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THE GENERALIZED 3-CONNECTIVITY AND 4-CONNECTIVITY OF CROSSED CUBE

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Abstract

The generalized connectivity, an extension of connectivity, provides a new reference for measuring the fault tolerance of networks. For any connected graph G, let $S \subseteq V(G)$ and $2 \leq |S| \leq V(G)$; $\kappa_G(S)$ refers to the maximum number of internally disjoint trees in G connecting S. The generalized k-connectivity of G, $\kappa_k(G)$, is defined as the minimum value of $\kappa_G(S)$ over all $S \subseteq V(G)$ with $|S| = k$. The n-dimensional crossed cube CQ_n, as a hypercube-like network, is considered as an attractive alternative to hypercube network because of its many good properties. In this paper, we study the generalized 3-connectivity and the generalized 4-connectivity of CQ_n and obtain $\kappa_3(CQ_n) = \kappa_4(CQ_n) = n - 1$, where $n \geq 2$.

Keywords: crossed cube, internally disjoint trees, generalized k-connectivity, fault tolerance.

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1. Introduction

A graph $G = (V(G), E(G))$ is often used to simulate an interconnection network, and in the process of simulation, vertex set and edge set of G refer to the processor set and the communication link set between the processors, respectively. Connectivity is an important parameter to measure the fault tolerance capability of an interconnection network.

The generalized k-connectivity was proposed by Hager in 1985 [4]. As an extension of connectivity, it is also widely used in the study of internet topology model and become a reference to measure the reliability and fault tolerance of networks. For $S \subseteq V(G)$ with $2 \leq |S| \leq V(G)$, $\kappa_G(S)$ refers to the maximum number r of internally disjoint trees $T_1, \ldots, T_r$ in G connecting S where $V(T_i) \cap V(T_j) = S$ and $E(T_i) \cap E(T_j) = \emptyset$ for any $1 \leq i \neq j \leq r$ [4]. The generalized k-connectivity of G, $\kappa_k(G)$, is defined as the minimum value of $\kappa_G(S)$ over all $S \subseteq V(G)$ and $|S| = k$ [4]. For a graph G, its connectivity $\kappa(G)$ is the smallest number of vertices in a vertex set F that makes $G - V(F)$ disconnected or trivial. Then, the equivalent definition of connectivity is given by Whitney, i.e., $\kappa(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = 2\}$ [14].

Since concepts of generalized connectivity were put forward, more and more research results have been published, such as the generalized 3-connectivity of some graphs, including Cartesian product graphs [3], graph products [7], Cayley graphs on symmetric groups generated by trees and cycles [9], star graphs S_n and bubble-sort graphs B_n [10], the Mycielskian of a graph [11], alternating group graphs and (n, k)-star graph [15], regular graphs with some special properties [17] and so on; the generalized 4-connectivity of some graphs, including hypercubes [12], exchanged hypercubes [16], hierarchical cubic networks [18] and so on. In this paper, we study the generalized 3-connectivity and the generalized 4-connectivity of n-dimensional crossed cube and obtain that $\kappa_3(CQ_n) = \kappa_4(CQ_n) = n - 1$, where $n \geq 2$.

This paper is divided into five sections. The first two sections are Introduction and Preliminaries, in the third section we introduce CQ_n, and in the fourth section we prove our main result. In the last section, it is Conclusion.

2. Preliminaries

Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G. If $xy \in E(G)$ and $x \neq y$, then we say x is a neighbor of y, or x is adjacent to y, and vice versa. The neighborhood of vertex x (vertex set X, respectively) in G is a set which contains all its neighbors in G except itself, that is, $N_G(x) = \{y \mid xy \in E(G), x \neq y\}$ ($N_G(X) = \bigcup_{x \in X} N_G(x) - X$, respectively).

For any edge $xy \in E(G)$, we say this edge is incident with vertices x and y. The degree $d_G(x)$ of x is the number of edges which are incident with it in G, and we use $\delta(G) = \min\{d_G(x) \mid x \in V(G)\}$ to denote the minimum degree of G. In this paper, we use P_{xy} or (x, y)-path to denote the path that begins and ends with x and y, respectively. For any two (x, y)-paths P_{xy} and Q_{xy}, if $V(P_{xy}) \cap V(Q_{xy}) = \{x, y\}$, then we say they are internally disjoint. For $X \subseteq V(G)$ and $Y \subseteq (V(G) \backslash X)$, (X, Y)-paths refer to a family of paths which are internally disjoint and all begin with the vertices of X and end with the vertices of Y. A k-fan refers to a family of (x, Y)-paths which begin with x and end with different vertices of Y, where $|Y| = k$.

3. DEFINITIONS OF CQ_n AND RELATED RESULTS

For any two-bit binary strings $x = x_1x_0$ and $y = y_1y_0$, if $(x, y) \in \{(00, 00), (10, 10), (11, 01), (01, 11)\}$, then we say they are pair-related, that is $x \sim y$ [2].

Definition [2]**.** CQ_1 is an edge with vertices 0 and 1. CQ_2 is a 4-cycle $\langle 10, 00, 01, 11, 10\rangle$. For $n \geq 3$, the structure of n-dimensional crossed cube CQ_n is recursive with two copies of CQ_{n-1}, CQ^0_{n-1} and CQ^1_{n-1}, whose vertex sets are $V(CQ^0_{n-1}) = \{0u_{n-2}\cdots u_1u_0 \mid u_i \in \{0,1\}, 0 \leq i \leq n-2\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}\cdots v_1v_0 \mid v_i \in \{0,1\},\ 0 \leq i \leq n-2\}$, respectively. For convenience, let $CQ_n = CQ^0_{n-1} \bigotimes CQ^1_{n-1}$. Moreover, for the vertices $u = 0u_{n-2}\cdots u_1u_0$ and $v = 1v_{n-2}\cdots v_1v_0$, if $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ for $0 \leq i < \left\lfloor \frac{n-1}{2} \right\rfloor$ and $u_{n-2} = v_{n-2}$ when n is even, then they are adjacent to each other. (CQ_3 and CQ_4 are shown in Figure 1.)

For any $u = u_{n-1}\cdots u_1u_0 \in V(CQ_n)$, its i-dimensional $(0 \leq i \leq n-1)$ neighbor $u^i = v_{n-1}\cdots v_1v_0$ is defined as follows: (1) $u_{n-1}\cdots u_{i+1} = v_{n-1}\cdots v_{i+1}$, (2) $u_i \neq v_i$, (3) if i is odd, then $u_{i-1} = v_{i-1}$, and (4) $u_{2j+1}u_{2j} \sim v_{2j+1}v_{2j}$, $0 \leq j < \left\lfloor \frac{i}{2} \right\rfloor$.

By Definition, we obtain that CQ_n is n-regular and $CQ_n = CQ^0_{n-1} \bigotimes CQ^1_{n-1} = \left(CQ^{00}_{n-2} \bigotimes CQ^{01}_{n-2}\right) \bigotimes \left(CQ^{10}_{n-2} \bigotimes CQ^{11}_{n-2}\right) = \cdots = \left(CQ^{00\cdots 00}_1 \bigotimes CQ^{00\cdots 01}_1\right) \bigotimes \cdots \bigotimes \left(CQ^{11\cdots 10}_1 \bigotimes CQ^{11\cdots 11}_1\right)$ where $CQ^0_{n-1} = CQ^{00}_{n-2} \bigotimes CQ^{01}_{n-2}$, $CQ^1_{n-1} = CQ^{10}_{n-2} \bigotimes CQ^{11}_{n-2}$ and $CQ^{00\cdots 00}_1$, $CQ^{00\cdots 01}_1$, $CQ^{11\cdots 10}_1$, $CQ^{11\cdots 11}_1$ are isomorphic to edges. For any $u = u_{n-1}\cdots u_1u_0$, $u \in CQ^{u_{n-1}\cdots u_{j+1}u_j}_j$, its j-dimensional $(1 \leq j \leq n-1)$ neighbor u^j is in $CQ^{u_{n-1}\cdots u_{j+1}\overline{u_j}}_j$ and 0-dimensional neighbor u^0 is in $CQ^{u_{n-1}\cdots u_1}_1$.

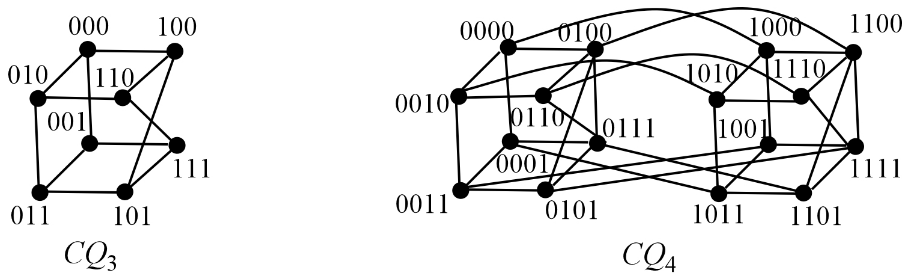

Figure 1. CQ_3 and CQ_4.

Lemma 1 [6]**.** *$\kappa(CQ_n) = n$, where $n \geq 1$.*

Lemma 2 [5]**.** *In CQ_n $(n \geq 2)$, the length of the cycle is at least 4.*

Lemma 3 [13]**.** *For any $u \in V(CQ_n)$, $u^{i,i+1}$ is the common neighborhood of u^i and u^{i+1} in CQ_n, where $0 \leq i \leq n-2$.*

Lemma 4 [8]**.** *If there is an edge $uv \in E(G)$ and $d_G(u) = d_G(v) = \delta(G)$, then $\kappa_k(G) \leq \delta(G) - 1$ for $3 \leq k \leq |V(G)|$.*

Lemma 5 [1]**.** *For a k-connected graph G, let $u \in V(G)$ and $Y \subseteq (V(G) \backslash u)$ with $|Y| \geq k$. Then there exists a k-fan in G which starts with u and ends with distinct vertices of Y.*

Lemma 6 [1]**.** *For a k-connected graph G, $\{u, v\} \subseteq V(G)$ and $u \neq v$, there exists a set of k internally disjoint paths in G to connect u and v.*

Lemma 7 [1]**.** *For a k-connected graph G, $X \subseteq V(G)$ with $|X| \geq k$ and $Y \subseteq (V(G) \backslash X)$ with $|Y| \geq k$, there exists a set of k pairwise disjoint (X, Y)-paths in G.*

Lemma 8 [12]**.** *For an r-regular graph G, if $\kappa_k(G) = r - 1$ for $k \geq 4$, then $\kappa_{k-1}(G) = r - 1$.*

Lemma 9. *For any $uv \in E\big(CQ^i_{n-1}\big)$, if $u \in V\big(CQ^{i0}_{n-2}\big)$ and $v \in V\big(CQ^{i1}_{n-2}\big)$, then $(n-1)$-dimensional neighbors of them must satisfy that one is in $CQ^{(1-i)0}_{n-2}$, the other is in $CQ^{(1-i)1}_{n-2}$ and be $(n-2)$-dimensional neighborhoods of each other, where $i \in \{0, 1\}$.*

Proof. Without loss of generality, let $uv \in E\big(CQ^0_{n-1}\big)$, $u = 00u_{n-3} \cdots u_1 u_0 \in V\big(CQ^{00}_{n-2}\big)$, $v = u^{n-2} = 01v_{n-3} \cdots v_1 v_0 \in V\big(CQ^{01}_{n-2}\big)$, and $(n-1)$-dimensional neighbors of them are $u^{n-1} = 1u'_{n-2}u'_{n-3} \cdots u'_1 u'_0$ and $v^{n-1} = 1v'_{n-2}v'_{n-3} \cdots v'_1 v'_0$, respectively.

If n is even, then $n-1$ is odd, by Definition, $u'_{n-2} = 0$ and $v'_{n-2} = 1$, that is, $u^{n-1} = 10u'_{n-3} \cdots u'_1 u'_0 \in V\big(CQ^{10}_{n-2}\big)$ and $v^{n-1} = 11v'_{n-3} \cdots v'_1 v'_0 \in V\big(CQ^{11}_{n-2}\big)$.

If n is odd, then $n-2$ is odd, by $v = u^{n-2}$ and Definition, we obtain $u_{n-3} = v_{n-3}$. Two cases will be discussed.

Combining Definition and the fact that $n-1$ is even, $0u_{n-3} \sim u'_{n-2}u'_{n-3}$ and $1v_{n-3} \sim v'_{n-2}v'_{n-3}$.

Case 1. $u_{n-3} = v_{n-3} = 0$. We have $0u_{n-3} = 00 \sim 00 = u'_{n-2}u'_{n-3}$ and $1v_{n-3} = 10 \sim 10 = v'_{n-2}v'_{n-3}$, that is, $u^{n-1} = 100u'_{n-4} \cdots u'_1 u'_0 \in V\big(CQ^{10}_{n-2}\big)$ and $v^{n-1} = 110v'_{n-4} \cdots v'_1 v'_0 \in V\big(CQ^{11}_{n-2}\big)$.

Case 2. $u_{n-3} = v_{n-3} = 1$. We have $0u_{n-3} = 01 \sim 11 = u'_{n-2}u'_{n-3}$ and $1v_{n-3} = 11 \sim 01 = v'_{n-2}v'_{n-3}$, that is, $u^{n-1} = 111u'_{n-4} \cdots u'_1 u'_0 \in V\big(CQ^{11}_{n-2}\big)$ and $v^{n-1} = 101v'_{n-4} \cdots v'_1 v'_0 \in V\big(CQ^{10}_{n-2}\big)$.

Therefore, if $u^{n-1} \in V\big(CQ^{10}_{n-2}\big)$, then $v^{n-1} \in V\big(CQ^{11}_{n-2}\big)$. And by Lemma 3, we know $v^{n-1} = u^{n-2,n-1}$ is the neighbor of u^{n-1}, so they are $(n-2)$-dimensional neighbors of each other. ■

Lemma 10. *For any* $\{u,v,w\} \subseteq V(CQ^i_{n-1})$*, we can find a path* P_1 *in* CQ^i_{n-1} *from* u *to* v*,* $n-2$ *internally disjoint trees* $T_2, T_3, \ldots, T_{n-1}$ *in* $(CQ^i_{n-1} \backslash P_1) \cup \{u,v\}$ *to connect* $\{u,v,w\}$*, and* $|V(T_k)| \geq 4$ *for any* $2 \leq k \leq n-1$*, where* $i \in \{0,1\}$*.*

Proof. Without loss of generality, let $\{u,v,w\} \subseteq V(CQ^0_{n-1})$, $u = 0u_{n-2}\cdots u_1u_0$, $v = 0v_{n-2}\cdots v_1v_0$ and $w = 0w_{n-2}\cdots w_1w_0$. We set u_l is the first bit (from left to right) that does not satisfy condition $u_l = v_l = w_l$ $(1 \leq l \leq n-2)$, that is, $0u_{n-2}\cdots u_{l+1} = 0v_{n-2}\cdots v_{l+1} = 0w_{n-2}\cdots w_{l+1}$. Without loss of generality, let $u_l = v_l = 0$, $w_l = 1$, then $\{u,v\} \subseteq V(CQ_l^{0u_{n-2}\cdots u_{l+1}0})$ and $\{w\} \subseteq V(CQ_l^{0u_{n-2}\cdots u_{l+1}1})$.

By Lemma 6 and $\kappa(CQ_l) = l$, we can find an internally disjoint (u,v)-path set $\mathcal{P} = \{P_1, \ldots, P_l\}$ in $CQ_l^{0u_{n-2}\cdots u_{l+1}0}$. Moreover, if u is adjacent to v, then let $P_1 = uv$. Hence, $|V(P_j)| \geq 3$ $(2 \leq j \leq l)$, and we take any vertex on P_j except u and v, and record it as x_j. Obviously, $w^l \in CQ_l^{0u_{n-2}\cdots u_{l+1}0}$, and two cases will be discussed.

Case 1. $w^l \notin \{u,v\}$. There is a path P in $CQ_l^{0u_{n-2}\cdots u_{l+1}0}$ between w^l and u since $CQ_l^{0u_{n-2}\cdots u_{l+1}0}$ is connected. Without loss of generality, let the first common vertex of P (here P starts at w^l) and $\mathcal{P}$ be t and $t \in V(P_l)$.

Let $X = \{x^l_j \mid 2 \leq j \leq l-1\} \cup \{u^l, v^l\}$ with $|X| = l$, where x^l_js are the l-dimensional neighbors of x_js. By Lemma 5, we can find l internally disjoint (w,X)-paths $Q_1, \ldots, Q_l$ in $CQ_l^{0u_{n-2}\cdots u_{l+1}1}$ where $u^l \in Q_1$, $x^l_j \in Q_j$ and $v^l \in Q_l$. Let $T_j = P_j \cup x_jx^l_j \cup Q_j$ for $2 \leq j \leq l-1$, $T_l = P_l \cup P_{w^lt} \cup ww^l$ where P_{w^lt} refers to the part from w^l to t on P, and $T_{l+1} = uu^l \cup Q_1 \cup Q_l \cup vv^l$. (See Figure 2.)

Case 2. $w^l \in \{u,v\}$. Suppose $w^l = u$. Let $X = \{x^l_j \mid 2 \leq j \leq l\} \cup \{v^l\}$ with $|X| = l$. By Lemma 5, we can find l internally disjoint (w,X)-paths $Q_1, \ldots, Q_l$ in $CQ_l^{0u_{n-2}\cdots u_{l+1}1}$ where $v^l \in Q_1$, $x^l_j \in Q_j$. Let $T_j = P_j \cup x_jx^l_j \cup Q_j$ for $2 \leq j \leq l$ and $T_{l+1} = uw \cup Q_1 \cup vv^l$. (See Figure 3.)

From the recursive structure of CQ_k $(l+1 \leq k \leq n-2)$, $CQ_k^{0u_{n-2}\cdots u_{k+1}u_k} = CQ_{k-1}^{0u_{n-2}\cdots u_ku_{k-1}} \bigotimes CQ_{k-1}^{0u_{n-2}\cdots u_k\overline{u_{k-1}}}$, then $\{u,v,w\} \subseteq CQ_k^{0u_{n-2}\cdots u_{k+1}u_k}$ and their k-dimensional neighbors u^k, v^k, w^k must be in $CQ_k^{0u_{n-2}\cdots u_{k+1}\overline{u_k}}$. $CQ_k^{0u_{n-2}\cdots u_{k+1}\overline{u_k}}$ is k-connected, so there is a tree T''_{k+1} in $CQ_k^{0u_{n-2}\cdots u_{k+1}\overline{u_k}}$ connecting them. Let $T_{k+1} = T''_{k+1} \cup uu^k \cup vv^k \cup ww^k$, where $l+1 \leq k \leq n-2$. (See Figures 2, 3.)

Clearly, $|V(T_k)| \geq 4$ for any $2 \leq k \leq n-1$. Hence, the lemma holds. ∎

4. The Generalized 4-Connectivity of Crossed Cube

Lemma 11. *For any* $S \subseteq V(CQ_n)$ *with* $|S| = 4$ *and* $n \geq 3$*, if* $|S \cap V(CQ^i_{n-1})| = 3$ *and* $|S \cap V(CQ^{1-i}_{n-1})| = 1$*, then we can find* $n-1$ *internally disjoint trees in* CQ_n *to connect* S*, where* $i \in \{0,1\}$*.*

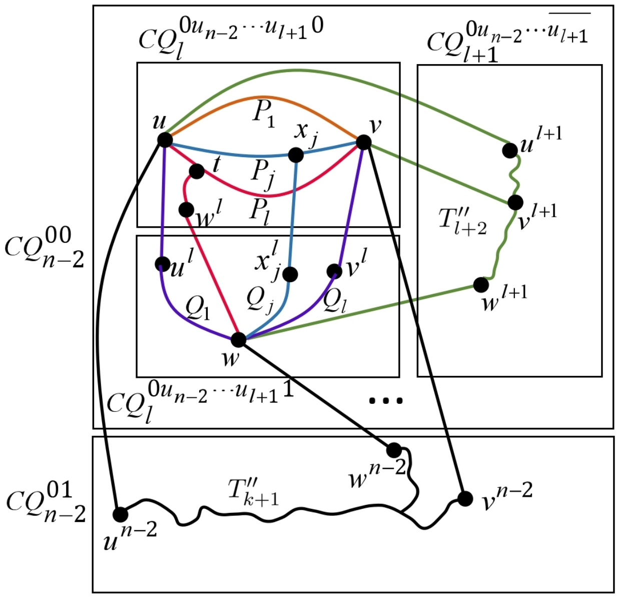

Figure 2. $w^l \notin \{u, v\}$.

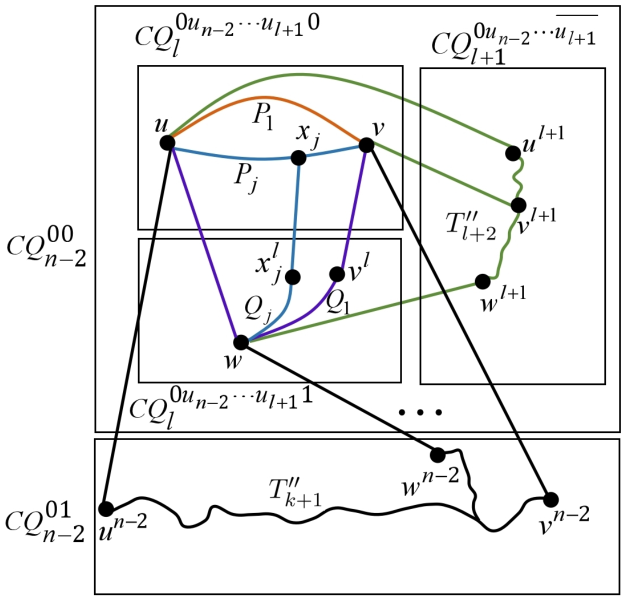

Figure 3. $w^l = u$.

Proof. Let $S = \{x, y, z, w\}$ and $S \cap V(CQ^i_{n-1}) = S^i$ $(i \in \{0, 1\})$. Without loss of generality, let $|S^0| = |\{x, y, z\}| = 3$ and $|S^1| = |\{w\}| = 1$, then $\{x^{n-1}, y^{n-1}, z^{n-1}\} \subseteq V(CQ^1_{n-1})$ and $w^{n-1} \in V(CQ^0_{n-1})$.

By Lemma 10, we can find a path P_1 in CQ^0_{n-1} from x to y, an internally disjoint tree set $\mathcal{T} = \{T_2, T_3, \ldots, T_{n-1}\}$ in $(CQ^0_{n-1} \backslash P_1) \cup \{x, y\}$ connecting S^0 and $|V(T_j)| \geq 4$ for any T_j $(2 \leq j \leq n-1)$. We take any vertex in $V(T_j) \backslash \{x, y, z\}$, and record it as o_j. Two cases will be considered.

Case 1. $w^{n-1} \notin \{x, y, z\}$. There is a path Q in CQ^0_{n-1} between w^{n-1} and x since CQ^0_{n-1} is connected. Let u be the first common vertex between $P_1 \cup \mathcal{T}$ and path Q, and here Q starts at w^{n-1}, then $u \in V(P_1)$ or $u \in V(\mathcal{T})$ (without loss of generality, let $u \in V(P_1)$ or $u \in V(T_{n-1})$).

Let $X = \{o^{n-1}_j \mid 2 \leq j \leq n-2\} \cup \{y^{n-1}, z^{n-1}\}$ where o^{n-1}_j is the $(n-1)$-dimensional neighbor of o_j and $o^{n-1}_j \in CQ^1_{n-1}$. By Lemma 5, we can find $n-1$ internally disjoint (w, X)-paths $R_1, \ldots, R_{n-1}$ in CQ^1_{n-1} where $z^{n-1} \in R_1$, $o^{n-1}_j \in R_j$ $(2 \leq j \leq n-2)$, and $y^{n-1} \in R_{n-1}$.

If $u \in V(P_1)$, we let $T'_1 = P_1 \cup Q_{w^{n-1}u} \cup ww^{n-1} \cup R_1 \cup zz^{n-1}$ where $Q_{w^{n-1}u}$ refers to the part from w^{n-1} to u on Q, $T'_j = T_j \cup o_j o^{n-1}_j \cup R_j$ $(2 \leq j \leq n-2)$, and $T'_{n-1} = T_{n-1} \cup yy^{n-1} \cup R_{n-1}$.

If $u \in V(T_{n-1})$, we let $T'_1 = P_1 \cup yy^{n-1} \cup R_{n-1} \cup R_1 \cup zz^{n-1}$, $T'_j = T_j \cup o_j o^{n-1}_j \cup R_j$ $(2 \leq j \leq n-2)$, and $T'_{n-1} = T_{n-1} \cup Q_{w^{n-1}u} \cup ww^{n-1}$.

Case 2. $w^{n-1} \in \{x, y, z\}$. Suppose $w^{n-1} = x$, and $X = \{o^{n-1}_j \mid 2 \leq j \leq n-1\} \cup \{z^{n-1}\}$ where o^{n-1}_j is the $(n-1)$-dimensional neighbor of o_j and $o^{n-1}_j \in CQ^1_{n-1}$. By Lemma 5, we can find $n-1$ internally disjoint (w, X)-paths $R_1, \ldots, R_{n-1}$ in CQ^1_{n-1} where $z^{n-1} \in R_1$, $o^{n-1}_j \in R_j$ $(2 \leq j \leq n-1)$.

Let $T'_1 = P_1 \cup xw \cup R_1 \cup zz^{n-1}$, $T'_j = T_j \cup o_j o^{n-1}_j \cup R_j$ $(2 \leq j \leq n-1)$.

Hence, the lemma holds. ■

Lemma 12. *For any $S \subseteq V(CQ_n)$ with $|S| = 4$ and $n \geq 3$, if $|S \cap V(CQ^{00}_{n-2})| = 2$ and $|S \cap V(CQ^{10}_{n-2})| = 2$, then we can find $n-1$ internally disjoint trees in CQ_n to connect S.*

Proof. For any $S = \{x, y, z, w\} \subseteq V(CQ_n)$, suppose $\{x, y\} \subseteq V(CQ^{00}_{n-2})$ and $\{z, w\} \subseteq V(CQ^{10}_{n-2})$.

Combining $\kappa(CQ^{00}_{n-2}) = \kappa(CQ^{10}_{n-2}) = n-2$ and Lemma 6, we can find two internally disjoint path sets, $\mathcal{P} = \{P_1, \ldots, P_{n-2}\}$ in CQ^{00}_{n-2} with x and y as ends, and $\mathcal{Q} = \{Q_1, \ldots, Q_{n-2}\}$ in CQ^{10}_{n-2} with z and w as ends. Moreover, if x is adjacent to y, then let $P_1 = xy$; if z is adjacent to w, then let $Q_1 = zw$.

We take vertex $u_1 \in V(P_1) \backslash \{x\}$ for $P_1 \neq xy$ and $u_1 = x$ for $P_1 = xy$ and vertex $u_j \in V(P_j) \backslash \{x, y\}$ for $2 \leq j \leq n-2$, then $u^{n-2}_j \in V(CQ^{01}_{n-2})$ for $1 \leq j \leq n-2$. We take vertex $v_1 \in V(Q_1)$ and vertex $v_j \in V(Q_j) \backslash \{z, w\}$ for

$2 \le j \le n-2$, if $v_j^{n-1} \in V\big(CQ_{n-2}^{01}\big)$ for $1 \le j \le n-2$, then let $o_j = v_j$ and $T_j = Q_j \cup o_j o_j^{n-1}$; if $v_j^{n-1} \in V\big(CQ_{n-2}^{00}\big)$ for $1 \le j \le n-2$, by $v_j^{n-2} \in V\big(CQ_{n-2}^{11}\big)$ and Lemma 9, $v_j^{n-2,n-1} \in V\big(CQ_{n-2}^{01}\big)$, let $o_j = v_j^{n-2}$ and $T_j = Q_j \cup v_j o_j \cup o_j o_j^{n-1}$.

Case 1. $\{x^{n-1}, y^{n-1}\} \subseteq V\big(CQ_{n-2}^{10}\big)$ and $\{z^{n-1}, w^{n-1}\} \subseteq V\big(CQ_{n-2}^{00}\big)$. If $\{x^{n-1}, y^{n-1}\} \ne \{z, w\}$, suppose $x^{n-1} \notin \{z, w\}$ and $z \notin \{x^{n-1}, y^{n-1}\}$, that is, $z^{n-1} \notin \{x, y\}$. (See Figure 4.) There is a path P in CQ_{n-2}^{00} between z^{n-1} and x since CQ_{n-2}^{00} is connected. Suppose u is the first common vertex between $\mathcal{P}$ and path P, and here P starts at z^{n-1}, that is, $u \in \mathcal{P}$. Let $u \in P_{n-2}$ and $P' = P_{z^{n-1}u}$ refers to the part from z^{n-1} to u on P. Similarly, there is a path Q in CQ_{n-2}^{10} between x^{n-1} and z since CQ_{n-2}^{10} is connected. Suppose v is the first common vertex between $\mathcal{Q}$ and path Q, and here Q starts at x^{n-1}, that is, $v \in \mathcal{Q}$. Let $v \in Q_{n-2}$ and $Q' = Q_{x^{n-1}v}$ refers to the part from x^{n-1} to v on Q. (If $\{x^{n-1}, y^{n-1}\} = \{z, w\}$, suppose $x^{n-1} = w$, $y^{n-1} = z$, $P' = yz$ and $Q' = xw$.)

Let $X = \big\{u_j^{n-2} \mid 2 \le j \le n-3\big\}$ with $|X| = n-4$, $Y = \big\{o_j \mid 2 \le j \le n-3\big\}$ and $Y' = Y \cap V\big(CQ_{n-2}^{11}\big)$, $|Y'| \le n-4$ and $CQ_{n-2}^{11} \backslash Y'$ is connected since $\kappa(CQ_{n-2}^{11}) = n-2$. So there is a tree T in $CQ_{n-2}^{11} \backslash Y'$ connecting $\big\{x^{n-2,n-1}, y^{n-2,n-1}, z^{n-2}, w^{n-2}\big\}$.

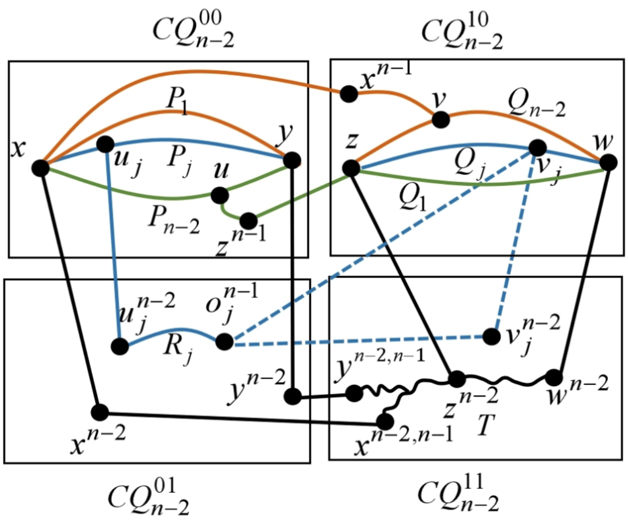

Figure 4. $x^{n-1} \notin \{z, w\}$ and $z \notin \{x^{n-1}, y^{n-1}\}$.

Note that since $\{x^{n-1}, z, w\} \cap Y = \emptyset$, by Lemma 9 and definition of neighborhood, $\big\{x^{n-2,n-1}, z^{n-2}, w^{n-2}\big\} \cap Y' = \emptyset$. And then we will prove that there is always Y' that makes $y^{n-2,n-1} \notin Y'$. Assume $y^{n-2,n-1} \in Y'$, without loss of generality, let $y^{n-2,n-1} = o_2$, then $y^{n-1} = v_2$. If $Q_1 = zw$, by Lemma 2, $|V(P_j)| \ge 4$ for all $2 \le j \le n-3$, we can retake vertex v_2 in $V(Q_2) \backslash \{z, w, y^{n-1}\}$ such that

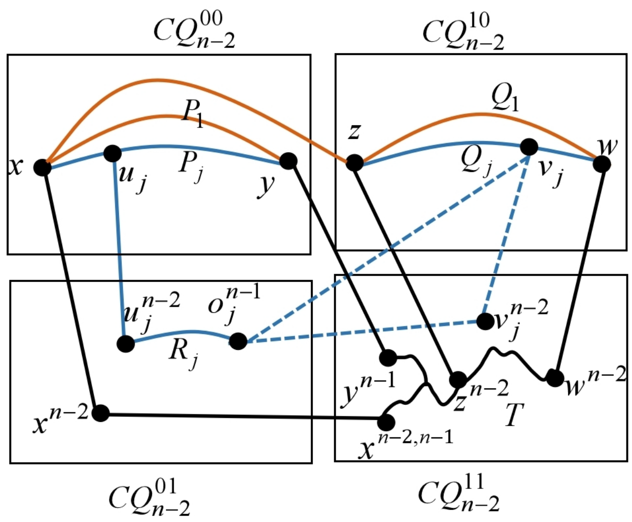

Figure 5. $x^{n-1} \in V(CQ^{10}_{n-2})$, $z^{n-1} \in V(CQ^{00}_{n-2})$ and $x^{n-1} = z$.

$y^{n-2,n-1} \notin Y'$. If $Q_1 \neq zw$, then $|V(P_j)| \geq 3$ for all $1 \leq j \leq n-3$, we can mark the original Q_2 as Q_1 and the original Q_1 as Q_2 such that $y^{n-2,n-1} \notin Y'$.

Let $Y'' = \{o_j^{n-1} \mid 2 \leq j \leq n-3\}$ with $|Y''| = n-4$. By Lemma 7, we can find $n-4$ pairwise disjoint (X, Y'')-paths $R_2, \ldots, R_{n-3}$ in $CQ^{01}_{n-2} \backslash \{x^{n-2}, y^{n-2}\}$ where $\{u_j^{n-2}, o_j^{n-1}\} \subseteq V(R_j)$. If $u_s^{n-2} = o_t^{n-1}$ and $2 \leq s \neq t \leq n-3$, the original Q_s is denoted as Q_t and the original Q_t is denoted as Q_s, we have $R_s = u_s^{n-2}$. (The following similar situations are handled in this way and will not be repeated one by one.)

Let $T_1' = P_1 \cup xx^{n-1} \cup Q' \cup Q_{n-2}$, $T_j' = P_j \cup u_j u_j^{n-2} \cup R_j \cup T_j$ $(2 \leq j \leq n-3)$, $T_{n-2}' = P_{n-2} \cup P' \cup zz^{n-1} \cup Q_1$, and $T_{n-1}' = xx^{n-2} \cup x^{n-2}x^{n-2,n-1} \cup yy^{n-2} \cup y^{n-2}y^{n-2,n-1} \cup zz^{n-2} \cup ww^{n-2} \cup T$. (See Figure 4.)

Case 2. $\{x^{n-1}, y^{n-1}\} \subseteq V(CQ^{10}_{n-2})$ and $\left|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})\right| = 1$, or $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| = 1$ and $\{z^{n-1}, w^{n-1}\} \subseteq V(CQ^{00}_{n-2})$.

Without loss of generality, let $\{x^{n-1}, y^{n-1}\} \subset V(CQ^{10}_{n-2})$ and $\big|\{z^{n-1}, w^{n-1}\}$ $\cap V(CQ^{00}_{n-2})\big| = |\{z^{n-1}\}| = 1$.

If $z^{n-1} \notin \{x, y\}$, then $x^{n-1} \notin \{z, w\}$. The proof is completely similar to Case 1.

If $z^{n-1} \in \{x, y\}$, suppose $z^{n-1} = y$. The proof is similar to Case 1 except that $P' = yz$ and $T_{n-2}' = P_{n-2} \cup P' \cup Q_1$.

Case 3. $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| = 1$ and $\left|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})\right| = 1$. Without loss of generality, let $x^{n-1} \in V(CQ^{10}_{n-2})$ and $z^{n-1} \in V(CQ^{00}_{n-2})$, then $y^{n-1} \in V(CQ^{11}_{n-2})$ and $w^{n-1} \in V(CQ^{01}_{n-2})$. Two cases will be considered.

Case 3.1. $x^{n-1} \neq z$. The proof is similar to Case 1 except that there is a tree T in $CQ^{11}_{n-2} \backslash Y'$ connecting $\{x^{n-2,n-1}, y^{n-1}, z^{n-2}, w^{n-2}\}$ and $T'_{n-1} = xx^{n-2} \cup x^{n-2}x^{n-2,n-1} \cup yy^{n-1} \cup ww^{n-2} \cup zz^{n-2} \cup T$.

Case 3.2. $x^{n-1} = z$. Suppose $X = \{u^{n-2}_j \mid 2 \leq j \leq n-2\}$ with $|X| = n-3$, $Y = \{o_j \mid 2 \leq j \leq n-2\}$ and $Y' = Y \cap V(CQ^{11}_{n-2})$, then $|Y'| \leq n-3$ and $CQ^{11}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{11}_{n-2}) = n-2$. So there is a tree T in $CQ^{11}_{n-2} \backslash Y'$ connecting $\{x^{n-2,n-1}, y^{n-1}, z^{n-2}, w^{n-2}\}$.

Let $Y'' = \{o^{n-1}_j \mid 2 \leq j \leq n-2\}$ with $|Y''| = n-3$. By Lemma 7, we can find $n-3$ pairwise disjoint (X, Y'')-paths $R_2, \ldots, R_{n-2}$ in $CQ^{01}_{n-2} \backslash \{x^{n-2}\}$ where $\{u^{n-2}_j, o^{n-1}_j\} \subseteq V(R_j)$.

Let $T'_1 = P_1 \cup xz \cup Q_1$, $T'_j = P_j \cup u_j u^{n-2}_j \cup R_j \cup T_j$ $(2 \leq j \leq n-2)$, $T'_{n-1} = xx^{n-2} \cup x^{n-2}x^{n-2,n-1} \cup yy^{n-1} \cup ww^{n-2} \cup zz^{n-2} \cup T$. (See Figure 5.)

Case 4. $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| \geq 1$ and $\left|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})\right| = 0$, or $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| = 0$ and $\left|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})\right| \geq 1$.

Without loss of generality, suppose $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| = 0$ and $z^{n-1} \in V(CQ^{00}_{n-2})$. (See Figure 6.) There is a path P in CQ^{00}_{n-2} between z^{n-1} and x since CQ^{00}_{n-2} is connected. Suppose u is the first common vertex between $\mathcal{P}$ and path P, and here P starts at z^{n-1}, that is, $u \in \mathcal{P}$. Let $u \in P_{n-2}$, $X = \{u^{n-2}_j \mid 1 \leq j \leq n-3\}$, $Y = \{o_j \mid 1 \leq j \leq n-3\}$ and $Y' = Y \cap V(CQ^{11}_{n-2})$, $|Y'| \leq n-3$ and $CQ^{11}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{11}_{n-2}) = n-2$. So there is a tree T in $CQ^{11}_{n-2} \backslash Y'$ connecting $\{x^{n-1}, y^{n-1}, z^{n-2}, w^{n-2}\}$.

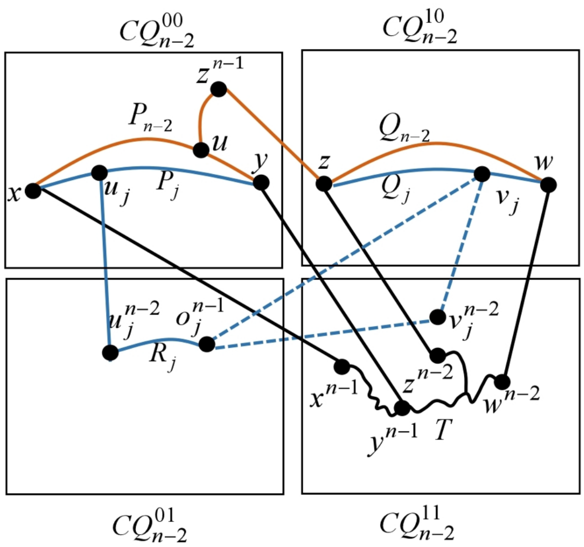

Figure 6. $|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})| = 0$ and $|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})| \geq 1$.

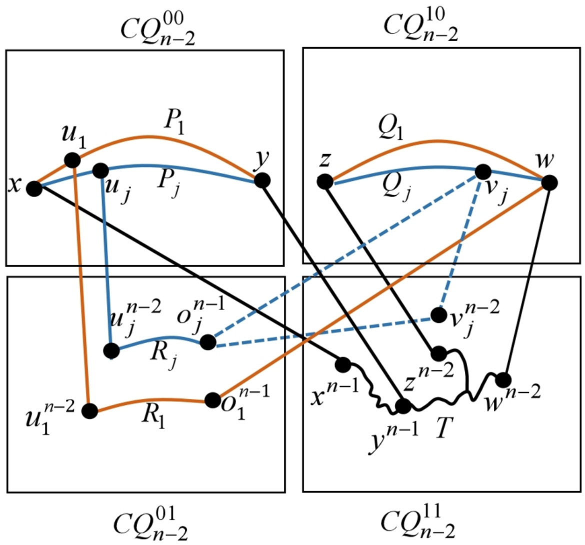

Figure 7. $|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})| = |\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})| = 0$.

Let $Y'' = \{o_j^{n-1} \mid 1 \leq j \leq n-3\}$, by Lemma 7, we can find $n-3$ pairwise disjoint (X, Y'')-paths $R_1, \ldots, R_{n-3}$ in CQ^{01}_{n-2} where $\{u_j^{n-2}, o_j^{n-1}\} \subseteq V(R_j)$.

Let $T'_j = P_j \cup u_j u_j^{n-2} \cup R_j \cup T_j$ $(1 \leq j \leq n-3)$, $T'_{n-2} = P_{n-2} \cup P_{z^{n-1}u} \cup zz^{n-1} \cup Q_{n-2}$, and $T'_{n-1} = xx^{n-1} \cup yy^{n-1} \cup zz^{n-2} \cup ww^{n-2} \cup T$. (See Figure 6.)

Case 5. $\left|\{x^{n-1}, y^{n-1}\} \cap V(CQ^{10}_{n-2})\right| = \left|\{z^{n-1}, w^{n-1}\} \cap V(CQ^{00}_{n-2})\right| = 0$. It is easy to see that $\{x^{n-1}, y^{n-1}\} \in V(CQ^{11}_{n-2})$ and $\{z^{n-1}, w^{n-1}\} \in V(CQ^{01}_{n-2})$. (See Figure 7.) Suppose $X = \{u_j^{n-2} \mid 1 \leq j \leq n-2\}$, $Y = \{o_j \mid 2 \leq j \leq n-2\}$ and $Y' = Y \cap V(CQ^{11}_{n-2})$, then $|Y'| \leq n-3$ and $CQ^{11}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{11}_{n-2}) = n-2$. So there is a tree T in $CQ^{11}_{n-2} \backslash Y'$ connecting $\{x^{n-1}, y^{n-1}, z^{n-2}, w^{n-2}\}$.

Let $Y'' = \{o_j^{n-1} \mid 1 \leq j \leq n-2\}$ where $o_1^{n-1} = w^{n-1}$. By Lemma 7, we can find $n-2$ pairwise disjoint (X, Y'')-paths $R_1, \ldots, R_{n-2}$ in CQ^{01}_{n-2} where $\{u_j^{n-2}, o_j^{n-1}\} \subseteq V(R_j)$.

Let $T'_j = P_j \cup u_j u_j^{n-2} \cup R_j \cup T_j$ $(1 \leq j \leq n-2)$, and $T'_{n-1} = xx^{n-1} \cup yy^{n-1} \cup zz^{n-2} \cup ww^{n-2} \cup T$. (See Figure 7.)

Hence, the lemma holds. ■

Lemma 13. *For any $S \subseteq V(CQ_n)$ and $|S| = 4$ and $n \geq 3$, if $\left|S \cap V(CQ^{00}_{n-2})\right| = 2$, $|S \cap V(CQ^{10}_{n-2})| = 1$ and $\left|S \cap V(CQ^{11}_{n-2})\right| = 1$, then we can find $n-1$ internally disjoint trees in CQ_n to connect S.*

Proof. For any $S = \{x, y, z, w\} \subseteq V(CQ_n)$, suppose $\{x, y\} \subseteq V(CQ^{00}_{n-2})$, $z \in V(CQ^{10}_{n-2})$ and $w \in V(CQ^{11}_{n-2})$.

Combining $\kappa(CQ^{00}_{n-2}) = n-2$ and Lemma 6, we can find an internally disjoint path set $\mathcal{P} = \{P_1, \ldots, P_{n-2}\}$ in CQ^{00}_{n-2} with x and y as ends. Assume $P_1 = xy$ if xy is an edge. And for any P_j $(2 \le j \le n-2)$, we take any vertex except x and y, record it as u_j. If $u_j^{n-1} \in V\big(CQ^{10}_{n-2}\big)$, then let $o_j = u_j$, and $T_j = P_j \cup o_j o_j^{n-1}$; if $u_j^{n-1} \in V\big(CQ^{11}_{n-2}\big)$, by $u_j^{n-2} \in V\big(CQ^{01}_{n-2}\big)$ and Lemma 9, $u_j^{n-2,n-1} \in V\big(CQ^{10}_{n-2}\big)$, let $o_j = u_j^{n-2}$ and $T_j = P_j \cup u_j o_j \cup o_j o_j^{n-1}$.

Case 1. $\big|\{x^{n-1}, y^{n-1}\} \cap \{z, w\}\big| = 0$. One of y^{n-1} and $y^{n-1,n-2}$ must belong to $V\big(CQ^{10}_{n-2}\big)$ and the other to $V\big(CQ^{11}_{n-2}\big)$, suppose $y^{n-1} \in V\big(CQ^{10}_{n-2}\big)$.

By $z \in V\big(CQ^{10}_{n-2}\big)$, $z^{n-2} \in V\big(CQ^{11}_{n-2}\big)$ and Lemma 9, z^{n-1} or $z^{n-2,n-1}$ is in $V\big(CQ^{00}_{n-2}\big)$, let $z^{n-1} \in V\big(CQ^{00}_{n-2}\big)$.

There is a path P in CQ^{00}_{n-2} between z^{n-1} and x since CQ^{00}_{n-2} is connected. Let u be the first common vertex between $\mathcal{P}$ and path P, and here P starts at z^{n-1}, that is, $u \in \mathcal{P}$. Let $u \in P_{n-2}$.

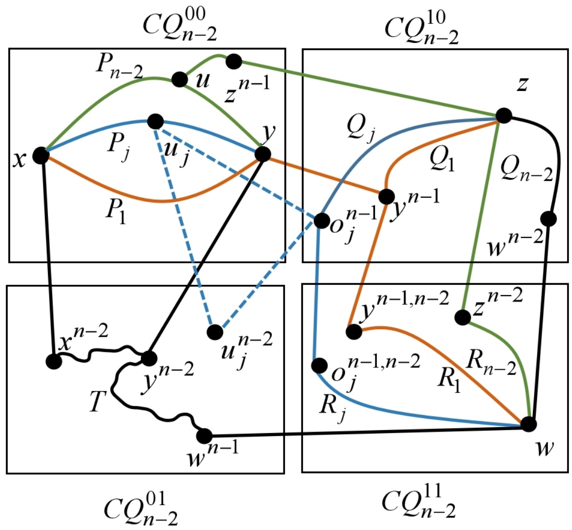

Figure 8. $|\{x^{n-1}, y^{n-1}\} \cap \{z, w\}| = 0$: $zw \notin E(CQ^1_{n-1})$.

Case 1.1. $zw \notin E\big(CQ^1_{n-1}\big)$. Since $w \in V\big(CQ^{11}_{n-2}\big)$, $w^{n-2} \in V\big(CQ^{10}_{n-2}\big)$, by Lemma 9, w^{n-1} or $w^{n-2,n-1}$ is in $V\big(CQ^{01}_{n-2}\big)$, suppose $w^{n-1} \in V\big(CQ^{01}_{n-2}\big)$. (See Figure 8.) Let $Y = \big\{o_j \mid 2 \le j \le n-3\big\}$ and $Y' = Y \cap V\big(CQ^{01}_{n-2}\big)$, then $|Y'| \le n-4$ and $CQ^{01}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{01}_{n-2}) = n-2$. So there is a tree T in $CQ^{01}_{n-2} \backslash Y'$ connecting $\big\{x^{n-2}, y^{n-2}, w^{n-1}\big\}$.

Let $Y'' = \big\{o_j^{n-1} \mid 2 \le j \le n-3\big\} \cup \big\{y^{n-1}, w^{n-2}\big\}$ and $Y''' = \big\{o_j^{n-1,n-2} \mid 2 \le j \le n-3\big\} \cup \big\{y^{n-1,n-2}, z^{n-2}\big\}$ with $|Y''| = |Y'''| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (z, Y'')-paths $Q_1, \ldots, Q_{n-2}$ in CQ^{10}_{n-2}

where $y^{n-1} \in Q_1$, $o_j^{n-1} \in Q_j$ $(2 \le j \le n-3)$, $w^{n-2} \in Q_{n-2}$, and $n-2$ internally disjoint (w, Y''')-paths $R_1, \ldots, R_{n-2}$ in CQ_{n-2}^{11} where $y^{n-1,n-2} \in R_1$, $o_j^{n-1,n-2} \in R_j$ $(2 \le j \le n-3)$, $z^{n-2} \in R_{n-2}$.

Let $T_1' = P_1 \cup yy^{n-1} \cup Q_1 \cup y^{n-1}y^{n-1,n-2} \cup R_1$, $T_j' = T_j \cup Q_j \cup o_j^{n-1}o_j^{n-1,n-2} \cup R_j$ $(2 \le j \le n-3)$, $T_{n-2}' = P_{n-2} \cup P_{z^{n-1}u} \cup zz^{n-1} \cup zz^{n-2} \cup R_{n-2}$ where $P_{z^{n-1}u}$ refers to the part from z^{n-1} to u on P, and $T_{n-1}' = xx^{n-2} \cup yy^{n-2} \cup ww^{n-1} \cup T \cup ww^{n-2} \cup Q_{n-2}$. (See Figure 8.)

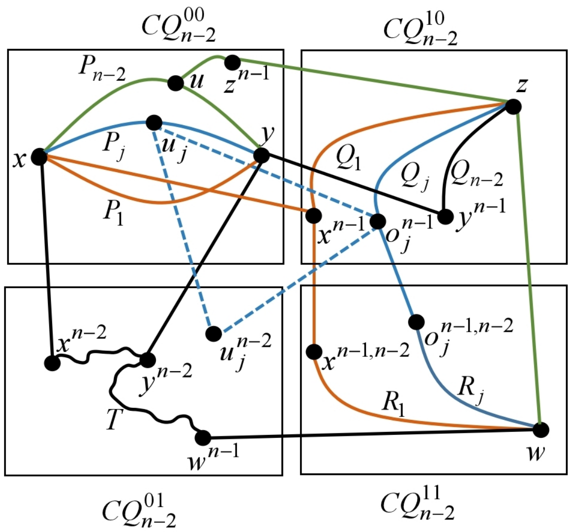

Figure 9. $|\{x^{n-1}, y^{n-1}\} \cap \{z, w\}| = 0$: $zw \in E(CQ_{n-1}^1)$.

Case 1.2. $zw \in E(CQ_{n-1}^1)$. By $z \in V\big(CQ_{n-2}^{10}\big)$, $w = z^{n-2} \in V\big(CQ_{n-2}^{11}\big)$, $z^{n-1} \in V\big(CQ_{n-2}^{00}\big)$ and Lemma 9, $w^{n-1} \in V\big(CQ_{n-2}^{01}\big)$. (See Figure 9.)

Let $Y = \big\{o_j \mid 2 \le j \le n-3\big\}$ and $Y' = Y \cap V\big(CQ_{n-2}^{01}\big)$, then $|Y'| \le n-4$ and $CQ_{n-2}^{01} \backslash Y'$ is connected since $\kappa\big(CQ_{n-2}^{01}\big) = n-2$. So there is a tree T in $CQ_{n-2}^{01} \backslash Y'$ connecting $\big\{x^{n-2}, y^{n-2}, w^{n-1}\big\}$.

Let $Y'' = \big\{o_j^{n-1} \mid 2 \le j \le n-3\big\} \cup \{x^{n-1}, y^{n-1}\}$ and $Y''' = \big\{o_j^{n-1,n-2} \mid 2 \le j \le n-3\big\} \cup \{x^{n-1,n-2}\}$ with $|Y''| = n-2$ and $|Y'''| = n-3$. Since $\kappa(CQ_{n-2}^{10}) = \kappa(CQ_{n-2}^{11}) = n-2$, by Lemma 5, we can find $n-2$ internally disjoint (z, Y'')-paths $Q_1, \ldots, Q_{n-2}$ in CQ_{n-2}^{10} where $x^{n-1} \in Q_1$, $o_j^{n-1} \in Q_j$ $(2 \le j \le n-3)$, $y^{n-1} \in Q_{n-2}$, and $n-3$ internally disjoint (w, Y''')-paths $R_1, \ldots, R_{n-3}$ in CQ_{n-2}^{11} where $x^{n-1,n-2} \in R_1$, $o_j^{n-1,n-2} \in R_j$ $(2 \le j \le n-3)$.

Let $T_1' = P_1 \cup xx^{n-1} \cup Q_1 \cup x^{n-1}x^{n-1,n-2} \cup R_1$, $T_j' = T_j \cup Q_j \cup o_j^{n-1}o_j^{n-1,n-2} \cup R_j$ $(2 \le j \le n-3)$, $T_{n-2}' = P_{n-2} \cup P_{z^{n-1}u} \cup zz^{n-1} \cup zw$ where $P_{z^{n-1}u}$ refers to the part from z^{n-1} to u on P, and $T'_{n-1} = xx^{n-2} \cup yy^{n-2} \cup ww^{n-1} \cup T \cup yy^{n-1} \cup Q_{n-2}$. (See Figure 9.)

Case 2. $|\{x^{n-1}, y^{n-1}\} \cap \{z, w\}| \geq 1$. Suppose $x^{n-1} = z$, then $y^{n-1} = w$ or $y^{n-1} \neq w$.

Case 2.1. $zw \notin E(CQ^1_{n-1})$. By $w \in V\big(CQ^{11}_{n-2}\big)$, $w^{n-2} \in V\big(CQ^{10}_{n-2}\big)$ and Lemma 9, we know w^{n-1} or $w^{n-2,n-1}$ is in $V\big(CQ^{01}_{n-2}\big)$, suppose $w^{n-1} \in V\big(CQ^{01}_{n-2}\big)$. Let $Y = \{o_j \mid 2 \leq j \leq n-2\}$ and $Y' = Y \cap V\big(CQ^{01}_{n-2}\big)$, then $|Y'| \leq n-3$ and $CQ^{01}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{01}_{n-2}) = n-2$. So there is a tree T in $CQ^{01}_{n-2} \backslash Y'$ connecting $\{x^{n-2}, y^{n-2}, w^{n-1}\}$.

Let $Y'' = \big\{o_j^{n-1} \mid 2 \leq j \leq n-2\big\} \cup \{w^{n-2}\}$ and $Y''' = \big\{o_j^{n-1,n-2} \mid 2 \leq j \leq n-2\big\} \cup \{z^{n-2}\}$ with $|Y''| = |Y'''| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (z, Y'')-paths $Q_1, \ldots, Q_{n-2}$ in CQ^{10}_{n-2} where $w^{n-2} \in Q_1$, $o_j^{n-1} \in Q_j$ $(2 \leq j \leq n-2)$, and $n-2$ internally disjoint (w, Y''')-paths $R_1, \ldots, R_{n-2}$ in CQ^{11}_{n-2} where $z^{n-2} \in R_1$, $o_j^{n-1,n-2} \in R_j$ $(2 \leq j \leq n-2)$.

Let $T'_1 = P_1 \cup xz \cup zz^{n-2} \cup R_1$, $T'_j = T_j \cup Q_j \cup o_j^{n-1} o_j^{n-1,n-2} \cup R_j$ $(2 \leq j \leq n-2)$, and $T'_{n-1} = xx^{n-2} \cup yy^{n-2} \cup ww^{n-1} \cup T \cup ww^{n-2} \cup Q_1$. (See Figure 10.)

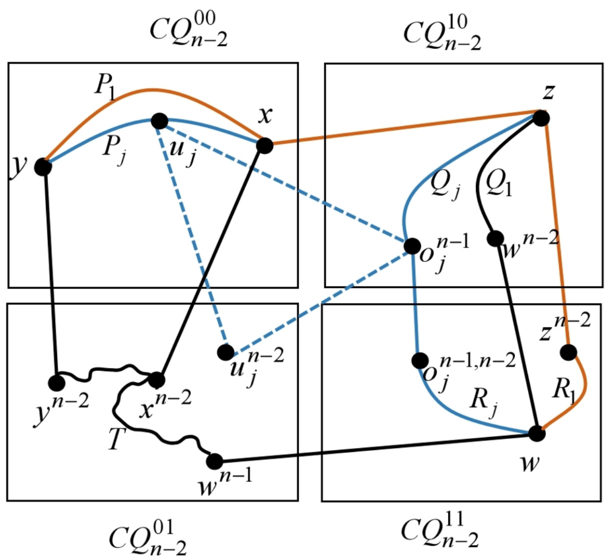

Figure 10. $x^{n-1} = z$: $zw \notin E(CQ^1_{n-1})$.

Case 2.2. $zw \in E\big(CQ^1_{n-1}\big)$. By $z \in V\big(CQ^{10}_{n-2}\big)$, $z^{n-2} = w \in V\big(CQ^{11}_{n-2}\big)$, $x = z^{n-1} \in V\big(CQ^{00}_{n-2}\big)$ and Lemma 9, then $w^{n-1} = z^{n-2,n-1} \in V\big(CQ^{01}_{n-2}\big)$ and $y^{n-1} \neq w$. One of y^{n-1} and $y^{n-1,n-2}$ must belong to $V\big(CQ^{10}_{n-2}\big)$ and the other to $V\big(CQ^{11}_{n-2}\big)$, suppose $y^{n-1} \in V\big(CQ^{10}_{n-2}\big)$. Let $Y = \{o_j \mid 2 \leq j \leq n-2\}$

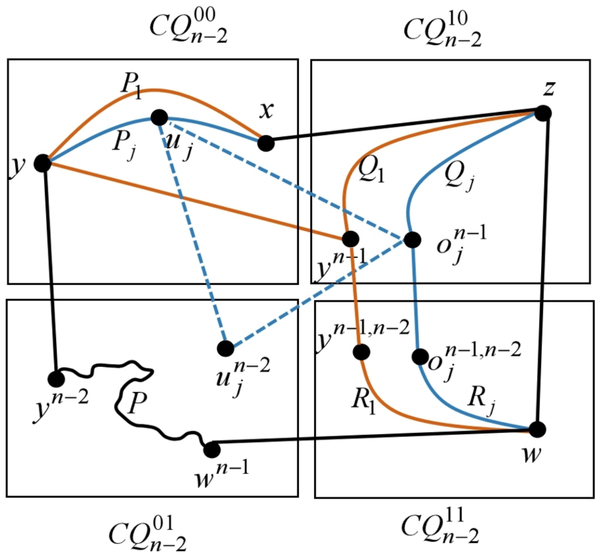

Figure 11. $x^{n-1} = z$: $zw \in E(CQ^1_{n-1})$.

and $Y' = Y \cap V(CQ^{01}_{n-2})$, then $|Y'| \leq n-3$ and $CQ^{01}_{n-2} \backslash Y'$ is connected since $\kappa(CQ^{01}_{n-2}) = n-2$. So there is a path P in $CQ^{01}_{n-2} \backslash Y'$ connecting $\{y^{n-2}, w^{n-1}\}$.

Let $Y'' = \{o_j^{n-1} \mid 2 \leq j \leq n-2\} \cup \{y^{n-1}\}$ and $Y''' = \{o_j^{n-1,n-2} \mid 2 \leq j \leq n-2\} \cup \{y^{n-1,n-2}\}$ with $|Y''| = |Y'''| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (z, Y'')-paths $Q_1, \ldots, Q_{n-2}$ in CQ^{10}_{n-2} where $y^{n-1} \in Q_1$, $o_j^{n-1} \in Q_j$ $(2 \leq j \leq n-2)$, and $n-2$ internally disjoint (w, Y''')-paths $R_1, \ldots, R_{n-2}$ in CQ^{11}_{n-2} where $y^{n-1,n-2} \in R_1$, $o_j^{n-1,n-2} \in R_j$ $(2 \leq j \leq n-2)$.

Let $T'_1 = P_1 \cup yy^{n-1} \cup Q_1 \cup y^{n-1}y^{n-1,n-2} \cup R_1$, $T'_j = T_j \cup Q_j \cup o_j^{n-1}o_j^{n-1,n-2} \cup R_j$ $(2 \leq j \leq n-2)$, $T'_{n-1} = yy^{n-2} \cup P \cup ww^{n-1} \cup zw \cup xz$. (See Figure 11.)

Hence, the lemma holds. ■

Lemma 14. *For any $S \subseteq V(CQ_n)$ with $|S| = 4$ and $n \geq 3$, if $|S \cap V(CQ^{00}_{n-2})| = |S \cap V(CQ^{01}_{n-2})| = |S \cap V(CQ^{10}_{n-2})| = |S \cap V(CQ^{11}_{n-2})| = 1$, then we can find $n-1$ internally disjoint trees in CQ_n to connect S.*

Proof. For any $S = \{x, y, z, w\} \subseteq V(CQ_n)$, suppose $x \in V(CQ^{00}_{n-2})$, $y \in V(CQ^{01}_{n-2})$, $z \in V(CQ^{10}_{n-2})$ and $w \in V(CQ^{11}_{n-2})$.

By $\kappa(CQ^0_{n-1}) = n-1$ and Lemma 6, we can find an internally disjoint path set $\mathcal{P} = \{P_1, \ldots, P_{n-1}\}$ in CQ^0_{n-1} with x and y as ends (if x is adjacent to y, then let $P_1 = xy$). And for any P_j, there is an edge $u_jv_j \in E(P_j)$ such that $u_j \in V(CQ^{00}_{n-2})$ and $v_j \in V(CQ^{01}_{n-2})$ since $x \in V(CQ^{00}_{n-2})$, $y \in V(CQ^{01}_{n-2})$, where $1 \leq j \leq n-1$. (If $P_1 = xy$, let $u_1 = x$ and $v_1 = y$.) By Lemma

9, $u_j^{n-1} \in V\big(CQ_{n-2}^{10}\big)$ or $v_j^{n-1} \in V\big(CQ_{n-2}^{10}\big)$. Without loss of generality, let $u_j^{n-1} \in V\big(CQ_{n-2}^{10}\big)$ and $T_j = P_j \cup u_j u_j^{n-1}$ where $1 \leq j \leq n-1$.

Case 1. $z^{n-1} \notin \{x, y\}$. There is a path P in CQ_{n-1}^{0} between z^{n-1} and x since CQ_{n-1}^{0} is connected. Let u be the first common vertex between $\mathcal{P}$ and path P, and here P starts at z^{n-1}, that is, $u \in \mathcal{P}$, suppose $u \in P_{n-1}$. Let $Y = \{u_j^{n-1} \mid 1 \leq j \leq n-2\}$ with $|Y| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (z, Y)-paths $Q_1, \dots, Q_{n-2}$ in CQ_{n-2}^{10} where $u_j^{n-1} \in Q_j$ $(1 \leq j \leq n-2)$. Two subcases will be discussed.

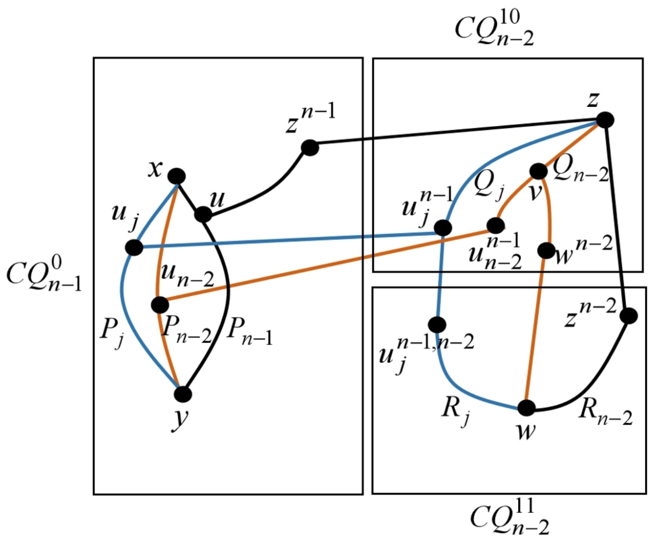

Figure 12. $z^{n-1} \notin \{x, y\}$: $w^{n-2} \neq z$.

Case 1.1. $w^{n-2} \neq z$. There is a path Q in CQ_{n-2}^{10} between w^{n-2} and z since CQ_{n-2}^{10} is connected. Let v be the first common vertex between $Q_1 \cup \dots \cup Q_{n-2}$ and path Q, and here Q starts at w^{n-2}, that is, $v \in Q_1 \cup \dots \cup Q_{n-2}$, suppose $v \in Q_{n-2}$. Let $Y' = \big\{u_j^{n-1,n-2} \mid 1 \leq j \leq n-3\big\} \cup \{z^{n-2}\}$ with $|Y'| = n-2$, by Lemma 5, we can find $n-2$ internally disjoint (w, Y')-paths $R_1, \dots, R_{n-2}$ in CQ_{n-2}^{11} where $u_j^{n-1,n-2} \in R_j$ $(1 \leq j \leq n-3)$ and $z^{n-2} \in R_{n-2}$.

Let $T_j' = T_j \cup Q_j \cup u_j^{n-1} u_j^{n-1,n-2} \cup R_j$ $(1 \leq j \leq n-3)$, $T_{n-2}' = T_{n-2} \cup Q_{n-2} \cup Q_{w^{n-2}v} \cup ww^{n-2}$, and $T_{n-1}' = P_{n-1} \cup P_{z^{n-1}u} \cup zz^{n-1} \cup zz^{n-2} \cup R_{n-2}$ where $Q_{w^{n-2}v}$ refers to the part from w^{n-2} to v on Q, $P_{z^{n-1}u}$ refers to the part from z^{n-1} to u on P. (See Figure 12.)

Case 1.2. $w^{n-2} = z$. Let $Y' = \{u_j^{n-1,n-2} \mid 1 \leq j \leq n-2\}$ with $|Y'| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (w, Y')-paths $R_1, \dots, R_{n-2}$ in CQ_{n-2}^{11} where $u_j^{n-1,n-2} \in R_j$ $(1 \leq j \leq n-2)$.

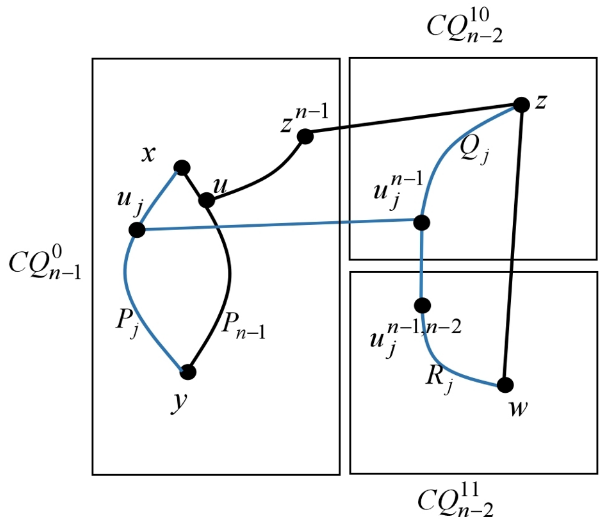

Figure 13. $z^{n-1} \notin \{x, y\}$: $w^{n-2} = z$.

Let $T'_j = T_j \cup Q_j \cup u_j^{n-1}u_j^{n-1,n-2} \cup R_j$ $(1 \leq j \leq n-2)$, and $T'_{n-1} = P_{n-1} \cup P_{z^{n-1}u} \cup zz^{n-1} \cup zw$. (See Figure 13.)

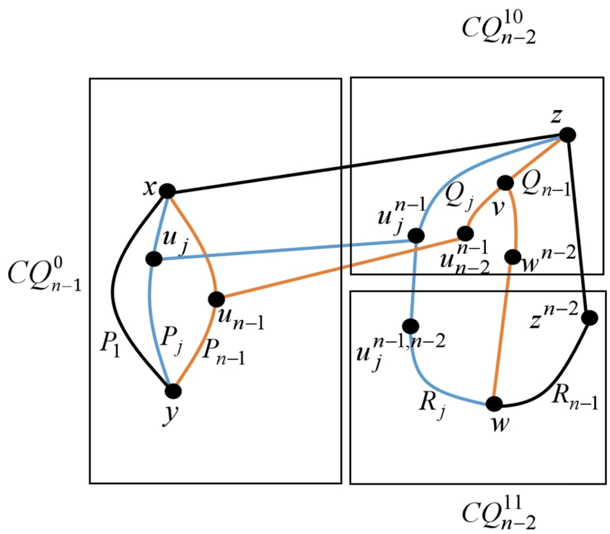

Figure 14. $z^{n-1} \in \{x, y\}$: $w^{n-2} \neq z$.

Case 2. $z^{n-1} \in \{x, y\}$. Let $x = z^{n-1}$ and $Y = \{u_j^{n-1} \mid 2 \leq j \leq n-1\}$ with $|Y| = n-2$. By Lemma 5, we can find $n-2$ internally disjoint (z, Y)-paths $Q_2, \ldots, Q_{n-1}$ in CQ^{10}_{n-2} where $u_j^{n-1} \in Q_j$ $(2 \leq j \leq n-1)$.

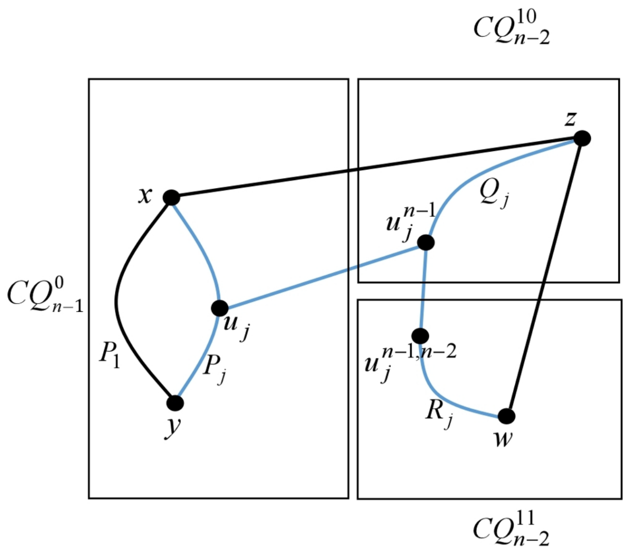

Figure 15. $z^{n-1} \in \{x, y\}$: $w^{n-2} = z$.

Case 2.1. $w^{n-2} \neq z$. There is a path Q in CQ^{10}_{n-2} between w^{n-2} and z since CQ^{10}_{n-2} is connected. Let v be the first common vertex between $Q_2 \cup \cdots \cup Q_{n-1}$ and path Q, and here Q starts at w^{n-2}, that is, $v \in Q_2 \cup \cdots \cup Q_{n-1}$, suppose $v \in Q_{n-1}$. Let $Y' = \{u_j^{n-1,n-2} \mid 2 \leq j \leq n-2\} \cup \{z^{n-2}\}$ with $|Y'| = n-2$, by Lemma 5, we can find $n-2$ internally disjoint (w, Y')-paths $R_2, \ldots, R_{n-1}$ in CQ^{11}_{n-2} where $u_j^{n-1,n-2} \in R_j$ $(2 \leq j \leq n-2)$ and $z^{n-2} \in R_{n-1}$.

Let $T_1' = P_1 \cup xz \cup zz^{n-2} \cup R_{n-1}$, $T_j' = T_j \cup Q_j \cup u_j^{n-1}u_j^{n-1,n-2} \cup R_j$ $(2 \leq j \leq n-2)$, and $T_{n-1}' = T_{n-1} \cup Q_{n-1} \cup Q_{w^{n-2}v} \cup ww^{n-2}$ where $Q_{w^{n-2}v}$ refers to the part from w^{n-2} to v on Q. (See Figure 14.)

Case 2.2. $w^{n-2} = z$. Let $Y' = \{u_j^{n-1,n-2} \mid 2 \leq j \leq n-1\}$ with $|Y'| = n-2$, by Lemma 5, we can find $n-2$ internally disjoint (w, Y')-paths $R_2, \ldots, R_{n-1}$ in CQ^{11}_{n-2} where $u_j^{n-1,n-2} \in R_j$ $(2 \leq j \leq n-1)$.

Let $T_1' = P_1 \cup xz \cup zw$, $T_j' = T_j \cup Q_j \cup u_j^{n-1}u_j^{n-1,n-2} \cup R_j$ where $2 \leq j \leq n-1$. (See Figure 15.)

Hence, the lemma holds. ■

Theorem 15. $\kappa_4(CQ_n) = n-1$ *for* $n \geq 2$.

Proof. By Lemma 4 and the fact that CQ_n $(n \geq 2)$ is n-regular, we have $\kappa_4(CQ_n) \leq \delta(CQ_n) - 1 = n-1$. Next, we only need to prove $\kappa_4(CQ_n) \geq n-1$ for $n \geq 2$. That is, for any $S \subset V(CQ_n)$ with $|S| = 4$, we can find $n-1$ internally disjoint trees in CQ_n to connect S. Let $S = \{x, y, z, w\}$ and $S \cap V(CQ^i_{n-1}) = S^i$ $(i \in \{0, 1\})$. For $n = 2$, by CQ_2 is 2-connected with only four vertices, it is easy to know that $\kappa_4(CQ_2) \geq 1$, then $\kappa_4(CQ_2) = 1$. For $n \geq 3$, three cases will be considered.

Case 1. $|S^0| = 4$ or $|S^1| = 4$. Without loss of generality, let $|S^0| = 4$, then $\{x^{n-1}, y^{n-1}, z^{n-1}, w^{n-1}\} \subset V(CQ^1_{n-1})$. We will prove it by induction hypothesis on n. The conclusion holds for $n = 2$, suppose it also holds while $3 \leq l \leq n-1$. By $CQ^0_{n-1} \cong CQ_{n-1}$ and induction hypothesis, $\kappa_4(CQ_{n-1}) = n-2$, that is, we can find $n-2$ internally disjoint trees $T_1, \ldots, T_{n-2}$ in CQ^0_{n-1} connecting S. There is a tree T in CQ^1_{n-1} connecting $\{x^{n-1}, y^{n-1}, z^{n-1}, w^{n-1}\}$ since CQ^1_{n-1} is connected. Let $T_{n-1} = xx^{n-1} \cup yy^{n-1} \cup zz^{n-1} \cup ww^{n-1} \cup T$, where $V(T_{n-1}) \cap V(T_j) = \{x, y, z, w\}$ and $E(T_{n-1}) \cap E(T_j) = \emptyset$ for any $1 \leq j \leq n-2$.

Case 2. $|S^i| = 3$ and $|S^{1-i}| = 1$, where $i = 0$ or 1. By Lemma 11, the theorem is true.

Case 3. $|S^0| = |S^1| = 2$. By $CQ^{00}_{n-2} \cong CQ^{01}_{n-2} \cong CQ^{10}_{n-2} \cong CQ^{11}_{n-2} \cong CQ_{n-2}$, we only need to consider three subcases.

Case 3.1. $|S^0 \cap V(CQ^{00}_{n-2})| = |S^1 \cap V(CQ^{10}_{n-2})| = 2$. By Lemma 12, the theorem is true.

Case 3.2. $|S^0 \cap V(CQ^{00}_{n-2})| = 2$, $|S^1 \cap V(CQ^{10}_{n-2})| = |S^1 \cap V(CQ^{11}_{n-2})| = 1$. By Lemma 13, the theorem is true.

Case 3.3. $|S^0 \cap V(CQ^{00}_{n-2})| = |S^0 \cap V(CQ^{01}_{n-2})| = |S^1 \cap V(CQ^{10}_{n-2})| = |S^1 \cap V(CQ^{11}_{n-2})| = 1$. By Lemma 14, the theorem is true.

Hence, the theorem holds. ■

By Lemma 8 and Theorem 15, there is the following theorem.

Theorem 16. $\kappa_3(CQ_n) = n-1$ *for* $n \geq 2$.

5. Conclusion

In this paper, we discuss the generalized 3-connectivity and the generalized 4-connectivity of n-dimensional crossed cube CQ_n and obtain $\kappa_3(CQ_n) = \kappa_4(CQ_n) = n-1$ for $n \geq 2$. This provides a new reference for measuring the fault tolerance of CQ_n.

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