P^-(y_0) = 1$  and  $d_P^-(y_{p+1}) - d_P^+(y_{p+1}) = 1$ . If for an integer  $k \ge 1$ ,  $d_P^+(y_0) = k + 1 \ge 2$ , then  $d_P^-(y_0) = k$ . Hence we can denote the ditrail  $P = y_0 \cdots y_0^1 \cdots y_0^2 \cdots y_0^k \cdots y_{p+1}$ , where  $y_0 = y_0^1 = y_0^2 = \cdots = y_0^k$ . For any  $h \in [k]$ ,  $P_{[y_0^h, y_{p+1}]} = y_0^h \cdots y_{p+1}$  is also a  $(y_0, y_{p+1})$ -ditrail which travels along S from  $y_0$  to  $y_{p+1}$ . If  $\left| V \left( P_{[y_0^h, y_{p+1}]} \right) \right| = |V(P)|$ , then  $\left| A \left( P_{[y_0^h, y_{p+1}]} \right) \right| < |A(P)|$ , contrary to the choice of P. If  $\left| V \left( P_{[y_0^h, y_{p+1}]} \right) \right| \neq |V(P)|, \text{ then } \left| V \left( P_{[y_0^h, y_{p+1}]} \right) \right| < |V(P)| \text{ and } \left| A \left( P_{[y_0^h, y_{p+1}]} \right) \right| < |A(P)|, \text{ contrary to the choice of } P. \text{ Hence } k = 0, d_P^+(y_0) = 1 \text{ and } d_P^-(y_0) = 0. \text{ By similar arguments, we can get that } d_P^-(y_{p+1}) = 1 \text{ and } d_P^+(y_{p+1}) = 0. \text{ Therefore,}$ 

(2) 
$$d_P^+(y_0) = d_P^-(y_{p+1}) = 1 \text{ and } d_P^-(y_0) = d_P^+(y_{p+1}) = 0.$$

First for any  $i \in [t]$  we have  $d_W(x_i) = 0$  by the choice of T and (1). If  $d_W^+(x_i) > 0$  or  $d_W^-(x_i) > 0$ , then without loss of generality, we may assume that  $d_W^+(x_i) > 0$ , and so there exists a vertex  $y_j \in W$   $(j \in [p])$  such that  $(x_i, y_j) \in A(D)$ . If  $2 \leq j \leq p$ , then we can get another (S, S)-dipath T' with the initial vertex  $y_0$  and the terminal vertex  $y_j$  such that the length of  $(y_0, y_j)$ -ditrail P' in S is less then the length of P in S, contrary to the choice of T above. If j = 1, then  $S \cup T_{[y_0,x_i]} + (x_i, y_1) - (y_0, y_1)$  is a closed ditrail with  $|S \cup T_{[y_0,x_i]} + (x_i, y_1) - (y_0, y_1)| > |S|$ , contrary to (1). Therefore  $d_W^+(x_i) = 0$ . The proof for  $d_W^-(x_i) = 0$  is similar. In particular,  $x_i$  and  $y_j$  are nonadjacent, for  $i \in [t]$  and  $j \in [p]$ .

By the definition of P',  $x_i \notin V(P')$ . If for some  $x_i \in V(T)$ , D contains a  $(y_{p+1}, y_0)$ -ditrail S' with vertex set  $V(P') \cup \{x_i\}$ , then  $S' \cup P$  is a closed ditrail with  $|S' \cup P| > |S|$ , contrary to (1). Thus D does not have a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P') \cup \{x_i\}$ , for any  $x_i \in V(T)$ . By Corollary 10 and  $d_W(x_i) = 0$ , we can deduce that

(3) 
$$d_S(x_i) = d_{P'}(x_i) = d_{P'-W}(x_i) \le |P'| - c' = s - p.$$

Obviously,

(4) 
$$d_{W-P_1}(y_i) \le 2(p-c-1).$$

Furthermore, by the choice of T and (1), there is no vertex  $z \in R$  satisfying  $\{(y_j, z), (z, x_i)\} \subseteq A(D)$  or  $\{(x_i, z), (z, y_j)\} \subseteq A(D)$ , for  $i \in [t]$  and  $j \in [p]$ . Accordingly,

(5) 
$$d_R(x_i) + d_R(y_i) \le 2(n-s-1)$$

Let  $y_k$  be any vertex in W such that  $(y_0, y_k) \in A(D)$ . Combining (3)–(5) with the fact that the pair of nonadjacent vertices  $\{x_1, y_k\}$  is dominated by  $y_0$  and the pair of nonadjacent vertices  $\{x_t, y_p\}$  dominates  $y_{p+1}$ , we get

$$2n - 3 \le d(x_1) + d(y_k) \le d_{P_1}(y_k) + 2n - |P_1| - 4 - c$$

and

$$2n - 3 \le d(x_t) + d(y_p) \le d_{P_1}(y_p) + 2n - |P_1| - 4 - c.$$

Accordingly,

(6) 
$$d_{P_1}(y_k) \ge |P_1| + 1 + c \text{ and } d_{P_1}(y_p) \ge |P_1| + 1 + c.$$

Now we consider two cases in the following.

Case 1.  $c = |W \cap P_1| = 0$ . In this case, we have  $W \cap P_1 = \emptyset$  and  $S = P + P_1$ . By (1), D does not have a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup V(P_{[y_1, y_p]})$ . Then by Lemma 9, we get

(7) 
$$d_{P_1}^-(y_1) + d_{P_1}^+(y_p) \le |P_1| = s - p.$$

By symmetry, assume that  $y_1 = y_k$ . If  $y_1 = y_p$ , then  $d_{P_1}(y_1) \leq |P_1|$ , contrary to (6). Thus  $y_1 \neq y_p$ . By (6),  $d_{P_1}(y_1) \geq |P_1| + 1 + c$ . There must exist vertices  $y_a, y_c \in V(P_1)$  such that  $\{(y_a, y_1), (y_1, y_a), (y_p, y_c), (y_c, y_p)\} \subseteq A(D)$ . Since  $W \cap P_1 = \emptyset$  and  $S = P + P_1$ , we have  $y_1, y_p \in W, y_1, y_p \notin V(P_1), y_a, y_c \in V(P_1)$  and  $y_a, y_c \notin W$ . Then  $(y_a, y_1), (y_1, y_a), (y_p, y_c), (y_c, y_p) \notin A(P_1)$ .

By (2), we have  $d_P^-(y_{p+1}) = |\{(y_p, y_{p+1})\} \cap A(D)| = 1$  and  $d_P^+(y_{p+1}) = 0$ . If  $y_a = y_{p+1}$ , then as  $y_p \neq y_1$ , we have  $(y_{p+1}, y_1), (y_1, y_{p+1}) \notin A(P)$ . Therefore  $(y_a, y_1), (y_1, y_a) \notin A(S)$ . But then we can get a closed ditrail  $S' = S \cup$  $\{(y_a, y_1), (y_1, y_a)\}$  with |A(S')| > |A(S)|, contrary to (1). Thus  $y_a \neq y_{p+1}$ . Similarly, we can get that  $y_c \neq y_0$ .

If  $y_a \neq y_0$ , then  $(y_1, y_a), (y_a, y_1) \notin A(P)$ . Therefore  $(y_a, y_1), (y_1, y_a) \notin A(S)$ . But then we can get a closed ditrail  $S' = S \cup \{(y_a, y_1), (y_1, y_a)\}$  with |A(S')| > |A(S)|, contrary to (1). Thus  $y_a = y_0$ . Similarly, we can get that  $y_c = y_{p+1}$ . Then  $\{(y_0, y_1), (y_1, y_0), (y_p, y_{p+1}), (y_{p+1}, y_p)\} \subseteq A(D) - A(P_1)$ .

By (1) with (6), for any  $y_i \in V(P_1) - y_0$  and  $y_j \in V(P_1) - y_{p+1}$ , we have  $|\{(y_i, y_1), (y_1, y_i) \cap A(D)\}| = 1$  and  $|\{(y_j, y_p), (y_p, y_j)\} \cap A(D)| = 1$ . By (2),  $(y_1, y_0), (y_{p+1}, y_p) \notin A(P)$ . Then  $(y_1, y_0), (y_{p+1}, y_p) \notin A(S)$ . If  $(y_m, y_1) \in A(D)$ , note that  $(y_m, y_1) \notin A(S)$ , then we can get a closed ditrail  $S' = S + (y_m, y_1) + (y_1, y_0) - (y_m, y_0)$  with |A(S')| > |A(S)|, contrary to (1). Thus  $(y_m, y_1) \notin A(D)$  and  $(y_1, y_m) \in A(D)$ . Continuing this process, we finally conclude that for any  $y_i \in V(P_1) - y_0, (y_i, y_1) \notin A(D)$  and  $(y_1, y_i) \in A(D)$ . Similarly, we can get that for any  $y_j \in V(P_1) - y_{p+1}, (y_p, y_j) \notin A(D)$  and  $(y_j, y_p) \in A(D)$ . In particular,  $(y_{p+1}, y_1) \notin A(D)$ .

Now we have  $d_{P_1}^+(y_1) = |P_1| = d_{P_1}^-(y_p)$  and  $d_{P_1}^-(y_1) = 1 = d_{P_1}^+(y_p)$ . Combining (3) with the fact that  $d_R(x_1) + d_R(y_1) \le 2(n - s - 1)$  and the assumption of the theorem, note that the pair of nonadjacent vertices  $\{x_1, y_1\}$  is dominated by  $y_0$ , we obtain  $d_S(y_1) \ge s + p - 1$ . Since  $d_{P_1}(y_1) = |P_1| + 1 = s - p + 1$ ,  $d_W(y_1) \ge 2(p - 1)$ . That is, for any  $y_j \in W$ ,  $(y_j, y_1), (y_1, y_j) \in A(D)$ . But then we can get a closed ditrail  $S' = T \cup P_1 \cup \{(y_j, y_1), (y_1, y_j)\} \cup \{(y_0, y_1), (y_1, y_0)\}$ , for every  $y_j \in W$ , with |S'| > |S|, contrary to (1).

Case 2.  $c = |W \cap P_1| \ge 1$ . From (6) and Corollary 10 it follows that D contains a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_k\}$ . Note that  $V(P_1) = V(P') \cup \{y_0, y_1, \ldots, y_{r-1}\}$ . By (1),  $r-1 \le p-1$ . Then there exists an integer r with  $2 \le r \le p$  such that D contains a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_n, y_n, \ldots, y_{r-1}\}$ .

 $\{y_{r-1}\}$  but does not contain a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_r\}$ . In particular,  $y_{r-1} \in V(P_1)$ . Therefore D does not contain a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup V(S_{[y_{r-1}, y_r]})$ .

By Corollary 10 and the fact that D does not contain a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_r\}, d_{P_1}(y_r) \leq |P_1| = s - p + c$ . By (6),  $y_r \neq y_k$ , that is  $(y_0, y_r) \notin A(D)$ . From this with (3)–(5), we obtain  $d(y_r) + d(x_1) \leq 2n - 5$ . By assumption, note that  $\{x_1, y_1\}$  is a pair of dominated nonadjacent vertices, we have  $d(x_1) \geq n - 2$ . Therefore, we get

$$(8) d(y_r) \le n-3.$$

Note that  $y_{r-1} \in V(P_1)$  and  $y_r \notin V(P_1)$ . Let  $P_1 = y_{p+1}y_{a-d}y_{a-d+1}\cdots y_{a-2}y_{r-1}$  $y_a y_{a+1} \cdots y_{a+l} y_0$ .

Therefore by (8) and the assumption of the theorem, as the pair  $\{y_r, y_a\}$  is dominated by  $y_{r-1}$ , we have that  $y_r$  and  $y_a$  are adjacent. If  $(y_r, y_a) \in A(D)$ , then D contains a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup V(S_{[y_{r-1}, y_r]})$ , a contradiction. So assume that  $(y_a, y_r) \in A(D)$ . Then the pair  $\{y_{a+1}, y_r\}$  is dominated by  $y_a$ , we similarly conclude that  $(y_{a+1}, y_r) \in A(D)$ . Continuing this process, we can deduce that  $(y_0, y_r) \in A(D)$ , which contradicts the conclusion above  $(y_0, y_r) \notin A(D)$ . This proves Theorem 5.

**Proof of Theorem 6.** By similar arguments as in the proof of Theorem 5, we obtain that  $d_W(x_i) = 0$  for any  $i \in [t]$  and  $d_{P_1}(y_1) \ge |P_1| + 1$ . From this and Corollary 10 it follows that D contains a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_1\}$ . Since D does not contain a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{y_r\}$ ,  $|W| \ge 2$ . This together with  $d_W(x_1) = 0$  implies

$$d(x_1) \le 2(n-3) = 2n - 6.$$

Since  $\{y_1, x_1\}$  is a pair of dominated nonadjacent vertices and by assumption, we get that  $d(y_1) \ge 2n - 5$  and  $d(x_1) \ge 2$ . By the choice of T,  $y_1$  and any vertex of  $T_{[x_1,x_t]}$  are not adjacent. This together with  $d(y_1) \ge 2n - 5$ , we obtain t = 1. As we known the pair of nonadjacent vertices  $\{y_1, x_1\}$  satisfying  $d(y_1) \le 2n - 4$ and  $d(x_1) \le 2n - 4$ . Then we have  $2n - 5 \le d(y_1) \le 2n - 4$ . If  $d(y_1) = 2n - 4$ , then we get  $(z, y_1), (y_1, z) \in A(D)$  for any  $z \in V(D) - \{y_1, x_1\}$ . Thus, for all  $j \in [m]$  and  $j \notin \{0, 1, p + 1\}, S' = y_0 x_1 y_{p+1} y_1 y_0 \cup \{(y_1, y_j), (y_j, y_1)\}$  is a closed ditrail with |S'| > |S|, contrary to (1). So assume that  $d(y_1) = 2n - 5$ . We can assume that  $d(y_p) = 2n - 5$  by similar arguments as above.

If  $\{(y_1, y_j), (y_j, y_1)\} \subseteq A(D)$  for any  $j \in [m]$ , then, for all  $i \in [m]$  and  $i \notin \{0, 1, p+1\}, S' = y_0 x_1 y_{p+1} y_1 y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$  is a closed ditrail with |S'| > |S|, a contradiction. Thus, we consider two cases in the following.

Case 1.  $(y_1, y_j) \notin A(D)$  for some  $j \in [m]$ . In this case,  $(y_j, y_1) \in A(D)$ and  $\{(y_i, y_1), (y_1, y_i)\} \subseteq A(D)$ , where  $i \in [m]$  and  $i \notin \{1, j\}$ . If j = 0, then,

for all  $i \in [m]$  and  $i \notin \{0, 1, p+1, m\}$ ,  $S' = y_0 x_1 y_{p+1} y_1 y_m y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with |S'| > |S|, a contradiction. Thus  $j \neq 0$ . If j < p+1, then, for all  $i \in [m]$  and  $i \notin \{0, 1, \ldots, j, p+1\}$ ,  $S' = y_0 x_1 y_{p+1} S_{[y_1, y_j]} y_1 y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$  is a closed ditrail with |S'| > |S|, contrary to (1). So  $j \ge p+1$ . Then  $S' = y_0 x_1 y_{p+1} S_{[y_{p+1}, y_j]} y_1 y_2 y_1 y_3 \cdots y_1 y_p y_1 y_{j+1} y_1 y_{j+2} \cdots y_1 y_m y_1 y_0$  is a closed ditrail with |S'| > |S|, contrary to (1).

Case 2.  $(y_j, y_1) \notin A(D)$  for some  $j \in [m]$ . In this case,  $(y_1, y_j) \in A(D)$ and  $\{(y_i, y_1), (y_1, y_i)\} \subseteq A(D)$ , where  $i \in [m]$  and  $i \notin \{1, j\}$ . If j = p, that is  $(y_p, y_1) \notin A(D)$ , then by  $d(y_p) = 2n - 5$ , we have  $\{(y_p, y_{p+1}), (y_{p+1}, y_p)\} \subseteq$ A(D). Then  $S' = y_0 x_1 y_{p+1} y_p y_{p+1} y_1 y_0 y_1 y_2 \cdots y_1 y_{p-1} y_1 y_{p+2} y_1 y_{p+3} \cdots y_1 y_m y_1 y_0$  is a closed ditrail with |S'| > |S|, contrary to (1). If j = p + 1, then, for all  $i \in [m]$  and  $i \notin \{0, 1, p + 1, p + 2\}$ ,  $S' = y_0 x_1 y_{p+1} y_{p+2} y_1 y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$  is a closed ditrail with |S'| > |S|, contrary to (1). If  $j \leq p - 1$ , then, for all  $i \in [m]$ and  $i \notin \{0, 1, \dots, j + 1, p + 1\}$ ,  $S' = y_0 x_1 y_{p+1} S_{[y_1, y_{j+1}]} y_1 y_0 \cup \{(y_1, y_i), (y_i, y_1)\}$ is a closed ditrail with |S'| > |S|, contrary to (1). Thus  $j \geq p + 2$ , then  $S' = y_0 x_1 y_{p+1} y_1 y_1 S_{[y_j, y_0]} y_1 y_2 y_1 y_3 \cdots y_1 y_p y_1$  is a closed ditrail with |S'| > |S|, contrary to (1). Thus  $j \geq p + 2$ , then  $S' = y_0 x_1 y_{p+1} y_1 y_1 S_{[y_j, y_0]} y_1 y_2 y_1 y_3 \cdots y_1 y_p y_1$  is a closed ditrail with |S'| > |S|, contrary to (1). Thus  $j \geq p + 2$ , then  $S' = y_0 x_1 y_{p+1} y_1 y_1 S_{[y_j, y_0]} y_1 y_2 y_1 y_3 \cdots y_1 y_p y_1$  is a closed ditrail with |S'| > |S|, contrary to (1). Thus  $y_1 \geq |S|$ , contrary to (1). This proves Theorem 6.

**Proof of Theorem 7.** By the same arguments as in the proof of Theorem 6, we obtain

$$d(x_1) \le 2(n-3) = 2n - 6.$$

By assumption and since  $\{y_1, x_1\}$  is a pair of dominated nonadjacent vertices, we see that  $d(y_1) \ge 2n-4$  and  $d(x_1) \ge 1$ . This implies that  $(z, y_1), (y_1, z) \in A(D)$ for any  $z \in V(D) - \{y_1, x_1\}$ . By the choice of  $T, y_1$  and any vertex of  $T_{[x_1, x_t]}$  are not adjacent. This together with  $d(y_1) \ge 2n - 4$ , we obtain t = 1. Then, for all  $j \in [m]$  and  $j \notin \{0, 1, p + 1\}, S' = y_0 x_1 y_{p+1} y_1 y_0 \cup \{(y_1, y_j), (y_j, y_1)\}$  is a closed ditrail with |S'| > |S|, contrary to (1). This proves Theorem 7.

**Proof of Theorem 8.** First, we show that D contains an (S, S)-dipath T with V(T) = 3.

Since D is strong, there exists a vertex  $r \in V(R)$  and a vertex  $y_i \in V(S)$ such that  $(y_i, r) \in A(D)$ , and a dipath P from r to a vertex  $y_j \in V(S)$  such that  $V(P) \cap V(S) = \{y_j\}$ , where  $i, j \in \{0, 1, \ldots, m\}$ . If  $(y_a, r) \in A(D)$  for all  $a \in \{0, 1, \ldots, m\}$ , then  $S' = S \cup P + (y_j, r)$  is a closed ditrail with |S'| > |S|, contrary to the maximality of S. So there exists a vertex  $y_a \in V(S)$  such that  $(y_a, r) \notin A(D)$ . Using this together with the fact that there exists a vertex  $y_i \in V(S)$  such that  $(y_i, r) \in A(D)$  we can conclude, without loss of generality, that  $(y_i, r) \in A(D)$  but  $(y_{i+1}, r) \notin A(D)$ . If  $(r, y_{i+1}) \in A(D)$ , then  $S' = S + (y_i, r) + (r, y_{i+1}) - (y_i, y_{i+1})$ is a closed ditrail with |S'| > |S|, contrary to the maximality of S. Therefore r and  $y_{i+1}$  are nonadjacent. By assumption, as the pair  $\{r, y_{i+1}\}$  is dominated by  $y_i$ , we get  $d^+(r) + d^-(y_{i+1}) \ge n - 1$ . Then we can obtain that there exists a vertex  $z \in V(D) - \{r, y_{i+1}\}$  such that  $\{(r, z), (z, y_{i+1})\} \subseteq A(D)$ . By (1), we have that  $z \in V(S)$ . Then,  $T = y_i r z$  is an (S, S)-dipath with three vertices.

Now choose an (S, S)-dipath T with V(T) = 3 such that the length of the ditrail P is minimum in S, where P is a shortest (u, v)-ditrail which travels along S from u to v such that the initial vertex u of P is the initial vertex of T and the terminal vertex v of P is the terminal vertex of T. Without loss of generality, write  $T = y_0 x_1 y_{p+1}$ , that is  $u = y_0$  and  $v = y_{p+1}$ . Let  $W = \{y_1, y_2, \ldots, y_p\}$  be the set of internal vertices of P,  $P_1$  be a longest ditrail from  $y_{p+1}$  to  $y_0$  in S. Then  $|P_1| = s - p + c$ , where  $c = |W \cap P_1|$ .

By the choice of T and (1), we have

(9) 
$$d_W(x_1) = 0.$$

If D contains a  $(y_{p+1}, y_0)$ -ditrail P' with vertex set  $V(P_1) \cup V(P_{[y_1, y_p]})$ , then  $S' = P' \cup T$  is a closed ditrail with |S'| > |S|, contrary to the maximality of S. Thus D does not have a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup V(P_{[y_1, y_p]})$ . Then by Lemma 9, we have

(10) 
$$d_{P_1}^+(y_p) + d_{P_1}^-(y_1) \le |P_1|.$$

From (1) and Corollary 10, we have that D does not have a  $(y_{p+1}, y_0)$ -ditrail with vertex set  $V(P_1) \cup \{x_1\}$  and

(11) 
$$d_{P_1}(x_1) \le |P_1|.$$

By (1), there is no vertex  $u \in R$  satisfying  $\{(y_p, u), (u, x_1)\} \subseteq A(D)$  or  $\{(x_1, u), (u, y_1)\} \subseteq A(D)$ . Then

(12) 
$$d_R^-(x_1) + d_R^+(y_p) + d_R^+(x_1) + d_R^-(y_1) \le 2(n-s-1).$$

It is obvious that

(13) 
$$d_{W-P_1}^+(y_p) + d_{W-P_1}^-(y_1) \le 2(p-c-1).$$

By adding (9)–(13), note that  $\{x_1, y_1\}$  is a pair of dominated nonadjacent vertices and  $\{x_1, y_p\}$  is a pair of dominating nonadjacent vertices, we get

$$2n - 2 \le d^+(x_1) + d^-(y_1) + d^+(y_p) + d^-(x_1) \le 2n - 4,$$

a contradiction. This proves Theorem 8.

It is obvious that Theorems 6 and 7 are best possible in some sense, since the condition of Theorem 6 or 7 cannot be weakened by more than a constant, otherwise, it is not strong and impossible to be superculerian. Now we demonstrate an example with the condition  $d^+(u) + d^-(v) = d^+(v) + d^-(u) = n - 2$ for a pair of dominated (dominating) nonadjacent vertices  $\{u, v\}$  which does not necessarily imply being superculerian. Theorems 5 and 8 are thus best possible. **Example 11.** We construct a strong digraph D with  $V(D) = \{u, v, w, w'\} \cup V(K_{n-4}^*)$  and the arcs of D are shown in (i) and (ii) below. (See Figure 2.)

- (i)  $w' \cup K_{n-4}^*$  is a complete digraph.
- (ii)  $(w', w) \in A(D), N^+(w) = \{u, v\} \cup V(K^*_{n-4}) \text{ and } N^+(u) = \{w'\} \cup V(K^*_{n-4}) = N^+(v).$



Figure 2. The strong digraph D.

It is not difficult to show that the digraph D of Figure 2 is nonsupereulerian. In fact, as  $d_D^-(u) = d_D^-(v) = 1$  in D, any spanning eulerian subdigraph S (if it exists) of D has to contain the arcs (w, u), (w, v), that is  $d_S^+(w) = d_S^-(w) \ge 2$  in any spanning eulerian subdigraph S (if it exists) of D. And  $d_D^-(w) \ge d_S^-(w)$ . However  $d_D^-(w) = 1$ , so such a spanning eulerian subdigraph does not exist. Consequently, it is obvious that  $d_D^+(u) + d_D^-(v) = d_D^+(v) + d_D^-(u) = n - 2$  and  $\{u, v\}$  is the only pair of dominated or dominating nonadjacent vertices. Therefore, Example 11 demonstrates that there are infinitely many nonsupereulerian digraphs satisfying  $d_D^+(u) + d_D^-(v) = d_D^+(v) + d_D^-(u) = n - 2$  for a pair of dominated or dominating nonadjacent vertices  $\{u, v\}$ . Thus conditions of Theorems 5 and 8 cannot be weakened by more than a constant. So the result of Theorems 5 and 8 are sharp.

### 3. Concluding Remarks

The remaining case of Problem 4 is  $1 \le t \le n-5$ .

**Conjecture 12.** There exists an integer t with  $1 \le t \le n-5$  so that any strong digraph with n vertices satisfying the condition  $C_t$  is supereulerian.

We believe this can be generalized to the following. If the result is ture, then it, together with the fact that any strong semicomplete digraph is supereulerian, can be seen as a generalization of Theorem 2. **Conjecture 13.** If a strong digraph with n vertices satisfies  $d(u) + d(v) \ge 2n - 3$  for any pair of dominated or dominating nonadjacent vertices  $\{u, v\}$ , then it is supereulerian.

By Theorems 6 and 7, we propose the following.

**Conjecture 14.** There exists an integer t with  $0 \le t \le n-5$  so that any strong digraph with n vertices satisfying the condition  $C_t$ , for any pair of dominated nonadjacent vertices  $\{u, v\}$ , is supereulerian.

**Conjecture 15.** If a strong digraph with n vertices satisfies  $d(u) + d(v) \ge 2n - 3$  for any pair of dominated nonadjacent vertices  $\{u, v\}$ , then it is superculerian.

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