

ON WALK DOMINATION: WEAKLY TOLL DOMINATION, l_2 AND l_3 DOMINATION

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Abstract

In this paper we study domination between different types of walks connecting two non-adjacent vertices of a graph. In particular, we center our attention on weakly toll walk and l_k -path for $k \in \{2, 3\}$. A walk between two non-adjacent vertices in a graph G is called a weakly toll walk if the first and the last vertices in the walk are adjacent, respectively, only to the second and second-to-last vertices, which may occur more than once in the walk. And an l_k -path is an induced path of length at most k between two non-adjacent vertices in a graph G . We study the domination between weakly toll walks, l_k -paths ($k \in \{2, 3\}$) and different types of walks connecting two non-adjacent vertices u and v of a graph (shortest paths, induced paths, paths, tolled walks, weakly toll walks, l_k -paths for $k \in \{3, 4\}$), and show how these give rise to characterizations of graph classes.

Keywords: domination, paths, geodesic, chordal graphs, interval graphs.

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1. INTRODUCTION

Walks in graphs are subgraphs that tell us about topological structure of graphs.

From the trivial connectivity till no-trivial geometries [2–6, 10], the walks in graphs are studied. For example, chordal graphs and ptolemaic graphs have been

characterized as convex geometries with respect to the monophonic convexity and the geodesic convexity, respectively [5]. Similarly, interval graphs have been characterized as convex geometries with respect to the toll convexity [2]; and proper interval graphs have been characterized as convex geometries with respect to the weakly toll convexity [4].

In the present paper, we treat a different aspect that comes from interval graphs, walk domination.

A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line.

Let G be an interval graph, and let P and Q be two induced paths in G between two non-adjacent vertices of G . Then, in every interval representation $(I_v)_{v \in V(G)}$ of G , each internal vertex of P is adjacent or equal to some internal vertex of Q and vice versa. Inspired in this property, Alc3n [1] studied domination between different types of walks connecting two non-adjacent vertices of a graph.

Given two non-adjacent vertices u and v , a uv -walk W dominates a uv -walk W' if every internal vertex of W' is adjacent to some internal vertex of W or belongs to W .

A class of walks \mathbf{A} dominates a class of walks \mathbf{B} if every uv -walk of \mathbf{A} dominates every a uv -walk of \mathbf{B} , for all pair of non-adjacent vertices of the graphs.

Given a class of graph it is natural to ask if for every graph in the class, certain kind of walks dominate others.

In walk domination context not only this question is studied but if a class of graphs is characterized for this property for certain types of walks.

In [1], Alc3n considered walks, tolled-walks, paths, induced-paths or shortest-paths. Therefore she found characterizations of standard graph classes like **Chordal**, **Interval** and **Superfragile**.

In this paper, we study the domination between weakly toll walks [4], l_k -paths for $k \in \{2, 3\}$ [6], and different types of walks connecting two non-adjacent vertices u and v of a graphs (shortest paths, induced paths, paths, tolled walks, weakly toll walks, l_k -paths for $k \in \{2, 3\}$), and show how these give rise to characterizations of graph classes.

The paper is organized as follows. In Section 2, we give necessary definitions, in Section 3, it is presented the main results. In particular, we obtain a characterization for

Interval \cap {**chair**, **dart**}-free, **Chordal** \cap {**chair**, **dart**, $F_4(6)$ }-free,

Chordal \cap { F_2 , $F_3(n)_{n \geq 6}$, $F_4(n)_{n \geq 6}$, $F_5(n)_{n \geq 8}$ }-free,

Chordal \cap { F_2 , $F_4(n)_{n \geq 6}$, $F_5(n)_{n \geq 8}$, $F_6(n)_{n \geq 7}$, $F_7(n)_{n \geq 7}$ }-free,

{ C_4 , C_5 , C_6 }-free, { C_4 }-free and a new characterization of standard graph classes like **Chordal** and **Superfragile**.

Conclusions are developed in Section 4.

2. PRELIMINARIES

In this section, we recall the definitions of the most used notions in this paper.

All the graphs in this paper are finite, undirected, simple, and connected. We use standard graph terminology [12].

Let G be a graph. The subgraph induced in G by a subset $S \subseteq V(G)$ is denoted by $G[S]$. For any vertex v of G , the *neighborhood* of v is denoted by $N[v] = \{u \in V(G) \mid uv \text{ is an edge of } G\} \cup \{v\}$.

For any pair of non-adjacent vertices $u, v \in V(G)$ let us introduce the following definitions. A uv -walk is a sequence $W : u = v_0, v_1, \dots, v_{k-1}, v_k = v$ whose terms are vertices, not necessarily distinct, such that u is adjacent to v_1 , v_i is adjacent to v_{i+1} for $i \in \{1, \dots, k-2\}$, and v_{k-1} is adjacent to v . The vertices u and v are called *ends of the walk*, and the vertices v_1, \dots, v_{k-1} are its *internal vertices*.

The integer k is the *length of the walk*. The *distance* $d(u, v)$ between vertices u and v is the length of a shortest uv -walk.

A uv -path is a uv -walk with all its vertices distinct. The walk 1, 2, 3, 7 is a 17-path (Figure 1).

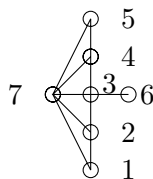


Figure 1.

A uv -induced path (or *monophonic path* [5]) is a uv -path such that two of its vertices are adjacent if and only if are consecutive. The 15-path: 1, 2, 3, 4, 5 is a 15-induced path (Figure 1). Observe that the 17-path: 1, 2, 3, 7 is not a 17-induced path.

A uv -shortest path (or *geodesic* [5]) is a uv -path of length $d(u, v)$. The induced path 1, 7, 5 is a 15-shortest path (Figure 1). Observe that the induced path 1, 2, 3, 4, 5 is not a 15-shortest path.

A uv -weakly toll walk is a uv -walk such that u is adjacent only to the vertex v_1 , with possibly $\{v_1\} \cap \{v_2, \dots, v_{k-1}\} \neq \emptyset$, and v is adjacent only to the vertex v_{k-1} , with possibly $\{v_{k-1}\} \cap \{v_1, \dots, v_{k-2}\} \neq \emptyset$ [4]. Note that v_1 may be v_{k-1} . The walk 2, 3, 6, 3, 4 is a 24-weakly toll walk (Figure 1).

A uv -tolled walk is a uv -walk satisfying that u is adjacent only to the vertex v_1 , v is adjacent only to the vertex v_{k-1} , $\{v_1\} \cap \{v_2, \dots, v_{k-1}\} = \emptyset$ and $\{v_{k-1}\} \cap \{v_1, \dots, v_{k-2}\} = \emptyset$ [2]. Note that v_1 may be v_{k-1} , but if $v_1 = v_{k-1}$, then $k = 2$.

The walk $1, 2, 3, 6, 3, 4, 5$ is an 15-tolled walk (Figure 1). Observe that $2, 3, 6, 3, 4$ is not a 24-tolled walk.

A uv - l_k -path is a uv -induced path with length at most k . The 67-induced path: $6, 3, 7$ is a 67- l_k -path for $k \in \{2, 3\}$.

Notice that every shortest path is an induced path, every induced path is a tolled walk, and a tolled walk is a weakly toll walk. Also every l_k -path is an induced path. However, paths and tolled-walks are incomparable just like l_k -paths and shortest paths.

Definition 1. The walk $W : v_0, v_1, \dots, v_m$ contains the walk $W' : v'_0, v'_1, \dots, v'_n$ if there exists an strict increasing function $\Theta : \{0, 1, 2, \dots, n\} \rightarrow \{0, 1, 2, \dots, m\}$ such that $v'_i = v_{\Theta(i)}$ for $0 \leq i \leq n$.

It is known that every uv -walk contains some uv -path and that every uv -path contains some uv -induced path [12]. However, not every uv -induced path contains some uv -shortest path.

Definition 2. The uv -walk $W : u, v_1, \dots, v_{m-1}, v$ dominates the uv -walk $W' : u, v'_1, \dots, v'_{n-1}, v$ if every internal vertex of W' is adjacent to some internal vertex of W or belongs to W .

Now, we introduce the notation **SP**, **IP**, **P**, **TW**, **W**, **WTW** and l_k for $k = 2, 3$ to refer to the set of different types of walks connecting two non-adjacent vertices u and v of a graph G .

$$\begin{aligned} \mathbf{SP}(u, v) &= \{W : W \text{ is a } uv\text{-shortest path}\}, \\ \mathbf{IP}(u, v) &= \{W : W \text{ is a } uv\text{-induced path}\}, \\ \mathbf{P}(u, v) &= \{W : W \text{ is a } uv\text{-path}\}, \\ \mathbf{TW}(u, v) &= \{W : W \text{ is a } uv\text{-tolled walk}\}, \\ \mathbf{W}(u, v) &= \{W : W \text{ is a } uv\text{-walk}\}, \\ \mathbf{WTW}(u, v) &= \{W : W \text{ is a } uv\text{-weakly toll walk}\}. \end{aligned}$$

In case of induced paths with bounded length, we use the following notation.

$$l_k(u, v) = \{W : W \text{ is a } uv\text{-}l_k\text{-path}\} \text{ for } k = 2, 3.$$

The following two remarks summarize the relation between the different types of walks we have considered.

Remark 1. $\mathbf{SP}(u, v) \subseteq \mathbf{IP}(u, v) \subseteq \mathbf{P}(u, v) \subseteq \mathbf{W}(u, v)$,

$$\mathbf{SP}(u, v) \subseteq \mathbf{IP}(u, v) \subseteq \mathbf{TW}(u, v) \subseteq \mathbf{WTW}(u, v) \subseteq \mathbf{W}(u, v),$$

$$l_2(u, v) \subseteq l_3(u, v) \subseteq \mathbf{IP}(u, v) \subseteq \mathbf{P}(u, v) \subseteq \mathbf{W}(u, v),$$

$$l_2(u, v) \subseteq l_3(u, v) \subseteq \mathbf{IP}(u, v) \subseteq \mathbf{TW}(u, v) \subseteq \mathbf{WTW}(u, v) \subseteq \mathbf{W}(u, v).$$

Remark 2. If $W \in \mathbf{W}(u, v)$, then W contains some $W' \in \mathbf{IP}(u, v)$.

A *cycle* of length k in a graph G is a path $C : v_1, v_2, \dots, v_k$ plus an edge between v_1 and v_k . Each edge of G between two non-consecutive vertices of C is called a *chord*. The cycle of length k without chords is denoted by C_k .

A graph is *chordal* if every cycle of length at least 4 has a chord. Let **Chordal** denote the class of chordal graphs. Note that **Chordal** = $\{C_k : k > 3\}$ -free

A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line. Let **Interval** denote the class of interval graphs.

An *asteroidal triple* of a graph G is a set of 3 non-adjacent vertices of G such that each pair is connected by a path avoiding the neighborhood of the third vertex.

Lekkerkerker and Boland [8] proved the following.

1. For any graph G , G is an interval graph if and only if G is chordal and contains no asteroidal triple.
2. **Interval** = **Chordal** \cap $\{F_1, F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}\}$ -free (see Figure 2).

Let **Interval**⁺ be the class of those chordal graphs G that contain none of the graphs F_2 or $F_4(n)_{n \geq 6}$ as induced subgraph, and satisfy the following condition. If G has an induced subgraph H isomorphic to F_1 ($F_3(n)_{n \geq 6}$), then the distance in G between the vertices of F_1 ($F_3(n)_{n \geq 6}$) labelled u and v in Figure 2 is 2, and any vertex of G adjacent to both u and v is universal to F_1 ($F_3(n)_{n \geq 6}$) [1].

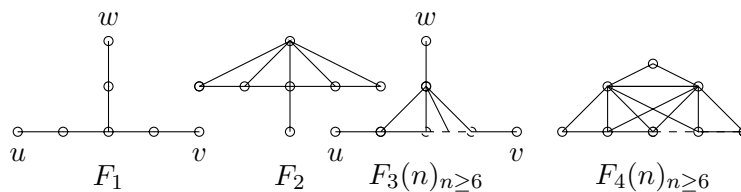


Figure 2. Chordal forbidden induced subgraphs for interval graphs.

The *closed geodesic interval* for two vertices u and v of a graph G is the set of all vertices lying on some uv -shortest-path of G .

Let **g – Chordal** denote the class of graphs G in which any closed geodesic interval induces a chordal subgraph [1].

A graph is *ptolematic* if for every four vertices v_1, v_2, v_3 , and v_4 , $d(v_1, v_2) \cdot d(v_3, v_4) \leq d(v_1, v_2) \cdot d(v_2, v_4) + d(v_1, v_4) \cdot d(v_2, v_3)$. It is well known that the class of ptolemaic graphs, denoted by **Ptolemaic**, corresponds to the class of gem-free chordal graphs [7] (see Figure 3). Following the notation [1], in this article we also consider **Ptolemaic**[−] as the class of those ptolemaic graphs which contain none of the graphs co-chair or dart in Figure 3.

A graph is *superfragile* if it has a vertex elimination order with respect to the two rules below, such that at each stage every vertex is eligible for elimination.

Rule 1. If v does not appear as an end vertex in an induced P_3 , then v may be removed.

Rule 2. If v does not appear as an internal vertex in an induced P_3 , then v may be removed.

Let **Superfragile** denote the class of superfragile graphs. Note that **Superfragile** = $\{C_4, P_4, \text{dart}\}$ -free [11] (see Figure 3).

Figure 3 shows some special graphs used to describe the graphs classes considered in our results.

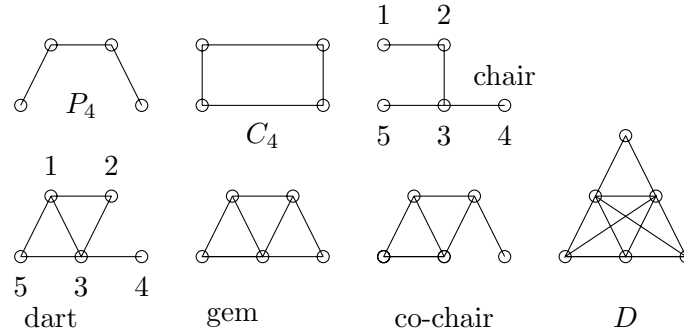


Figure 3. Graphs used to describe the graphs classes considered in our results.

Definition 3. Let $\mathbf{A}, \mathbf{B} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}, l_2, l_3\}$. \mathbf{A}/\mathbf{B} is the class formed by those graphs G such that for every pair of non-adjacent vertices u and v of G , every $W \in \mathbf{A}(u, v)$ dominates every $W' \in \mathbf{B}(u, v)$ i.e., $W \in \mathbf{A}(u, v)$ and $W' \in \mathbf{B}(u, v)$ implies W dominates W' .

Some important classes of graphs have been characterized by domination between different types of walks [1]. These results are summarized in Table 1.

	SP	IP	P	TW	W
SP	g – Chordal	Chordal	Ptolematic [−]		Superfragile
IP	Chordal	Chordal	Ptolematic [−]	Interval	Superfragile
P	Chordal	Chordal	Ptolematic [−]	Interval	Superfragile
TW	Chordal	Chordal	Ptolematic [−]	Interval	Superfragile
W	Chordal	Chordal	Ptolematic [−]	Interval	Superfragile

Table 1. With $\mathbf{A} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{W}\}$ in the first column and $\mathbf{B} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{W}\}$ in the first row, the table describes each one of the graph classes \mathbf{A}/\mathbf{B} . Observe that \mathbf{SP}/\mathbf{TW} has a partial characterization, $\mathbf{SP}/\mathbf{TW} \subseteq \mathbf{Interval}^+$ [1].

3. MAIN RESULTS

The aim of the present paper is to describe the graph classes \mathbf{A}/\mathbf{B} with $\mathbf{A}, \mathbf{B} \in \{l_2, l_3, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}\}$. Our main results are summarized in Table 2. As Table 1, with $\mathbf{A} \in \{l_2, l_3, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}\}$ in the first column and $\mathbf{B} \in \{l_2, l_3, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{WTW}, \mathbf{W}\}$ in the first row, the table describes each one of the graph classes \mathbf{A}/\mathbf{B} .

-	l_2	l_3	\mathbf{SP}	\mathbf{IP}	\mathbf{P}	\mathbf{TW}	\mathbf{WTW}	\mathbf{W}
l_2	$\{C_4\}$ -free	\mathbf{Ch}	$\{C_4\}$ -free	\mathbf{Ch}	\mathbf{Pt}^-	$\mathbf{Ch} \cap F_{2,4,5,6,7}$ -free	$\mathbf{Ch} \cap \{\mathbf{chair}, \mathbf{dart}, F_4(6)\}$ -free	\mathbf{S}
l_3	$\{C_4, C_5, C_6\}$ -free	\mathbf{Ch}		\mathbf{Ch}	\mathbf{Pt}^-	$\mathbf{Ch} \cap F_{2,3,4,5}$ -free	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	\mathbf{S}
\mathbf{SP}	$\{C_4\}$ -free	$\{C_4, C_5, C_6\}$ -free	T_1	T_1	T_1		$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	T_1
\mathbf{IP}	\mathbf{Ch}	\mathbf{Ch}	T_1	T_1	T_1	T_1	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	T_1
\mathbf{P}	\mathbf{Ch}	\mathbf{Ch}	T_1	T_1	T_1	T_1	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	T_1
\mathbf{TW}	\mathbf{Ch}	\mathbf{Ch}	T_1	T_1	T_1	T_1	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	T_1
\mathbf{WTW}	\mathbf{Ch}	\mathbf{Ch}	\mathbf{Ch}	\mathbf{Ch}	\mathbf{Pt}^-	\mathbf{Int}	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	\mathbf{Sup}
\mathbf{W}	\mathbf{Ch}	\mathbf{Ch}	T_1	T_1	T_1	T_1	$\mathbf{Int} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free	T_1

Table 2. T_1 results Table 1, we denote by \mathbf{Ch} the class of chordal graphs, by \mathbf{Int} the class of interval graphs, by \mathbf{Sup} the class of superfragile graphs, by \mathbf{Pt}^- the class $\mathbf{Ptolemaic}^-$, by $F_{2,4,5,6,7} = \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$, and by $F_{2,3,4,5} = \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$.

In [1], Alc3n presented the following lemma.

Lemma 1 [1]. *For $\mathbf{A} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{W}\}$, the following statements hold.*

1. $\mathbf{A}/\mathbf{W} \subseteq \mathbf{A}/\mathbf{P} \subseteq \mathbf{A}/\mathbf{IP} \subseteq \mathbf{A}/\mathbf{SP}$.
2. $\mathbf{A}/\mathbf{W} \subseteq \mathbf{A}/\mathbf{TW} \subseteq \mathbf{A}/\mathbf{IP} \subseteq \mathbf{A}/\mathbf{SP}$.
3. $\mathbf{TW}/\mathbf{W} \subseteq \mathbf{TW}/\mathbf{P} \subseteq \mathbf{TW}/\mathbf{IP} \subseteq \mathbf{TW}/\mathbf{SP}$.
4. $\mathbf{TW}/\mathbf{W} \subseteq \mathbf{TW}/\mathbf{TW} \subseteq \mathbf{TW}/\mathbf{IP} \subseteq \mathbf{TW}/\mathbf{SP}$.
5. $\mathbf{W}/\mathbf{A} = \mathbf{TW}/\mathbf{A} = \mathbf{P}/\mathbf{A} = \mathbf{IP}/\mathbf{A} \subseteq \mathbf{SP}/\mathbf{A}$.
6. $\mathbf{W}/\mathbf{TW} = \mathbf{TW}/\mathbf{TW} = \mathbf{P}/\mathbf{TW} = \mathbf{IP}/\mathbf{TW} \subseteq \mathbf{SP}/\mathbf{TW}$.

Now, we consider l_k for $k \in \{2, 3\}$ and \mathbf{WTW} .

Note that for example $\mathbf{W}/\mathbf{WTW} \subseteq \mathbf{IP}/\mathbf{WTW}$, since every induced path is a walk (Remark 1). On the other hand, as every walk contains some induced paths (Remark 2), it follows that $\mathbf{IP}/\mathbf{WTW} \subseteq \mathbf{W}/\mathbf{WTW}$.

Thus, using Remark 1, Remark 2 and Lemma 1, the proof of the next lemma does not represent difficulty and is left to the reader.

Lemma 2. *For $\mathbf{A}, \mathbf{B} \in \{\mathbf{IP}, \mathbf{P}, \mathbf{W}\}$ and for $k = 2, 3$, the following statements hold.*

1. $\mathbf{A}/\mathbf{W} \subseteq \mathbf{A}/\mathbf{WTW} \subseteq \mathbf{A}/\mathbf{TW} \subseteq \mathbf{A}/\mathbf{IP} \subseteq \mathbf{A}/l_3 \subseteq \mathbf{A}/l_2$, and $\mathbf{W}/l_k \subseteq \mathbf{P}/l_k \subseteq \mathbf{IP}/l_k \subseteq l_3/l_k \subseteq l_2/l_k$.

2. $\mathbf{WTW}/\mathbf{W} \subseteq \mathbf{WTW}/\mathbf{TW} \subseteq \mathbf{WTW}/\mathbf{IP} \subseteq \mathbf{WTW}/\mathbf{l}_k$ and $\mathbf{WTW}/\mathbf{IP} \subseteq \mathbf{WTW}/\mathbf{SP}$.
3. $\mathbf{W}/\mathbf{A} = \mathbf{WTW}/\mathbf{A} = \mathbf{TW}/\mathbf{A} = \mathbf{IP}/\mathbf{A} \subseteq \mathbf{SP}/\mathbf{A}$ and $\mathbf{IP}/\mathbf{A} \subseteq \mathbf{l}_k/\mathbf{A}$.
4. $\mathbf{l}_k/\mathbf{W} \subseteq \mathbf{l}_k/\mathbf{WTW} \subseteq \mathbf{l}_k/\mathbf{TW} \subseteq \mathbf{l}_k/\mathbf{IP} \subseteq \mathbf{l}_k/\mathbf{l}_3 \subseteq \mathbf{l}_k/\mathbf{l}_2$.
5. $\mathbf{A}/\mathbf{B} \subseteq \mathbf{l}_k/\mathbf{B}$, $\mathbf{TW}/\mathbf{B} \subseteq \mathbf{l}_k/\mathbf{B}$ and $\mathbf{WTW}/\mathbf{B} \subseteq \mathbf{l}_k/\mathbf{B}$.
6. $\mathbf{W}/\mathbf{l}_k = \mathbf{WTW}/\mathbf{l}_k = \mathbf{TW}/\mathbf{l}_k = \mathbf{IP}/\mathbf{l}_k = \mathbf{SP}/\mathbf{l}_k \subseteq \mathbf{l}_3/\mathbf{l}_k \subseteq \mathbf{l}_2/\mathbf{l}_k$.
7. $\mathbf{IP}/\mathbf{WTW} \subseteq \mathbf{l}_k/\mathbf{WTW}$.
- $\mathbf{W}/\mathbf{WTW} = \mathbf{WTW}/\mathbf{WTW} = \mathbf{TW}/\mathbf{WTW} = \mathbf{IP}/\mathbf{WTW} = \mathbf{P}/\mathbf{WTW}$.
8. $\mathbf{W}/\mathbf{TW} = \mathbf{IP}/\mathbf{TW} = \mathbf{WTW}/\mathbf{TW}$.

Note that the previous lemma implies that the last five rows of Table 2 must be the same.

The following theorem provides a characterization for $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free, obtained in terms of domination between induced paths versus weakly toll walks.

Theorem 3. $\mathbf{IP}/\mathbf{WTW} = \mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free.

Proof. By Lemma 2, $\mathbf{IP}/\mathbf{WTW} \subseteq \mathbf{IP}/\mathbf{TW}$, and by Table 1, $\mathbf{IP}/\mathbf{TW} = \mathbf{Interval}$. Thus $\mathbf{IP}/\mathbf{WTW} \subseteq \mathbf{Interval}$.

On the other hand, as it is shown in Figure 3, chair has a pair of non-adjacent vertices 4 and 5, a 45-weakly toll walk: 4, 3, 2, 1, 2, 3, 5 which is not dominated by the 45-induced path. Also, dart has a pair of non-adjacent vertices 2 and 5, a 25-weakly toll walk: 2, 3, 4, 3, 5 which is not dominated by the 25-induced path: 2, 1, 5 (see Figure 3).

Thus the class \mathbf{IP}/\mathbf{WTW} is contained in $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free.

We will now prove that $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free $\subseteq \mathbf{IP}/\mathbf{WTW}$.

Let G be an $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}$ -free. Suppose, in order to derive a contradiction, that $G \notin \mathbf{IP}/\mathbf{WTW}$. Then there exist two non-adjacent vertices u and v , a uv -induced path $W : u = x_0, \dots, x_n = v$ and a uv -weakly toll walk $W' : u = x'_0, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W .

Let k be the first index such that x'_k is neither a vertex of W nor adjacent to any interval vertex of W . Let also $I_u = [y_u, w_u]$ and $I_v = [y_v, w_v]$ with $y_u < w_u < y_v < w_v$ be the intervals corresponding to vertices u and v in a given interval representation of G .

Clearly the segment of line $[w_u, y_v]$ is contained in the union of the intervals corresponding to the internal vertices of W , then we can assume that the interval $I_{x'_k}$ is contained in (w_v, ∞) .



Figure 4.

Then since W' is a uv -weakly toll walk, $I_v \subset I_{x'_{m-1}} = [y_{x'_{m-1}}, w_{x'_{m-1}}]$ with $w_v < w_{x'_{m-1}}$ (Figure 4).

By the choice of k , x'_{k-1} is not a vertex of W , in particular $x'_{k-1} \neq x_{n-1}$. It follows that x'_{k-1} is adjacent to an internal vertex of W .

To show that G has a chair or dart as induced subgraph, we consider the following cases.

Case 1. $I_{x'_{k-1}} \cap I_v = \emptyset$. Then the interval $I_{x'_{k-1}}$ is contained in $(w_v, +\infty)$. By the choice of k , x_{n-1} must be adjacent to x'_{k-1} . Thus the interval I_v is contained in the interval $I_{x_{n-1}}$. Since W is a uv -induced path, there exists x_{n-2} , $w_{x_{n-2}} < y_v < w_v < y_{x'_{k-1}}$. It follows that x_{n-2} is not adjacent to x'_{k-1} . Note that x_{n-2} may be u . Thus $G[x_{n-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a chair, which contradicts our assumption.

Case 2. $I_{x'_{k-1}} \cap I_v \neq \emptyset$. Since W' is a uv -weakly toll walk, it follows that $x'_{k-1} = x'_{m-1}$. Observe that there exists x'_{k-2} , which may be u , and $w_{x'_{k-2}} < y_v$. Clearly x_{n-1} is adjacent to x'_{k-1} . Since W is a uv -induced path, there exists x_{n-2} . Note that x_{n-2} may be a vertex of $W' - \{x'_{k-1}\}$.

Case 2.1. Suppose that x_{n-2} is not a vertex of W' . If x_{n-2} is adjacent to x'_{k-1} , then $G[x_{n-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart, in contradiction with our assumption. If x_{n-2} is not adjacent to x'_{k-1} , then $w_{x_{n-2}} < y_{x'_{k-1}}$. And so x'_{k-2} must be adjacent to x_{n-1} . Note that x'_{k-2} is not x_{n-1} because x'_{k-2} is not adjacent to v , since $W' \in \mathbf{WTW}$. Thus $G[x'_{k-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart, a contradiction.

Case 2.2. Suppose now that x_{n-2} is a vertex of W' . If it is adjacent to x'_{k-1} , then $G[x_{n-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart, a contradiction. If x_{n-2} is not adjacent to x'_{k-1} , then there exists x'_{k-2} such that $w_{x_{n-2}} < y_{x'_{k-1}} \leq w_{x'_{k-2}}$ and also x'_{k-2} is not adjacent to v , since $W' \in \mathbf{WTW}$. Observe that as $y_{x'_{k-1}} \leq w_{x'_{k-2}} < y_v$ results x'_{k-2} must be adjacent to x_{n-1} . Thus $G[x'_{k-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart, a contradiction.

Hence $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}\text{-free} \subseteq \mathbf{IP}/\mathbf{WTW}$. Therefore $\mathbf{IP}/\mathbf{WTW} = \mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}\text{-free}$. ■

In [1] it was proved that \mathbf{SP}/\mathbf{TW} is not hereditary, and $\mathbf{SP}/\mathbf{TW} \subseteq$

Interval⁺. However, note that **SP/WTW** is hereditary as we will show in Theorem 4. Moreover, **SP/WTW** = **IP/WTW**.

Theorem 4. **SP/WTW** = **Interval** \cap {**chair**, **dart**}-free.

Proof. By Lemma 2, **SP/WTW** \subseteq **SP/TW**, and by [1], **SP/TW** \subseteq **Interval**⁺. So **SP/WTW** does not contain none of the graphs F_2 or $F_4(n)_{n \geq 6}$ as induced subgraph.

On the other hand, as it is shown in Figure 3, chair has a pair of non-adjacent vertices 4 and 5, a 45-weakly toll walk: 4, 3, 2, 1, 2, 3, 5 which is not dominated by the 45-shortest path. Thus chair is not in **SP/WTW**. Since a chair is an induced subgraph of F_1 , F_1 is not in **SP/WTW**.

Also dart has a pair of non-adjacent vertices 2 and 5, a 25-weakly toll walk: 2, 3, 4, 3, 5 which is not dominated by the 25-shortest path: 2, 1, 5 (see Figure 3). Thus dart is not in **SP/WTW**.

Note that dart is an induced subgraph of $F_3(n)_{n \geq 6}$. Thus $F_3(n)_{n \geq 6} \notin$ **SP/WTW**. On the other hand, it is easy to check that $F_3(6) \notin$ **SP/WTW**.

By before exposed, $F_3(n)_{n \geq 6}$ is not in **SP/WTW**. Hence **SP/WTW** does not contain none of the graphs chair, dart F_i for $i = 1, 2$, $F_j(n)_{n \geq 6}$ for $j = 3, 4$, as induced subgraph. Therefore, **SP/WTW** \subseteq **Interval** \cap {**chair**, **dart**}-free.

In that follows, we will show that **Interval** \cap {**chair**, **dart**}-free \subseteq **SP/WTW**.

Let G be an **Interval** \cap {**chair**, **dart**}-free. Assume, in order to obtain a contradiction that G is not in **SP/WTW**. Then there exist two non-adjacent vertices u and v , a uv -shortest path $W : u = x_0, \dots, x_n = v$ and a uv -weakly toll walk $W' : u = x'_0, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W .

As in the proof of Theorem 3, let k be the first index such that x'_k is neither a vertex of W nor adjacent to any interval vertex of W . Let $I_u = [y_u, w_u]$ and $I_v = [y_v, w_v]$ with $y_u < w_u < y_v < w_v$ be the intervals corresponding to vertices u and v in a given interval representation of G . Clearly the segment of line $[w_u, y_v]$ is contained in the union of the intervals corresponding to the internal vertices of W , then we can assume that the interval $I_{x'_k}$ is contained in (w_v, ∞) .

Then since W' is a uv -weakly toll walk, $I_v \subset I_{x'_{m-1}} = [y_{x'_{m-1}}, w_{x'_{m-1}}]$. By the choice of k , x'_{k-1} is not a vertex of W , in particular $x'_{k-1} \neq x_{n-1}$. However it is adjacent to an internal vertex of W .

Suppose that $I_{x'_{k-1}}$ is contained in (w_v, ∞) , it follows that x'_{k-1} is adjacent to x_{n-1} . Since $W \in$ **SP**, x_{n-2} is not adjacent to v . Thus $G[x_{n-2}, x_{n-1}, v, x'_{k-1}, x'_k]$ is a chair, a contradiction.

Suppose that $I_{x'_{k-1}}$ is not contained in (w_v, ∞) . We can assume that x'_{k-1} is adjacent to v , and since $W' \in$ **WTW** results $I_v \subset I_{x'_{k-1}}$. Note that $x_{n-1} \notin W'$. If x_{n-2} is adjacent to x'_{k-1} , since $W \in$ **SP** results $w_{x_{n-2}} < y_v$, and then

$G[x_{n-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart in contradiction with our assumption. If x_{n-2} is not adjacent to x'_{k-1} , since W' is a uv -weakly toll walk, then x'_{k-2} is not x_{n-1} . Thus $G[x'_{k-2}, x_{n-1}, x_n = v, x'_{k-1}, x'_k]$ is a dart, a contradiction.

Hence $\mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}\text{-free} \subseteq \mathbf{SP}/\mathbf{WTW}$. Therefore $\mathbf{SP}/\mathbf{WTW} = \mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}\text{-free}$. ■

As a consequence of Lemma 2, Theorem 3, and Theorem 4, we obtain the following.

Corollary 5. $\mathbf{SP}/\mathbf{WTW} = \mathbf{IP}/\mathbf{WTW} = \mathbf{P}/\mathbf{WTW} = \mathbf{W}/\mathbf{WTW} = \mathbf{TW}/\mathbf{WTW} = \mathbf{WTW}/\mathbf{WTW} = \mathbf{Interval} \cap \{\mathbf{chair}, \mathbf{dart}\}\text{-free}$.

In the rest of this section we will study the domination between l_k -paths for $k \in \{2, 3\}$ and different types of walks.

Lemma 6. *For every $k \in \{2, 3\}$ the following statements hold.*

1. $l_k/\mathbf{IP} \subseteq \mathbf{Chordal}$.
2. $\mathbf{IP}/l_k \subseteq \mathbf{Chordal}$.

Proof. 1. Let $G \in l_k/\mathbf{IP}$, and suppose, in order to derive a contradiction, that $G \notin \mathbf{Chordal}$. Thus G contains as induced subgraph a cycle C_n with $n \geq 4$. Let $C_n : x_1, x_2, \dots, x_n, x_1$. The x_1x_3 - l_k -path: x_1, x_2, x_3 does not dominate the x_1x_3 -induced path: $x_1, x_k, x_{n-1}, \dots, x_3$, a contradiction. Hence $l_k/\mathbf{IP} \subseteq \mathbf{Chordal}$.

2. The proof is similar to the proof of 1, taking x_1x_3 -induced path: $x_1, x_n, x_{n-1}, \dots, x_3$, which does not dominate the x_1x_3 - l_k -path: x_1, x_2, x_3 . Thus $\mathbf{IP}/l_k \subseteq \mathbf{Chordal}$. ■

The following theorem allows us to find a new characterization of chordal graphs in terms of domination between l_k -path, tolled walk, weakly toll walk, induced path and walk.

Theorem 7. *For $k = 2, 3$, the followings statements hold.*

1. $l_k/\mathbf{IP} = \mathbf{Chordal}$.
2. $\mathbf{IP}/l_k = \mathbf{Chordal}$.

Proof. 1. By Table 1, $\mathbf{Chordal} = \mathbf{IP}/\mathbf{IP}$ and by Lemma 2, $\mathbf{IP}/\mathbf{IP} \subseteq l_k/\mathbf{IP}$. Thus $\mathbf{Chordal} = \mathbf{IP}/\mathbf{IP} \subseteq l_k/\mathbf{IP}$. By Lemma 6, $l_k/\mathbf{IP} \subseteq \mathbf{Chordal}$. Hence $l_k/\mathbf{IP} = \mathbf{Chordal}$.

2. By Lemma 2, $\mathbf{IP}/\mathbf{IP} \subseteq \mathbf{IP}/l_k$. Thus $\mathbf{Chordal} \subseteq \mathbf{IP}/l_k$. By Lemma 6, $\mathbf{IP}/l_k \subseteq \mathbf{Chordal}$. Hence $\mathbf{IP}/l_k = \mathbf{Chordal}$. ■

Lemma 2, and Theorem 7, imply the following.

Corollary 8. *For $k = 2, 3$, the followings statement holds. $\mathbf{IP}/l_k = \mathbf{W}/l_k = \mathbf{WTW}/l_k = \mathbf{TW}/l_k = \mathbf{Chordal}$.*

Corollary 9. *For $k = 2, 3$, the followings statements hold.*

1. $\text{Interval} \subseteq l_k/\text{TW} \subseteq \text{Chordal}$.
2. $\text{Interval} \cap \{\text{chair}, \text{dart}\}\text{-free} \subseteq l_k/\text{WTW} \subseteq \text{Chordal}$.

Proof. 1. By Lemma 2, $\text{IP}/\text{TW} \subseteq l_k/\text{TW}$ and $l_k/\text{TW} \subseteq l_k/\text{IP}$.

By [1], $\text{Interval} = \text{IP}/\text{TW}$, then $\text{Interval} \subseteq l_k/\text{TW}$. By Theorem 7, $l_k/\text{TW} \subseteq l_k/\text{IP} = \text{Chordal}$.

2. By Lemma 2, Theorem 3, and Theorem 7, it follows that $\text{IP}/\text{WTW} = \text{Interval} \cap \{\text{chair}, \text{dart}\}\text{-free} \subseteq l_k/\text{WTW} \subseteq l_k/\text{IP} = \text{Chordal}$. ■

Superfragile and **Ptolematic**[−] can be also characterized in terms of domination between l_k -paths, paths, and walks, which are given in the following theorem.

Theorem 10. *For $k = 2, 3$, the followings statements hold.*

1. $l_k/\mathbf{W} = \text{Superfragile}$.
2. $l_k/\mathbf{P} = \text{Ptolematic}^-$.

Proof. 1. By Table 1, and Lemma 2, $\text{Superfragile} = \text{IP}/\mathbf{W} \subseteq l_k/\mathbf{W}$.

It is easy to check that C_4 and dart have a pair of non-adjacent vertices u and v and a uv -walk which is not dominated by a uv - l_k -path, then C_4 and dart are not in l_k/\mathbf{W} . In the case of $P_4 : x_0, x_1, x_2, x_3$, we consider $u = x_0, v = x_2$, the uv -walk: u, x_1, x_2, x_3, v which is not dominated by the uv - l_k -walk: u, x_1, v . So, P_4 is not in l_k/\mathbf{W} .

Suppose that $G \in l_k/\mathbf{W}$ and G is not superfragile. Since G is not superfragile, G contains P_4 or C_4 or dart as induced subgraph, a contradiction.

2. By Table 1, and Lemma 2, $\text{Ptolematic}^- = \text{IP}/\mathbf{P} \subseteq l_k/\mathbf{P}$.

It is easy to show that D , gem and co-chair are not in l_k/\mathbf{P} , see Figure 5. Thus $l_k/\mathbf{P} = \text{Ptolematic}^-$. ■

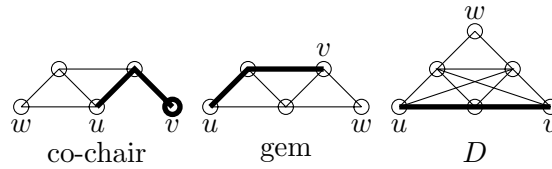


Figure 5. In each graph above, the vertex labelled w belongs to a uv -path and it is adjacent to no internal vertex of the bold uv - l_k -path.

We now study domination between l_k -path and tolled walk.

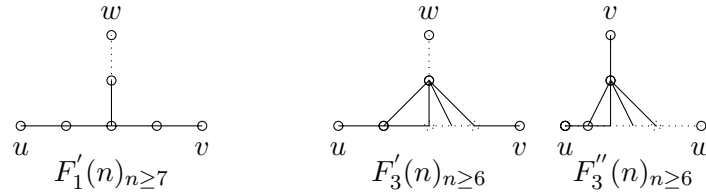


Figure 6. Graphs $F'_1(n)_{n \geq 7}$, and $F'_3(n)_{n \geq 6}$ are obtained by increasing one and only one path of F_1 and $F_3(n)_{n \geq 6}$, respectively.

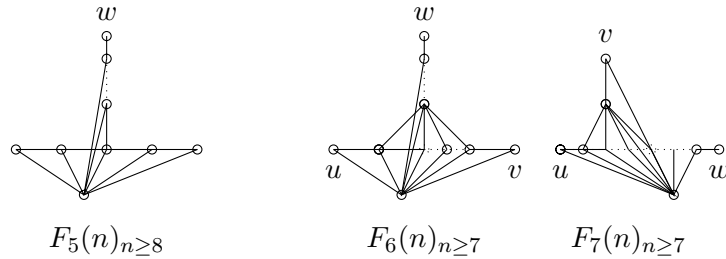


Figure 7. $F_5(n)_{n \geq 8}$ is obtained from $F'_1(n)$ by adding a universal vertex to $F'_1(n) - w$, $F_6(n)_{n \geq 7}$ is obtained from $F'_3(n)$ by adding a universal vertex to $F'_3(n) - w$, and $F_7(n)_{n \geq 7}$ is obtained from $F''_3(n)$ by adding a universal vertex to $F''_3(n) - w$.

Observation 11. Let G be a graph, and $W' : u = x'_0, x'_1, \dots, x'_m = v$ a uv -tolled walk. It follows from the definition of tolled walk that if there exists a vertex in W' such that $x'_k \notin N[u] \cup N[v]$, then u and x'_k are in the same connected component of $G[W'] - N[v]$, and also v and x'_k are in the same connected component of $G[W'] - N[u]$.

Theorem 12. $l_3/\text{TW} = \text{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free.

Proof. By Corollary 9, $l_3/\text{TW} \subseteq \text{Chordal}$.

It is easy to verify that $F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}$, and $F_5(n)_{n \geq 8}$ are not in l_3/TW . Thus $l_3/\text{TW} \subseteq \text{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free.

Now, we will prove that $\text{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5\}$ -free $\subseteq l_3/\text{TW}$.

Let $G \in \text{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free, and suppose, in order to derive a contradiction, that $G \notin l_3/\text{TW}$.

As $G \notin l_3/\text{TW}$ there exist two non-adjacent vertices u and v , a uv - l_3 -walk $W : u = x_0, \dots, x_n = v$ (observe that $n = 2$ or $n = 3$) and a uv -tolled-walk $W' : u = x'_0, x'_1, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus,

there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W . Let x'_k be a vertex of $W' - W$ such that it is not adjacent to any vertex of W . We can assume that $k \neq 1, m-1$, otherwise $G[W \cup W']$ contains as induced subgraph a cycle of size at least four.

Let P be a shortest path in $G[W']$ from u to v . Since W' is a uv -tolled walk, x'_1 and x'_{m-1} are vertices of P . Note that $x'_1 \neq x'_{m-1}$, and then $|V(P)| \geq 4$.

Claim 13. $x'_k \notin P$.

Proof. Suppose that $x'_k \in P$. Let us consider two cases, depending of the length n of W .

Case a. $n = 2$. Clearly $P \cap \{x_1\} = \emptyset$ since $|V(P)| \geq 4$. On the other hand $G[\{x_1\} \cup P]$ is a chordal graph, it follows that x_1 is adjacent to every vertex of P . Thus x_1 is adjacent to x'_k , a contradiction.

Case b. $n = 3$. Let us consider two cases.

Case b.1. $P \cap \{x_1\} \neq \emptyset$. Since W' is a uv -tolled walk, $x'_1 = x_1$. On the other hand, $x_2 \notin P$ since by our assumption $x'_k \in P$. As $G[\{x_2\} \cup P]$ is a chordal graph, it follows that x_2 is adjacent to x'_k , a contradiction.

Case b.2. $P \cap \{x_1\} = \emptyset$. We can also assume that $P \cap \{x_2\} = \emptyset$. Since $G[\{x_1, x_2\} \cup P]$ is a chordal graph, there exist chords between vertices of $\{x_1, x_2\}$ and P . Then x_1 is adjacent to x'_k or x_2 is adjacent to x'_k , a contradiction. Therefore $x'_k \notin P$. \square

In what follows we will analyze two cases, depending of x'_k is or is not adjacent to vertices of P .

Case 1. x'_k is adjacent to some vertex of P . Note that if x'_k is adjacent to two non-consecutive vertices of P , let x'_a and x'_b , since $G[P[x'_a, x'_b] \cup \{x'_k\}] \neq C_r$ (for some $r > 3$), it follows that x'_k is adjacent to every vertex of $P[x'_a, x'_b]$.

Let us consider two cases, depending of the length n of W .

Case 1.1. $n = 2$. Clearly $x_1 \notin P$. Since $G[\{x_1\} \cup P]$ is a chordal graph, x_1 is adjacent to every vertex of P .

Case 1.1.2. Suppose that x'_k is adjacent to two consecutive vertices x'_i, x'_{i+1} of P . By the choice of k , and since $G[x'_{i-1}, x'_i, x'_{i+1}, x'_{i+2}, x'_k, x_1] \neq F_4(6)$, results x'_{i-1} or x'_{i+2} is adjacent to x'_k . Thus $G[x'_{i-1}, x'_{i+1}, x'_k, x_1] = C_4$ or $G[x'_i, x'_{i+2}, x'_k, x_1] = C_4$, a contradiction.

Case 1.1.3. Suppose that x'_k is adjacent to one and only one vertex of P . Let us consider two cases: x'_k is or is not adjacent to x'_1 (by symmetry x'_k is or is not adjacent to x'_{m-1}).

First, suppose that x'_k is adjacent to x'_1 (by symmetry x'_k is adjacent to x'_{m-1}). Since W' is a uv -tolled walk by Observation 11, there exists a shortest path P_1 between v and x'_k in $G[W']$ such that $V(P_1) \cap N[u] = \emptyset$. And as W' is a uv -tolled walk, results $v, x'_{m-1} \in V(P) \cap V(P_1)$. Let x'_h be a vertex of P_1 such that it is adjacent to x_1 , and minimizes the distance in P_1 between x'_k and x'_h . Note that $x'_h \neq x'_k$ since x'_h is adjacent to x_1 .

As G is chordal, every vertex of $P_1[x'_h, v]$ must be adjacent to x_1 . By choice of x'_k and x'_h , and as $G[\{x'_1, x_1\} \cup P_1[x'_k, x'_h]] \neq C_r$ (for some $r > 3$), it follows that every vertex of $P_1[x'_k, x'_h]$ must be adjacent to x'_1 .

Let $x'_i \in V(P_1[x'_h, x'_k])$ such that $x'_i x'_h$ is an edge of P_1 . Since x'_i is adjacent to x'_1 but it is not adjacent to x_1 , results $G[\{u, x'_1, x_1\} \cup P_1[x'_i, v]]$ contains $F_4(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction.

Suppose that x'_k is not adjacent to x'_1 neither x'_{m-1} . Then $G[P \cup \{x'_k, x_1\}]$ contains F_2 as induced subgraph, a contradiction.

Case 1.2. $n = 3$. Let us consider two cases, depending of x_1 is or is not a vertex of P .

Case 1.2.1. Suppose that $x_1 \in P$. By the choice of P , which is a shortest path in $G[W']$, and since x'_k is adjacent to some vertex of P , results $x_2 \notin P$. Observe that since W' is a uv -tolled walk, then $x'_1 = x_1$.

Since $G[P[x_1, v] \cup \{x_2\}]$ is a chordal graph, x_2 must be adjacent to every vertex of $P[x_1, v]$. And then, if x'_k is not adjacent to x'_2 following the proof of Case 1.1.3, $G[P[x'_1, v] \cup \{x_2, x'_k\}]$ contains F_2 , or $F_4(n)$ (for some $n \geq 6$), so we derive to a contradiction.

Suppose that x'_k is adjacent to x'_2 . Since $G[W \cup \{x'_2, x'_k\}] \neq F_3(n)$ (for some $n \geq 6$), it follows that x_2 must be adjacent to x'_k , a contradiction.

Case 1.2.2. Suppose that $x_1 \notin P$. We can assume that $x_2 \notin P$. If x'_k is adjacent to two consecutive vertices of P or it is only adjacent to x'_1 , respectively x'_{m-1} , then by the same arguments developing in Case 1.1, we can conclude that $G[W \cup P]$ contains C_r (for some $r > 3$), or F_2 , or $F_4(n)$ (for some $n \geq 6$), or $F_3(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction.

If x'_k is adjacent to one and only one vertex x'_i of $P - \{x'_1, x'_{m-1}\}$, then G contains F_2 or $F_3(n)$ (for some $n \geq 6$) as induced subgraph according to x_1 is adjacent or is not to x'_{i+2} (x_2 is adjacent or is not to x'_{i-2}), a contradiction.

Case 2. x'_k is adjacent to no vertex of P . Note that $P \cap N[x'_k] = \emptyset$ by Claim 13.

Since W' is a uv -tolled walk, by Observation 11, there exists an induced path between u and x'_k in $G[W']$ avoiding the neighborhood of v , and also there exists an induced path between v and x'_k in $G[W']$ avoiding the neighborhood of u . Those paths, together with P allow us to state that u, v, x'_k is an asteroidal triple.

Hence, we assume that there exist three induced paths of $G[W']$: P between u and v ; P_1 between u and x'_k ; P_2 between v and x'_k ; and three vertices $x'_{a_i} \in V(P) \cap V(P_1)$; $x'_{a_j} \in V(P) \cap V(P_2)$; and $x'_{a_h} \in V(P_1) \cap V(P_2)$; such that: $V(P) \cap N[x'_k] = \emptyset$, $V(P_1) \cap N[v] = \emptyset$, $V(P_2) \cap N[u] = \emptyset$, and the distance between x'_{a_i} and u , between x'_{a_j} and v , and between x'_{a_h} and x'_k in the respective paths is maximum.

Note that $u, x'_1 \in P_1$ and $v, x'_{m-1} \in P_2$, since W' is an uv -tolled walk. Without loss of generality, we can assume that $(P \cap P_1)[u, x'_{a_i}]$, $(P \cap P_2)[x'_{a_j}, v]$ and $(P_1 \cap P_2)[x'_{a_h}, x'_k]$ are induced paths.

In that follows, we will show that $G[P \cup P_1 \cup P_2]$ must be a tree. Note that we will apply arguments similar to those developed in [8].

In order to derive to a contradiction suppose that $x'_{a_p} \neq x'_{a_q}$ for every $p \neq q$ with $p, q \in \{i, j, h\}$. First, we will show that by our choice of $x'_{a_i}, x'_{a_j}, x'_{a_h}$, P, P_1 and P_2 , we can assume that $x'_k \neq x'_{a_h}$.

Suppose by contradiction, that $x'_k = x'_{a_h}$. Since x'_k is not adjacent to no vertex of P , $|V(P_1[x'_{a_i}, x'_{a_h}])| > 2$ and $|V(P_2[x'_{a_j}, x'_{a_h}])| > 2$. As $G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]]$ is a chordal graph, then the neighborhood of x'_{a_h} in $P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]$ must be a complete set of three vertices.

If x'_{a_i} is adjacent to x'_{a_j} , then $G[P \cup P_1 \cup P_2]$ contains $F_3(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction. Thus x'_{a_i} is not adjacent to x'_{a_j} , and by our choice of them, the neighborhood of x'_{a_p} (for $p \in \{i, j, h\}$) in $P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]$ must be a complete set of three vertices. Moreover, there exists $p \in \{i, j, h\}$ such that $N[x'_{a_p}] \cap N[x'_{a_q}] \cap V[G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]] \neq \emptyset$ for $q \in \{i, j, h\} - p$. Hence $G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]]$ contains $F_4(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction. By before exposed, $x'_k \neq x'_{a_h}$ and then $|V((P_1 \cap P_2)[x'_{a_h}, x'_k])| > 1$.

If at least two pair of vertices of $\{x'_{a_i}, x'_{a_j}, x'_{a_h}\}$ are adjacent, then since $G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]]$ is a chordal graph, it follows that $G[P \cup P_1 \cup P_2]$ contains $F_3(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction. Thus these vertices are not adjacent, and by our choice of them, it follows that neighborhood of x'_{a_p} (for $p \in \{i, j, h\}$) in $P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]$ must be a complete set of three vertices. Moreover there exists $p \in \{i, j, h\}$ such that $N[x'_{a_p}] \cap N[x'_{a_q}] \cap V[G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]] \neq \emptyset$ for $q \in \{i, j, h\} - p$. Hence $G[P_1[x'_{a_i}, x'_{a_h}] \cup P_2[x'_{a_h}, x'_{a_j}] \cup P[x'_{a_i}, x'_{a_j}]]$ contains $F_4(n)$ (for some $n \geq 6$) as induced subgraph, a contradiction. By before exposed, $G[P \cup P_1 \cup P_2]$ is a tree, and since u, v, x'_k is an asteroidal triple of $G[P \cup P_1 \cup P_2]$ results $|V(P)| \geq 5$ and $|V(P_i)| \geq 5$ for $i = 1, 2$.

Without loss of generality, we can assume that $G[P \cup P_1] = T_1 (= G[P \cup P_1 \cup P_2] = G[P \cup P_2] = G[P_1 \cup P_2])$ is an induced tree with one and only one vertex of degree exactly 3 different from x'_1 and x'_{m-1} , and then T_1 contains F_1 as induced

subgraph.

Note that if $x_1 \in V(T_1)$, then $x'_1 = x_1$, $n = 3$ and $x_2 \notin V(T_1)$. And we repeat the procedure done when study the Case 1.2.

Thus, without loss of generality, we can suppose that x_1 and x_2 are not vertices of T_1 .

In the following cases, let x'_{a_i} be the vertex of degree three of T_1 , and x'_h be the vertex of $P_1 - P$ adjacent to x'_{a_i} .

Case 2.1. $n = 2$. Since $x_1 \notin T_1$, x_1 is adjacent to u and v , and G is chordal, we have that x_1 is adjacent to each vertex of P . As $G[P \cup \{x_1\} \cup P_1]$ does not contain $F_5(n)$ (for $n \geq 8$) as induced subgraph, it follows that x_1 must be adjacent to every vertex of P_1 . Thus x_1 is adjacent to x'_k , a contradiction.

Case 2.2. $n = 3$. Let us consider as $P : u, x'_1, x'_{a_2}, \dots, x'_{a_{i-1}}, x'_{a_i}, x'_{a_{i+1}}, \dots, x'_{a_p}, x'_{m-1}, v$, and $P_1[x'_{a_i}, x'_k] : x'_{a_i}, x'_h, x'_{b_{h+1}}, \dots, x'_{b_q}, x'_k$. Since x_1 is adjacent to u , x_2 is adjacent to v , and G is chordal, we have that there exist chords between vertices of W and P .

Let $j \in \{1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_p, m-1\}$ be the first index such that $x_1x'_j$ and $x_2x'_j$ are chords.

Case 2.2.1. Suppose that $j \leq a_{i-2}$. Since G is chordal, there exist chords x_2y for $y \in \{x'_{a_{i-2}}, x'_{a_{i-1}}, x'_{a_i}, x'_{a_{i+1}}, \dots, x'_{a_p}, x'_{m-1}\}$. Also, as $G[P_1[x'_{a_i}, x'_k] \cup P[x'_{a_{i-2}}, v] \cup \{x_2\}]$ does not contain F_2 as induced subgraph, then there exist chords x_2z for some $z \in V(P_1[x'_{a_i}, x'_k]) \setminus \{x'_{a_i}\}$.

If $z \neq x'_h$, then x_2 is adjacent to each vertex of $P_1[x'_h, z]$, otherwise $G[P_1[x'_{a_i}, z] \cup \{x_2\}] = C_r$ (for some $r > 3$) contradicting that G is chordal.

Since $G[P_1[x'_{a_i}, x'_k] \cup P[x'_{a_{i-2}}, v] \cup \{x_2\}]$ does not contain $F_5(n)$ (for some $n \geq 8$) as induced subgraph, results x_2 is adjacent to each vertex of $P_1[x'_{a_i}, x'_k]$, in particular it is adjacent to x'_k , a contradiction.

Case 2.2.2. Suppose that $j \geq a_{i+2}$. This case can be treated similarly to Case 2.2.1 by symmetry.

Case 2.2.3. Suppose that $j \in \{a_{i-1}, a_i, a_{i+1}\}$. Note that a_{i-2} and a_{i+2} may not exist, in this case $|V(P)| = 5$, and then $x'_{a_{i-1}} = x'_1$ and $x'_{a_{i+1}} = x'_{m-1}$.

Case 2.2.3.1. $j = a_{i-1}$.

a. Assume that x_1 is not adjacent to x'_{a_i} . Then it is not adjacent to $x'_{a_{i+1}}$ neither $x'_{a_{i+2}}$, otherwise $G[x_1, x'_{a_{i-1}}, x'_{a_i}, x'_{a_{i+1}}] = C_4$ or $G[x_1, x'_{a_{i-1}}, x'_{a_i}, x'_{a_{i+1}}, x'_{a_{i+2}}] = C_5$. Since $G[x'_{a_{i-1}}, x'_{a_i}, x'_{a_{i+1}}, x'_{a_{i+2}}, x_2] \neq C_5$, x_2 is adjacent to each vertex of $P[x'_{a_i}, v]$. Since $G[P[x'_j, v] \cup P_1[x'_{a_i}, x'_k] \cup \{x_1, x_2\}]$ does not contains F_2 as induced subgraph, x_1 must be adjacent to some vertex of $P_1[x'_{a_i}, x'_k] - x'_{a_i}$ or x_2 must be adjacent to some vertex of $P_1[x'_{a_i}, x'_k] - x'_{a_i}$.

a.1. Suppose that x_1 is adjacent to some vertex of $P_1[x'_{a_i}, x'_k] - x'_{a_i}$. Let w be a vertex of $P_1[x'_{a_i}, x'_k] - \{x'_{a_i}\}$ minimizing the distance to x'_{a_i} . Then, since $G[\{x_1, x'_{a_{i-1}}\} \cup P_1[x'_{a_i}, w]] \neq C_r$ (for some $r > 3$), x_1 is adjacent to every vertex of $P_1[x'_{a_i}, w]$, in particular it is adjacent to x'_{a_i} , a contradiction.

a.2. If x_1 is not adjacent to any vertex of $P_1[x'_{a_i}, x'_k] - x'_{a_i}$, then as $G[W \cup P_1[x'_{a_i}, x'_k] \cup P[x'_k, v]]$ does not contain F_2 neither $F_5(n)$ as induced subgraph (for some $n \geq 8$), then x_2 must be adjacent to each vertex of P_1 , in particular it is adjacent to x'_k , a contradiction.

b. Now, suppose that x_1 is adjacent to x'_{a_i} . Since $G[W \cup P_1[x'_{a_i}, x'_k]]$ does not contain $F_3(n)$ (for some $n \geq 6$) as induced subgraph, it follows that x_1 or x_2 must be adjacent to every vertex of $P_1[x'_{a_i}, x'_k]$, in particular to x'_k , a contradiction.

Case 2.2.3.2. $j = i$. Thus x_2 is not adjacent to $x'_{a_{i-1}}$. Since $G[W \cup P_1[x'_{a_i}, x'_k]]$ does not contain $F_3(n)$ (for some $n \geq 6$) as induced subgraph, it follows that x_1 or x_2 must be adjacent to every vertex of $P_1[x'_{a_i}, x'_k]$, in particular to x'_k , a contradiction.

Case 2.2.3.3. $j = a_{i+1}$. This case can be treated similarly to Case 2.2.3.1 by symmetry.

By the before exposed, $\mathbf{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free $= l_3/\mathbf{TW}$. ■

Theorem 14. $l_2/\mathbf{TW} = \mathbf{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$ -free.

Proof. By Corollary 9, $l_2/\mathbf{TW} \subseteq \mathbf{Chordal}$.

It is easy to see that $F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}$, and $F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}$ are not in l_2/\mathbf{TW} . Thus $l_2/\mathbf{TW} \subseteq \mathbf{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$ -free.

On the other hand, by Lemma 2 it follows that $l_3/\mathbf{TW} \subseteq l_2/\mathbf{TW}$, and then by Theorem 12, $\mathbf{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free $\subseteq l_2/\mathbf{TW}$.

In what follows, we will prove that $\mathbf{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$ -free $\subseteq l_2/\mathbf{TW}$.

In order to derive a contradiction, we suppose that $G \in \mathbf{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$ -free and $G \notin l_2/\mathbf{TW}$. Since $G \notin l_2/\mathbf{TW}$, there exist two non-adjacent vertices u and v , a uv - l_2 -walk $W : u = x_0, \dots, x_n = v$ (observe that $n = 2$) and a uv -tolled-walk $W' : u = x'_0, x'_1, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W . Let x'_k be a vertex of $W' - W$ such that it is not adjacent to any vertex of W . We can

assume that $k \neq 1, m-1$, otherwise $G[W \cup W']$ contains as induced subgraph a cycle of size at least four.

Let P be a shortest path in $G[W']$ from u to v . Since W' is a uv -tolled walk, x'_1 and x'_{m-1} are vertices of P . By Claim 13 Case a, results $x'_k \notin P$.

In what follows we will analyze two cases, depending of x'_k is or is not adjacent to vertices of P . Note that $x_1 \notin P$.

Case 1. x'_k is adjacent to some vertex of P . This case can be treated similarly to Case 1 of Theorem 12.

Case 2. x'_k is adjacent to no vertex of P . Thus $P \cap N[x'_k] = \emptyset$. Since W' is a uv -tolled walk, by Observation 11, there exists an induced path between u and x'_k in $G[W']$ avoiding the neighborhood of v , and also there exists an induced path between v and x'_k in $G[W']$ avoiding the neighborhood of u . Those paths, together with P allow us to state that u, v, x'_k is an asteroidal triple.

Hence, let us consider $P, P_1, P_2, x'_{a_i}, x'_{a_j}, x'_{a_h}$ as in Case 2 of Theorem 12.

Case 2.1. $G[P \cup P_1 \cup P_2]$ is an induced tree. This case can be treated similarly to Case 2.1 of Theorem 12.

Case 2.2. $G[P \cup P_1 \cup P_2]$ is not an induced tree. We can assume that it does not contains $F'_1(n)$ with leaves x'_k, u , and v (for some $n \geq 7$) as induced subgraph. Then $x'_{a_p} \neq x'_{a_q}$ for $p \neq q$ with $p, q \in \{i, j, h\}$.

Note that at least two pairs of $\{x'_{a_i}, x'_{a_j}, x'_{a_h}\}$ must be adjacent, otherwise by the exposed in Case 2 of Theorem 12, it follows that $G[P \cup P_1 \cup P_2]$ contains $F_4(n)$ (for some $n \geq 6$), a contradiction.

Since $G[P \cup \{x_1\}]$ is not an induced cycle of length at least four, results x_1 is adjacent to every vertex of P . Note that $x'_k \neq x'_{a_h}$, since x'_{a_h} is adjacent to x'_{a_i} or x'_{a_j} and x'_k is adjacent to no vertex of P . Thus x'_k and x'_{a_h} are vertices of $P_1 \cap P_2$. Hence, the intersection between every pair of $\{P, P_1, P_2\}$ has at least two vertices.

Let us consider as $P : u, x'_1, x'_{a_2}, \dots, x'_{a_i}, \dots, x'_{a_j}, \dots, x'_{m-1}, v$, $P_1 : u, x'_1, x'_{a_2}, \dots, x'_{a_i}, \dots, x'_{a_h}, \dots, x'_k$ and $P_2 : x'_k, \dots, x'_{a_h}, \dots, x'_{a_j}, \dots, x'_{m-1}, v$.

Case 2.2.1. x'_{a_i} and x'_{a_j} are adjacent to x'_{a_h} . Since G is chordal graph, x'_{a_h} must be adjacent to each vertex of $P[x'_{a_i}, x'_{a_j}]$, otherwise $G[\{x'_{a_h}\} \cup P[x'_{a_i}, x'_{a_j}]]$ contains C_r as induced subgraph (for some $r > 3$).

Case 2.2.1.1. Suppose that x'_{a_i} is adjacent to x'_{a_j} . Since $G[\{x'_{a_{i-1}}, x'_{a_i}, x'_{a_j}, x'_{a_{j+1}}, x_1, x'_{a_h}\}] \neq F_4(6)$, we have that x_1 must be adjacent to x'_{a_h} . Moreover, as $G[P[x'_{a_{i-1}}, x'_{a_{j+1}}] \cup P_1[x'_{a_h}, x'_k]]$ does not contain $F_6(n)$ (for some $n \geq 7$) as induced subgraph, then there exist chords x_1z for every vertex z of $P_1[x'_{a_i}, x'_k]$, in particular, x_1 must be adjacent to x'_k , which contradicts our assumption.

Case 2.2.1.2. Suppose that x'_{a_i} is not adjacent to x'_{a_j} . Since $G[x'_{a_i}, x'_{a_j}, x_1, x'_{a_h}] \neq C_4$, we have that x_1 must be adjacent to x'_{a_h} . Moreover, as $G[P[x'_{a_{i-1}}, x'_{a_{j+1}}] \cup P_1[x'_{a_h}, x'_k]]$ does not contain $F_6(n)$ (for some $n \geq 7$) as induced subgraph, then there exist chords x_1z for every z vertex of $P_1[x'_{a_i}, x'_k]$, in particular x_1 must be adjacent to x'_k , which contradicts our assumption.

Case 2.2.2. x'_{a_j} and x'_{a_h} are adjacent to x'_{a_i} (by symmetry x'_{a_i} and x'_{a_h} are adjacent to x'_{a_j}).

Suppose that x'_{a_j} is not adjacent to x'_{a_h} , otherwise following Case 2.2.1.1 we arrive to a contradiction.

Since $G[\{x'_{a_i}\} \cup P_2[x'_{a_j}, x'_{a_h}]] \neq C_r$ (for some $r > 3$), we have that x'_{a_i} is adjacent to every vertex of $P_2[x'_{a_j}, x'_{a_h}]$. Let x'_{a_q} be the vertex of $P_2[x'_{a_j}, x'_k]$ adjacent to x'_{a_j} . As $G[\{x'_{a_{i-1}}, x'_{a_i}, x'_{a_j}, x'_{a_q}, x'_{a_{j+1}}, x_1\}] \neq F_4(6)$, then there exists the chord $x_1x'_{a_q}$. Also $G[\{x'_{a_{i-1}}, x'_{a_i}, x'_{a_{j+1}}, x_1\} \cup P_2[x'_{a_j}, x'_{a_h}]]$ does not contain $F_4(n)$ (for some $n \geq 6$), then x_1 must be adjacent to every vertex of $P_2[x'_{a_j}, x'_{a_h}]$.

On the other hand, $G[\{x'_{a_{i-1}}, x'_{a_i}, x'_{a_j}, x'_{a_{j+1}}, x_1\} \cup P_2[x'_{a_j}, x'_k]]$ does not contain $F_7(n)$ (for some $n \geq 7$), it follows that x_1 must be adjacent to every vertex of $P_2[x'_{a_j}, x'_k]$, in particular, it is adjacent to x'_k contradicting our assumption. Therefore

$l_2/\text{TW} = \text{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}$ -free. ■

From Lemma 2, and Theorem 12, we obtain the following characterization.

Theorem 15. $l_3/\text{WTW} = \text{Interval} \cap \{\text{chair}, \text{dart}\}$ -free.

Proof. From Lemma 2, $l_3/\text{WTW} \subseteq l_3/\text{TW}$, and by Theorem 12, $l_3/\text{WTW} \subseteq \text{Chordal} \cap \{F_2, F_3(n)_{n \geq 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}\}$ -free.

One can readily verify that F_1 is not in l_3/WTW . Then $l_3/\text{WTW} \subseteq \text{Interval}$. Also dart and chair are not in l_3/WTW . Thus $l_3/\text{WTW} \subseteq \text{Interval} \cap \{\text{chair}, \text{dart}\}$ -free.

Also by Lemma 2, $\text{IP}/\text{WTW} \subseteq l_3/\text{WTW}$. Thus $\text{Interval} \cap \{\text{chair}, \text{dart}\}$ -free $\subseteq l_3/\text{WTW}$

Hence $l_3/\text{WTW} = \text{Interval} \cap \{\text{chair}, \text{dart}\}$ -free. ■

From the Lemma 2, and Theorem 15, we get the following.

Corollary 16. $\text{SP}/\text{WTW} = \text{IP}/\text{WTW} = \text{P}/\text{WTW} = \text{W}/\text{WTW} = \text{TW}/\text{WTW} = \text{WTW}/\text{WTW} = l_3/\text{WTW} = \text{Interval} \cap \{\text{dart}, \text{chair}\}$ -free.

The following theorem provides a characterization of $\text{Chordal} \cap \{\text{chair}, \text{dart}, F_4(6)\}$ -free.

Theorem 17. $l_2/\text{WTW} = \text{Chordal} \cap \{\text{chair}, \text{dart}, F_4(6)\}$ -free.

Proof. By Lemma 2, $l_3/\mathbf{WTW} \subseteq l_2/\mathbf{WTW}$, and by Theorem 15, we obtain $\mathbf{Interval} \cap \{\mathbf{dart}, \mathbf{chair}\}\text{-free} \subseteq l_2/\mathbf{WTW}$. Also by Lemma 2, $l_2/\mathbf{WTW} \subseteq l_2/\mathbf{TW}$, and by Theorem 14, results $l_2/\mathbf{WTW} \subseteq \mathbf{Chordal} \cap \{F_2, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}\text{-free}$.

Note that F_1 and $F_3(n)_{n > 6}$ are not in l_2/\mathbf{WTW} but $F_3(6) \in l_2/\mathbf{WTW}$. Thus $l_2/\mathbf{WTW} \subseteq \mathbf{Chordal} \cap \{F_1, F_2, F_3(n)_{n > 6}, F_4(n)_{n \geq 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}\}\text{-free}$.

On the other hand, $F_1, F_2, F_3(n)_{n > 6}, F_4(n)_{n > 6}, F_5(n)_{n \geq 8}, F_6(n)_{n \geq 7}, F_7(n)_{n \geq 7}$ contain as induced subgraph a chair or a dart. Note that chair, dart and $F_4(6)$ are not in l_2/\mathbf{WTW} . Thus $l_2/\mathbf{WTW} \subseteq \mathbf{Chordal} \cap \{\mathbf{chair}, \mathbf{dart}, F_4(6)\}\text{-free}$.

Now, we will prove that $\mathbf{Chordal} \cap \{\mathbf{chair}, \mathbf{dart}, F_4(6)\}\text{-free} \subseteq l_2/\mathbf{WTW}$. In that follows, we suppose that $G \in \mathbf{Chordal} \cap \{\mathbf{chair}, \mathbf{dart}, F_4(6)\}\text{-free}$ and $G \notin l_2/\mathbf{WTW}$. Then there exist two non-adjacent vertices u and v , a uv - l_2 -walk $W : u = x_0, x_1, x_2 = v$ and a uv -weakly toll $W' : u = x'_0, x'_1, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W . Let x'_k be a vertex of $W' - W$ such that it is not adjacent to any vertex of W .

Let us consider the following exhaustive cases.

Case 1. $W' \cap \{x_1\} \neq \emptyset$. Then, by definition of uv -weakly toll walk, one has that $x_1 = x'_1 = x'_{m-1}$ and that u and v are not adjacent to those vertices in W' which are different from x_1 . Then let P be a shortest path in $G[W']$ from x_1 to x'_k . Thus, $G[P \cup \{u, v\}]$ contains as induced subgraph a chair, a contradiction.

Case 2. $W' \cap \{x_1\} = \emptyset$.

Case 2.1. $x'_1 = x'_{m-1}$. Then, by definition of a uv -weakly toll walk, one has that u and v are non-adjacent vertices to those vertices in W' which are different from $x'_1 = x'_{m-1}$. Since G is chordal, x_1 is adjacent to $x'_1 = x'_{m-1}$. Then let P be a shortest path in $G[W']$ from $x'_1 = x'_{m-1}$ to x'_k , then $G[P \cup \{u, v, x_1\}]$ contains an induced dart (if $x'_1 = x'_{m-1}$ is adjacent to x'_k) or an induced chair (otherwise), a contradiction.

Case 2.2. $x'_1 \neq x'_{m-1}$. Then, by definition of a uv -weakly toll walk, one has that u is non-adjacent to those vertices in W' which are different from x'_1 , and that v is non-adjacent to those vertices in W' which are different from x'_{m-1} . Let P be a shortest path in $G[W']$ from x'_1 to x'_{m-1} . Since G is chordal, x_1 is adjacent to every vertex of P . Then let P_1 be a shortest path in $G[W']$ from P to x'_k , in particular, let x'_h be the vertex of $P_1 - P$ which is adjacent to some vertex of P . Observe that x'_h may be x'_k .

If x'_1 is adjacent to x'_{m-1} (that is P is formed just by such two vertices), then if x'_h is adjacent to both x'_1 and x'_{m-1} , then $G[u, v, x'_1, x'_{m-1}, x'_h]$ is a chair, a contradiction. If x'_h is adjacent only to x'_{m-1} , then $G[u, v, x'_1, x'_{m-1}, x'_k]$ is a chair, a contradiction.

If x'_1 is non-adjacent to x'_{m-1} (that is P is not formed just by such two vertices), then let z be the vertex of P adjacent to x'_1 . Then $G[u, v, x_1, x'_1, z]$ is a dart, a contradiction.

Hence $\mathbf{l}_2/\mathbf{WTW} = \mathbf{Chordal} \cap \{\mathbf{chair}, \mathbf{dart}, \mathbf{F}_4(6)\}\text{-free}$. ■

For last, we study the domination between l_k -paths for $k \in \{2, 3\}$ versus l_h -paths for $h \in \{2, 3\}$ and walks.

Theorem 18. $\mathbf{l}_3/\mathbf{l}_3 = \{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}\text{-free}$.

Proof. Clearly, C_4, C_5 and C_6 are not in $\mathbf{l}_3/\mathbf{l}_3$. Thus $\mathbf{l}_3/\mathbf{l}_3 \subseteq \{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}\text{-free}$.

In that follows, we will prove that $\{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}\text{-free} \subseteq \mathbf{l}_3/\mathbf{l}_3$. Suppose that $G \in \{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}\text{-free}$ and $G \notin \mathbf{l}_3/\mathbf{l}_3$. Then there exist two non-adjacent vertices u and v , a uv - l_3 -walk $W : u = x_0, \dots, x_n = v$ (observe that $n = 2$ or $n = 3$) and a uv - l_3 -walk $W' : u = x'_0, x'_2, \dots, x'_m = v$ ($m = 2$ or $m = 3$) satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W .

We analyze two situations depending on the length of W and W' .

1. $n = 2$. Clearly, $m = 2$ or $m = 3$, in both cases $W \cap W' = \{u, v\}$ and $W \cup W' = C_4$ or $W \cup W' = C_5$, a contradiction.
2. $n = 3$. Observe that m may be 2 or 3, and $W \cap W' = \{u, v\}$. Thus $W \cup W' = C_5$ or $W \cup W' = C_6$, in both cases a contradiction. ■

Theorem 19. $\mathbf{l}_2/\mathbf{l}_2 = \{\mathbf{C}_4\}\text{-free}$.

Proof. Clearly C_4 is not in $\mathbf{l}_2/\mathbf{l}_2$. Thus $\mathbf{l}_2/\mathbf{l}_2 \subseteq \{\mathbf{C}_4\}\text{-free}$.

Now, we will prove the other contention. Suppose that $G \in \{\mathbf{C}_4\}\text{-free}$ and $G \notin \mathbf{l}_2/\mathbf{l}_2$. Then there exist two non-adjacent vertices u and v , a uv - l_2 -walk $W : u = x_0, x_1, x_2 = v$ and a uv - l_2 -walk $W' : u = x'_0, x'_1, x'_2 = v$ satisfying that W does not dominate W' . Thus, there is x'_1 internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W . Thus $W \cap W' = \{u, v\}$ and $W \cup W' = C_4$, a contradiction. ■

Lemma 20. For $k = 2, 3$, the followings statements hold.

1. $\mathbf{Chordal} \subseteq \mathbf{l}_k/\mathbf{SP}$.
2. $\mathbf{Chordal} \subseteq \mathbf{SP}/\mathbf{l}_k$.

Proof. By Lemma 2 $\mathbf{l}_k/\mathbf{IP} \subseteq \mathbf{l}_k/\mathbf{SP}$, and by Theorem 12, $\mathbf{Chordal} \subseteq \mathbf{l}_k/\mathbf{SP}$.

On the other hand, $\mathbf{SP}/\mathbf{P} \subseteq \mathbf{SP}/\mathbf{l}_k$, and by Table 1, $\mathbf{Chordal} \subseteq \mathbf{SP}/\mathbf{l}_k$. ■

Theorem 21. The followings statements hold.

1. $\mathbf{l}_2/\mathbf{SP} = \{\mathbf{C}_4\}\text{-free}$ and $\mathbf{SP}/\mathbf{l}_2 = \{\mathbf{C}_4\}\text{-free}$.
2. $\mathbf{SP}/\mathbf{l}_3 = \{\mathbf{C}_4, \mathbf{C}_5, \mathbf{C}_6\}\text{-free}$.

Proof. 1. Clearly C_4 is not in l_2/\mathbf{SP} . Thus $l_2/\mathbf{SP} \subseteq \{C_4\}$ -free.

In order to prove the other contention, suppose that $G \in \{C_4\}$ -free and $G \notin l_2/\mathbf{SP}$. Thus there exist two non-adjacent vertices u and v , a uv - l_2 -walk $W : u = x_0, x_1, x_2 = v$ and a uv -shortest walk $W' : u = x'_0, x'_1, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W .

Since $W' \in \mathbf{SP}$, $|W'| = 3$. But then u, x_1, v, x'_1, u is C_4 , a contradiction. Hence $l_2/\mathbf{SP} = \{C_4\}$ -free.

We leave to the reader the proof of $\mathbf{SP}/l_2 = \{C_4\}$ -free.

2. Clearly C_i does not belong to \mathbf{SP}/l_3 for $i \in \{4, 5, 6\}$. Thus $\mathbf{SP}/l_3 \subseteq \{C_4, C_5, C_6\}$ -free.

In that follows, we will prove that $\{C_4, C_5, C_6\}$ -free $\subseteq \mathbf{SP}/l_3$. Suppose that $G \in \{C_4, C_5, C_6\}$ -free and $G \notin \mathbf{SP}/l_3$. Thus there exist two non-adjacent vertices u and v , a uv -shortest path $W : u = x_0, \dots, x_n = v$ and a uv - l_3 -walk $W' : u = x'_0, x'_1, \dots, x'_m = v$ satisfying that W does not dominate W' . Thus, there is some internal vertex of W' that is neither a vertex of W nor adjacent to any internal vertex of W . Note that $m = 2$ or $m = 3$.

In case that $m = 2$ then $n = 2$. Thus $W \cup W' = C_4$, a contradiction.

In case that $m = 3$ then $n = 2$ or $n = 3$. If $m = 3$ and $n = 2$, $W \cup W' = C_5$, a contradiction. If $m = n = 3$, then $W \cup W' = C_6$, a contradiction. Hence $\mathbf{SP}/l_3 = \{C_4, C_5, C_6\}$ -free. ■

Theorem 18, Theorem 19, and Theorem 21, give the following characterization of $\{C_4\}$ -free and $\{C_4, C_5, C_6\}$ -free in terms of dominations between walks.

Corollary 22. $l_2/\mathbf{SP} = \mathbf{SP}/l_2 = l_2/l_2 = \{C_4\}$ -free and $\mathbf{SP}/l_3 = l_3/l_3 = \{C_4, C_5, C_6\}$ -free

Corollary 23. $l_3/\mathbf{SP} \subseteq \{C_4, C_5\}$ -free.

Proof. By Lemma 2, $l_3/\mathbf{SP} \subseteq l_2/\mathbf{SP}$ and by Theorem 21, results $l_3/\mathbf{SP} \subseteq \{C_4\}$ -free. Clearly C_5 is not in l_3/\mathbf{SP} . Thus $l_3/\mathbf{SP} \subseteq \{C_4, C_5\}$ -free. ■

Note that $C_6 \notin l_3/\mathbf{SP}$. However, C_6 plus a universal vertex belongs to l_3/\mathbf{SP} . One consequence of the above is that l_3/\mathbf{SP} is not hereditary class.

Let us observe one final thing $l_3/\mathbf{SP} \neq \mathbf{SP}/\mathbf{SP}$ because $C_5 \in \mathbf{SP}/\mathbf{SP} - l_3/\mathbf{SP}$, and $C_8 \in l_3/\mathbf{SP} - \mathbf{SP}/\mathbf{SP}$.

4. CONCLUSIONS

Alcón proved that the notion of domination between different types of walks plays an central role in characterizations of graph classes. We continue the study

of domination between different types of walks focus on weakly toll walks and induced paths with bounded length. We have obtained characterization of the graphs in which, for every pair of non-adjacent vertices u and v , every uv -walk, weakly toll walk, tolled walk, path, l_k -path $k \in \{2, 3\}$, induced path, or shortest path dominates every uv -weakly toll walk, and every uv -weakly toll walk dominates every uv -walk, weakly toll walk, tolled walk, path, l_k -path $k \in \{2, 3\}$, induced path, or shortest path. Some of them, give rise to characterization of standard graph classes.

As the anonymous referee observe, it is impossible to characterize proper interval graph in this sense, since every walk in the claw graph $K_{1,3}$ between non-adjacent vertices contains the universal vertex. Thus, all the special walks dominate all the other special walks.

In the context of convexity theory, chordal and ptolemaic graphs have been characterized as convex geometries with respect to the monophonic convexity and the geodesic convexity, respectively [5]. Similarly, weak polarizable graphs [9] (**{hole, house, domino}-free**) have been characterized as convex geometries with respect to the m^3 -convexity [3] (convexity defined with m_3 -paths, which are induced paths of length at least 3); interval graphs have been characterized as convex geometries with respect to the toll convexity [2]; and proper interval graphs have been characterized as convex geometries with respect to the weakly toll convexity [4].

In [1] it was proved that the class of interval graphs is **IP/TW**. Surprisingly, proper interval graph is not **A/WTW** for $\mathbf{A} \in \{\mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{W}\}$. Furthermore, this does not define any subclass of proper interval graphs.

Natural question arise.

1. Is there any special uv -walks that dominates any special uv -walks, which allow to characterize weak polarizable graphs?
2. Do \mathbf{A}/\mathbf{m}_3 and \mathbf{m}_3/\mathbf{A} , for $\mathbf{A} \in \{\mathbf{l}_k, \mathbf{m}_3, \mathbf{SP}, \mathbf{IP}, \mathbf{P}, \mathbf{TW}, \mathbf{TWT}, \mathbf{W}\}$ give rise to characterize class of graphs?

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